REMARKS ON THE RELATIVISTIC HARTREE EQUATIONS

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ABSTRACT. We study the global well-posedness (GWP) and small data scattering of radial solutions of the relativistic Hartree type equations with nonlocal nonlinearity $F(u) = \lambda(|\cdot|^{-\gamma} * |u|^2)u$, $\lambda \in \mathbb{R} \setminus \{0\}$, $0 < \gamma < n$, $n \geq 3$. We establish a weighted L^2 Strichartz estimate applicable to non-radial functions and some fractional integral estimates for radial functions.

1. INTRODUCTION

In this paper, we consider the Cauchy problems concerning the relativistic Hartree equations:

$$\begin{cases} i\partial_t u = \sqrt{1-\Delta}u + F(u) \ \text{ in } \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 3, \\ u(0) = \varphi, \end{cases} \tag{1}$$

$$\begin{cases} \partial_t^2 u + (1-\Delta)u = F(u) \ \text{ in } \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 3, \\ u(0) = \varphi_1, \quad \partial_t u(0) = \varphi_2. \end{cases} \tag{2}$$

The nonlinear part $F(u)$ is of Hartree type such that $F(u) = V_\gamma(u)u$, where

$$V_\gamma(u)(x) = \lambda(|\cdot|^{-\gamma} * |u|^2)(x) = \lambda \int_{\mathbb{R}^n} \frac{|u(y)|^2}{|x-y|^\gamma}\, dy.$$

Here λ is a non-zero real number and γ is a positive number less than the space dimension n.

The first equation (1) is called the semi-relativistic equation which describes the Boson stars [6, 7, 13] and the second one (2) is the well-known Klein-Gordon equation whose physical model is originated from the helium atom [10, 14, 17]. For the simplicity of presentation, the mass, speed of light and Planck constant of both equations have been normalized.

The equations (1) and (2) can be rewritten in the form of the integral equations

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \tag{3}$$

$$u(t) = (\cos t\omega)\varphi_1 + \omega^{-1}(\sin t\omega)\varphi_2 - \int_0^t \omega^{-1}(\sin(t-t')\omega)F(u)\, dt', \tag{4}$$

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where $\omega = \sqrt{1-\Delta}$ and the associated unitary group $U(t)$ is realized by the Fourier transform as

$$U(t)\varphi = (e^{-it\omega}\varphi)(x) \equiv \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-it\sqrt{1+|\xi|^2}}\widehat{\varphi}(\xi)\,d\xi,$$

where $\widehat{g}$ denotes the Fourier transform of g defined by $\widehat{g}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}g(x)\,dx$. The operators $\cos t\omega$ and $\sin t\omega$ are defined by replacing $e^{-it\sqrt{1+|\xi|^2}}$ with

$$\cos(t\sqrt{1+|\xi|^2}) \quad \text{and} \quad \sin(t\sqrt{1+|\xi|^2}),$$

respectively.

If the solution u of (1) or (3) has a decay at infinity and smoothness, it satisfies two conservation laws:

$$\begin{aligned} &\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \\ &E_1(u) \equiv K_1(u) + V(u) = E_1(\varphi), \\ &K(u) = \frac{1}{2}\langle\sqrt{1-\Delta}\,u, u\rangle, V(u) = \frac{1}{4}\langle F(u), u\rangle, \end{aligned} \tag{5}$$

where $\langle , \rangle$ is the complex inner product in L^2. Also the solution of (2) or (4) satisfies the conservation law:

$$\begin{aligned} &E_2(u, \partial_t u) \equiv K_2(u, \partial_t u) + V(u) = E_2(\varphi_1, \varphi_2), \\ &K_2(u, \partial_t u) = \frac{1}{2}(\langle\partial_t u, \partial_t u\rangle + \langle\sqrt{1-\Delta}u, \sqrt{1-\Delta}u\rangle). \end{aligned} \tag{6}$$

The main concern of this paper is to establish the global well-posedness and scattering of radial solutions of the equations (1) and (2).

The study of the global well-posedness (GWP) and scattering for the semi-relativistic equation (1) has not been long before. In [15], GWP was considered with a three dimensional Coulomb type potential which corresponds to $\gamma = 1$. In [4], the first and second authors of the present paper showed GWP for $0 < \gamma \le 1$ if $n \ge 2$ and $0 < \gamma < 1$ if $n = 1$, for $0 < \gamma < \frac{2n}{n+1}$ if $n \ge 2$, and small data scattering for $\gamma > 2$ if $n \ge 3$. In this paper we tried to fill the gap $1 < \gamma \le 2$ for GWP under the assumption of radial symmetry. For further study like blowup of solutions, solitary waves, mean field limit problem for semi-relativistic equation, see the references [13, 6, 7, 8, 9].

The first result is on the GWP for radial solutions of (3).

Theorem 1. *Let $1 < \gamma < \frac{3}{2}$ for $n = 3$ and $1 < \gamma < 2$ for $n \ge 4$. Let $\varphi \in H^{\frac{1}{2}}$ be radially symmetric and assume that $\|\varphi\|_{L^2}$ is sufficiently small if $\lambda < 0$. Then there exists a unique radial solution $u \in C_bH^{\frac{1}{2}}$ such that $|x|^{-1}u \in L^2_{loc}L^2$ of (3) satisfying the energy and L^2 conservations (5).*

Here $C_b = C \cap L^\infty$, $H^s_r = (1-\Delta)^{-s/2}L^r$ and $\dot{H}^s_r = (-\Delta)^{-s/2}L^r$ are the usual and homogeneous Sobolev spaces, respectively. We mean H^s_2 by H^s and $\dot{H}^s_2$ by $\dot{H}^s$. Hereafter, the space $L^q_T(B)$ denotes $L^q(-T, T; B)$ for $T > 0$ and $\|\cdot\|_{L^q_T B}$ its norm for some Banach space B. If $T = \infty$, we use $L^q(B)$ for $L^q(\mathbb{R}; B)$ with norm $\|\cdot\|_{L^q B}$, $1 \le q \le \infty$. We also denote $v \in L^q_T(B)$ for all $T < \infty$ by $v \in L^q_{loc}(B)$.

The next result is on the small data scattering of radial solutions of (3) for $n \ge 4$.

Theorem 2. *Let $\gamma = 2$ and $n \geq 4$. Let $\varphi \in H^1$ be radially symmetric. If $\|\varphi\|_{H^1}$ is sufficiently small, then there exists a unique radial solution $u \in C_bH^1$ such that $|x|^{-1}u \in L^2L^2$ to* (3)*. Moreover, there exist radial functions φ^+ and φ^- such that*

$$\|u(t) - U(t)\varphi^{\pm}\|_{H^1} \to 0 \quad \text{as} \quad t \to \pm\infty.$$

In [4], the authors used the L^qL^r type Strichartz estimates of the Klein-Gordon equation to prove GWP and scattering for the equation (1). Contrary to the case of Klein-Gordon equation, semi-relativistic equation preserves a regularity in a contraction argument based on the Strichartz estimate, from which the gap ($\frac{2n}{n+1} \leq \gamma \leq 2$) arises naturally in the range of γ for GWP. To tide over this difficulty, we assume the radial symmetry for data and solutions, which enables us to estimate fractional integrals associated with the nonlinearity $V_\gamma(u)u$. Then we establish an L^2 Strichartz estimate for $n \geq 2$ with weight $|x|^{-a}$ which is useful to treat radial functions but also applicable to non-radial functions (a gain of angular regularity is achieved in the non-radial case). The one dimensional analog is attainable. See Remark 4 below.

For GWP we use a fractional integral estimate on the unit sphere such that

$$\int_{S^{n-1}} |re_1 - \rho\sigma|^{-\theta}\, d\sigma \leq M(r, \rho) < \infty,$$

where $e_1 = (1, 0, \cdots, 0)$. The result of Theorem 1 corresponds to the case $\theta = \gamma + \frac{1}{2}$. If $n = 3$, then the finiteness of integralenforces γ to be less than $\frac{3}{2}$ as in Theorem 1, since the integral is finite only when $n - 2 - \theta > -1$. For details see Lemma 2 and Lemma 3. In Theorem 2, we treated the case $\theta = 2$ for which the integral is not finite if $n = 3$. However, the three dimensional GWP can be slightly improved up to $\frac{5}{3}$ by using another Strichartz estimate on a hybrid Sobolev space (for this see [5]). It will be worthy of trying to fill the gap $\frac{5}{3} \leq \gamma \leq 2$ for $n = 3$.

