

# Higher Multiplicity in the One-Dimensional Allen-Cahn Action Functional

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## Abstract

We prove the  $\Gamma$ -convergence of the Allen-Cahn action functional in the sharp-interface limit. In previous work, good lower bounds were developed under the assumption of single-multiplicity, but the bounds deteriorated in the case of higher-multiplicity interfaces. We develop improved bounds by working directly with the limiting energy measures.

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## 1 Introduction

In this paper we complete the analysis of the sharp-interface limit ( $\varepsilon \rightarrow 0$ ) of the functional

$$S_\varepsilon(u) = \frac{1}{4} \int_0^T \int_0^1 \varepsilon \dot{u}^2 + \varepsilon^{-1} (\varepsilon u_{xx} - \varepsilon^{-1} V'(u))^2 dx dt, \quad (1.1)$$

where for simplicity we choose the standard potential

$$V(u) = \frac{(1 - u^2)^2}{4}. \quad (1.2)$$

(The analysis can be carried out for any nondegenerate double-well potential  $V$ , changing only the value of the constant  $c_0$  defined below.) The work in [18] stopped short of a complete analysis because of the issue of multiplicity in the limiting energy. We now resolve that issue.

The functional  $S_\varepsilon$  will be defined on the space:

$$\mathcal{A} = \left\{ u \in C^\infty([0, 1] \times [0, T]) \mid u(\cdot, 0) \equiv -1, u(\cdot, T) \equiv +1, \right. \\ \left. u_x(0, t) = u_x(1, t) = 0 \right\}.$$

We think of  $\mathcal{A}$  as the space of pathways that connect  $u \equiv -1$  at time  $t = 0$  to  $u \equiv +1$  at time  $t = T$ . See Subsection 1.1 for a discussion of the related stochastic problem.

In order to introduce the limit functional and space, we need to introduce some measures. Let

$$\begin{aligned} \mathcal{M} := & \left\{ \text{measures } \mu \text{ on } [0, 1] \times [0, T] \mid \mu = \int_0^T d\mu^t dt \right. \\ & \text{and } \exists \{T_k\}_{k=1}^M = \{0 \leq T_1 < \dots < T_M \leq T\} \\ & \text{such that } \forall t \notin \{T_k\}_{k=1}^M, \mu^t = c_0 \sum_{j=1}^{N(k)} \delta_{g_j(t)} \\ & \text{where } 0 \leq g_1(t) \leq \dots \leq g_{N(k)}(t) \leq 1, \sup_k N(k) < \infty, \\ & \left. \text{and } g_j \in C((T_k, T_{k+1})), \dot{g}_j \in L^2((T_k, T_{k+1})) \forall j, k \right\}. \end{aligned}$$

Notice that the points  $g_j$  are allowed to be equal; we say that  $g_j$  has multiplicity  $J$  if  $J$  is the maximal number such that there exists a set  $\{g_{i+1}, \dots, g_{i+J}\} \ni g_j$  with

$$g_{i+1} = g_{i+2} = \dots = g_{i+J}.$$

Let

$$\mu^{T_k^+} = \lim_{t \downarrow T_k} \mu^t, \quad \mu^{T_k^-} = \lim_{t \uparrow T_k} \mu^t,$$

where we set by definition

$$\mu^t = 0 \quad \text{for } t < 0 \text{ or } t > T, \quad (1.3)$$

so that in particular  $\mu^{T_1^-} = \mu^{T_M^+} = 0$ . For  $\mu \in \mathcal{M}$ , define

$$S^{\mathcal{M}}(\mu) = \frac{1}{2} \sum_{k=1}^M \left| \mu^{T_k^+} - \mu^{T_k^-} \right|_{TV} + \frac{c_0}{4} \sum_{k=1}^{M-1} \int_{(T_k, T_{k+1})} \sum_{j=1}^{N(k)} (\dot{g}_j)^2 dt. \quad (1.4)$$

Here  $|\cdot|_{TV}$  denotes the total variation norm and the constant  $c_0$  denotes

$$c_0 = \int_{-1}^{+1} \sqrt{2V(u)} du \stackrel{(1.2)}{=} \frac{2\sqrt{2}}{3}.$$

In some sense, (1.4) represents the limit of (1.1), but although the functional  $S_\varepsilon$  is defined on *functions*, the limiting object  $S^{\mathcal{M}}$  is defined on *measures*. For the usual  $\Gamma$ -convergence framework, we would like instead to measure the limiting cost in terms of the function  $u$  to which a sequence  $\{u_\varepsilon\}$  converges. We make that connection below.

First let us define the set of admissible functions in the following way.

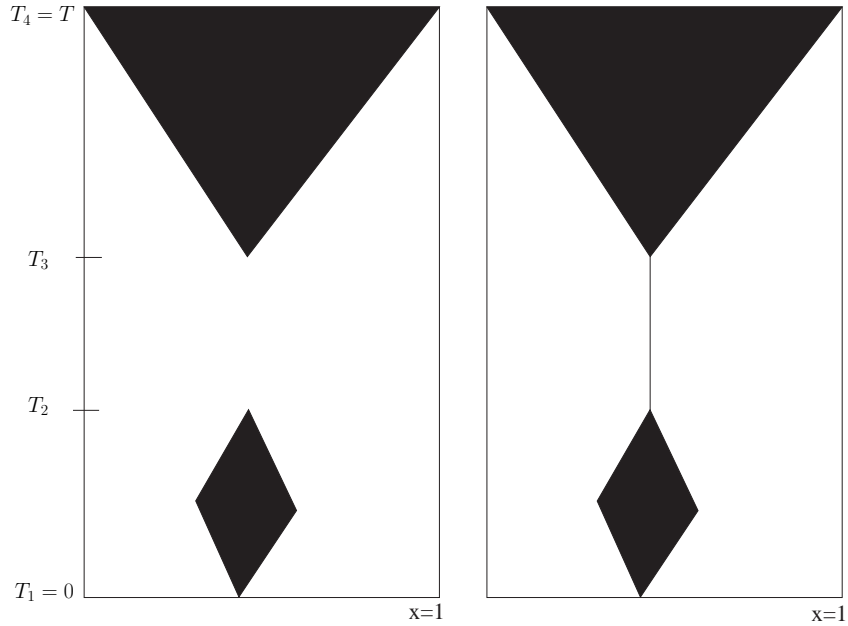


Figure 1: The figure on the left depicts a function  $u(x, t)$  that is equal to  $+1$  in the shaded regions and  $-1$  outside. A measure  $\mu_u$  that is supported *only* at the discontinuities of  $u$  incurs a cost at  $T_2$  and  $T_3$  in the first term of  $S^M$ . A measure  $\mu$  that has a multiplicity two interface for  $t \in (T_2, T_3)$  as depicted in the figure on the right, however, incurs no such cost.

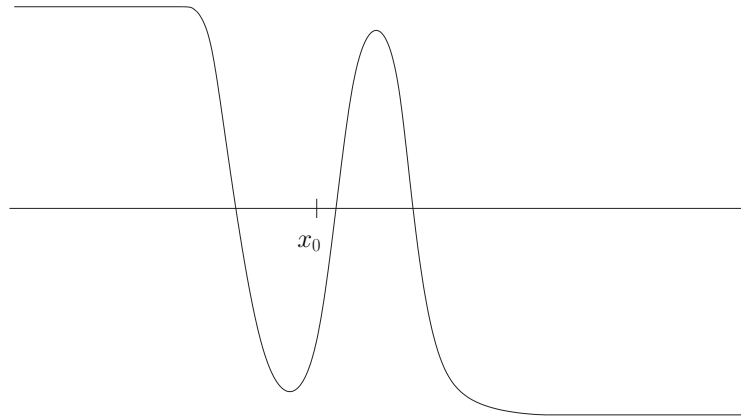


Figure 2: A sequence  $\{u_\varepsilon\}$  in which three interfaces accumulate at the same point  $x_0$  in the limit  $\varepsilon \rightarrow 0$  leads to a multiplicity three delta mass at  $x_0$ . (See Subsection 1.2.1 for details.) The single-multiplicity assumption that was used in [18] prohibited such behavior. Theorem 1.1 removes the single-multiplicity assumption.

**Definition 1 (Admissible functions).** We call the function  $u_0 : [0, 1] \times [0, T] \rightarrow \mathbb{R}$  admissible if

(i) The function  $u_0 = \pm 1$  a.e. and the number of jump discontinuities of  $u_0(\cdot, t)$  is uniformly bounded.

(ii) The boundary between a region of  $u_0 = +1$  and  $u_0 = -1$  is a continuous function of time. (More precisely, the boundary of  $u_0 = +1$  is contained by the graphs of finitely many continuous functions of time.)

(iii) For any  $x_0 \in [0, 1]$  and  $r > 0$ , the function

$$m(\{x \in B_r(x_0) \mid u(x, t) = +1\})$$

is a continuous function of time. Here  $m(A)$  denotes Lebesgue measure and  $B_r(x_0)$  denotes a ball of radius  $r$  around  $x_0$ .

(iv) The function satisfies  $u_0(\cdot, 0) \equiv -1$  and  $u_0(\cdot, T) \equiv +1$ .

From [18] it follows:

**Lemma 1.1.** For any sequence of functions  $u_\varepsilon \in \mathcal{A}$  such that  $S_\varepsilon(u_\varepsilon)$  is uniformly bounded, there exists a subsequence and a limit function  $u_0$  such that  $u_0$  is an admissible function and

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|u_\varepsilon(\cdot, t) - u_0(\cdot, t)\|_{L^2([0, 1])} = 0.$$

Therefore it makes sense to consider the set of admissible functions. We associate to each admissible  $u$  the measures  $\mu_u$  that are compatible in the following sense.

**Definition 2 (Compatible measures).** Suppose  $u(\cdot, t) = \pm 1$  a.e. and  $u(\cdot, t)$  has jump discontinuities  $0 < y_1 < y_2 < \dots < y_{n(t)} < 1$ . We say that the measure  $\mu_u \in \mathcal{M}$  is compatible with  $u$  if

(a) For all  $t \notin \{T_k\}_{k=1}^M$ ,

$$\{y_j\}_{j=1}^{n(t)} \subset \{g_j\}_{j=1}^{N(k)}.$$

(b) At points where  $\mu_u$  has a delta mass with odd multiplicity,  $u$  has a jump discontinuity, and at points where  $\mu_u$  has a delta mass with even multiplicity,  $u$  has no jump discontinuity.

We define

$$\mathcal{A}_0 = \left\{ u \text{ is admissible and there exists a compatible measure } \mu_u \in \mathcal{M} \right\}.$$

Notice that for a given function  $u \in \mathcal{A}_0$ , there are many compatible measures  $\mu_u$ . We will show the  $\Gamma$ -convergence of  $S_\varepsilon$  to the functional  $S_0 : \mathcal{A}_0 \rightarrow \mathbb{R}$  defined via:

$$S_0(u) := \inf_{\mu_u} S^{\mathcal{M}}(\mu_u). \quad (1.5)$$

In other words, the limit functional chooses the “best” admissible measure. See Figure 1 for an example in which the best measure is *not* just the one with delta masses at the discontinuities of the limit function.

There are two ingredients for the  $\Gamma$ -convergence. The first is:

**Proposition 1.1 (Upper bound).** *For every  $u_0 \in \mathcal{A}_0$ , there exists a sequence  $u_\varepsilon \in \mathcal{A}$  with  $u_\varepsilon \rightarrow u_0$  in  $L^\infty(L^2)$  such that*

$$\limsup_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) \leq S_0(u_0).$$

We prove Proposition 1.1 in Section 2. The first task is to show that the infimum in (1.5) is attained; the rest of the proof relies on reducing to the construction from [17].

The lower bound is subtle since it requires proving that no sequence can do better than the constructions used in Proposition 1.1. In [18], a lower bound is proved *under the assumption of single-multiplicity*. This assumption prohibits different interfaces of the finite  $\varepsilon$  problem from accumulating at the same position in the limit  $\varepsilon \rightarrow 0$ ; cf. Figure 2 and the discussion in Subsection 1.2.2. In this paper we prove the lower bound with no extra assumptions:

**Theorem 1.1 (Lower bound).** *Given any sequence  $u_\varepsilon \in \mathcal{A}$  with*

$$\limsup_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) < \infty \quad \text{and} \quad u_\varepsilon \rightarrow u_0 \text{ in } L^\infty(L^2),$$

*we have that  $u_0 \in \mathcal{A}_0$  and moreover,*

$$\liminf_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) \geq S_0(u_0). \quad (1.6)$$

We briefly review the stochastic motivation behind (1.1). Then in Subsection 1.2 we summarize the results from [18] and discuss the method of this paper. In Subsection 1.3 we set notation and collect some interpretations and generalizations of our result.

## 1.1 Rare events in stochastic differential equations

**Ordinary differential equations.** The Wentzell-Freidlin theory of large deviations [12] links the study of “rare events” with the variational problem of action minimization, as we now explain. Consider the finite-dimensional stochastic gradient flow

$$\begin{aligned} \dot{X} &= -\nabla V(X) + \sqrt{2\gamma}\dot{B} & t > 0, \\ X &= x_- & t = 0, \end{aligned} \tag{1.7}$$

where  $X \in \mathbb{R}^n$ ,  $\dot{B}$  represents white noise, and  $V$  is a double-well potential with minima  $x_-$  and  $x_+$ . Under the deterministic dynamics ( $\gamma = 0$ )  $x_-$  is stable, but under noisy dynamics ( $\gamma > 0$ ) the solution is eventually driven out of the basin of attraction of  $x_-$  and into  $B_\epsilon(x_+)$ , a ball of radius  $\epsilon$  around  $x_+$ . Let  $\mathcal{B}$  denote the set of functions that “switch” in time  $T$ ,

$$\mathcal{B} := \left\{ x \mid x(0) = x_-, x(T) \in B_\epsilon(x_+) \right\}.$$

Wentzell-Freidlin theory estimates the exponential factor in the probability of switching as

$$\lim_{\gamma \rightarrow 0} \gamma \log \text{Prob}(X \in \mathcal{B}) = -\inf\{S(x) \mid x \in \mathcal{B}\}, \tag{1.8}$$

where  $S(\cdot)$  is the so-called large deviation action functional,

$$S(x) = \frac{1}{4} \int_0^T |\dot{x} + \nabla V(x)|^2 dt. \tag{1.9}$$

Notice that the minimization problem on the right-hand side of (1.8) is deterministic.

In addition to estimating the *probability* of switching, large deviation theory estimates the *mechanism* of switching. Suppose that  $x^*$  is the unique minimizer of  $S$  over  $\mathcal{B}$ . Then for any  $\delta > 0$ ,

$$\lim_{\gamma \rightarrow 0} \frac{\text{Prob}(X \in \bar{\mathcal{B}} \text{ and } \text{dist}(X, x^*) < \delta)}{\text{Prob}(X \in \bar{\mathcal{B}})} = 1.$$

(See [12], Chapter 3, Theorem 3.4.) In other words, given that switching is achieved, the stochastic trajectory stays within a small neighborhood of  $x^*$  with probability one in the zero-noise limit. For this reason  $x^*$  is called the most-likely switching pathway.