The Klein-Gordon equation (2) was initially studied by [26] (see also [18]). In [21], the GWP is considered for $\lambda < 0$ and $0 < \gamma \leq 4$. It was proved in [26, 20, 23] that the scattering operator for (2) is well-defined on some domain if $n \geq 2$, $4/3 < \gamma \leq 4n/(n+1)$ and $\gamma < n$. Furthermore, [19] showed that if $n \geq 3$, $2 \leq \gamma \leq 4$ and $\gamma < n$, then the scattering operator can be defined on some neighborhood near zero in the energy space.

In this paper the small data scattering of radial solutions is successfully treated below energy space, provided $\frac{3}{2} < \gamma < 2$. To state precisely, let us define a weighted spaces $W_{s,\varepsilon}$ and a data space $D_{\alpha,\beta}$ by

$$W_{s,\varepsilon} = \{\psi \in L^2 : \|\psi\|_{W_{s,\varepsilon}}^2 \equiv \||\cdot|^{-s-\varepsilon}\psi\|_{L^2(|x|\leq 1)}^2 + \||\cdot|^{-s+\varepsilon}\psi\|_{L^2(|x|>1)}^2 < \infty\}$$

$$\text{and} \quad D_{\alpha,\beta} = H^{\alpha - \frac{1}{2}} \cap L^{\frac{2n}{n+2-2\beta}},$$

respectively, where $\varepsilon > 0$ is sufficiently small.

Theorem 3. *Let $\frac{3}{2} < \gamma < 2$ for $n = 3$ and $\frac{3}{2} < \gamma \leq 2$ for $n \geq 4$. Then there is a real number s and ε such that*

$$\frac{1}{2} < s < \frac{\gamma}{2}, \quad 0 < \varepsilon < \min\left(\frac{\gamma}{2} - s, s - \frac{1}{2}\right), \quad 1 + s - \varepsilon < \gamma < 1 + s + \varepsilon. \tag{7}$$

For fixed such s and ε, let $(\varphi_1, \varphi_2) \in D_{s+\varepsilon,s+\varepsilon} \times D_{s+\varepsilon-1,s+\varepsilon}$ be radially symmetric data. Then if $\|\varphi_1\|_{D_{s+\varepsilon,s+\varepsilon}} + \|\varphi_2\|_{D_{s+\varepsilon-1,s+\varepsilon}}$ is sufficiently small, then there exists a unique radial solution $u \in C_b H^{s-\frac12+\varepsilon} \cap L^2 W_{s,\varepsilon}$ to (4). Moreover, there exist radial functions $\varphi_1^{\pm} \in H^{s-\frac12+\varepsilon}$ and $\varphi_2^{\pm} \in H^{s-\frac32+\varepsilon}$ such that

$$\|u(t) - u^{\pm}(t)\|_{H^{s-\frac12+\varepsilon}} \to 0 \quad \text{as} \quad t \to \pm\infty,$$

where $u^{\pm}$ is the solutions to the Cauchy problem

$$\begin{cases} \partial_t^2 u^{\pm} + (1-\Delta)u^{\pm} = 0, \\ u^{\pm}(0) = \varphi_1^{\pm}, \partial_t u^{\pm}(0) = \varphi_2^{\pm}. \end{cases} \tag{8}$$

In the definition of initial data space $D_{\alpha,\beta}$ the space $L^{\frac{2n}{n+2-2\beta}}$ can be slightly weakened by the homogeneous Sobolev space $\dot H^{-(1-\beta)}$. In fact, $L^{\frac{2n}{n+2-2\beta}} \hookrightarrow \dot H^{-(1-\beta)}$ for $0 < \beta < 1$. See the proof of Theorem 3 below. Let $\widetilde D_{\alpha,\beta}$ be the weakened space $H^{\alpha-\frac12} \cap \dot H^{-(1-\beta)}$. Then one can easily show that the solution $u \in C_b(\mathbb{R}; \dot H^{-(1-(s-\varepsilon))})$ and then the existence of scattering operator of (2) on a small neighborhood of the origin in $\widetilde D_{s+\varepsilon,s-\varepsilon} \times \widetilde D_{s+\varepsilon-1,s-\varepsilon}$. For details see Remark 6 below.

The lower bound $\frac32$ of γ is caused by the condition (7) which follows from the relation between the weight $|x|^{-a}$ and the L^2 estimate of Bessel function such that

$$\int_0^\infty r^{1-2a} |J_{\frac{n-2}{2}}(r)|^2\, dr < \infty.$$

For the finiteness, the assumption $\frac12 < a < \frac n2$ is inevitable because $J_{\frac{n-2}{2}}(r) = O(r^{\frac{n-2}{2}})$ as $r \to 0$ and $J_{\frac{n-2}{2}}(r) = O(r^{-\frac12})$ as $r \to \infty$. For more explicit formula, see the identity (13) below. Hence for the present it seems hard to improve the range of γ for the small data scattering. From the perspective of negative result for the scattering[1], it will be very interesting to show the scattering up to the value of γ greater than 1.

This paper is organized as follows. In Section 2 we introduce a weighted Strichartz estimate for $n \ge 2$. In Section 3 some fractional integral estimates are considered under radial symmetry. All the proofs of theorems are shown in Section 4.

If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \le CB$ and $A \ge C^{-1}B$, respectively. Different positive constants possibly depending on n, λ, γ and a might be denoted by the same letter C. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

2. Weighted L^2 Strichartz estimate

In this section, we show the following weighted L^2 Strichartz estimate.

[1]The non-existence of the asymptotically free solutions occurs when $\gamma \le 1$. For instance see the last section of [4].

Proposition 1. *Let $\frac{1}{2} < a < \frac{n}{2}$ and $n \ge 2$. Then for any $\varphi \in H^s$ and $F \in L^1_T H^s$, $s \ge 0$, we have*

$$\|U(\cdot)\varphi\|_{L^2_T(\widetilde{H}^s_a H^{a-\frac{1}{2}}_\sigma)} \lesssim \|\varphi\|_{H^s},$$

$$\Big\|\int_0^\bullet U(\cdot - t')F(t')\,dt'\Big\|_{L^2_T(\widetilde{H}^s_a H^{a-\frac{1}{2}}_\sigma)} \lesssim \|F\|_{L^1_T H^s}. \tag{9}$$

The constants in the estimates can be chosen independently of T.

Here we denote the weighted Sobolev space $\widetilde{H}^s_a$ by

$$\widetilde{H}^s_a = \{v : \|v\|_{\widetilde{H}^s_a} \equiv \||\cdot|^{-a} L_a(\Delta)(1-\Delta)^{\frac{s}{2}} v\|_{L^2} < \infty\},$$

where a is a positive real number and $L_a(\Delta) = (-\Delta)^{\frac{1}{2}(1-a)}(1-\Delta)^{-\frac{1}{4}}$. The Sobolev space $H^\alpha_\sigma = (1-\Delta_\sigma)^{-\frac{\alpha}{2}} L^2(S^{n-1})$ is defined on the unit sphere S^{n-1}, where Δ_σ is the Laplace-Beltrami operator on the unit sphere S^{n-1} (see [11, 16] for instance). The mixed norm $\widetilde{H}^s_a H^\alpha_\sigma$ is defined as follows.

$$\|v\|^2_{\widetilde{H}^s_a H^\alpha_\sigma} = \int_{\mathbb{R}^n} |x|^{-2a} \left|L_a(\Delta)(1-\Delta)^{\frac{s}{2}}(1-\Delta_\sigma)^{\frac{\alpha}{2}} v\right|^2 dx.$$

Remark 1. If φ and F are radially symmetric, then the angular regularity H^α_σ is not necessary.

Remark 2. If we use Theorem 3.4 of [4] for small data GWP, then from the Strichartz estimates above, we readily observe that if $2 < \gamma < n, n \ge 3$, $s > \frac{\gamma}{2} - \frac{n-2}{2}$, and $\|\varphi\|_{H^s}$ is sufficiently small, then for $\frac{1}{2} < a < \frac{n}{2}$ the solution u of (1) is in $L^2(\widetilde{H}^s_a H^{a-\frac{1}{2}}_\sigma)$. In fact, in view of the proof of Theorem 3.4 of [4] we have $\|F(u)\|_{L^1 H^s} \lesssim \|\varphi\|_{H^s}$ and hence

$$\|u\|_{L^2(\widetilde{H}^s_a H^{a-\frac{1}{2}}_\sigma)} \lesssim \|\varphi\|_{H^s} + \|F(u)\|_{L^1 H^s} \lesssim \|\varphi\|_{H^s}.$$

Similarly, we have by using Lemma 2.4 of [23] that if $\frac{4}{3} < \gamma < 2$, $n \ge 2$ and $\|\varphi_1\|_{H^s} + \|\varphi_2\|_{H^{s-1}}$ is sufficiently small for $s \ge 1$, then the solution u of (2) is in $L^2(\widetilde{H}^s_a H^{a-\frac{1}{2}}_\sigma)$ for any $\frac{1}{2} < a < \frac{n}{2}$.