**Partial differential equations.** Large deviation theory generalizes to the case of infinite-dimensional stochastic gradient flows, i.e., stochastically perturbed *partial* differential equations. The simplest interesting example of a stochastic PDE that is well-posed and for which the large deviation action functional has been identified is the stochastic Allen-Cahn equation,

$$\begin{aligned} \dot{U} &= U_{xx} - V'(U) + \sqrt{2\gamma}\eta & t > 0, x \in [0, L], \\ U &= u_- & t = 0, x \in [0, L], \end{aligned} \quad (1.10)$$

where  $\eta$  is a space-time white noise and  $u_-$  is an energy minimizer; see below. Assume for simplicity that  $V$  is the standard double-well potential,

$$V(u) = \frac{(1 - u^2)^2}{4}.$$

The deterministic PDE (i.e.,  $\gamma = 0$ ) is the  $L^2$  gradient flow for the energy functional

$$E(u) = \int_0^L \frac{1}{2}(u_x)^2 + V(u) dx, \quad (1.11)$$

which admits two global minimizers  $u_-$  and  $u_+$  (as long as  $L \geq 2\pi$ ). For Neumann or periodic boundary conditions  $u_{\pm} \equiv \pm 1$ ; for Dirichlet boundary conditions  $u_{\pm} \approx \pm 1$  on most of  $[0, L]$  with modification near the boundary. For simplicity, we will focus in this section on homogeneous Dirichlet boundary conditions.

Faris and Jona-Lasinio [9] prove that

$$S(u) = \frac{1}{4} \int_0^T \int_0^L (\dot{u} - u_{xx} + V'(u))^2 dx dt \quad (1.12)$$

is the action functional for the Allen-Cahn equation on  $C_0^-$ , the space of continuous functions on  $[0, L] \times [0, T]$  with  $u(0, t) = u(L, t) = 0$  and  $u(\cdot, 0) = u_-$ , with the topology inherited from the sup norm.

In analogy with the finite dimensional system (1.7),  $u_-$  is stable under the deterministic dynamics, but for  $\gamma > 0$  there is a positive probability of switching to  $B_{\epsilon}(u_+)$ , a ball of radius  $\epsilon$  around the symmetric minimum. Let  $\mathcal{B}$  denote the set of functions in  $C_0^-$  that transform in time  $T$ :

$$\mathcal{B} := \left\{ u \mid u(0, t) = u(L, t) = 0, u(\cdot, 0) = u_-, u(\cdot, T) \in B_{\epsilon}(u_+) \right\}. \quad (1.13)$$

Notice that  $\mathcal{B}$  is a “regular set” in the sense of Wentzell and Freidlin [12], i.e.,

$$\inf_{u \in \mathcal{B}} S(u) = \inf_{u \in \bar{\mathcal{B}}} S(u) =: s. \quad (1.14)$$



In particular, it follows from [9] that

$$\lim_{\gamma \rightarrow 0} \gamma \log \text{Prob}(U \in \mathcal{B}) = -s. \quad (1.15)$$

As in the finite dimensional case, we have in addition that action minimizers are the most-likely switching pathways. We do not expect the action minimizer to be unique, so let us define

$$\mathcal{B}_* := \{u \mid u \in \overline{\mathcal{B}}, S(u) = s\}. \quad (1.16)$$

(Here  $\overline{\mathcal{B}}$  denotes the closure of  $\mathcal{B}$ .) From the work of Faris and Jona-Lasinio [9] it follows:

**Lemma 1.2.** *The set  $\mathcal{B}_*$  is nonempty and for any  $\delta > 0$ ,*

$$\lim_{\gamma \rightarrow 0} \frac{\text{Prob}(U \in \overline{\mathcal{B}} \text{ and } \text{dist}(U, \mathcal{B}_*) < \delta)}{\text{Prob}(U \in \overline{\mathcal{B}})} = 1. \quad (1.17)$$

We include a proof of Lemma 1.2 at the end of the introduction. Now we turn to the analysis of the action functional. For an interpretation of our results in terms of the stochastic equation see Remark 5 in Subsection 1.3.

## 1.2 Analysis of the sharp-interface limit

Because the action minimization problem is complicated, it is natural to ask whether there are limiting regimes in which the analysis simplifies. It is well-known that for the finite-dimensional problem in the limit  $T \rightarrow \infty$ , the most-likely pathway is the one that follows the time-reversed gradient flow

$$\dot{x} = \nabla V(x)$$

to flow from  $x_-$  to the saddle point with lowest energy, and the forward gradient flow

$$\dot{x} = -\nabla V(x)$$

to flow from this saddle point to  $B_\epsilon(x_+)$ . Faris and Jona-Lasinio proved that the same holds for the infinite dimensional problem [9]. This saddle-point problem that controls the action minimization problem in the long-time limit  $T \rightarrow \infty$  is important; indeed, if phase transformation is studied on the natural time-scale of the system, then this pathway is the most likely.

Our focus is different. We are interested in rare events that are observed when one conditions on switching being achieved within a finite time  $T$ . Such

events are physically relevant: For instance they explain phenomena observed in magnetic switching experiments [19]. They are also analytically interesting: We will see that a competition between the time- and length-scales of the system leads to a family of action minimizing pathways with increasing spatial structure.

**Introduction of the sharp-interface limit.** We consider the action functional in the limit  $L \rightarrow \infty$ ,  $T \rightarrow \infty$ ,  $L/\sqrt{T} = \text{constant}$ . (For a general discussion of the different parameter regimes of the action problem, see [17].) Letting  $\varepsilon := 1/L$ ,  $x \rightarrow \varepsilon x$ ,  $t \rightarrow \varepsilon^2 t$ , and  $T \rightarrow \varepsilon^2 T$ , (1.12) can be reexpressed as:

$$S_\varepsilon(u) = \frac{1}{4} \int_0^T \int_0^1 \left( \varepsilon^{1/2} \dot{u} + \varepsilon^{-1/2} (\varepsilon u_{xx} - \varepsilon^{-1} V'(u)) \right)^2 dx dt, \quad (1.18)$$

where we have grouped terms in order to isolate

$$f_\varepsilon(u) := \varepsilon u_{xx} - \varepsilon^{-1} V'(u),$$

the first variation of the rescaled energy

$$E_\varepsilon(u) = \int_0^1 \frac{\varepsilon}{2} (u_x)^2 + \frac{V(u)}{\varepsilon} dx. \quad (1.19)$$

Recall the boundary conditions

$$u_x(0, t) = u_x(1, t) = 0 \quad (1.20)$$

and the initial and final conditions

$$u(x, 0) = u_-, \quad u(x, T) = u_+. \quad (1.21)$$

(See Remark 2 in Subsection 1.3 for alternate formulations.) Notice that (1.21) implies

$$E_\varepsilon(u(\cdot, 0)) = E_\varepsilon(u(\cdot, T)) = 0. \quad (1.22)$$

Squaring the integrand of (1.18), using (1.20) to integrate by parts, and applying (1.22), we arrive at the functional  $S_\varepsilon$  defined in (1.1).

In addition, observe that for any sequence  $u_\varepsilon$  with action bounded by  $\bar{C}$  and for any time  $t \leq T$ , we have

$$\bar{C} \geq S_\varepsilon(u) \geq \frac{1}{2} \left| \int_0^t \int_0^1 \dot{u} f_\varepsilon(u) dx dt \right| \stackrel{(1.20), (1.22)}{=} \frac{1}{2} |E_\varepsilon(u(\cdot, t))|. \quad (1.23)$$

Hence the energy is uniformly bounded in time, and (1.19) implies that  $u_\varepsilon$  must converge to  $\pm 1$  almost everywhere in space with *sharp interfaces* dividing regions of  $u = +1$  from regions of  $u = -1$ . The uniform energy bound (1.23) is critical for the analysis.

**Related sharp-interface limits.** The sharp-interface limit of the action functional  $S_\varepsilon$  fits naturally into a “family” of sharp-interface problems that are well-known in the calculus of variations and PDE communities. The convergence of the energy functional (1.19) to the perimeter functional was analyzed in [21] (see also [20] and [25]). Subsequently, the sharp-interface limit of the gradient flow dynamics was investigated; see [3, 14, 2] for the case  $d = 1$  and [24, 6, 7, 16] for the case  $d > 1$ . Another related problem in  $d > 1$  is a conjecture of DeGiorgi on which there has been recent progress [23, 22] and which says, roughly speaking,

$$\varepsilon^{-1} \int_{\Omega} (\varepsilon \Delta u - \varepsilon^{-1} V'(u))^2 dx \rightarrow c_0 \int_{\Gamma} \kappa^2 d\sigma,$$

where  $\Gamma$  is the interface and  $\kappa$  is the mean curvature. While closely related to these sharp-interface problems, the action functional is unique as a *time-dependent variational problem*.

### 1.2.1 Summary of previous results for the action functional

The numerical study of the sharp-interface limit of the Allen-Cahn action functional for  $d = 1, 2$  in [8] suggested two competing action costs: A nucleation cost to form interfaces, and a propagation cost to move them. (See also [11] for  $d = 1$  and [17] for  $d \geq 1$ .) The numerically observed minimizers formed an optimal number  $N$  of interfaces at  $t = 0$  and moved the interfaces across the system with constant velocity. It was observed that the optimal number of interfaces increases as the experiment time  $T$  decreases, which was understood to reflect the fact that moving a single wall across the system in a short time incurs a high propagation cost.

To what degree can these observations be made rigorous? This question was raised in [18] (for  $d = 1$ ). It was observed that insight into the action functional could be gained by exploiting earlier results on the time-independent problem

$$\varepsilon \Delta u - \varepsilon^{-1} V'(u) = f_\varepsilon,$$

where  $f_\varepsilon$  is a sequence of functions satisfying a given bound (cf. [15, 27, 26]). For the convenience of the reader, we recall the main results from [18].

Consider the energy measures defined by

$$d\mu_\varepsilon := \left( \frac{\varepsilon}{2}(u_x)^2 + \frac{V(u)}{\varepsilon} \right) dx dt, \quad (1.24)$$

and the action measures  $\nu_\varepsilon$  defined by

$$d\nu_\varepsilon := \frac{1}{4} (\varepsilon \dot{u}^2 + \varepsilon^{-1} f_\varepsilon^2) dx dt. \quad (1.25)$$

The first result is a continuity theorem for the limiting energy measures:

**Theorem 1.2 (Continuity, Theorem 1.1 in [18]).** *Consider any sequence of smooth functions on  $[0, 1] \times [0, T]$  that have uniformly bounded action, bounded initial and final energy, and Neumann boundary conditions. Choose any subsequence such that the corresponding measures  $\mu_\varepsilon$  and  $\nu_\varepsilon$  converge as measures to  $\mu$  and  $\nu$  in the limit  $\varepsilon \rightarrow 0$ . Let  $E$  be the set of times at which*

$$\eta := \int_{[0,1]} d\nu \quad (1.26)$$

has a point mass. Then

- (1) For all  $t$  in  $[0, T] \setminus E$ ,  $\mu_\varepsilon^t$  converges as a measure to a limit,  $\mu^t$ .
- (2) For all  $t$  in  $[0, T] \setminus E$ ,  $\mu^t$  is continuous as a function of  $t$  with values in  $(W^{1,\infty})^*$ .
- (3)  $\mu(\Psi) = \int_0^T \mu^t(\Psi) dt$  for all  $\Psi \in C([0, T] \times [0, 1])$ .

The second result identifies the structure of the limit measures. It can be expressed:

**Theorem 1.3 (Structure, Theorem 1.2 in [18]).** *For any subsequence as in Theorem 1.2, there exists a finite set of “singular times”*

$$\mathcal{T}_{sing} := \{T_k\}_{k=1}^M = \{0 \leq T_1 < T_2, \dots < T_M \leq T\}$$

such that for all  $t \in [0, T] \setminus \mathcal{T}_{sing}$ ,

$$\mu_\varepsilon^t \xrightarrow{\varepsilon \rightarrow 0} \mu^t = c_0 \sum_{j=1}^{N(k)} \delta_{g_j(t)}, \quad (1.27)$$

where

$$0 \leq g_1(t) \leq g_2(t) \leq \dots \leq g_{N(k)}(t) \leq 1. \quad (1.28)$$

Here  $\delta_{g_j}$  is the delta-function at  $g_j$  and  $N(k) < \infty$ . Moreover,

$$\forall j = 1, \dots, N(k), \quad g_j(t) \text{ is a continuous function of time.}$$

Time-integrated equipartition of energy follows as a corollary:

**Corollary 1 (Equipartition, Corollary 1 in [18]).** *Consider any subsequence as in Theorem 1.3 and any interval  $[s, t] \subset [0, T]$  that does not contain any singular time of  $\mu$ . Then*

$$\lim_{\varepsilon \rightarrow 0} \int_s^t \int_0^1 \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 dx dt' = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_s^t \int_0^1 \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \frac{V(u_\varepsilon)}{\varepsilon} \right) dx dt'.$$

As far as the action functional  $S_\varepsilon$ , [18] identifies the limit of the *minimum value* of the action, but stops short of a true  $\Gamma$ -convergence argument for the functional itself; see below.

### 1.2.2 Progress and method

In the case of higher multiplicity, the method of [18] produces too weak a lower bound. For an interface  $g$  of multiplicity  $J$  one expects the propagation cost to be  $J$  times that of a single-interface construction. Instead the lower bound from [18] is of the form

$$\frac{c_0}{J} \int_0^T (\dot{g})^2 dt.$$

While valid as a bound, it fails to capture the extra cost of moving a  $J$ -multiplicity wall.

The difficulty is that in order to get a sharp lower bound for the action functional one needs information about not just the limit *function*  $u_0$ , but also the limit *measure*  $\mu$ . (The single-multiplicity assumption hides this difficulty since then  $u_0$  uniquely identifies the measure  $\mu$ .) A loss of information in the case of higher-multiplicity interfaces is familiar: In the case of the energy functional (1.19), the limiting energy is proportional to the multiplicity, but the perimeter functional (the  $\Gamma$ -limit) throws away this extra cost. To get a good bound for the action, however, we need to track the limiting energy measures, not just the perimeter.

The main ingredient for Theorem 1.1 is:

**Proposition 1.2.** *Let  $\{u_\varepsilon\}$  be any sequence of smooth functions with Neumann boundary conditions and uniformly bounded action. Assume without loss that  $\mu_\varepsilon$  and  $\nu_\varepsilon$  converge. Suppose that  $[0, T]$  contains no singular time.*

*Then the interfaces  $\{g_j\}_{j=1}^N$  are such that  $\dot{g}_j \in L^2([0, T]) \forall j = 1 \dots N$  and moreover,*

$$c_0 \int_0^T \sum_{j=1}^N (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt. \quad (1.29)$$

To illustrate the idea, we roughly sketch the argument for an isolated interface  $g$  that has multiplicity  $J$  on  $[0, T]$ , so that in particular for any subinterval  $[t_1, t_2]$  we have

$$\mu^t = c_0 J \delta_{g(t)} \quad \forall t \in [t_1, t_2]. \quad (1.30)$$

Notice also that by Corollary 1, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \int_0^1 \varepsilon (u_{\varepsilon, x})^2 dx dt = c_0 J \Delta t, \quad \text{where } \Delta t := t_2 - t_1. \quad (1.31)$$

Suppose that  $g$  moves monotonically to the right and let  $\Delta g := g(t_2) - g(t_1)$ . Define the piecewise linear test function:

$$\phi(x) = \begin{cases} 0 & x < g(t_1) \\ x - g(t_1) & g(t_1) \leq x \leq g(t_2) \\ \Delta g & x > g(t_2) \end{cases}$$

and observe that

$$|\phi'| \leq 1 \text{ a.e.} \quad \text{and} \quad |\phi|_\infty \leq \Delta g. \quad (1.32)$$

We compute formally:

$$\begin{aligned} c_0 J \Delta g &\stackrel{(1.30)}{=} \int_0^1 \phi d\mu^{t_2} - \int_0^1 \phi d\mu^{t_1} \\ &= \lim_{\varepsilon \rightarrow 0} \left( \int_0^1 \phi d\mu_\varepsilon^{t_2} - \int_0^1 \phi d\mu_\varepsilon^{t_1} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \frac{d}{dt} \int_0^1 \phi \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \int_0^1 \phi (\varepsilon u_{\varepsilon, x} \dot{u}_{\varepsilon, x} + \varepsilon^{-1} W'(u_\varepsilon) \dot{u}_\varepsilon) dx dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \int_0^1 (-\phi' \varepsilon u_{\varepsilon, x} \dot{u}_\varepsilon + \phi f_\varepsilon \dot{u}_\varepsilon) dx dt \\ &\leq \lim_{\varepsilon \rightarrow 0} \left( \int_{t_1}^{t_2} \int_0^1 (\phi')^2 \varepsilon (u_{\varepsilon, x})^2 dx dt \int_{t_1}^{t_2} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt \right)^{1/2} \\ &\quad + \frac{1}{2} |\phi|_\infty \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} f_\varepsilon^2 dx dt \\ &\stackrel{(1.32)}{\leq} \left( c_0 J \Delta t \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt \right)^{1/2} + 2 \Delta g \int_{t_1}^{t_2} d\eta. \quad (1.33) \end{aligned}$$

The idea is that if  $\eta([t_1, t_2])$  is small, then (1.33) gives

$$(c_0 J - \delta) \frac{(\Delta g)^2}{\Delta t} \leq \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{t_2} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt$$

for some small  $\delta > 0$ . Taking a Riemann sum and letting  $\delta \rightarrow 0$  leads to

$$c_0 J \int_0^T (\dot{g})^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt.$$

In the actual proof, we will replace  $\Delta g$  by the oscillation of  $g$  on subintervals and use the fact:

**Lemma 1.3.** *Let  $g$  be any continuous function on  $[0, T]$ . Let  $\Sigma$  denote the family of all finite partitions  $\sigma$  of  $[0, T]$ ,*

$$\sigma = \{0 = t_0 < t_1 < \dots < t_n = T\},$$

and

$$|\sigma| := \max_{0 \leq k \leq n-1} |t_{k+1} - t_k|.$$

Suppose that

$$\lim_{\delta \rightarrow 0} \left( \sup_{\substack{\sigma \in \Sigma, \\ |\sigma| < \delta}} \sum_{k=0}^{n-1} \frac{(\text{osc}_{[t_k, t_{k+1}]} g)^2}{t_{k+1} - t_k} \right) =: M < \infty.$$

Then  $\dot{g} \in L^2([0, T])$  and

$$\int_0^T (\dot{g})^2 dt = M.$$

### 1.3 Remarks and notation

It is helpful to introduce some language.

**Definition 3 (Multiplicity).** *Given an interface  $g : [0, T] \rightarrow [0, 1]$ , the set*

$$\{\text{Mult } g = J\}$$

*refers to the set of times  $t \in [0, T]$  for which  $J$  is the maximal number such that there exists a set of interfaces  $\{g_\ell\}_{\ell=j+1}^{j+J} \ni g$  with*

$$g_{j+1}(t) = g_{j+2}(t) = \dots = g_{j+J}(t).$$

*We say that on this set  $g$  “has multiplicity  $J$ .” When we are referring to a group of  $J$  interfaces, we will sometimes shorthand*

$$\{\text{Mult} = J\} \stackrel{\text{short}}{=} \{\text{Mult } g = J\}.$$

**Definition 4 (Isolated group of interfaces).** Let  $1 \leq j_1 \leq j_2 \leq N$ . We will call  $\{g_j\}_{j=j_1}^{j_2}$  an isolated group of interfaces on  $(t_1, t_2)$  if there does not exist  $t \in (t_1, t_2)$  such that  $g_{j_1-1}(t) = g_{j_1}(t)$  or  $g_{j_2}(t) = g_{j_2+1}(t)$ .

**Definition 5 (Consecutive group of interfaces).** By a consecutive group of  $J$  interfaces we will mean a set  $g_{j+1}, g_{j+2}, \dots, g_{j+J}$ .

**Remark 1 (Elementary bounds).** We will often refer to the fact that if a sequence  $u_\varepsilon$  has uniformly bounded action, i.e.,

$$\limsup_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) \leq \bar{C},$$

then it follows from (1.25) and (1.26) that:

$$\int_0^T d\eta \leq \bar{C} \quad (1.34)$$

and from (1.23) that

$$\sup_{t \in [0, T]} E_\varepsilon(u(\cdot, t)) \leq 2\bar{C}.$$

**Remark 2 (Different initial and final conditions).** The initial condition  $u(\cdot, 0) \equiv -1$  in the definition of  $\mathcal{A}$  is not necessary. It can be replaced by a condition of uniformly bounded initial energy, or  $S_\varepsilon(u)$  can be replaced by

$$E_\varepsilon(u) + S_\varepsilon(u).$$

We work with  $u(\cdot, 0) \equiv -1$  for simplicity and because of the switching problem that motivates our study of the action functional.

One can also remove the end condition  $u(\cdot, T) \equiv +1$  and work instead with the original functional

$$S_\varepsilon(u) = \frac{1}{4} \int_0^T \int_0^1 \left( \varepsilon^{1/2} \dot{u} + \varepsilon^{-1/2} (\varepsilon u_{xx} - \varepsilon^{-1} V'(u)) \right)^2 dx dt. \quad (1.35)$$

This form is more natural from a probabilistic point of view, but since we are working mainly with the variational problem we use (1.1) (which has an attractive symmetry). So that we can apply our results to the stochastic problem, let us consider how the sharp-interface limits of (1.35) and (1.1) are related.

In the sharp-interface limit of (1.35), the total variation in  $S^\mathcal{M}$  is replaced by

$$\sup_{0 \leq \phi \leq 1} \left( \mu_k^{T^+}(\phi) - \mu_k^{T^-}(\phi) \right)^+, \quad (1.36)$$



where  $(\cdot)^+$  denotes positive part. This relies mainly on the fact that for any singular time  $T_j$  we have

$$\begin{aligned} & \int_{T_j-\delta}^{T_j+\delta} \int_0^1 (\varepsilon^{1/2} \dot{u} - \varepsilon_i^{-1/2} f_\varepsilon(u))^2 dx dt \\ & \geq \int_{T_j-\delta}^{T_j+\delta} \int_0^1 (\varepsilon^{1/2} \dot{u} - \varepsilon^{-1/2} f_\varepsilon(u))^2 \phi(x) dx dt \\ & \geq 4 \left( \int_{T_j-\delta}^{T_j+\delta} \int_0^1 -\dot{u} f_\varepsilon(u) \phi(x) dx dt \right)^+. \end{aligned}$$

(See [18], proof of Theorem 1.4.) Heuristically, (1.36) reflects that instead of paying half the nucleation cost to form or annihilate delta masses, one pays the full cost to form them and nothing to annihilate them.

The second term in the sharp-interface limit of (1.35) is the same as for (1.1). To see that the propagation cost is unchanged, notice that on any interval  $(t_1, t_2)$  containing no singular time, we have by Theorem 1.3 that  $\mu^t([0, 1])$  is constant for all  $t \in (t_1, t_2)$ . It follows in particular that

$$\int_{t_1}^{t_2} \int_0^1 (\varepsilon^{1/2} \dot{u} + \varepsilon^{-1/2} f_\varepsilon(u))^2 dx dt = \int_{t_1}^{t_2} \int_0^1 \varepsilon (\dot{u})^2 + \varepsilon^{-1} (f_\varepsilon(u))^2 dx dt.$$

One modification is that with the end condition removed, we need to add  $(T_M, T)$  to the intervals over which we integrate in the second term on the right-hand side of (1.4). When  $u(\cdot, T) \equiv +1$  is enforced, either  $T$  is a singular time or the interfaces have all annihilated at  $T_M < T$ , so that there is no propagation cost on  $(T_M, T)$ . When  $u(\cdot, T) \equiv +1$  is not enforced,  $T$  is by definition never a singular time, however interfaces may propagate on  $(T_M, T)$ .

**Remark 3 (Different boundary conditions).** Neumann boundary conditions are simplest, but the  $\Gamma$ -convergence of the action functional can also be proved for the case of homogeneous Dirichlet boundary conditions. In this case the limit of the initial energy is  $c_0$  instead of zero, and  $\mu^0$  consists of two delta masses with weight  $c_0/2$  at  $x = 0$  and  $x = 1$ .

**Remark 4 (Minimizers are what we expect).** The  $\Gamma$ -convergence of  $S_\varepsilon$  implies in particular that minimizing configurations are what we expect. Consider the limit problem

$$\inf_{u \in \mathcal{A}_0} S_0(u).$$

The work in [18] identified the minimum value, but not the minimizer(s). What one expects is that for a given value of  $T$ , a minimizer should have an optimal

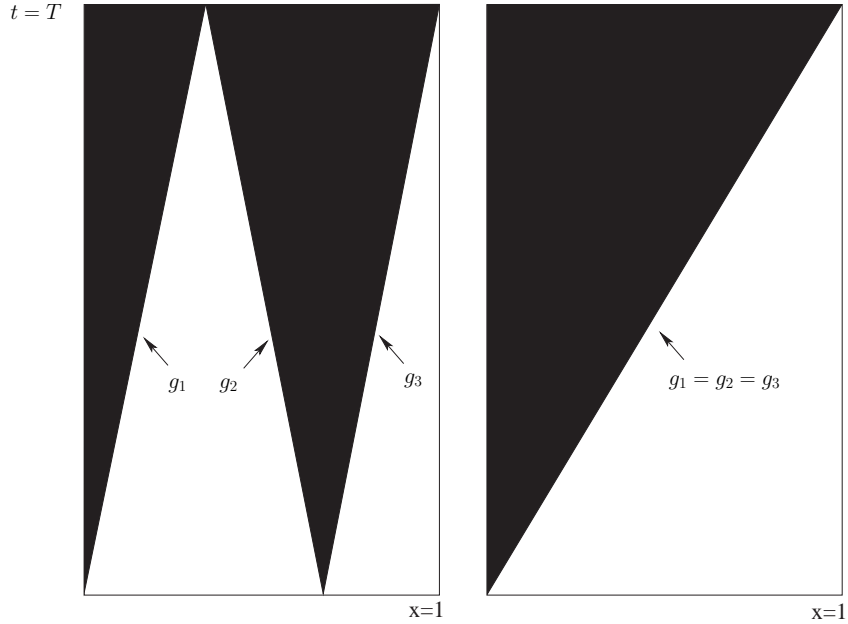


Figure 3: When  $T$  is sufficiently small, achieving the minimal action requires three delta masses (cf. Remark 4). While [18] showed that the single-multiplicity configuration depicted on the left attains the minimal action (in this regime), it could not rule out the higher-multiplicity configuration depicted on the right. Theorem 1.1 resolves this issue. Minimizing configurations *must* have single-multiplicity and constant velocities. The only minimizers are the one shown on the left and its reflection.

*number  $N$  of jump discontinuities, which form at  $t = 0$  and move with constant velocity across the system (cf. Figure 3). Moreover, one expects the associated optimal measure to have single-multiplicity. While [18] showed that such a configuration achieved the minimal action, it did not show that this was the ONLY way to achieve the minimal action. In particular, it could not rule out higher-multiplicity configurations (cf. Figure 3, right-hand figure). The lower bound in Theorem 1.1 and the structure of  $S^M$  reveal that indeed, optimal measures are what we expect: There is an optimal number  $N$  of delta masses (depending on  $T$ ), and to minimize the second term in  $S^M$  they must have single-multiplicity and move with constant velocity across the system.*

**Remark 5 (Stochastic interpretation of our results).** *By considering the limit of (1.35), the action minimization problem is reduced to the question of how best to place and move points! In this remark we consider the implications of the  $\Gamma$ -convergence for the stochastic equations. Notice that the rescaled*

functional (1.35) is the action functional corresponding to

$$\varepsilon \dot{U} = \varepsilon U_{xx} - \varepsilon^{-1} V'(U) + \sqrt{2\gamma\varepsilon}\eta. \quad (1.37)$$

We can also obtain (1.37) by rescaling space and time in the S-PDE (1.10).

In order to connect our results with the large deviation estimates of [9], consider the S-PDE with homogeneous Dirichlet boundary conditions. Fix  $T$  and let  $s_0$  denote the minimum of the limit action (modified according to Remarks 2 and 3 above). The large deviation estimate (1.15) and the  $\Gamma$ -convergence of (1.35) imply that for the solution of (1.37) we have

$$s_0 - o(1)_{\varepsilon \rightarrow 0} \leq - \lim_{\gamma \rightarrow 0} \gamma \log \text{Prob}(U \in \mathcal{B}) \leq s_0 + o(1)_{\varepsilon \rightarrow 0}.$$

As far as the most-likely switching pathways, Lemma 1.2 says that trajectories stay within a  $\delta$ -neighborhood of action minimizers (in the sup norm), but it doesn't tell us what the action minimizers are. The  $\Gamma$ -convergence of (1.35) gives us a way to approximate switching trajectories. Consider a sequence of action minimizers  $u_\varepsilon^*$  with  $u_\varepsilon^* \rightarrow u^*$  as  $\varepsilon \rightarrow 0$ . Consider the corresponding functions  $\tilde{u}_\varepsilon \rightarrow u^*$  constructed as in Proposition 1.1 with modification for the boundary conditions. Finally, consider the stochastic trajectories  $U$  that are within a  $\delta$ -neighborhood of  $u_\varepsilon^*$  in the sup norm on  $[0, T] \times [0, 1]$ . Noting that

$$\sup_{t \in [0, T]} \|U - u_\varepsilon^*\|_{L^2([0, 1])} \leq \delta,$$

we observe that  $U$  is well-approximated by the constructions  $\tilde{u}_\varepsilon$  in the sense that

$$\begin{aligned} & \sup_{t \in [0, T]} \|U - \tilde{u}_\varepsilon\|_{L^2([0, 1])} \\ & \leq \sup_{t \in [0, T]} \|U - u_\varepsilon^*\|_{L^2([0, 1])} + \sup_{t \in [0, T]} \|u_\varepsilon^* - u^*\|_{L^2([0, 1])} + \sup_{t \in [0, T]} \|u^* - \tilde{u}_\varepsilon\|_{L^2([0, 1])} \\ & \leq \delta + o(1)_{\varepsilon \rightarrow 0}. \end{aligned}$$

It is tempting to conjecture that the limiting functional is in fact the action functional of a sharp-interface stochastic process, i.e., a stochastic process taking values in  $\{-1, +1\}$ . To prove the conjecture requires taking the sharp-interface limit of the stochastic equation (1.10). While there are some results concerning sharp-interface limits of stochastic equations (cf. [13, 10, 1]), there are many open questions. In particular, rare events in stochastic, sharp-interface models are not yet understood.