Proof of Proposition 1. Without loss of generality we may assume $s = 0$. Let us first define an operator $W_\nu(t)$ by

$$(W_\nu(t)f)(r) = r^{-a}\int_0^\infty e^{-it\sqrt{1+\rho^2}}(r\rho)^{\frac{1}{2}} J_\nu(r\rho)\frac{\rho^{1-a}}{(1+\rho^2)^{\frac{1}{4}}} f(\rho)\,d\rho, \tag{10}$$

where ν is a real number greater than equal to $\frac{n-2}{2}$ and J_ν is the Bessel function of order ν. We claim that for any $\nu \ge \frac{n-2}{2}$ and $\frac{1}{2} < a < \frac{n}{2}$

$$\|W_\nu(\cdot)f\|_{L^2(\mathbb{R}_+\times(-T,T))} \le \|W_\nu(\cdot)f\|_{L^2(\mathbb{R}^2_+)} \lesssim (1+\nu)^{-(a-\frac{1}{2})}\|f\|_{L^2}. \tag{11}$$

In fact, using the change of variables $\sqrt{1+\rho^2} \mapsto \rho$, we have

$$(W_\nu(t)f)(r)$$
$$= r^{\frac12 - a} \int_{-\infty}^{\infty} e^{-it\rho} \chi_{(1,\infty)}(\rho) J_\nu(r\sqrt{\rho^2-1})\sqrt{\rho}(\rho^2-1)^{\frac14 - \frac{a}{2}} f(\sqrt{\rho^2-1})\, d\rho,$$

where $\chi_{(1,\infty)}$ is the characteristic function on the interval $(1,\infty)$. Now from the Plancherel theorem with respect to the time variable and the change of variables $\sqrt{\rho^2-1} \mapsto \rho$, it follows that

$$\begin{aligned} \|W_\nu(\cdot)f\|^2_{L^2(\mathbb{R}^2_+)} &= 2\pi \int_0^\infty \rho^{2-2a}|f(\rho)|^2 \int_0^\infty r^{1-2a}|J_\nu(r\rho)|^2\, dr\, d\rho \\ &= 2\pi \left(\int_0^\infty r^{1-2a}|J_\nu(r)|^2\, dr \right) \|f\|^2_{L^2}. \end{aligned} \tag{12}$$

For the estimate of inner integral, we use the known formula on Bessel function (see p. 402 of [27]) such that for any $\nu \ge \frac{n-2}{2}$ and $\frac12 < a < \frac{n}{2}$

$$\int_0^\infty r^{1-2a}|J_\nu(r)|^2\, dr = \frac{\Gamma(2a-1)\Gamma(\nu+1-a)}{2^{2a-1}\Gamma(a)^2\Gamma(\nu+a)}, \tag{13}$$

which has no singularity at $\nu \ge \frac{n-2}{2}$. We note that the numerator on the RHS is finite as far as $\frac12 < a < \frac{n}{2} \le \nu + 1$. By Stirling's formula such that $\Gamma(s) \sim s^{s-\frac12}e^{-(s-1)}$ for large s (for instance, see [2]), we get (11).

From now on, we prove the proposition. We expand φ with radial functions and spherical harmonic functions as follows:

$$\varphi(r\sigma) = \sum_{k\ge 0} \sum_{1\le l\le d(k)} g_{k,l}(r) Y_{k,l}(\sigma), \quad (r,\sigma) \in (0,\infty)\times S^{n-1},$$

where $g_{k,l}$ are radial functions such that

$$\int_0^\infty |g_{k,l}(r)|^2 r^{n-1}\, dr < \infty,$$

$Y_{k,l}$ are orthonormal spherical harmonics of order k, and $d(k)$ is the dimension of the space of spherical harmonics of order k. See [3, 11, 16].

By the othornormality, we have

$$\|\varphi\|^2_{L^2} \sim \sum_{k\ge 0} \sum_{1\le l\le d(k)} \int_0^\infty |g_{k,l}(r)|^2 r^{n-1}\, dr.$$

Using the Fourier transform of spherical harmonic functions (see for instance [25]), we have

$$\widehat{g_{k,l}Y_{k,l}}(\rho\sigma) = G_{k,l}(\rho) Y_{k,l}(\sigma),$$

where

$$G_{k,l}(\rho) = c_n \int_0^\infty g_{k,l}(r) r^{n-1} (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho)\, dr,$$
$$\text{and} \quad \nu(k) = \frac{2k+n-2}{2}.$$

The constant c_n is independent of k. By the Plancherel theorem, one can easily observe that

$$\int_0^\infty |g_{k,l}(r)|^2 r^{n-1}\,dr = \int_0^\infty |G_{k,l}(\rho)|^2\rho^{n-1}\,d\rho.$$

Now let us define functions $f_{k,l}$ by $f_{k,l}(\rho) = G_{k,l}(\rho)\rho^{\frac{n-1}{2}}$. Then from the Fourier transform of spherical harmonic functions, we have

$$r^{-a}L_a(\Delta)U(t)\varphi(r\sigma) = r^{-\frac{n-1}{2}}\sum_{k,l}(W_{\nu(k)}(t)f_{k,l})(r)Y_{k,\,l}(\sigma).$$

By the fact $-\Delta_\sigma Y_{k,\,l} = k(k+n-2)Y_{k,\,l}$, the orthonormality of spherical harmonics and the estimate (11), we get

$$\begin{aligned}
\|U(\cdot)\varphi\|^2_{L^2\widetilde{H}^0_a H^{a-\frac12}_\sigma} &\lesssim \sum_{k,\,l}(1+\nu(k))^{2a-1}\|W_{\nu(k)}(\cdot)f_{k,l}\|^2_{L^2(\mathbb{R}^2_+)}\\
&\lesssim \sum_{k,\,l}\|f_{k,\,l}\|^2_{L^2} = \sum_{k,l}\int_0^\infty |G_{k,l}(\rho)|^2\rho^{n-1}\,d\rho\\
&= \sum_{k,l}\int_0^\infty |g_{k,l}(r)|^2 r^{n-1}\,dr\\
&\sim \|\varphi\|^2_{L^2}.
\end{aligned}$$

For the proof of the second inequality of (9) we introduce a lemma for low-diagonal operator estimate (see [1, 24]).

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces. Let K be an operator such that $\|KG\|_{L^q_T(\mathcal{A})} \le C\|G\|_{L^p_T(\mathcal{B})}$ for $1 \le p \le q \le \infty$ with kernel k defined by $KG(t) = \int_0^T k(t-t')G(t')\,dt'$, where C does not depend on T. If $p < q$, then the low-diagonal operator $\widetilde{K}$ defined by $\widetilde{K}G = \int_0^t k(t-t')G(t')\,dt'$ satisfies that $\|\widetilde{K}G\|_{L^q_T(\mathcal{A})} \le \widetilde{C}\|G\|_{L^p_T(\mathcal{B})}$, where $\widetilde{C}$ is C times a constant depending only on p, q.*

In view of Lemma 1 with kernel $k(t) = U(t)$, $\mathcal{A} = \widetilde{H}^0_a H^{a-\frac12}_\sigma$ and $\mathcal{B} = L^2$, it suffices to show that

$$\left\|\int_0^T U(\cdot - t')F(t')\,dt'\right\|_{L^2_T(\widetilde{H}^0_a H^{a-\frac12}_\sigma)} \lesssim \|F\|_{L^1_T L^2}. \tag{14}$$

In fact, by the first Strichartz estimate (9), we have

$$\begin{aligned}
\left\|\int_0^T U(\cdot - t')F(t')\,dt'\right\|_{L^2_T(\widetilde{H}^0_a H^{a-\frac12}_\sigma)} &= \left\|U(\cdot)\int_0^T U(-t')F(t')\,dt'\right\|_{L^2_T(\widetilde{H}^0_a H^{a-\frac12}_\sigma)}\\
&\lesssim \|F\|_{L^1_T L^2}.
\end{aligned}$$

This yields the second inequality of (9). □

Remark 3. From the proof of Proposition 1, one can see the identity

$$\|e^{-it\omega}\varphi\|^2_{L^2\widetilde{H}^s_a} = \frac{2\pi\Gamma(2a-1)\Gamma(\frac{n}{2}-a)}{2^{2a-1}\Gamma(a)^2\Gamma(\frac{n-2}{2}+a)}\|\varphi\|^2_{H^s}$$

for any radial function $\varphi \in H^s$ for some $s \geq 0$. Thus the weighted Strichartz estimate is sharp as far as $\frac{1}{2} < a < \frac{n}{2}$.