## 1.4 Proof of Lemma 1.2

*Proof.* As in Subsection 1.1 above, let  $C_0^-$  denote the space of continuous functions on  $[0, L] \times [0, T]$  that satisfy  $u(0, t) = u(L, t) = 0$  and  $u(\cdot, 0) = u_-$ , equipped with the sup norm. By [9, Proposition 6.3], we have for any  $C < \infty$  that

$$\{u \mid S(u) \leq C\} \quad \text{is compact in } C_0^-. \quad (1.38)$$

Consider the set  $\mathcal{B} \subset C_0^-$  defined in (1.13) and a minimizing sequence  $\{u_n\}$  in  $\overline{\mathcal{B}}$  such that  $\lim_{n \rightarrow \infty} S(u_n) = s$ , the minimal action, cf. (1.14). Since  $\overline{\mathcal{B}}$  is a closed subset of a compact space, we can extract a convergent subsequence. Let  $u^*$  denote the limit. By [9, Proposition 6.2],

$$S(\cdot) \quad \text{is lower semi-continuous on } C_0^-. \quad (1.39)$$

Hence  $S(u^*) = s$  and  $\mathcal{B}_*$  defined in (1.16) is not empty.

We now turn to the proof of (1.17). Since  $\mathcal{B}$  is an open set in  $C_0^-$ , it follows from [9, Theorem 6.1] that for every  $\zeta > 0$  there exists  $\gamma_0 > 0$  such that for  $\gamma \leq \gamma_0$  we have

$$\text{Prob}(U \in \overline{\mathcal{B}}) \geq \text{Prob}(U \in \mathcal{B}) \geq \exp\left(-\frac{1}{\gamma}(s + \zeta)\right). \quad (1.40)$$

Next, we claim that for any  $\delta > 0$ , we have

$$\bar{s} := \inf_{\substack{u \in \overline{\mathcal{B}} \\ \text{dist}(u, \mathcal{B}_*) \geq \delta}} S(u) > s. \quad (1.41)$$

Indeed, suppose to the contrary that there exists a sequence  $\{u_n\}$  in  $\overline{\mathcal{B}}$  with

$$\begin{aligned} \lim_{n \rightarrow \infty} S(u_n) &= s \\ \text{dist}(u_n, \mathcal{B}_*) &\geq \delta. \end{aligned}$$

We may assume without loss that  $S(u_n) \leq C$  and hence by (1.38), we can extract a convergent subsequence. Let  $\bar{u}$  denote the limit. It follows that

$$\text{dist}(\bar{u}, \mathcal{B}_*) \geq \delta. \quad (1.42)$$

On the other hand, by (1.39) we have in particular

$$s \leq S(\bar{u}) \leq \lim_{n \rightarrow \infty} S(u_n) = s,$$

which implies that  $\bar{u} \in \mathcal{B}_*$ , contradicting (1.42).

Hence (1.41) is established. Moreover, since

$$\{u \mid u \in \bar{\mathcal{B}}, \text{dist}(u, \mathcal{B}_*) \geq \delta\}$$

is a closed set in  $C_0^-$ , it follows from [9, Theorem 6.1] that for every  $\zeta > 0$ , there exists a  $\gamma_0 > 0$  such that for  $\gamma \leq \gamma_0$  we have

$$\text{Prob}\left(U \in \bar{\mathcal{B}} \text{ and } \text{dist}(U, \mathcal{B}_*) \geq \delta\right) \leq \exp\left(-\frac{1}{\gamma}(\bar{s} - \zeta)\right). \quad (1.43)$$

Let  $\zeta < (\bar{s} - s)/2$ . Then the combination of (1.43) and (1.40) implies (1.17).  $\square$

## 1.5 Organization

We begin in Section 2 by proving the lower bound (Theorem 1.1) given Proposition 1.2. Then in Subsection 2.2 we prove the upper bound (Proposition 1.1). The heart of the paper is Section 3, where we prove Proposition 1.2. The proof is by induction. We prove the base case (Proposition 3.1) in Subsection 3.1. The induction step requires more work: We illustrate the main idea in Subsection 3.2 by showing how to go from  $J = 1$  to  $J = 2$  (Lemma 3.3). Then in Subsection 3.3 we prove the induction step (Proposition 3.2). Finally, the proofs of the auxiliary lemmas appear in Subsection 3.4

## 2 The upper and lower bounds

### 2.1 Proof of Theorem 1.1

Given Proposition 1.2, the proof of Theorem 1.1 is straightforward and follows the method from [18].

*Proof.* Choose a subsequence such that

$$\liminf_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon)$$

is attained. Choose a subsequence such that  $\mu_\varepsilon$  and  $\nu_\varepsilon$  converge as measures. We consider separately the intervals  $[T_k - \delta, T_k + \delta] \cap [0, T]$  “near singular times” and the intervals  $[T_k + \delta, T_{k+1} - \delta]$  “away from singular times,” where  $\delta > 0$  satisfies

$$\delta < \frac{1}{2} \min_k |T_{k+1} - T_k|.$$

Consider any interval  $[T_k - \delta, T_k + \delta]$ . Choose any  $\phi \in C^1([0, 1])$  with  $0 \leq \phi(x) \leq 1$ . We estimate:

$$\begin{aligned}
& \frac{1}{4} \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\
& \geq \frac{1}{2} \left| \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \dot{u}_\varepsilon (\varepsilon u_{\varepsilon,xx} - \varepsilon^{-1} V'(u_\varepsilon)) \phi dx dt \right| \\
& \geq \frac{1}{2} \left| \int_{T_k - \delta}^{T_k + \delta} \int_0^1 (\varepsilon \dot{u}_{\varepsilon,x} u_{\varepsilon,x} - \varepsilon^{-1} \dot{u}_\varepsilon V'(u_\varepsilon)) \phi dx dt \right| \\
& \quad - \frac{1}{2} \left| \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \varepsilon \dot{u}_\varepsilon u_{\varepsilon,x} \phi'(x) dx dt \right|. \tag{2.1}
\end{aligned}$$

We observe that on the one hand,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{T_k - \delta}^{T_k + \delta} \int_0^1 (\varepsilon \dot{u}_{\varepsilon,x} u_{\varepsilon,x} - \varepsilon^{-1} \dot{u}_\varepsilon V'(u_\varepsilon)) \phi dx dt \\
& = \lim_{\varepsilon \rightarrow 0} \int_{T_k - \delta}^{T_k + \delta} \frac{d}{dt} \mu_\varepsilon^t(\phi) dt \\
& = \lim_{\varepsilon \rightarrow 0} \left( \mu_\varepsilon^{T_k + \delta}(\phi) - \mu_\varepsilon^{T_k - \delta}(\phi) \right) \\
& = \mu^{T_k + \delta}(\phi) - \mu^{T_k - \delta}(\phi),
\end{aligned}$$

and on the other hand, letting  $\bar{C}$  denote the bound on the action, we have

$$\begin{aligned}
& \left| \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \varepsilon \dot{u}_\varepsilon u_{\varepsilon,x} \phi'(x) dx dt \right| \\
& \leq \|\phi\|_{C^1} \left( \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \varepsilon \dot{u}_\varepsilon^2 dx dt \right)^{1/2} \left( \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \varepsilon u_{\varepsilon,x}^2 dx dt \right)^{1/2} \\
& \leq \|\phi\|_{C^1} \sqrt{4\bar{C}} \sqrt{2\bar{C}2\delta} \\
& = \|\phi\|_{C^1} O(\sqrt{\delta}).
\end{aligned}$$

Together with (2.1), this yields

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{1}{4} \int_{T_k - \delta}^{T_k + \delta} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\
& \geq \frac{1}{2} |\mu^{T_k + \delta}(\phi) - \mu^{T_k - \delta}(\phi)| - \|\phi\|_{C^1} O(\sqrt{\delta}). \tag{2.2}
\end{aligned}$$

Notice that the “boundary intervals” are special: For  $T_1 = 0$ , the interval is  $[0, \delta]$  and recalling  $\mu_\varepsilon^0 = 0 \forall \varepsilon$ , (2.2) becomes

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{4} \int_0^\delta \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\ & \geq \frac{1}{2} |\mu^\delta(\phi)| - \|\phi\|_{C^1} O(\sqrt{\delta}) \\ & = \frac{1}{2} |\mu^\delta(\phi) - \mu^{-\delta}(\phi)| - \|\phi\|_{C^1} O(\sqrt{\delta}). \end{aligned}$$

Similarly, for  $T_M = T$  we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{4} \int_{T-\delta}^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\ & \geq \frac{1}{2} |\mu^{T-\delta}(\phi)| - \|\phi\|_{C^1} O(\sqrt{\delta}) \\ & = \frac{1}{2} |\mu^{T+\delta}(\phi) - \mu^{T-\delta}(\phi)| - \|\phi\|_{C^1} O(\sqrt{\delta}). \end{aligned}$$

Away from singular times, we will use Proposition 1.2. Consider any interval  $[T_k + \delta, T_{k+1} - \delta]$ . Then  $g_j$  for  $j = 1, \dots, N(k)$  are well-defined and moreover by (1.29),

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{4} \int_{(T_k+\delta, T_{k+1}-\delta)} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt \geq \frac{c_0}{4} \int_{(T_k+\delta, T_{k+1}-\delta)} \sum_{j=1}^{N(k)} (\dot{g}_j)^2 dt. \quad (2.3)$$

Combining (2.2) and (2.3) and summing over  $k$ , we deduce

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \frac{1}{4} \int_0^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\ & \geq \frac{1}{2} \sum_{k=1}^M |\mu^{T_k+\delta}(\phi) - \mu^{T_k-\delta}(\phi)| + \frac{c_0}{4} \sum_{k=1}^{M-1} \int_{(T_k+\delta, T_{k+1}-\delta)} \sum_{j=1}^{N(k)} (\dot{g}_j)^2 dt \\ & \quad - M \|\phi\|_{C^1} O(\sqrt{\delta}). \end{aligned}$$

Sending  $\delta \rightarrow 0$  on the right-hand side gives

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \frac{1}{4} \int_0^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\ & \geq \frac{1}{2} \sum_{k=1}^M |\mu^{T_k^+}(\phi) - \mu^{T_k^-}(\phi)| + \frac{c_0}{4} \sum_{k=1}^{M-1} \int_{(T_k, T_{k+1})} \sum_{j=1}^{N(k)} (\dot{g}_j)^2 dt, \end{aligned}$$

and taking the supremum over  $0 \leq \phi \leq 1$  yields

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \frac{1}{4} \int_0^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} (f_\varepsilon(u_\varepsilon))^2 dx dt \\ & \geq \frac{1}{2} \sum_{k=1}^M \left| \mu^{T_k^+} - \mu^{T_k^-} \right|_{TV} + \frac{c_0}{4} \sum_{k=1}^{M-1} \int_{(T_k, T_{k+1})} \sum_{j=1}^{N(k)} (\dot{g}_j)^2 dt, \end{aligned}$$

i.e.,

$$\liminf_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) \geq S^{\mathcal{M}}(\mu).$$

Since for all  $t \notin \{T_k\}_{k=1}^M$  the support of  $\mu$  includes the discontinuities of  $u_0$  and delta masses have odd multiplicity where  $u_0$  has a jump discontinuity and even multiplicity otherwise, it follows that  $\mu$  is an admissible measure and we have in particular that

$$\liminf_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) \geq S_0(u_0).$$

□

## 2.2 Proof of Proposition 1.1

*Proof.* We begin with two simplifying lemmas. First we argue that the minimal action is attained:

**Lemma 2.1.** *For every  $u_0 \in \mathcal{A}_0$  there exists a compatible measure  $\mu_*$  such that*

$$S^{\mathcal{M}}(\mu_*) = \inf_{\mu_{u_0}} S^{\mathcal{M}}(\mu_{u_0}).$$

The idea is to build a construction  $u_\varepsilon$  with  $u_\varepsilon \rightarrow u_0$  in  $L^\infty(L^2)$  such that the associated energy measures  $\mu_\varepsilon$  converge to the minimizing measure  $\mu_*$ . The basic building block is the construction from [17], however a new complication is that  $\mu_*$  may exhibit higher-multiplicity interfaces. We can reduce to the simpler case using:

**Lemma 2.2.** *Any compatible measure  $\mu$  with higher-multiplicity interfaces may be approximated by compatible measures  $\mu_\alpha$  with single-multiplicity interfaces in  $(0, 1)$  in such a way that*

$$\lim_{\alpha \rightarrow 0} S^{\mathcal{M}}(\mu_\alpha) = S^{\mathcal{M}}(\mu).$$



Hence given any  $u_0 \in \mathcal{A}_0$  we may assume that the compatible measure  $\mu_*$  minimizes  $S^{\mathcal{M}}$  and that its interfaces have single-multiplicity and take values in  $(0, 1)$ . Now we follow the construction from [17]. We briefly sketch the main ideas.

First we build the construction locally around a pair of interfaces and then show that the localized constructions can be merged. (The interfaces appear and annihilate in pairs because of condition (iii) in Definition 1.) Suppose that a pair of interfaces appear at location  $x_0$  at time  $T_1$ . Without loss, suppose that  $u = -1$  near  $x_0$  just before  $T_1$  and  $u = +1$  near  $x_0$  just after  $T_1$ . Define the “nucleated state”

$$u_n(x) = \begin{cases} \tanh\left(\frac{x - x_0 + \ell}{\sqrt{2\varepsilon}}\right) & x \leq x_0 \\ \tanh\left(\frac{-x + x_0 + \ell}{\sqrt{2\varepsilon}}\right) & x \geq x_0, \end{cases}$$

where  $\ell$  is a free parameter. We connect  $u = -1$  to  $u_n(x)$  at time  $t = T_1 + \tau$  in the following way. If  $\ell$  is not too large, then there is an orbit connecting it via the dynamics  $\varepsilon^{1/2}\dot{u} = \varepsilon^{-1/2}(\varepsilon u_{xx} - \varepsilon^{-1}V'(u))$  to  $u = -1$  in infinite time. On  $[T_1, T_1 + \tau]$  we first interpolate from  $u = -1$  to a point arbitrarily nearby (in  $L^2$ ) and on this orbit. Then we follow the time-reversed gradient flow to  $u_n$ . (For annihilation of interfaces, we use instead the forward gradient flow.) As in [17], one can check that the time required for such a pathway is of order  $\tau \sim \varepsilon \exp(c\ell/\varepsilon)$  and that for any tolerance  $\beta > 0$  there exists such a path with action bounded by

$$c_0 + \beta = \frac{1}{2} \left| \mu^{T_1^+} - \mu^{T_1^-} \right|_{TV} + \beta.$$

Hence by choosing  $\ell \ll \varepsilon$ , we can reach  $u_n$  in a time  $\tau \ll 1$  with a good action bound.