Remark 4. If $n = 1$, then a modified weighted Strichartz estimate is possible. To state precisely, we take any L^2 function w as a weight and define a weighted Sobolev space $\widetilde{H}^s_w$ as follows

$$\widetilde{H}^s_w = \{v : \|v\|_{\widetilde{H}^s_w} \equiv \|wL_{\frac{1}{2}}(\Delta)(1-\Delta)^{\frac{s}{2}}v\|_{L^2} < \infty\}.$$

Then for any $T > 0$ we have

$$\|U(\cdot)\varphi\|_{L^2_T\widetilde{H}^s_w} \lesssim \|\varphi\|_{H^s}$$

$$\left\|\int_0^{\bullet} U(\cdot - t')F(t')\,dt'\right\|_{L^2_T\widetilde{H}^s_w} \lesssim \|F\|_{L^1_T H^s}.$$

To prove these estimates we have only to show that $\|W(\cdot)f\|_{L^2L^2} \lesssim \|f\|_{L^2}$, where

$$W(t)f(x) = w(x)\int_{-\infty}^{\infty} e^{i(x\xi - t\sqrt{1+\xi^2})}\frac{|\xi|^{\frac{1}{2}}}{(1+\xi^2)^{\frac{1}{4}}}f(\xi)\,d\xi.$$

By the change of variables $\xi \mapsto \sqrt{1+\xi^2}$ and by applying Plancherel theorem w.r.t. the time variable, and then using the change of variables $\xi \mapsto \sqrt{\xi^2-1}$ again, one can readily have that

$$\|W(\cdot)f\|_{L^2_TL^2} \leq \|W(\cdot)f\|_{L^2L^2} \leq \sqrt{2}\|w\|_{L^2}\|f\|_{L^2}.$$

The inhomogeneous Strichartz estimate can be treated by the same way as in the proof of Proposition 1.

3. Fractional integral estimates for radial functions

We prove some fractional integral estimates for radial functions.

Lemma 2. *Let $n \geq 3$ and $0 < \gamma < n-1$.*

(*i*) *If f and g are radial functions with $f, |x|^{-\delta}f, |x|^{-(\gamma-\delta)}g \in L^2$ for some $0 < \delta \leq \gamma$, then*

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy \lesssim \||\cdot|^{-\delta}f\|_{L^2}\||\cdot|^{-(\gamma-\delta)}g\|_{L^2}. \tag{15}$$

(*ii*) *If f, g are radially symmetric and $f, |x|^{-(\gamma-\delta)}g \in L^2$ for some $0 < \delta \leq \gamma$, then*

$$\sup_{x\in\mathbb{R}^n}|x|^\delta\int_{\mathbb{R}^n}\frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy \lesssim \|f\|_{L^2}\||\cdot|^{-(\gamma-\delta)}g\|_{L^2}. \tag{16}$$

Remark 5. In [8] Fröhlich and Lenzmann showed that if $\varphi \in C_0^\infty(\mathbb{R}^3)$ is radially symmetric and satisfies $E_1(\varphi) < 0$, then the radial solution u of (1) with $\gamma = 1$ blows up within a finite time. To lead to the blowup they used a variance type estimate where the estimates $|V_1(u)(x)| \le \|\varphi\|_{L^2}^2/|x|$ and $|\nabla V_1(u)(x)| \le \|\varphi\|_{L^2}^2/|x|^2$ are crucial. The lemma above leads us to the same estimates for $n \ge 4$ and hence to the finite time blowup.

Proof of Lemma 2. We revisit the proof of Lemma 3 of [5]. Fixing x, we divide the integration into three parts as follows

$$\int_{\mathbb{R}^n} \frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy = \int_{|y|>2|x|} + \int_{\frac{|x|}{2}\le|y|\le2|x|} + \int_{|y|<\frac{|x|}{2}} \equiv I + II + III.$$

For I, since $|x-y| \ge \frac{|y|}{2}$ for $|y| > 2|x|$, we have

$$\begin{aligned} I &\lesssim \int_{|y|>2|x|} \frac{|f(y)|}{|y|^\delta}\frac{|g(y)|}{|y|^{\gamma-\delta}}\,dy \\ &\lesssim \||\cdot|^{-\delta} f\|_{L^2}\||x|^{-(\gamma-\delta)}g\|_{L^2} \quad \text{or} \quad |x|^{-\delta}\|f\|_{L^2}\||x|^{-(\gamma-\delta)}g\|_{L^2}. \end{aligned}$$

Since f and g are radially symmetric, we may assume that $x = |x|e_1 = re_1 = r(1,0,\cdots,0,0)$. Using the spherical coordinates $(\rho,\theta_1,\theta_2,\cdots,\theta_{n-1}) \in (0,\infty)\times[0,\pi]\times[0,\pi]\times\cdots\times[0,2\pi]$ for y variable, the integrals II and III are converted into

$$II + III = \left(\int_{\frac{r}{2}}^{2r} + \int_0^{\frac{r}{2}}\right)\rho^{n-1}|f(\rho)||g(\rho)|\Omega(r,\rho)d\rho, \tag{17}$$

where

$$\begin{aligned} \Omega(r,\rho) = \int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi &(r^2+\rho^2-2r\rho\cos\theta_1)^{-\frac{\gamma}{2}} \\ &\times \sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\,d\theta_1\cdots d\theta_{n-2}d\theta_{n-1}. \end{aligned}$$

If $\frac{r}{2} \le \rho \le 2r$, then by the fact $n-2-\gamma > -1$ and

$$\sqrt{r^2+\rho^2-2r\rho\cos\theta_1} \ge \rho\sin\theta_1$$

we have

$$\Omega(r,\rho) \lesssim \rho^{-\gamma}\int_0^\pi \sin^{n-2-\gamma}\theta_1\,d\theta_1 \lesssim \rho^{-\gamma}. \tag{18}$$

If $\rho < \frac{r}{2}$, then

$$\Omega(r,\rho) \lesssim r^{-\gamma}, \tag{19}$$

since $r^2+\rho^2-2r\rho\cos\theta_1 \ge r^2$. Therefore by the Hölder inequality we have

$$II + III \lesssim \||\cdot|^{-\delta}f\|_{L^2}\||\cdot|^{-(\gamma-\delta)}g\|_{L^2} \quad \text{or} \quad |x|^{-\delta}\|f\|_{L^2}\||\cdot|^{-(\gamma-\delta)}g\|_{L^2}.$$

This completes the proof. $\square$

Lemma 3. *Let $1<\gamma<2$ if $n\ge 4$ and $1<\gamma<\frac{3}{2}$ if $n=3$. Let ε satisfy $0<\varepsilon<\min(\gamma-1,2-\gamma)$. Then for any radial functions $f,g\in L^2$ with $|x|^{-1}f,|x|^{-1}g\in L^2$*

$$\begin{aligned}&\Big\||\cdot|^{-\gamma-\frac12}*(fg)\Big\|_{L^{2n}}\\ &\lesssim (\|f\|_{L^2}^{2-\gamma-\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1+\varepsilon}+\|f\|_{L^2}^{2-\gamma+\varepsilon}\||\cdot|^{-1}f\|_{L^2})\||\cdot|^{-1}g\|_{L^2}.\end{aligned}\tag{20}$$

Proof of Lemma 3. As in the proof of Lemma 2 we split the fractional integral and estimate pointwise as

$$\begin{aligned}\int\frac{|f(y)g(y)|}{|x-y|^{\gamma+\frac12}}\,dy&=\int_{|x|>2|y|}+\int_{\frac12|y|\le|x|\le 2|y|}+\int_{|x|<\frac12|y|}\\ &=I+II+III.\end{aligned}$$

In case that $|x|<1$, from similar estimates to (17), (18) and (19), and from the Hölder inequality and interpolation it follows that for small ε with $\gamma+\varepsilon<2$

$$I\lesssim|x|^{-\frac12+\varepsilon}\int\frac{|f(y)g(y)|}{|y|^{\gamma+\varepsilon}}dy\lesssim|x|^{-\frac12+\varepsilon}\|f\|_{L^2}^{2-\gamma-\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1+\varepsilon}\||\cdot|^{-1}g\|_{L^2},$$

$$II+III\lesssim|x|^{-\frac12+\varepsilon}\|f\|_{L^2}^{2-\gamma-\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1+\varepsilon}\||\cdot|^{-1}g\|_{L^2}\int_0^\pi\sin^{n-\frac52-\gamma}\theta\,d\theta.$$