To move the interface, we use

$$u_\varepsilon(x, t) = \begin{cases} \tanh\left(\frac{x - g_1(t) + \ell}{\sqrt{2\varepsilon}}\right) & x \leq x_0 \\ \tanh\left(\frac{-x + g_2(t) + \ell}{\sqrt{2\varepsilon}}\right) & x \geq x_0, \end{cases}$$

where  $g_1(T_1 + \tau) = g_2(T_1 + \tau) = x_0$ . Then the second term in the integrand of (1.1) vanishes identically and for the first we can show the bound

$$\int_{(T_1+\tau, T_2)} \int_0^1 \varepsilon \dot{u}^2 dx dt \leq c_0 \int_{(T_1+\tau, T_2)} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt.$$

Sending  $\tau \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we find

$$\limsup_{\varepsilon \rightarrow 0} S_\varepsilon(u_\varepsilon) \leq \frac{1}{2} \left| \mu^{T_1^+} - \mu^{T_1^-} \right|_{TV} + \frac{c_0}{4} \int_{(T_1, T_2)} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt.$$

Consecutive interfaces can be added. The construction  $u_\varepsilon$  is defined piecewise from the midpoint between one pair of interfaces to the midpoint between the next pair. It remains to show that the discontinuities in  $u_{\varepsilon, x}$  at every time can be smoothed with only a small action cost. We refer the reader to [17] for details.  $\square$

### 2.2.1 Proofs of Lemmas

*Proof of Lemma 2.1.* Choose a minimizing sequence of compatible measures  $\mu_n$  such that

$$S^{\mathcal{M}}(\mu_n) \downarrow S_0(u_0) \text{ as } n \rightarrow \infty.$$

It follows from the definition of  $S^{\mathcal{M}}$  that the  $\mu_n$  are uniformly bounded, and hence we may choose a subsequence that converges in the sense of measures to some limit,  $\mu_*$ . That  $\mu_*$  is compatible with  $u_0$  follows from the fact that  $u_0$  is admissible.

We will now argue that

$$S^{\mathcal{M}}(\mu_*) = \inf_{\mu_u} S^{\mathcal{M}}(\mu_u). \quad (2.4)$$

Let  $\mathcal{T} = \{T_1, \dots, T_M\}$  and  $\mathcal{T}^n = \{T_1^n, \dots, T_{M_n}^n\}$  denote the singular times of  $\mu_*$  and  $\mu_n$ , respectively. By the convergence of  $\mu_n$  to  $\mu_*$ , we have that  $\mathcal{T}^n \rightarrow \mathcal{T}$ . Hence for  $c > 0$  sufficiently small and any  $k \in \{1, \dots, M\}$ , taking  $n$  sufficiently large implies that there are no singular times of  $\mu_n$  on  $(T_k + \delta, T_{k+1} - \delta)$ . Let  $\{g_1, \dots, g_{N(k)}\}$  and  $\{g_1^n, \dots, g_{N^n(k)}^n\}$  denote the locations of the delta masses of  $\mu_*$  and  $\mu_n$ , respectively. From the convergence of  $\mu_n$  it follows that  $N^n(k) = N(k)$  and for each  $j \in \{1, \dots, N(k)\}$ ,  $g_j^n$  converges uniformly to  $g_j$ . Moreover by the uniform boundedness of each  $g_j^n$  in  $W^{1,2}$ , we may conclude that  $g_j^n \rightharpoonup g_j$  weakly in  $W^{1,2}$ . By the weak lower semicontinuity of the  $L^2$  norm, it follows that

$$\int_{(T_k + \delta, T_{k+1} - \delta)} \sum_{j=1}^{N(k)} (\dot{g}_j)^2 dt \leq \liminf_{n \rightarrow \infty} \int_{(T_k + \delta, T_{k+1} - \delta)} \sum_{j=1}^{N(k)} (\dot{g}_j^n)^2 dt \quad (2.5)$$

This is the first half of the proof of (2.4).

Now consider an interval  $(T_k - \delta, T_k + \delta)$  around one of the singular times of  $\mu_*$ . (As in the proof of Theorem 1.1, we use  $(T_k - \delta, T_k + \delta) \cap [0, T]$  for

the “boundary intervals.”) On this interval the measure  $\mu_n$  has singular times  $T_{k,1}^n, \dots, T_{k,J}^n$  for some  $J \geq 1$ . By the convergence of  $\mu_n$ , we have

$$\nu_n := \sum_{j=1}^J \mu_n^{(T_{k,j}^n)^+} - \mu_n^{(T_{k,j}^n)^-} \rightarrow \mu_*^{T_k^+} - \mu_*^{T_k^-}.$$

Hence by the weak lower semicontinuity of the total variation with respect to measure convergence, it follows that

$$\begin{aligned} |\mu_*^{T_k^+} - \mu_*^{T_k^-}|_{TV} &\leq \liminf_{n \rightarrow \infty} \left| \sum_{j=1}^J \mu_n^{(T_{k,j}^n)^+} - \mu_n^{(T_{k,j}^n)^-} \right|_{TV} \\ &\leq \liminf_{n \rightarrow \infty} \sum_{j=1}^J \left| \mu_n^{(T_{k,j}^n)^+} - \mu_n^{(T_{k,j}^n)^-} \right|_{TV}. \end{aligned} \quad (2.6)$$

The combination of (2.6) and (2.5) (repeated for each  $T_k \in \mathcal{T}$ ) establishes

$$S^{\mathcal{M}}(\mu_*) \leq \liminf_{n \rightarrow \infty} S^{\mathcal{M}}(\mu_n).$$

Since  $\mu_n$  is a minimizing sequence of  $S^{\mathcal{M}}$  over the set of compatible measures, this completes the proof of (2.4).  $\square$

*Proof of Lemma 2.2.* Consider  $(T_k + \delta, T_{k+1} - \delta)$  where  $T_k$  is an arbitrary singular time of  $\mu$ . We will separate the interfaces in the following way. Let  $\{g_j\}_{j=1}^N$  denote the delta masses of  $\mu^t$  on this interval. Add to the set  $g_0 \equiv 0$  and  $g_{N+1} \equiv 1$ . To separate the interfaces, we introduce:

$$\begin{aligned} \text{for } j = 1, \dots, N, \quad \tilde{g}_j &:= g_j + \min \left\{ \alpha^2, \frac{g_{j+1} - g_j}{4} \right\} \phi \left( \frac{g_j - g_{j-1}}{\alpha} \right) \\ &\quad - \min \left\{ \alpha^2, \frac{g_j - g_{j-1}}{4} \right\} \phi \left( \frac{g_{j+1} - g_j}{\alpha} \right), \end{aligned}$$

where  $\phi(x)$  is a smooth function that satisfies

$$\phi(0) = 1, \quad 0 \leq \phi \leq 1, \quad |\phi'| \leq 2, \quad \text{and} \quad \phi(x) = 0 \text{ for } x \geq 1. \quad (2.7)$$

It is not hard to check that if  $g_j(t) \neq g_{j+1}(t)$ , then  $\tilde{g}_j(t) \neq \tilde{g}_{j+1}(t)$ , while on the other hand if there is a group of interfaces with multiplicity  $J$ , then in the new variables this group is reduced to a group of multiplicity  $J - 2$ . (The “ends” have been separated from the group.) Since  $N$  is finite, after repeating this

procedure a finite number of times all of the interfaces have single multiplicity. We denote the new interfaces by  $\{g_j^\alpha\}_{j=1}^N$ .

We repeat the procedure on each interval  $(T_k + \delta, T_{k+1} - \delta)$ . To complete the construction, we linearly interpolate between the original and new positions of the delta masses. More precisely, on  $[T_k, T_k + \delta]$ , we linearly interpolate between

$$\lim_{t \downarrow T_k} g_j(t) \quad \text{and} \quad g_j^\alpha(T_k + \delta),$$

and similarly, on  $[T_k - \delta, T_k]$ , we linearly interpolate between

$$\lim_{t \uparrow T_k} g_j(t) \quad \text{and} \quad g_j^\alpha(T_k - \delta).$$

This completes the construction of  $\{g_j^\alpha\}$  on  $[0, T]$  and hence of  $\mu_\alpha$ . It remains to show that the associated action is close to that of  $\mu$ . It is easy to see that

$$\sum_{k=1}^M \left| \mu^{T_k^+} - \mu^{T_k^-} \right|_{TV} = \sum_{k=1}^M \left| \mu_\alpha^{T_k^+} - \mu_\alpha^{T_k^-} \right|_{TV}.$$

Hence, it suffices to show

$$\int_{T_k + \delta}^{T_{k+1} - \delta} (\dot{g}_j^\alpha)^2 dt = \int_{T_k + \delta}^{T_{k+1} - \delta} (\dot{g}_j)^2 dt + o(1)_{\alpha \rightarrow 0}. \quad (2.8)$$

Moreover, since the iterative separation scheme is completed in a finite number of steps, it is enough to show (2.8) for a single step in the scheme. We compute the derivative explicitly:

$$\begin{aligned} & \frac{d}{dt} \tilde{g}_j \\ &= \dot{g}_j + \alpha(\dot{g}_j - \dot{g}_{j-1})\phi' \mathbf{1}_{\{\alpha^2 \leq (g_{j+1} - g_j)/4\}} \\ &+ \left( \frac{1}{4}(\dot{g}_{j+1} - \dot{g}_j)\phi + \frac{1}{4\alpha}(g_{j+1} - g_j)(\dot{g}_j - \dot{g}_{j-1})\phi' \right) \mathbf{1}_{\{0 < (g_{j+1} - g_j)/4 < \alpha^2\}} \\ &- \alpha(\dot{g}_j - \dot{g}_{j-1})\phi' \mathbf{1}_{\{\alpha^2 \leq (g_j - g_{j-1})/4\}} \\ &- \left( \frac{1}{4}(\dot{g}_j - \dot{g}_{j-1})\phi + \frac{1}{4\alpha}(g_j - g_{j-1})(\dot{g}_{j+1} - \dot{g}_j)\phi' \right) \mathbf{1}_{\{0 < (g_j - g_{j-1})/4 < \alpha^2\}}, \end{aligned}$$

where  $\mathbf{1}_A$  denotes the characteristic function of the set  $A$ . Thus we have

$$\int_{T_k + \delta}^{T_{k+1} - \delta} \left( \frac{d}{dt} \tilde{g}_j \right)^2 dt = \int_{T_k + \delta}^{T_{k+1} - \delta} (\dot{g}_j)^2 dt + \text{error}$$

where the error can be controlled by three different kinds of estimates. First, we have terms of the following form, in which  $\alpha$  multiplies a bounded integral:

$$\begin{aligned} & \alpha \left| \int_{T_k+\delta}^{T_{k+1}-\delta} \dot{g}_\ell \dot{g}_m \phi' \mathbf{1}_{\{\alpha^2 \leq (g_{j+1}-g_j)/4\}} dt \right| \\ & \stackrel{(2.7)}{\leq} 2\alpha \left( \int_{(T_k, T_{k+1})} (\dot{g}_\ell)^2 dt \int_{(T_k, T_{k+1})} (\dot{g}_m)^2 dt \right)^{1/2}. \end{aligned}$$

Second, we have terms for which we use the smallness of  $(g_\ell - g_m)$  on the set over which it is integrated:

$$\begin{aligned} & \left| \int_{T_k+\delta}^{T_{k+1}-\delta} \frac{1}{\alpha} (g_\ell - g_m) \dot{g}_r \phi' \mathbf{1}_{\{0 < (g_\ell - g_m)/4 < \alpha^2\}} dt \right| \\ & \stackrel{(2.7)}{\leq} 2\alpha T^{1/2} \left( \int_{(T_k, T_{k+1})} (\dot{g}_r)^2 dt \right)^{1/2}. \end{aligned}$$

Finally, we have terms in which it is the smallness of the set that gives us control:

$$\begin{aligned} & \left| \int_{T_k+\delta}^{T_{k+1}-\delta} \dot{g}_\ell \dot{g}_m \phi \mathbf{1}_{\{0 < (g_r - g_{r-1})/4 < \alpha^2\}} dt \right| \\ & \stackrel{(2.7)}{\leq} \left( \int_{(T_k, T_{k+1})} (\dot{g}_\ell)^2 \mathbf{1}_{\{0 < (g_r - g_{r-1})/4 < \alpha^2\}} dt \int_{(T_k, T_{k+1})} (\dot{g}_m)^2 \mathbf{1}_{\{0 < (g_r - g_{r-1})/4 < \alpha^2\}} dt \right)^{1/2} \\ & = o(1)_{\alpha \rightarrow 0}, \end{aligned}$$

since each  $g_j$  is uniformly bounded in  $W^{1,2}$  and

$$\bigcap_{\alpha} \{0 < (g_r - g_{r-1})/4 < \alpha^2\} = \emptyset.$$

□

### 3 Propagation estimate

*Proof of Proposition 1.2.* We prove a slightly stronger statement, namely, that for any  $J \leq N$ , we have that  $\dot{g}_1, \dots, \dot{g}_N \in L^2(\{\text{Mult} \leq J\})$  and moreover that for any isolated group of  $J$  consecutive interfaces and any open set

$$O \Subset \{\text{Mult} \leq J\},$$

we have:

$$c_0 \int_O \sum_{j=1}^J (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_O \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt, \quad (3.1)$$

where  $\mathbf{1}^r([g_1, g_J])(t)$  denotes the characteristic function of  $[g_1(t) - r, g_J(t) + r] \cap [0, 1]$  for any  $r > 0$  sufficiently small. Proposition 1.2 follows from (3.1) with  $J = N$  by observing

$$\begin{aligned} c_0 \int_O \sum_{j=1}^N (\dot{g}_j)^2 dt &\leq \lim_{\varepsilon \rightarrow 0} \int_O \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_N]) dx dt \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 dx dt, \end{aligned}$$

and letting  $O \uparrow [0, T]$ .

The proof of (3.1) is by induction. The base case requires deriving an estimate on sets of single multiplicity that is localized near the graph of the interface. This localized result takes the form:

**Proposition 3.1.** *Consider any interface  $g$  and any open set  $O_1$  with*

$$O_1 \Subset \{\text{Mult } g = 1\}.$$

*Fix any  $r > 0$  sufficiently small, and let  $\mathbf{1}^r(g)(t)$  denote the characteristic function of  $[g(t) - r, g(t) + r] \cap [0, 1]$ . Then*

$$c_0 \int_{O_1} (\dot{g})^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_{O_1} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt. \quad (3.2)$$

*We remark that it follows:*

$$\dot{g} \in L^2(\{\text{Mult } g = 1\}). \quad (3.3)$$

Then for the induction step, we show:

**Proposition 3.2.** *Suppose that*

$$\dot{g}_j \in L^2(\{\text{Mult } g_j \leq J - 1\}), \quad j = 1, \dots, N, \quad (3.4)$$

*and that for any isolated group of  $J - 1$  consecutive interfaces  $g_1, \dots, g_{J-1}$  and any open set  $O_{J-1}$  with*

$$O_{J-1} \Subset \{\text{Mult} \leq J - 1\},$$

we have:

$$c_0 \int_{O_{J-1}} \sum_{j=1}^{J-1} (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_{O_{J-1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_{J-1}]) dx dt, \quad (3.5)$$

for any  $r > 0$  sufficiently small. Then

$$\dot{g}_j \in L^2(\{\text{Mult } g_j \leq J\}), \quad j = 1, \dots, N, \quad (3.6)$$

and for any isolated group of  $J$  consecutive interfaces and any open set  $O_J$  with

$$O_J \Subset \{\text{Mult} \leq J\},$$

we have:

$$c_0 \int_{O_J} \sum_{j=1}^J (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_{O_J} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt, \quad (3.7)$$

for any  $r > 0$  sufficiently small.