Since $n-\frac52-\gamma>-1$ if $1<\gamma<2$ for $n\ge 4$ and $1<\gamma<\frac32$ for $n=3$, the last integral is finite. Hence

$$\||\cdot|^{-\gamma-\frac12}*(fg)\|_{L^{2n}(|x|<1)}\lesssim\|f\|_{L^2}^{2-\gamma-\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1+\varepsilon}\||\cdot|^{-1}g\|_{L^2}.$$

If $|x|\ge 1$, then choosing ε such that $\gamma-\varepsilon>1$, by the same argument as above we have

$$I+II+III\lesssim|x|^{-\frac12-\varepsilon}\|f\|_{L^2}^{2-\gamma+\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1-\varepsilon}\||\cdot|^{-1}g\|_{L^2}.$$

Hence

$$\||\cdot|^{-\gamma-\frac12}*(fg)\|_{L^{2n}(|x|\ge1)}\lesssim\|f\|_{L^2}^{2-\gamma+\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1-\varepsilon}\||\cdot|^{-1}g\|_{L^2}.$$

The proof has been completed. □

Lemma 4. *Let $n\ge 3$ and $1<\gamma<n-1$. Let $f,g\in W_{s,\varepsilon}$ be radial functions for some s,ε satisfying the condition* (7)*. Then it follows that*

$$\Big\||\cdot|^{-\gamma}*(fg)\Big\|_{L^{\frac{n}{1-(s-\varepsilon)}}\cap L^{\frac{n}{1-(s+\varepsilon)}}}\lesssim\|f\|_{W_{s,\varepsilon}}\|g\|_{W_{s,\varepsilon}}.\tag{21}$$

Proof of Lemma 4. By the same spirit as in the proof of Lemma 3, we split the fractional integral into three parts I,II,III and estimate them using radial symmetry. We also divide each part into two regions of x; inside the unit ball and its outside

If $|x|<1$, then since $\varepsilon<\frac{\gamma}{2}-s$, we have

$$I+II+III\lesssim|x|^{-(\gamma-2(s+\varepsilon))}\||\cdot|^{-(s+\varepsilon)}f\|_{L^2}\||\cdot|^{-(s+\varepsilon)}g\|_{L^2}.$$

Since $\||\cdot|^{-(s+\varepsilon)}f\|_{L^2} \le \|f\|_{W_{s,\varepsilon}}$ and $(\gamma - 2(s+\varepsilon))\frac{n}{1-(s\pm\varepsilon)} < n$, we have

$$\||\cdot|^{-\gamma} * (fg)\|_{(L^{\frac{n}{1-(s-\varepsilon)}} \cap L^{\frac{n}{1-(s+\varepsilon)}})(|x|<1)} \lesssim \|f\|_{W_{s,\varepsilon}}\|g\|_{W_{s,\varepsilon}}.$$

If $|x| \ge 1$, then

$$I + II + III \lesssim |x|^{-(\gamma-2(s-\varepsilon))}\||\cdot|^{-(s-\varepsilon)}f\|_{L^2}\||\cdot|^{-(s-\varepsilon)}g\|_{L^2}.$$

Since $(\gamma - 2(s-\varepsilon))\frac{n}{1-(s\pm\varepsilon)} > n$, we have

$$\||\cdot|^{-\gamma} * (fg)\|_{(L^{\frac{n}{1-(s-\varepsilon)}} \cap L^{\frac{n}{1-(s+\varepsilon)}})(|x|\ge 1)} \lesssim \|f\|_{W_{s,\varepsilon}}\|g\|_{W_{s,\varepsilon}}.$$

This completes the proof of the lemma. □

4. Proofs of the theorems

4.1. **Proof of Theorem 1.** We only consider the positive time because the proof for negative time can be treated in the same way.

Let us first define a complete metric space $X_{T,\rho}$ with metric $d(u,v) = \|u-v\|_{X_T}$, where $X_T = C([0,T]; H^{\frac12}) \cap L_T^2\widetilde{H}_1^{\frac12}$ by

$$X_{T,\rho} \equiv \{v \in X_T :\ v \text{ is radially symmetric and } \|v\|_{X_T} \le \rho\}.$$

Here let us observe that the space $\widetilde{H}_1^{\frac12}$ is exactly the same as $\{v : \||\cdot|^{-1}v\|_{L^2} < \infty\}$.

Now we define a mapping $N : u \mapsto N(u)$ on $X_{T,\rho}$ by

$$N(u)(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')\,dt'. \tag{22}$$

For any $u \in X_{T,\rho}$, $N(u)$ is radially symmetric. By the Strichartz estimate (9) with $a = 1, b = 0$ and $s = \frac12$, we have

$$\|N(u)\|_{X_{T,\rho}} \lesssim \|\varphi\|_{H^{\frac12}} + \|F\|_{L^1H^{\frac12}}.$$

For the second term, we use the generalized Leibniz rule (see Lemma A1 $\sim$ Lemma A4 in Appendix of [12]) such that for any $s \ge 0$

$$\|D^s(uv)\|_{L^r} \lesssim \|D^s u\|_{L^{r_1}}\|v\|_{L^{q_2}} + \|u\|_{L^{q_1}}\|D^s v\|_{L^{r_2}}, \tag{23}$$

$$\text{where } D^s = (-\Delta)^{s/2}$$

$$\textit{and } \frac1r = \frac1{r_1} + \frac1{q_2} = \frac1{q_1} + \frac1{r_2}, \quad r_i \in (1,\infty),\ q_i \in (1,\infty], \quad i = 1,2.$$

From (23), we have

$$\begin{aligned}&\|N(u)\|_{X_{T,\rho}}\\ &\lesssim \|\varphi\|_{H^{\frac12}} + \|V_\gamma(u)\|_{L_T^1L^\infty}\|u\|_{L_T^\infty H^{\frac12}} + \|(-\Delta)^{\frac14}V_\gamma(u)\|_{L_T^1L^{2n}}\|u\|_{L_T^\infty L^{\frac{2n}{n-1}}}.\end{aligned} \tag{24}$$

To estimate the last two terms, we use Lemma 2. Using (15) with $f = g = u$ and interpolation we have

$$\|V_\gamma(u)\|_{L^\infty} \lesssim \|u\|_{L^2}^{2-\gamma}\|u\|_{\widetilde{H}_1^{\frac12}}^{\gamma}.$$

and using (21) with $f = g = u$ for some small ε as in Lemma 3

$$\begin{aligned}\|(-\Delta)^{\frac14}V_\gamma(u)\|_{L^{2n}} &\lesssim \|V_{\gamma+\frac12}(u)\|_{L^{2n}}\\ &\lesssim \|u\|_{L^2}^{2-\gamma-\varepsilon}\|u\|_{\widetilde{H}_1^{\frac12}}^{\gamma+\varepsilon} + \|u\|_{L^2}^{2-\gamma+\varepsilon}\|u\|_{\widetilde{H}_1^{\frac12}}^{\gamma-\varepsilon}.\end{aligned}$$

Hence the nonlinear estimate (24) has the following: for some positive number ε

$$\begin{aligned}&\|N(u)\|_{X_{T,\rho}}\\ &\lesssim \|\varphi\|_{H^{\frac12}} + (T^{\frac{2}{2-\gamma}} + T^{\frac{2}{2-\gamma-\varepsilon}} + T^{\frac{2}{2-\gamma+\varepsilon}})\|u\|^2_{L_T^\infty H^{\frac12}}\|u\|_{L_T^2\widetilde{H}_1^{\frac12}}\\ &\lesssim \|\varphi\|_{H^{\frac12}} + (T^{\frac{2}{2-\gamma}} + T^{\frac{2}{2-\gamma-\varepsilon}} + T^{\frac{2}{2-\gamma+\varepsilon}})\rho^3.\end{aligned}$$

From the choice of T and ρ satisfying that

$$\begin{aligned}C\|\varphi\|_{H^1} &\le \frac{\rho}{2},\\ C(T^{\frac{2}{2-\gamma}} + T^{\frac{2}{2-\gamma-\varepsilon}} + T^{\frac{2}{2-\gamma+\varepsilon}})\rho^3 &\le \frac{\rho}{2}\end{aligned}$$

for some constant C, it follows that N maps $X_{T,\rho}$ to itself.

For any $u, v \in X_{T,\rho}$, we have

$$\begin{aligned}d(N(u),N(v)) &\lesssim \|F(u)-F(v)\|_{L_T^1H^{\frac12}}\\ &\lesssim \|(V_\gamma(u)-V_\gamma(v))u\|_{L_T^1H^{\frac12}} + \|V_\gamma(v)(u-v)\|_{L_T^1H^{\frac12}}.\end{aligned}$$

Using Lemma 2, Lemma 3 and the Leibniz rule (23) again, we have that

$$\begin{aligned}&\|(V_\gamma(u)-V_\gamma(v))u\|_{L_T^1H^{\frac12}}\\ &\lesssim \|(V_\gamma(u)-V_\gamma(v))u\|_{L_T^1L^2} + \|(-\Delta)^{\frac14}((V_\gamma(u)-V_\gamma(v))u)\|_{L_T^1L^2}\\ &\lesssim \Big(\|V_\gamma(u)-V_\gamma(v)\|_{L_T^1L^\infty} + \|(-\Delta)^{\frac14}(V_\gamma(u)-V_\gamma(v))\|_{L_T^1L^{2n}}\Big)\|u\|_{L_T^\infty H^{\frac12}}\\ &\lesssim \int_0^T(\|u\|_{L^2}+\|v\|_{L^2})^{2-\gamma}\|u-v\|^\gamma_{\widetilde{H}_1^{\frac12}}\,dt\|u\|_{L_T^\infty H^{\frac12}}\\ &\qquad + \||\cdot|^{-(\gamma+\frac12)} * (|u|^2-|v|^2)\|_{L_T^1L^{2n}}\|u\|_{L_T^\infty H^{\frac12}}\\ &\lesssim (T^{\frac{2}{2-\gamma}} + T^{\frac{2}{2-(\gamma+\varepsilon)}} + T^{\frac{2}{2-(\gamma-\varepsilon)}})\rho^2\|u-v\|_{L_T^2\widetilde{H}_1^{\frac12}}\end{aligned}$$

and

$$\begin{aligned}&\|V_\gamma(v)(u-v)\|_{L_T^1H^{\frac12}}\\ &\lesssim \|V_\gamma(v)\|_{L_T^1L^\infty}\|u-v\|_{L_T^\infty H^{\frac12}} + \|(-\Delta)^{\frac14}V_\gamma(v)\|_{L_T^1L^{2n}}\|u\|_{L_T^\infty H^{\frac12}}\\ &\lesssim (T^{\frac{2}{2-\gamma}} + T^{\frac{2}{2-(\gamma+\varepsilon)}} + T^{\frac{2}{2-(\gamma-\varepsilon)}})\rho^2\|u-v\|_{L_T^\infty H^{\frac12}}.\end{aligned}$$

Thus by the choice of T, ρ as above

$$d(N(u),N(v)) \le C(T^{\frac{2}{2-\gamma}} + T^{\frac{2}{2-\gamma-\varepsilon}} + T^{\frac{2}{2-\gamma+\varepsilon}})\rho^2 d(u,v) \le \frac12 d(u,v). \tag{25}$$

Therefore N becomes a contraction mapping on $X_{T,\rho}$. This proves the local existence. The energy and L^2 conservations follow from the Strichartz estimate (9) and the method of [22].

Now we consider the global well-posedness. To do so, we need the following energy inequality that for $\lambda > 0$,

$$\frac{1}{2}\|u(t)\|_{H^{\frac12}}^2 \le E(u) = E(\varphi).$$

If $\lambda < 0$, then for any φ with $\|\varphi\|_{L^2} \le 1$

$$\begin{aligned}\|u(t)\|_{\dot H^{\frac12}}^2 &\le 2|E(u)| + 2|V(u)|\\ &\le 2|E(\varphi)| + C\|u\|_{L^{\frac{2n}{n-\gamma+1}}}^2\|u\|_{\dot H^{\frac12}}^2\\ &\le C(1+\|\varphi\|_{\dot H^{\frac12}}^2)^\gamma + C\|\varphi\|_{L^2}^{4-2\gamma}\|u\|_{\dot H^{\frac12}}^{2\gamma}.\end{aligned}$$

Here for the third inequality we used the fact that

$$2|E(\varphi)| \le 1 + \|\varphi\|_{\dot H^{\frac12}}^2 + C\|\varphi\|_{L^2}^{4-2\gamma}\|\varphi\|_{\dot H^{\frac12}}^{2\gamma} \le C(1+\|\varphi\|_{\dot H^{\frac12}}^2)^\gamma.$$

Hence especially in the case of negative λ, with assumption of the smallness of $\|\varphi\|_{L^2}$ for $\gamma < 2$ such that

$$\|\varphi\|_{L^2} \le \min\left(1, (8^\gamma C^\gamma(1+\|\varphi\|_{\dot H^{\frac12}}^2)^{\gamma(\gamma-1)})^{-\frac{1}{4-2\gamma}}\right),$$

we have

$$\|u(t)\|_{H^{\frac12}}^2 \le 2C(1+\|\varphi\|_{\dot H^{\frac12}}^2)^\gamma. \tag{26}$$

Now let us denote $E(\varphi)$ for $\lambda < 0$ and $(1+\|\varphi\|_{\dot H^{\frac12}}^2)^\gamma$ for $\lambda < 0$ by $\mathcal{E}(\varphi)$. Then from the Strichartz estimate (9), (26) and Lemma 2, 3, we have for some small time $0 < \delta < 1$ and small $\varepsilon > 0$

$$\begin{aligned}&\|u\|_{L_\delta^2\widetilde H_{1,0}^{\frac12}}\\ &\lesssim (1+\mathcal{E}(\varphi))^{\frac12} + \delta^{\frac{2}{2-\gamma-\varepsilon}}(1+\|u\|_{L_\delta^\infty H^{\frac12}})^{3-\gamma+\varepsilon}\|u\|_{L_\delta^2\widetilde H_1^{\frac12}}\\ &\lesssim (1+\mathcal{E}(\varphi))^{\frac12} + \delta^{\frac{2}{2-\gamma-\varepsilon}}(1+\mathcal{E}(\varphi))^{\frac{3-\gamma+\varepsilon}{2}}\|u\|_{L_\delta^2\widetilde H_1^{\frac12}}.\end{aligned}$$

Thus for some δ so small that $\frac14 \le \delta^{\frac{2}{2-\gamma-\varepsilon}}(1+\mathcal{E}(\varphi))^{\frac{3-\gamma+\varepsilon}{2}} \le \frac12$, we have

$$\|u\|_{L^2(T_{j-1},T_j;\widetilde H_1^{\frac12})} \le C(1+\mathcal{E}(\varphi))^{\frac12},$$

where $T_j - T_{j-1} = \delta$ for $j \le k-1$, $T_k = T$ and $T_k - T_{k-1} \sim \delta$. This implies that

$$\begin{aligned}\|u\|_{L_T^2\widetilde H_1^{\frac12}}^2 &\le \sum_{1\le j\le k}\|u\|_{L^2(T_{j-1},T_j;\widetilde H_1^{\frac12})}^2\\ &\lesssim k\delta(1+\mathcal{E}(\varphi))^{1+\frac{(3-\gamma+\varepsilon)(2-\gamma-\varepsilon)}{4}}\\ &\lesssim T(1+\mathcal{E}(\varphi))^{1+\frac{(3-\gamma+\varepsilon)(2-\gamma-\varepsilon)}{4}}.\end{aligned} \tag{27}$$

From (26) and (27) we conclude that $u \in C_b(\mathbb{R}_+; H^{\frac12}) \cap L_{loc}^2\widetilde H_1^{\frac12}$. This completes the proof of Theorem 1.

4.2. **Proof of Theorem 2.** Let Y_ρ be a complete metric space with metric $d(u,v) = \|u-v\|_Y$, where $Y = C_b(\mathbb{R}; H^1) \cap L^2 \widetilde{H}_1^{\frac12}$ by

$$Y_\rho \equiv \{v \in Y : \ v \ \text{ is radially symmetric and } \ \|v\|_Y \le \rho\}.$$

Then we claim that the map N defined as (22) is a contraction on Y, provided ρ is sufficiently small.