This completes the proof of Proposition 1.2.  $\square$

### 3.1 Proof of Proposition 3.1

Because of the single-multiplicity, (3.2) follows already from the proof of Theorem 1.4 in [18]. For completeness and to illustrate the method of this paper, however, we include a proof.

*Proof of Proposition 3.1.* Since  $O_1$  is open, it can be decomposed into the countable union of nonintersecting open intervals, and it is enough to consider the case  $O_1 = (a, b)$ . Since  $O_1$  is compactly contained within the set of single multiplicity, there exists  $\delta > 0$  such that the distance between  $g$  and the neighboring interfaces is at least  $\delta$ . We consider  $r < \delta$  so that  $g(t)$  is the only point in  $[g(t) - r, g(t) + r] \cap [0, 1]$  in the support of  $\mu^t$  for  $t \in [a, b]$ . In particular,

$$\mu^t = c_0 \delta_{g(t)} \quad \forall t \in [a, b]. \quad (3.8)$$

Let  $\sigma$  be an arbitrary, finite partition of  $(a, b)$ :

$$\sigma = \{a = t_0 < t_1 < \dots < t_n = b\} \quad \text{with} \quad \Delta t_k := t_{k+1} - t_k.$$

Consider  $[t_k, t_{k+1}]$  and define

$$t_* := \operatorname{argmin}_{[t_k, t_{k+1}]} g, \quad t_{**} := \operatorname{argmax}_{[t_k, t_{k+1}]} g. \quad (3.9)$$

Assume without loss that  $t_* < t_{**}$ . We would like to build a test function  $\Phi(x, t)$  such that

$$\Phi(g(t_{**})) - \Phi(g(t_*)) = \underset{[t_k, t_{k+1}]}{\text{osc}} g, \quad (3.10)$$

while at the same time, the following properties hold:

$$|\Phi|_\infty \leq \underset{[t_k, t_{k+1}]}{\text{osc}} g \quad (3.11)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \int_0^1 \dot{\Phi} \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt = 0, \quad (3.12)$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 |\Phi_x \varepsilon u_{\varepsilon, x} \dot{u}_\varepsilon| dx dt \\ \leq \left( c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt \right)^{1/2}. \end{aligned} \quad (3.13)$$

Given such a test function, we calculate (similarly to in (1.33)):

$$\begin{aligned} c_0 \underset{[t_k, t_{k+1}]}{\text{osc}} g \\ \stackrel{(3.8), (3.10)}{=} \int_0^1 \Phi d\mu^{t_{**}} - \int_0^1 \Phi d\mu^{t_*} \\ = \lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \frac{d}{dt} \int_0^1 \Phi \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt \\ \leq \lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \int_0^1 \dot{\Phi} \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt \\ + \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 |\Phi_x \varepsilon u_{\varepsilon, x} \dot{u}_\varepsilon| dx dt \\ + \frac{1}{2} |\Phi|_\infty \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 + \varepsilon^{-1} f_\varepsilon^2 dx dt \\ \stackrel{(3.11), (3.12), (3.13)}{\leq} \left( c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt \right)^{1/2} \\ + \frac{1}{2} \underset{[t_k, t_{k+1}]}{\text{osc}} g \int_{t_k}^{t_{k+1}} d\eta. \end{aligned} \quad (3.14)$$

The idea for the construction is to define

$$\Phi(x, t) = \phi(x) \Psi(x, t),$$



where  $\phi$  is the piecewise linear function

$$\phi(x) = \begin{cases} 0 & x < g(t_*) \\ x - g(t_*) & g(t_*) \leq x \leq g(t_{**}) \\ \text{osc}_{[t_k, t_{k+1}]} g & x > g(t_{**}), \end{cases}$$

and  $\Psi$  is a cut-off function such that  $0 \leq \Psi \leq 1$  and

$$\Psi(x, t) = \begin{cases} 1 & g(t) - r/2 \leq x \leq g(t) + r/2 \\ 0 & x \leq g(t) - r \text{ or } x \geq g(t) + r. \end{cases}$$

Notice in particular that we have

$$\int_0^1 \phi \dot{\Psi} d\mu^t = 0 \quad (3.15)$$

$$\int_0^1 (\Psi_x)^2 d\mu^t = 0 \quad (3.16)$$

$$|\phi'(x)| \leq 1 \text{ a.e.} \quad (3.17)$$

It is straightforward to check that  $\Phi$  satisfies (3.10) and also (3.11):

$$|\Phi|_\infty \leq |\phi|_\infty = \text{osc}_{[t_k, t_{k+1}]} g,$$

and (3.12):

$$\lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \int_0^1 \dot{\Phi} \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt = \int_{t_*}^{t_{**}} \int_0^1 \phi \dot{\Psi} d\mu^t dt \stackrel{(3.15)}{=} 0.$$

The problem with satisfying (3.13) is that  $\Phi_x$  is not defined at  $g(t_*)$  and  $g(t_{**})$  because of the discontinuity in  $\phi'$  there. However proceeding formally for the

moment, we calculate:

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 |\Phi_x \varepsilon u_{\varepsilon,x} \dot{u}_\varepsilon| dx dt \\
&= \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \phi' \Psi \varepsilon u_{\varepsilon,x} \dot{u}_\varepsilon dx dt + \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \phi \Psi_x \varepsilon u_{\varepsilon,x} \dot{u}_\varepsilon dx dt \\
&\leq \lim_{\varepsilon \rightarrow 0} \left( \int_{t_k}^{t_{k+1}} \int_0^1 (\phi')^2 \varepsilon (u_{\varepsilon,x})^2 dx dt \int_{t_k}^{t_{k+1}} \int_0^1 (\Psi)^2 \varepsilon (\dot{u}_\varepsilon)^2 dx dt \right)^{1/2} \\
&\quad + \lim_{\varepsilon \rightarrow 0} \left( \int_{t_k}^{t_{k+1}} \int_0^1 (\Psi_x)^2 \varepsilon (u_{\varepsilon,x})^2 dx dt \int_{t_k}^{t_{k+1}} \int_0^1 \phi^2 \varepsilon (\dot{u}_\varepsilon)^2 dx dt \right)^{1/2} \\
&\stackrel{(3.16)}{=} \lim_{\varepsilon \rightarrow 0} \left( \int_{t_k}^{t_{k+1}} \int_0^1 (\phi')^2 \varepsilon (u_{\varepsilon,x})^2 dx dt \int_{t_k}^{t_{k+1}} \int_0^1 (\Psi)^2 \varepsilon (\dot{u}_\varepsilon)^2 dx dt \right)^{1/2} \\
&\stackrel{(3.17)}{\leq} \left( c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt \right)^{1/2}.
\end{aligned}$$

To deal honestly with the discontinuity of  $\phi'$  at  $g(t_*)$  and  $g(t_{**})$ , we introduce a regularized function  $\phi_\alpha$  (cf. Figure 4) such that

$$\phi_\alpha(x) = x - g(t_*) \quad g(t_*) \leq x \leq g(t_{**}), \quad (3.18)$$

while at the same time

$$|\phi_\alpha|_\infty \leq \underset{[t_k, t_{k+1}]}{\text{osc}} g + \alpha, \quad |\phi'_\alpha| \leq 1.$$

Letting  $\Phi_\alpha := \phi_\alpha \Psi$ , (3.12) and (3.13) are satisfied, while (3.11) is replaced by

$$|\Phi_\alpha|_\infty \leq \underset{[t_k, t_{k+1}]}{\text{osc}} g + \alpha.$$

Repeating the calculation in (3.14) gives:

$$\begin{aligned}
c_0 \underset{[t_k, t_{k+1}]}{\text{osc}} g &\leq \left( c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt \right)^{1/2} \\
&\quad + \frac{1}{2} \left( \underset{[t_k, t_{k+1}]}{\text{osc}} g + \alpha \right) \int_{t_k}^{t_{k+1}} d\eta.
\end{aligned}$$

Sending  $\alpha$  to zero recovers (3.14).

We next introduce a lemma that says (3.14) implies (3.2). (In fact, we allow for a slightly more general estimate than (3.14) that will be useful later in the proof of Proposition 3.2.)

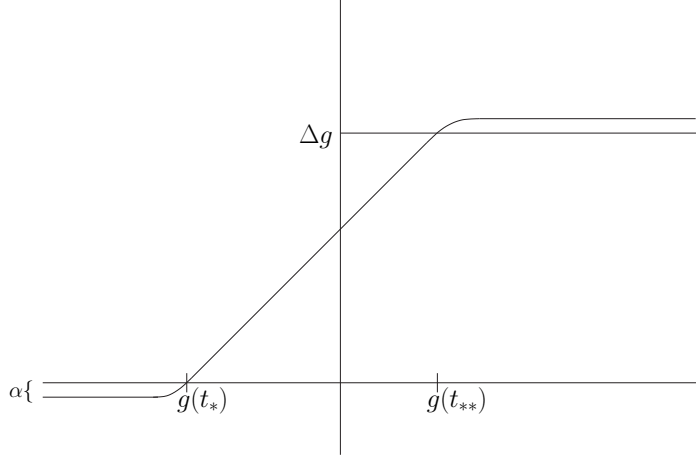


Figure 4: The regularized test function  $\phi_\alpha$ .

**Lemma 3.1.** *Let  $g$  be in  $C([a, b])$ ,  $h \geq 0$  be in  $L^1([a, b])$ ,  $\eta$  be a finite measure on  $[a, b]$ , and  $m_\varepsilon \geq 0$  be a sequence of functions in  $L^1([a, b] \times [0, 1])$  with*

$$\lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt < \infty. \quad (3.19)$$

Suppose that for any finite partition  $\sigma$  of  $(a, b)$ :

$$\sigma = \{a = t_0 < t_1 < \dots < t_n = b\} \quad \text{with} \quad \Delta t_k := t_{k+1} - t_k,$$

we have

$$\begin{aligned} \tilde{c} \operatorname{osc}_{[t_k, t_{k+1}]} g &\leq \left( \tilde{c} \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt \right)^{1/2} \\ &\quad + \left( \Delta t_k \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} + \frac{1}{2} \operatorname{osc}_{[t_k, t_{k+1}]} g \int_{t_k}^{t_{k+1}} d\eta. \end{aligned} \quad (3.20)$$

Then

$$\dot{g} \in L^2([a, b]) \quad (3.21)$$

and we have that for any  $\delta > 0$ ,

$$\tilde{c} \int_a^b (\dot{g})^2 dt \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_a^b \int_0^1 m_\varepsilon dx dt + \frac{1 + \delta^{-1}}{\tilde{c}} \int_a^b h dt. \quad (3.22)$$

Equation (3.14) is of the form (3.20) with

$$m_\varepsilon = \varepsilon(\dot{u}_\varepsilon)^2 \mathbf{1}^r(g), \quad \tilde{c} = c_0, \quad h = 0.$$

Thus, by Lemma 3.1 we deduce:

$$c_0 \int_a^b (\dot{g})^2 dt \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_a^b \int_0^1 \varepsilon(\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt,$$

and letting  $\delta \rightarrow 0$  concludes the proof of (3.2). To see (3.3), we coarsely estimate:

$$\begin{aligned} c_0 \int_{O_1} (\dot{g})^2 dt &\leq \lim_{\varepsilon \rightarrow 0} \int_{O_1} \int_0^1 \varepsilon(\dot{u}_\varepsilon)^2 \mathbf{1}^r(g) dx dt \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^1 \varepsilon(\dot{u}_\varepsilon)^2 dx dt \stackrel{(1.34)}{\leq} \bar{C}. \end{aligned}$$

Letting  $O_1 \uparrow \{\text{Mult } g = 1\}$  implies (3.3). □

## 3.2 From single multiplicity to higher multiplicity

The main idea is that on sets of single multiplicity, we use Proposition 3.1, while on sets of higher multiplicity (plus a small neighborhood), we prove estimates for the Dirichlet integral of the mean:

$$g_m := \frac{1}{J}(g_1 + \dots + g_J).$$

On the set of multiplicity  $J$ , estimates for  $g_m$  give exactly the right control since then

$$g_m = g_1 = g_2 = \dots = g_J. \tag{3.23}$$

To convert estimates for the mean into estimates for the interfaces, we will need the lemma:

**Lemma 3.2.** *Suppose that  $g_1 \leq g_2 \leq \dots \leq g_J$  are continuous functions on  $[0, T]$ . If*

$$\dot{g}_j \in L^2(\{\text{Mult} < J\}) \quad \text{for } j = 1, \dots, J \tag{3.24}$$

and

$$\frac{d}{dt} \left( \sum_{j=1}^J g_j \right) \in L^2(\{\text{Mult} \leq J\}), \tag{3.25}$$

then

$$\dot{g}_j \in L^2(\{\text{Mult} \leq J\}) \quad \text{for } j = 1, \dots, J. \tag{3.26}$$

In Subsection 3.3 we will show how these ingredients can be used to prove Proposition 3.2. Because the proof is somewhat technical, we first introduce the following simpler lemma. It illustrates the main idea for the case  $J = 2$ .

**Lemma 3.3.** *Given Proposition 3.1, consider any open set  $O_2$  with*

$$O_2 \Subset \{g_2(t) < g_3(t)\}. \quad (3.27)$$

We have that

$$\dot{g}_j \in L^2(O_2) \quad \text{for } j = 1, 2 \quad (3.28)$$

and moreover,

$$c_0 \int_{O_2} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_{O_2} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt. \quad (3.29)$$

### 3.2.1 Proof of Lemma 3.3

*Proof.* As in the proof of Proposition 3.1, it is enough to consider the case in which  $O_2$  is a single interval  $(a, b)$ . By (3.27), we may choose  $r > 0$  sufficiently small so that

$$r < \frac{1}{2} \inf_{t \in (a, b)} |g_3(t) - g_2(t)|.$$

From now on, we work entirely on  $[a, b]$  and ignore  $g_j$  for  $j = 3 \dots N$ .

We remark that

$$G_2 := \{g_1 = g_2\}$$

is closed and  $[a, b] \setminus G_2$  is (relatively) open with single multiplicity. Define

$$V_\ell := \{t; d(t, G_2) < 1/\ell\}, \quad (3.30)$$

so that

$$\bigcap_{\ell=1}^{\infty} V_\ell = G_2.$$

Let us denote the complement of a set  $A$  by  $A^c$ . Choose open sets  $U_{\ell,1}, U_{\ell,2}$  such that

$$U_{\ell,1} \supset V_\ell^c, \quad U_{\ell,2} \supset G_2, \quad d(U_{\ell,1}, U_{\ell,2}) \geq \frac{1}{2\ell}. \quad (3.31)$$

Notice in particular that

$$(U_{\ell,1} \cup G_2)^c \subset V_\ell \setminus G_2, \quad (3.32)$$

$$U_{\ell,2} \setminus G_2 \subset V_\ell \setminus G_2 \quad (3.33)$$

$$U_{\ell,1} \Subset G_2^c. \quad (3.34)$$

First, by Proposition 3.1 we have that

$$\dot{g}_1, \dot{g}_2 \in L^2(\{\text{Mult } g_1 = \text{Mult } g_2 = 1\}). \quad (3.35)$$

Moreover, by (3.34), we may choose  $r > 0$  sufficiently small so that

$$r < \frac{1}{2} \inf_{t \in U_{\ell,1}} |g_1(t) - g_2(t)|.$$

With this choice, the supports of  $\mathbf{1}^r(g_1)$  and  $\mathbf{1}^r(g_2)$  are disjoint on  $U_{\ell,1}$ , and (3.2) implies the estimate

$$\begin{aligned} & c_0 \int_{U_{\ell,1}} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt \\ & \leq \lim_{\varepsilon \rightarrow 0} \int_{U_{\ell,1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 (\mathbf{1}^r(g_1) + \mathbf{1}^r(g_2)) dx dt. \end{aligned} \quad (3.36)$$

Also, we will show that  $g_m := (g_1 + g_2)/2$  satisfies

$$\dot{g}_m \in L^2(U_{\ell,2}) \quad (3.37)$$

with the bound

$$\begin{aligned} & 2c_0 \int_{U_{\ell,2}} (\dot{g}_m)^2 dt \\ & \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_{U_{\ell,2}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt + C_\delta \times o(1)_{\ell \rightarrow \infty}. \end{aligned} \quad (3.38)$$

Notice that (3.37) together with (3.35) implies in particular that

$$\dot{g}_m \in L^2([a, b]). \quad (3.39)$$

This will complete the proof, as we now explain: First, (3.35) and (3.39) imply by Lemma 3.2 that  $\dot{g}_1$  and  $\dot{g}_2$  are in  $L^2([a, b])$  which establishes (3.28). Notice that together with (3.32) and (3.30), (3.28) implies that

$$\int_{(U_{\ell,1} \cup G_2)^c} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt \leq o(1)_{\ell \rightarrow \infty}. \quad (3.40)$$

Second, we observe:

$$\begin{aligned}
& c_0 \int_a^b \sum_{j=1}^2 (\dot{g}_j)^2 dt \\
& \stackrel{(3.40)}{\leq} c_0 \int_{U_{\ell,1}} \sum_{j=1}^2 (\dot{g}_j)^2 dt + c_0 \int_{G_2} \sum_{j=1}^2 (\dot{g}_j)^2 dt + o(1)_{\ell \rightarrow \infty} \\
& \stackrel{(3.23)}{=} c_0 \int_{U_{\ell,1}} \sum_{j=1}^2 (\dot{g}_j)^2 dt + 2c_0 \int_{G_2} (\dot{g}_m)^2 dt + o(1)_{\ell \rightarrow \infty} \\
& \stackrel{(3.39),(3.31)}{\leq} c_0 \int_{U_{\ell,1}} \sum_{j=1}^2 (\dot{g}_j)^2 dt + 2c_0 \int_{U_{\ell,2}} (\dot{g}_m)^2 dt + o(1)_{\ell \rightarrow \infty} \\
& \stackrel{(3.36),(3.38)}{\leq} \lim_{\varepsilon \rightarrow 0} \int_{U_{\ell,1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 (\mathbf{1}^r(g_1) + \mathbf{1}^r(g_2)) dx dt \\
& \quad + (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_{U_{\ell,2}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt + C_\delta \times o(1)_{\ell \rightarrow \infty} \\
& \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_a^b \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt + C_\delta \times o(1)_{\ell \rightarrow \infty}. \quad (3.41)
\end{aligned}$$

Sending first  $\ell \rightarrow \infty$  and then  $\delta \rightarrow 0$  completes the proof of (3.29). Thus, we need only show (3.37) and (3.38).

As usual, it suffices to consider the case that  $U_{\ell,2}$  is a single interval  $(c, d)$ . Let  $\sigma$  be an arbitrary, finite partition of  $(c, d)$ :

$$\sigma = \{c = t_0 < t_1 < \dots < t_n = d\} \quad \text{with} \quad \Delta t_k := t_{k+1} - t_k.$$

*Case 1:*  $[t_k, t_{k+1}] \cap G_2 = \emptyset$ . Then by Proposition 3.1,  $\dot{g}_1$  and  $\dot{g}_2$  are  $L^2([t_k, t_{k+1}])$  and

$$\begin{aligned}
2c_0 \operatorname{osc}_{(t_k, t_{k+1})} g_m & \leq c_0 \left( \operatorname{osc}_{(t_k, t_{k+1})} g_1 + \operatorname{osc}_{(t_k, t_{k+1})} g_2 \right) \\
& \leq c_0 \left( \int_{t_k}^{t_{k+1}} |\dot{g}_1| + |\dot{g}_2| dt \right) \\
& \leq \sqrt{2} c_0 \left( \Delta t_k \int_{t_k}^{t_{k+1}} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt \right)^{1/2}, \quad (3.42)
\end{aligned}$$

the last line following from Hölder's inequality and the inequality

$$\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}.$$

*Case 2:*  $[t_k, t_{k+1}] \cap G_2 \neq \emptyset$ . All operations below (argmin, infimum, etc...) are over the interval  $[t_k, t_{k+1}]$ , and for ease of notation, we omit writing the interval. In analogy with the proof of Proposition 3.1, we define

$$t_* := \operatorname{argmin} g_m, \quad t_{**} := \operatorname{argmax} g_m, \quad (3.43)$$

and assume without loss that  $t_* < t_{**}$ . We would like to build a test function  $\Phi(x, t)$  such that

$$\int_0^1 \Phi(x, t) d\mu^t = c_0 \left( \Phi(g_1(t), t) + \Phi(g_2(t), t) \right) \quad \forall t \in (a, b), \quad (3.44)$$

$$\Phi(g_2(t_{**})) + \Phi(g_1(t_{**})) - \left( \Phi(g_2(t_*)) + \Phi(g_1(t_*)) \right) = 2 \operatorname{osc} g_m, \quad (3.45)$$

while at the same time, the following properties hold:

$$\begin{aligned} |\Phi|_\infty &\leq \sup g_2 - \inf g_1 \\ \lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \int_0^1 \dot{\Phi} \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt &= 0, \\ \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 |\Phi_x \varepsilon u_{\varepsilon, x} \dot{u}_\varepsilon| dx dt & \\ &\leq \left( 2c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt \right)^{1/2}. \end{aligned} \quad (3.46)$$

Given such a test function, we calculate (similarly to in (1.33) or (3.14)):

$$\begin{aligned} 2c_0 \operatorname{osc} g_m &\stackrel{(3.44)(3.45)}{=} \int_0^1 \Phi d\mu^{t_{**}} - \int_0^1 \Phi d\mu^{t_*} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \frac{d}{dt} \int_0^1 \Phi \left( \frac{\varepsilon}{2} (u_{\varepsilon, x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt \\ &\stackrel{(3.46)}{\leq} \left( 2c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt \right)^{1/2} \\ &\quad + \frac{1}{2} (\sup g_2 - \inf g_1) \int_{t_k}^{t_{k+1}} d\eta. \end{aligned} \quad (3.47)$$

The idea for the test function is

$$\Phi(x, t) = \phi(x) \Psi(x, t),$$

where now  $\phi$  is the function

$$\phi(x) = \begin{cases} \inf g_1 - \inf g_m & x < \inf g_1 \\ x - \inf g_m & \inf g_1 \leq x \leq \sup g_2 \\ \sup g_2 - \inf g_m & x > \sup g_2, \end{cases}$$



and  $\Psi$  is an approximate identity such that  $0 \leq \Psi \leq 1$  and

$$\Psi(x, t) = \begin{cases} 1 & g_1(t) - r/2 \leq x \leq g_2(t) + r/2 \\ 0 & x \leq g_1(t) - r \text{ or } x \geq g_2(t) + r. \end{cases}$$

As in the proof of Proposition 3.1 (cf. (3.18) and the lines below it), introducing a regularization  $\phi_\alpha$  of  $\phi$  and considering the test function  $\Phi_\alpha = \phi_\alpha \Psi$  produces a controlled estimate that in the limit  $\alpha \rightarrow 0$  leads to precisely (3.47).

The estimate (3.47) is of the form (3.14) except that  $\sup g_2 - \inf g_1$  appears on the right-hand side instead of  $\text{osc } g_m$ . To control this term, we introduce:

**Lemma 3.4.** *Let  $g_1, g_2$  be continuous on  $[t_k, t_{k+1}]$  with  $\Delta t_k := t_{k+1} - t_k$  and*

$$0 \leq g_1 \leq g_2 \leq 1,$$

and suppose

$$\{t \in [t_k, t_{k+1}]; g_1 = g_2\} \neq \emptyset. \quad (3.48)$$

Suppose moreover that

$$\dot{g}_1, \dot{g}_2 \in L^2([t_k, t_{k+1}] \setminus \{g_1 = g_2\}).$$

Then

$$\begin{aligned} & \sup_{[t_k, t_{k+1}]} g_2 - \inf_{[t_k, t_{k+1}]} g_1 \\ & \leq \text{osc}_{[t_k, t_{k+1}]} g_m + \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = g_2\}} (\dot{g}_1)^2 dt \right)^{1/2} \\ & \quad + \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = g_2\}} (\dot{g}_2)^2 dt \right)^{1/2}. \end{aligned} \quad (3.49)$$

Applying Lemma 3.4 to the last term in (3.47) yields:

$$\begin{aligned} 2c_0 \text{osc } g_m & \leq \left( 2c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt \right)^{1/2} \\ & \quad + \frac{1}{2} (\text{osc } g_m + e_r) \int_{t_k}^{t_{k+1}} d\eta, \end{aligned} \quad (3.50)$$

with an error term  $e_r$  given by:

$$\begin{aligned} e_r & = \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = g_2\}} (\dot{g}_1)^2 dt \right)^{1/2} + \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = g_2\}} (\dot{g}_2)^2 dt \right)^{1/2} \\ & \leq \left( 2\Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = g_2\}} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt \right)^{1/2}. \end{aligned}$$

Recalling (1.34), we rewrite (3.50) as

$$\begin{aligned}
& 2c_0 \operatorname{osc} g_m \\
& \leq \left( 2c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt \right)^{1/2} \\
& + \frac{\bar{C}}{\sqrt{2}} \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = g_2\}} (\dot{g}_1)^2 + (\dot{g}_2)^2 dt \right)^{1/2} + \frac{1}{2} \operatorname{osc} g_m \int_{t_k}^{t_{k+1}} d\eta. \quad (3.51)
\end{aligned}$$

Combining Case 1 equation (3.42) and Case 2 equation (3.51), we have

$$\begin{aligned}
& 2c_0 \operatorname{osc} g_m \\
& \leq \left( 2c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt \right)^{1/2} \\
& + \left( \Delta t_k \int_{t_k}^{t_{k+1}} C((\dot{g}_1)^2 + (\dot{g}_2)^2) \operatorname{Id}_{U_{\ell_2} \setminus G_2} dt \right)^{1/2} + \frac{1}{2} \operatorname{osc} g_m \int_{t_k}^{t_{k+1}} d\eta,
\end{aligned}$$

where  $C = \max\{2c_0^2, \bar{C}^2/2\}$  and  $\operatorname{Id}_{U_{\ell_2} \setminus G_2}$  denotes the characteristic function of  $U_{\ell_2} \setminus G_2$ . We now apply Lemma 3.1 with:

$$\begin{aligned}
m_\varepsilon &= \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]), & h &= C((\dot{g}_1)^2 + (\dot{g}_2)^2) \operatorname{Id}_{U_{\ell_2} \setminus G_2} \\
g &= g_m, & \tilde{c} &= 2c_0,
\end{aligned}$$

concluding

$$\begin{aligned}
2c_0 \int_c^d (\dot{g}_m)^2 dt & \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_c^d \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_2]) dx dt \\
& + \frac{1 + \delta^{-1}}{2c_0} \int_c^d C((\dot{g}_1)^2 + (\dot{g}_2)^2) \operatorname{Id}_{U_{\ell_2} \setminus G_2} dt,
\end{aligned}$$

which establishes (3.37) and (3.38).  $\square$

### 3.3 Proof of Proposition 3.2

We turn now to the proof of the general case, i.e. Proposition 3.2.

*Proof of Proposition 3.2.* The proof proceeds along the same lines as the proof of Lemma 3.3. (Because the main ideas have already appeared in that proof, we sometimes abbreviate our derivations.)

Note that to show (3.6), it is enough to show that for any isolated group of  $J$  interfaces and any  $j \in \{1, \dots, J\}$ , we have that  $\dot{g}_j \in L^2(\{\text{Mult } g_j \leq J\})$ . Thus consider an isolated group of  $J$  interfaces and let

$$O_J \in \{\text{Mult} \leq J\}.$$

As usual, it suffices to consider the case that  $O_J$  is the interval  $(a, b)$ . We define the closed set

$$G_J := \{g_1 = \dots = g_J\},$$

and the open set

$$V_\ell := \{t; d(t, G_J) < 1/\ell\}, \quad (3.52)$$

and we choose open sets  $U_{\ell,1}, U_{\ell,2}$  such that

$$U_{\ell,1} \supset V_\ell^c, \quad U_{\ell,2} \supset G_J, \quad d(U_{\ell,1}, U_{\ell,2}) \geq \frac{1}{2\ell}. \quad (3.53)$$

We remark first that on each connected component of  $U_{\ell,1}$ ,

$$d(\{g_1 = \dots = g_{J-1}\}, \{g_2 = \dots = g_J\}) > 0.$$

Thus for any connected component of  $U_{\ell,1}$  there is a partition

$$\{s_0 < s_1 < \dots < s_n\}$$

such that for each  $(s_k, s_{k+1})$ , either  $(s_k, s_{k+1})$  is compactly contained within the set where  $g_1, \dots, g_{J-1}$  form an isolated group of  $J - 1$  interfaces or  $(s_k, s_{k+1})$  is compactly contained within the set where  $g_2, \dots, g_J$  form an isolated group of  $J - 1$ . Consider without loss a subinterval  $(c, d)$  on which  $g_1, \dots, g_{J-1}$  form an isolated group and let

$$r < \frac{1}{2} \inf_{(c,d)} |g_J - g_{J-1}|.$$

By the inductive assumption (3.5),

$$c_0 \int_c^d \sum_{j=1}^{J-1} (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_c^d \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_{J-1}]) dx dt,$$

and by Proposition 3.1,

$$c_0 \int_c^d \dot{g}_J^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_c^d \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r(g_J) dx dt.$$

The combination of these two inequalities yields

$$c_0 \int_c^d \sum_{j=1}^J (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_c^d \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt.$$

Since the same holds for each of the subintervals into which  $U_{\ell,1}$  was partitioned, we conclude

$$c_0 \int_{U_{\ell,1}} \sum_{j=1}^J (\dot{g}_j)^2 dt \leq \lim_{\varepsilon \rightarrow 0} \int_{U_{\ell,1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt, \quad (3.54)$$

where  $r$  can be chosen as the largest of the radii over all the subintervals.