From the Strichartz estimate (9) and the fractional integral estimates (15) and (16), we have for any $u \in Y_\rho$

$$\begin{aligned}
\|u\|_Y &\lesssim \|\varphi\|_{H^1} + \|V_2(u)\|_{L^1L^\infty}\|u\|_{L^\infty H^1} + \|(\nabla V_2(u))u\|_{L^1L^2}\\
&\lesssim \|\varphi\|_{H^1} + \|u\|^2_{L^2\widetilde{H}_1^{\frac12}}\|u\|_{L^\infty H^1}\\
&\lesssim \|\varphi\|_{H^1} + \rho^3.
\end{aligned}$$

Hence choosing ρ so small that $C\|\varphi\|_{H^1} \le \frac{\rho}{2}$ and $C\rho^3 \le \frac{\rho}{2}$, the mapping N maps Y to itself. We also have

$$\begin{aligned}
d(N(u), N(v)) &\lesssim \|V_2(u)u - V_2(v)v\|_{L^1H^1}\\
&\lesssim \|(V_2(u) - V_2(v))u\|_{L^1H^1} + \|V_2(v)(u-v)\|_{L^1H^1}.
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
&\|(V_2(u) - V_2(v))u\|_{L^1H^1}\\
&\lesssim \|(V_2(u) - V_2(v))\|_{L^1L^\infty}\|u-v\|_{L^\infty H^1} + \|\nabla((V_2(u) - V_2(v))u)\|_{L^1L^2}\\
&\lesssim (\|u\|_{L^2\widetilde{H}_1^{\frac12}} + \|v\|_{L^2\widetilde{H}_1^{\frac12}})\|u-v\|_{L^2\widetilde{H}_1^{\frac12}}\|u\|_{L^\infty H^1}\\
&\qquad + (\|u\|_{L^\infty H^1} + \|v\|_{L^\infty H^1})\|u-v\|_{L^2\widetilde{H}_1^{\frac12}}\|u\|_{L^2\widetilde{H}_1^{\frac12}}
\end{aligned}$$

and

$$\begin{aligned}
&\|V_2(v)(u-v)\|_{L^1H^1}\\
&\lesssim \|V_2(v)\|_{L^1L^\infty}\|u-v\|_{L^1H^1} + \|\nabla(V_2(v)(u-v))\|_{L^1L^2}\\
&\lesssim \|v\|^2_{L^2\widetilde{H}_1^{\frac12}}\|u-v\|_{L^\infty H^1} + \|v\|_{L^\infty H^1}\|v\|_{L^2\widetilde{H}_1^{\frac12}}\|u-v\|_{L^2\widetilde{H}_1^{\frac12}}.
\end{aligned}$$

Hence from the condition of u and v

$$d(N(u), N(v)) \lesssim \rho^2 d(u,v).$$

Thus by the choice of ρ such that $C\rho^2 \le \frac12$, N becomes a contraction.

As for the scattering, let us define functions $\varphi_\pm$ by

$$\varphi_\pm = \varphi - i\int_0^{\pm\infty} U(-s)F(u)(s)\,ds.$$

Then clearly $\varphi_\pm \in H^1$ and one can show that

$$\begin{aligned}
\|u(t) - U(t)\varphi_\pm\|_{H^1} &\lesssim \|u\|^2_{L^2(I_t^\pm;\, \widetilde{H}_1^{\frac12})}\|u\|_{L^\infty H^1}\\
&\to 0 \quad \text{as} \quad t \to \pm\infty,
\end{aligned}$$

where $I_t^+ = (t,\infty)$ and $I_t^- = (-\infty, t)$. This proves Theorem 2.

4.3. **Proof of Theorem 3.** Let Z_ρ be a complete metric space with metric $d(u,v) = \|u-v\|_Z$, where $Z = C_b(\mathbb{R}; H^s) \cap L^2 W_{s,\varepsilon}$ by

$$Z_\rho \equiv \{v \in Y : \ v \ \text{ is radially symmetric and } \ \|v\|_Z \le \rho\}.$$

Here from the definition we easily observe that the space $W_{s,\varepsilon}$ contains the space

$$L_{s\pm\varepsilon}(\Delta)\widetilde{H}^0_{s\pm\varepsilon} = \{v : (L_{s\pm\varepsilon}(\Delta))^{-1} v \in \widetilde{H}^0_{s\pm\varepsilon}\}.$$

Now we define a nonlinear map $\widetilde{N}$ by

$$\widetilde{N}(u) = (\cos t\omega)\varphi_1 + \omega^{-1}(\sin t\omega)\varphi_2 - \int_0^t \omega^{-1}(\sin(t-t')\omega)F(u)\,dt'.$$

Then we claim that the map $\widetilde{N}$ is a contraction on Z, provided that ρ is sufficiently small.

We first observe from the Strichartz estimate (9) [2] and fractional integration that

$$\begin{aligned}\|(L_{s\pm\varepsilon}(\Delta))^{-1}(\cos(\cdot)\omega)\varphi_1\|_{L^2\widetilde{H}^0_{s\pm\varepsilon}} &\lesssim \|(L_{s\pm\varepsilon}(\Delta))^{-1}\varphi_1\|_{L^2}\\ &\lesssim \|(-\Delta)^{-\frac{1-(s\pm\varepsilon)}{2}}\varphi_1\|_{L^2} + \|(-\Delta)^{\frac{s-\frac12+\varepsilon}{2}}\varphi_1\|_{L^2}\\ &\lesssim \|\varphi_1\|_{L^{\frac{2n}{n+2-2(s-\varepsilon)}}\cap H^{s+\varepsilon-\frac12}}\end{aligned}$$

and

$$\begin{aligned}\|(L_{s\pm\varepsilon}(\Delta))^{-1}\omega^{-1}(\sin(\cdot)\omega)\varphi_2\|_{L^2\widetilde{H}^0_{s\pm\varepsilon}} &\lesssim \|(L_{s\pm\varepsilon}(\Delta))^{-1}\omega^{-1}\varphi_2\|_{L^2}\\ &\lesssim \|\varphi_2\|_{L^{\frac{2n}{n+2-2(s-\varepsilon)}}\cap H^{s+\varepsilon-\frac32}}\end{aligned}$$

Since

$$\|\psi\|_{W_{s,\varepsilon}} \lesssim \|\psi\|_{L_{s+\varepsilon}(\Delta)\widetilde{H}^0_{s+\varepsilon}} + \|\psi\|_{L_{s-\varepsilon}(\Delta)\widetilde{H}^0_{s-\varepsilon}},$$

we have

$$\|(\cos t\omega)\varphi_1 + \omega^{-1}(\sin t\omega)\varphi_2\|_{L^\infty H^s\cap L^2W_{s,\varepsilon}} \lesssim \|\varphi_1\|_{D_{s+\varepsilon,s+\varepsilon}} + \|\varphi_2\|_{D_{s+\varepsilon-1,s+\varepsilon}}.$$

Now we estimate the nonlinear part. From the Strichartz estimate (9) and the boundedness of $(-\Delta)^{\frac{1-(s\pm\varepsilon)}{2}}(1-\Delta)^{-\frac{\frac32-(s+\varepsilon)}{2}}$ in L^2, it follows that

$$\begin{aligned}&\left\|\int_0^t \omega^{-1}(\sin(t-t')\omega)F(u)\,dt'\right\|_{L^\infty H^{s-\frac12+\varepsilon}\cap L^2W_{s,\varepsilon}}\\ &\lesssim \|(-\Delta)^{-\frac{1-(s+\varepsilon)}{2}}F(u)\|_{L^1L^2} + \|(-\Delta)^{-\frac{1-(s-\varepsilon)}{2}}F(u)\|_{L^1L^2}\\ &\lesssim \|F(u)\|_{L^1L^{\frac{2n}{n+2-(s+\varepsilon)}}} + \|F(u)\|_{L^1L^{\frac{2n}{n+2-(s-\varepsilon)}}}\\ &\lesssim (\|V_\gamma(u)\|_{L^1L^{\frac{n}{1-(s+\varepsilon)}}} + \|V_\gamma(u)\|_{L^1L^{\frac{n}{1-(s-\varepsilon)}}})\|u\|_{L^\infty L^2}.\end{aligned} \tag{28}$$

Using Lemma 4 with $f = g = u$, the last term on the RHS of (28) is bounded by a constant multiple of

$$\|u\|^2_{L^2W_{s,\varepsilon}}\|u\|_{L^\infty L^2}. \tag{29}$$

Therefore, for any $u \in Z_\rho$ we have

$$\|\widetilde{N}(u)\|_Z \le C(\|\varphi_1\|_{D_{s+\varepsilon,s+\varepsilon}} + \|\varphi_2\|_{D_{s+\varepsilon-1,s+\varepsilon}}) + C\rho^3$$

[2] To apply (9) we need the condition $s-\varepsilon > \frac12$.

and this implies that $\widetilde{N}$ maps from Z_ρ to itself, provided ρ and the norm of the initial data are sufficiently small.

On the other hand, by the Strichartz estimate (9) and fractional integral estimate we have that

$$\begin{aligned}
&d(\widetilde{N}(u),\widetilde{N}(v))\\
&\lesssim \|(-\Delta)^{\frac{1-(s+\varepsilon)}{2}}(V_\gamma(u)u - V_\gamma(v)v)\|_{L^1L^2} + \|(-\Delta)^{\frac{1-(s-\varepsilon)}{2}}(V_\gamma(u)u - V_\gamma(v)v)\|_{L^1L^2}\\
&\lesssim \|(-\Delta)^{\frac{1-(s+\varepsilon)}{2}}((V_\gamma(u) - V_\gamma(v))u)\|_{L^1L^2} + \|(-\Delta)^{\frac{1-(s-\varepsilon)}{2}}((V_\gamma(u) - V_\gamma(v))u)\|_{L^1L^2}\\
&\qquad + \|(-\Delta)^{\frac{1-(s+\varepsilon)}{2}}V_\gamma(v)(u-v)\|_{L^1L^2} + \|(-\Delta)^{\frac{1-(s-\varepsilon)}{2}}V_\gamma(v)(u-v)\|_{L^1L^2}\\
&\lesssim \|(V_\gamma(u) - V_\gamma(v))u\|_{L^1L^{\frac{2n}{n+2-(s+\varepsilon)}}} + \|(V_\gamma(u) - V_\gamma(v))u\|_{L^1L^{\frac{2n}{n+2-(s-\varepsilon)}}}\\
&\qquad + \|V_\gamma(v)(u-v)\|_{L^1L^{\frac{2n}{n+2-(s+\varepsilon)}}} + \|V_\gamma(v)(u-v)\|_{L^1L^{\frac{2n}{n+2-(s-\varepsilon)}}}.
\end{aligned}$$

Applying Lemma 4 with $f = u - v, g = \overline{u}$ or $f = v, g = \overline{u - v}$ or $f = g = v$ to the last four terms in the above estimate, we have

$$\begin{aligned}
&d(\widetilde{N}(u),\widetilde{N}(v))\\
&\lesssim (\|u\|_{L^2W_{s,\varepsilon}} + \|v\|_{L^2W_{s,\varepsilon}})\|u-v\|_{L^2W_{s,\varepsilon}}\|u\|_{L^\infty L^2} + \|v\|_{L^2W_{s,\varepsilon}}^2\|u-v\|_{L^\infty L^2}\\
&\lesssim \rho^2 d(u,v).
\end{aligned}$$

Hence the smallness of ρ and of the norms of initial data makes $\widetilde{N}$ a contraction.

Now we consider the scattering. Let us define four functions $\varphi_i^\pm, i = 1, 2$ by

$$\begin{aligned}
\widehat{\varphi_1^\pm}(\xi) &= \widehat{\varphi_1}(\xi) + \int_0^{\pm\infty}(\sqrt{1+|\xi|^2})^{-1}\sin(t'\sqrt{1+|\xi|^2})\,\widehat{F(u)}(\xi,t')\,dt'\\
\widehat{\varphi_2^\pm}(\xi) &= \widehat{\varphi_2}(\xi) - \int_0^{\pm\infty}\cos(t'\sqrt{1+|\xi|^2})\,\widehat{F(u)}(\xi,t')\,dt'.
\end{aligned}$$

Then it follows from the regularity of the solution u that $\varphi_1^\pm \in H^{s-\frac12+\varepsilon}$ and $\varphi_2^\pm \in H^{s-\frac32+\varepsilon}$. Furthermore, since $u \in L^2W_{s,\varepsilon}$, for the linear solution $u^\pm$ of (8) we conclude from the estimate (28) and (29) that

$$\begin{aligned}
&\|u(t) - u^\pm(t)\|_{H^s}\\
&\quad\lesssim \|(-\Delta)^{-\frac{1-(s+\varepsilon)}{2}}F(u)\|_{L^1(I_t^\pm;L^2)} + \|(-\Delta)^{\frac{1-(s-\varepsilon)}{2}}F(u)\|_{L^1(I_t^\pm;L^2)}\\
&\quad\lesssim (\|V_\gamma(u)\|_{L^1(I_t^\pm;L^{\frac{n}{1-(s+\varepsilon)}})} + \|V_\gamma(u)\|_{L^1(I_t^\pm;L^{\frac{n}{1-(s-\varepsilon)}})})\|u\|_{L^\infty L^2}\\
&\quad\lesssim \|u\|_{L^2(I_t^\pm;W_{s,\varepsilon})}^2\|u\|_{L^\infty L^2} \to 0 \quad \text{as} \quad t \to \pm\infty.
\end{aligned}$$

This completes the proof of the theorem.

Remark 6. If the initial data $(\varphi_1, \varphi_2) \in \widetilde{D}_{s+\varepsilon,s-\varepsilon} \times \widetilde{D}_{s+\varepsilon-1,s-\varepsilon}$ and their norm is sufficiently small, then the solution u is in $C_b(\mathbb{R}; \dot{H}^{-(1-(s-\varepsilon))})$. In fact, the existence and uniqueness in $C_b(\mathbb{R}; H^{s-\frac12+\varepsilon}) \cap L^2W_{s,\varepsilon}$ follows immediately from the previous proof. Hence we have only to show $\|u\|_{L^\infty \dot{H}^{-(1-(s-\varepsilon))}} < \infty$. From (28) and (29) we have

$$\begin{aligned}
&\|u\|_{L^\infty \dot H^{-(1-(s-\varepsilon))}}\\
&\le \|\varphi_1\|_{\dot H^{-(1-(s-\varepsilon))}} + \|\varphi_2\|_{\dot H^{-(1-(s-\varepsilon))}} + \int_0^t \|(-\Delta)^{-\frac{1-(s-\varepsilon)}{2}} F(u)\|_{L^2}\, dt'\\
&\lesssim \|\varphi_1\|_{\dot H^{-(1-(s-\varepsilon))}} + \|\varphi_2\|_{\dot H^{-(1-(s-\varepsilon))}} + \|V_\gamma(u)\|_{L^1 L^{\frac{n}{1-(s-\varepsilon)}}} \|u\|_{L^\infty L^2}\\
&\lesssim \|\varphi_1\|_{\dot H^{-(1-(s-\varepsilon))}} + \|\varphi_2\|_{\dot H^{-(1-(s-\varepsilon))}} + \|u\|_{L^2 W_{s,\varepsilon}}^2 \|u\|_{L^\infty L^2} < \infty.
\end{aligned}$$

Let u^- be the radial solution of the linear equation (8) with radially symmetric initial data $(\varphi_1^-, \varphi_2^-) \in \widetilde{D}_{s+\varepsilon, s-\varepsilon} \times \widetilde{D}_{s+\varepsilon-1, s-\varepsilon}$. Then from the proof as above we can find a unique solution $u \in C_b(\mathbb{R}; H^{s+\varepsilon-\frac12} \cap \dot H^{-(1-(s-\varepsilon))}) \cap L^2 W_{s,\varepsilon}$ satisfying

$$u(t) = u^-(t) + \int_t^{-\infty} \omega^{-1} \sin((t-t')\omega) F(u)\, dt',$$

provided that $\|\varphi_1^-\|_{\widetilde{D}_{s+\varepsilon,s-\varepsilon}} + \|\varphi_2^-\|_{\widetilde{D}_{s+\varepsilon-1,s-\varepsilon}}$ is sufficiently small. Here the solution u satisfies (1) with initial data $\varphi \in \widetilde{D}_{s+\varepsilon, s-\varepsilon}$ such that

$$\varphi = u(0) = \varphi_1^- - \int_0^{-\infty} \omega^{-1} \sin(t'\omega) F(u)\, dt'.$$

Now in turn there are radial functions φ_1^+ and φ_2^+ as in Theorem 3. Actually, they are uniquely determined under a smallness condition of initial data. Hence we conclude that there exists a scattering operator S maps $(\varphi_1^-, \varphi_2^-)$ in a small neighborhood of $\widetilde{D}_{s+\varepsilon,s-\varepsilon} \times \widetilde{D}_{s+\varepsilon-1,s-\varepsilon}$ to $(\varphi_1^+, \varphi_2^+)$ in a small neighborhood of $\widetilde{D}_{s+\varepsilon,s-\varepsilon} \times \widetilde{D}_{s+\varepsilon-1,s-\varepsilon}$ and that S is injective.

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