Next, we will show that  $g_m := (g_1 + \dots + g_J)/J$  satisfies

$$\dot{g}_m \in L^2(U_{\ell,2}) \quad (3.55)$$

with the bound

$$\begin{aligned} & J c_0 \int_{U_{\ell,2}} \dot{g}_m^2 dt \\ & \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_{U_{\ell,2}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt + C_\delta \times o(1)_{\ell \rightarrow \infty}. \end{aligned} \quad (3.56)$$

Notice that (3.56) together with (3.4) implies in particular that

$$\dot{g}_m \in L^2(\{\text{Mult} \leq J\}). \quad (3.57)$$

This will complete the proof: First, by Lemma 3.2, (3.5) and (3.57) imply that  $\dot{g}_1, \dots, \dot{g}_J \in L^2(\{\text{Mult} \leq J\})$ . This proves (3.6) and has the consequence that

$$\int_{(U_{\ell,1} \cup G_J)^c} \sum_{j=1}^J (\dot{g}_j)^2 dt \leq o(1)_{\ell \rightarrow \infty}. \quad (3.58)$$

Second, exactly as in (3.41) in the proof of Lemma 3.3, we combine (3.23), (3.58), (3.57), (3.53), (3.54), and (3.56) to observe:

$$\begin{aligned} & c_0 \int_a^b \sum_{j=1}^J (\dot{g}_j)^2 dt \\ & \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_a^b \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt + C_\delta \times o(1)_{\ell \rightarrow \infty}. \end{aligned}$$

Letting  $\ell \rightarrow \infty$  and then  $\delta \rightarrow 0$  leads to (3.7) and completes the proof of Proposition 3.2. Thus, we need only show (3.55) and (3.56).

It suffices to consider the case that  $U_{\ell,2}$  is an open interval,  $(c, d)$ . Let  $\sigma$  be an arbitrary, finite partition of  $(c, d)$ . Consider the subinterval  $[t_k, t_{k+1}]$ .

*Case 1:*  $[t_k, t_{k+1}] \cap G_J = \emptyset$ . Then by (3.4),  $\dot{g}_1, \dots, \dot{g}_J \in L^2([t_k, t_{k+1}])$  and

$$\begin{aligned} J c_0 \operatorname{osc}_{(t_k, t_{k+1})} g_m &\leq c_0 \sum_{j=1}^J \operatorname{osc}_{(t_k, t_{k+1})} g_j \\ &\leq c_0 \sum_{j=1}^J \int_{t_k}^{t_{k+1}} |\dot{g}_j| dt \\ &\leq \sqrt{J} c_0 \left( \Delta t_k \int_{t_k}^{t_{k+1}} \sum_{j=1}^J (\dot{g}_j)^2 dt \right)^{1/2}, \end{aligned} \quad (3.59)$$

the last line following from Hölder's inequality and the inequality

$$\sum_{j=1}^J \sqrt{a_j} \leq \sqrt{J \sum_{j=1}^J a_j}. \quad (3.60)$$

*Case 2:*  $[t_k, t_{k+1}] \cap G_J \neq \emptyset$ . All operations below (argmin, infimum, etc...) are over the interval  $[t_k, t_{k+1}]$ , and for ease of notation, we omit writing the interval. We define

$$t_* := \operatorname{argmin} g_m, \quad t_{**} := \operatorname{argmax} g_m,$$

and assume without loss that  $t_* < t_{**}$ . We construct  $\Phi$  – as in the proof of Lemma 3.3 – as the limit of regularized functions such that

$$\sum_{j=1}^J \left( \Phi(g_j(t_{**})) - \Phi(g_j(t_*)) \right) = J \operatorname{osc} g_m, \quad (3.61)$$

while at the same time, the following properties hold:

$$\begin{aligned} |\Phi|_\infty &\leq \sup g_J - \inf g_1 \\ \lim_{\varepsilon \rightarrow 0} \int_{t_*}^{t_{**}} \int_0^1 \dot{\Phi} \left( \frac{\varepsilon}{2} (u_{\varepsilon,x})^2 + \varepsilon^{-1} W(u_\varepsilon) \right) dx dt &= 0, \\ \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 |\Phi_x \varepsilon u_{\varepsilon,x} \dot{u}_\varepsilon| dx dt & \\ &\leq \left( J c_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt \right)^{1/2}. \end{aligned} \quad (3.62)$$

Then, as in the proof of Lemma 3.3, we calculate

$$Jc_0 \operatorname{osc} g_m \leq \left( Jc_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt \right)^{1/2} + \frac{1}{2} (\sup g_J - \inf g_1) \int_{t_k}^{t_{k+1}} d\eta. \quad (3.63)$$

To control the term  $\sup g_J - \inf g_1$ , we introduce:

**Lemma 3.5.** *Let  $g_1, \dots, g_J$  be continuous on  $[t_k, t_{k+1}]$  with  $\Delta t_k := t_{k+1} - t_k$  and*

$$0 \leq g_1 \leq \dots \leq g_J \leq 1,$$

and suppose

$$\{t \in [t_k, t_{k+1}]; g_1 = \dots = g_J\} \neq \emptyset. \quad (3.64)$$

Suppose moreover that

$$\dot{g}_1, \dots, \dot{g}_J \in L^2([t_k, t_{k+1}] \setminus \{g_1 = \dots = g_J\}).$$

Then

$$\begin{aligned} & \sup_{[t_k, t_{k+1}]} g_J - \inf_{[t_k, t_{k+1}]} g_1 \\ & \leq \operatorname{osc}_{[t_k, t_{k+1}]} g_m + \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = \dots = g_J\}} (\dot{g}_1)^2 + (\dot{g}_J)^2 dt \right)^{1/2}. \end{aligned} \quad (3.65)$$

Applying Lemma 3.5 to the last term in (3.63) implies:

$$\begin{aligned} Jc_0 \operatorname{osc} g_m & \stackrel{(1.34), (3.60)}{\leq} \left( Jc_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt \right)^{1/2} \\ & + \frac{\bar{C}\sqrt{J}}{2} \left( \Delta t_k \int_{[t_k, t_{k+1}] \setminus \{g_1 = \dots = g_J\}} (\dot{g}_1)^2 + (\dot{g}_J)^2 dt \right)^{1/2} \\ & + \frac{1}{2} \operatorname{osc} g_m \int_{t_k}^{t_{k+1}} d\eta. \end{aligned} \quad (3.66)$$

Combining Case 1 equation (3.59) and Case 2 equation (3.66), we have

$$\begin{aligned} & Jc_0 \operatorname{osc} g_m \\ & \leq \left( Jc_0 \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 \varepsilon (\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt \right)^{1/2} \\ & + \left( \Delta t_k \int_{t_k}^{t_{k+1}} C \sum_{j=1}^J (\dot{g}_j)^2 \operatorname{Id}_{U_{\ell_2} \setminus G_J} dt \right)^{1/2} + \frac{1}{2} \operatorname{osc} g_m \int_{t_k}^{t_{k+1}} d\eta, \end{aligned}$$

where  $C = \max\{Jc_0^2, \bar{C}^2 J/4\}$ . We now apply Lemma 3.1 with:

$$\begin{aligned} m_\varepsilon &= \varepsilon(\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]), & h &= C \sum_{j=1}^J (\dot{g}_j)^2 \text{Id}_{U_{\ell_2} \setminus G_J} \\ g &= g_m, & \tilde{c} &= Jc_0, \end{aligned}$$

to conclude

$$\begin{aligned} Jc_0 \int_c^d (\dot{g}_m)^2 dt &\leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_c^d \int_0^1 \varepsilon(\dot{u}_\varepsilon)^2 \mathbf{1}^r([g_1, g_J]) dx dt \\ &\quad + \frac{1 + \delta^{-1}}{Jc_0} \int_c^d C \sum_{j=1}^J (\dot{g}_j)^2 \text{Id}_{U_{\ell_2} \setminus G_J} dt, \end{aligned}$$

which establishes (3.55) and (3.56).  $\square$

## 3.4 Proof of Auxiliary Lemmas

### 3.4.1 Proof of Lemma 3.1

Roughly speaking, the main idea is that if we have the implication:

$$\Delta t_k \ll 1 \quad \Rightarrow \quad \int_{t_k}^{t_{k+1}} d\eta \ll 1,$$

then for any partition  $\sigma$  with  $|\sigma|$  sufficiently small, we have by a rearrangement of (3.20) that for any subinterval of the partition,

$$\begin{aligned} &\tilde{c} \frac{(\text{osc}_{[t_k, t_{k+1}]} g)^2}{\Delta t_k} \\ &\lesssim \left( \left( \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt \right)^{1/2} + \frac{1}{\sqrt{\tilde{c}}} \left( \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} \right)^2 \\ &\leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt + \frac{1 + \delta^{-1}}{\tilde{c}} \int_{t_k}^{t_{k+1}} h dt. \end{aligned}$$

Considering the sum over  $k$  and invoking Lemma 1.3 leads to the result. We need to be a little careful in case  $\eta$  has point masses, but we will see that because of the continuity of  $g$ , this does not present a real problem.

*Proof of Lemma 3.1.* Fix any tolerance  $\gamma > 0$ . Since  $\eta$  is a bounded measure, we can express its point masses as a countable sum:

$$\sum_{\ell=1}^{\infty} c_\ell \delta_{m_\ell},$$

and there exists  $R < \infty$  such that

$$\sum_{\ell=R+1}^{\infty} c_{\ell} \leq \frac{\gamma}{2}.$$

Then for  $s > 0$  sufficiently small,

$$\Delta t_k \leq s \quad \Rightarrow \quad \int_{(t_k, t_{k+1}) \setminus \{m_{\ell}\}_{\ell=1}^R} d\eta \leq \gamma. \quad (3.67)$$

Consider any subinterval  $[t_k, t_{k+1}]$ , and let

$$t_* := \operatorname{argmin}_{[t_k, t_{k+1}]} g, \quad t_{**} := \operatorname{argmax}_{[t_k, t_{k+1}]} g.$$

Suppose that  $m_1, \dots, m_R$  all lie within  $(t_k, t_{k+1})$  (worst case scenario). We begin with a rough estimate to prove (3.21). The starting point is:

$$\begin{aligned} & \tilde{c}|g(t_{**}) - g(t_*)| \\ & \leq \tilde{c}(|g(t_*) - g(m_1^-)| + |g(m_1^-) - g(m_1^+)| + |g(m_1^+) - g(m_2^-)| \\ & \quad + \dots + |g(m_R^-) - g(m_R^+)| + |g(m_R^+) - g(t_{**})|) \\ & = \tilde{c}(|g(t_*) - g(m_1^-)| + |g(m_1^+) - g(m_2^-)| + \dots + |g(m_R^+) - g(t_{**})|), \end{aligned} \quad (3.68)$$

where  $g(m_{\ell}^-)$  (resp.  $g(m_{\ell}^+)$ ) denotes the limit of  $g$  as  $t \uparrow m_{\ell}$  (resp.  $t \downarrow m_{\ell}$ ). The first line is the triangle inequality and the second follows from continuity. We may assume without loss that

$$t_* < m_1 < \dots < m_R < t_{**}.$$

According to (3.20), we can bound the first term on the right-hand side of (3.68) by:

$$\begin{aligned} & \tilde{c}|g(t_*) - g(m_1^-)| \\ & \leq \left( \tilde{c}(m_1 - t_*) \lim_{\varepsilon \rightarrow 0} \int_{t_*}^{m_1} \int_0^1 m_{\varepsilon} dx dt \right)^{1/2} \\ & \quad + \left( (m_1 - t_*) \int_{t_*}^{m_1} h dt \right)^{1/2} + \frac{1}{2} \operatorname{osc}_{[t_*, m_1]} g \int_{t_*}^{m_1} d\eta \\ & \leq \left( \tilde{c} \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_{\varepsilon} dx dt \right)^{1/2} \\ & \quad + \left( \Delta t_k \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} + \frac{1}{2} \operatorname{osc}_{[t_k, t_{k+1}]} g \int_{t_*}^{m_1} d\eta \\ & \leq (C \Delta t_k)^{1/2} \left( \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_{\varepsilon} dx dt + \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} \\ & \quad + \frac{1}{2} \operatorname{osc}_{[t_k, t_{k+1}]} g \int_{t_*}^{m_1} d\eta, \end{aligned}$$



for  $C = 2 \max\{\tilde{c}, 1\}$ . Estimating the other terms similarly, we deduce from (3.68) that

$$\begin{aligned}
& \tilde{c} \operatorname{osc}_{[t_k, t_{k+1}]} g \\
& \leq (R+1)(C\Delta t_k)^{1/2} \left( \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt + \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} \\
& \quad + \frac{1}{2} \operatorname{osc}_{[t_k, t_{k+1}]} g \left( \int_{t_*}^{m_1} + \int_{m_1}^{m_2} + \dots + \int_{m_R}^{t_{**}} \right) d\eta \\
& \stackrel{(3.67)}{\leq} (R+1)(C\Delta t_k)^{1/2} \left( \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt + \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} \\
& \quad + \gamma \operatorname{osc}_{[t_k, t_{k+1}]} g.
\end{aligned}$$

Consequently for any  $\gamma < \tilde{c}$ ,

$$\begin{aligned}
& \frac{(\operatorname{osc}_{[t_k, t_{k+1}]} g)^2}{\Delta t_k} \\
& \leq C_\gamma (R+1)^2 \left( \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt + \int_{t_k}^{t_{k+1}} h dt \right), \quad (3.69)
\end{aligned}$$

with  $C_\gamma = C/(\tilde{c} - \gamma)^2$ . Taking the sum over  $k$  and recalling  $h \in L^1([a, b])$  and (3.19) leads by Lemma 1.3 to (3.21), i.e.

$$\dot{g} \in L^2([a, b]).$$

We now refine our estimates in order to prove (3.22). By (3.21),

$$\int_a^b (\dot{g})^2 dt = \int_a^{m_1^-} (\dot{g})^2 dt + \int_{m_1^+}^{m_2^-} (\dot{g})^2 dt + \dots + \int_{m_R^+}^b (\dot{g})^2 dt. \quad (3.70)$$

Consider any of the subintervals  $(a, m_1^-)$ ,  $(m_1^+, m_2^-)$ ,  $\dots$ ,  $(m_R^+, b)$ , and let  $\sigma$  be an arbitrary partition of the subinterval with  $|\sigma|$  sufficiently small. On any  $(t_k, t_{k+1})$  of the partition, we have

$$\begin{aligned}
\tilde{c} \operatorname{osc}_{(t_k, t_{k+1})} g & \stackrel{(3.20), (3.67)}{\leq} \left( \tilde{c} \Delta t_k \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt \right)^{1/2} \\
& \quad + \left( \Delta t_k \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} + \gamma \operatorname{osc}_{(t_k, t_{k+1})} g,
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{(\tilde{c} - \gamma)^2 (\text{osc}_{(t_k, t_{k+1})} g)^2}{\tilde{c} \Delta t_k} \\
& \leq \left( \left( \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt \right)^{1/2} + \left( (\tilde{c})^{-1} \int_{t_k}^{t_{k+1}} h dt \right)^{1/2} \right)^2 \\
& \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_{t_k}^{t_{k+1}} \int_0^1 m_\varepsilon dx dt + \frac{1 + \delta^{-1}}{\tilde{c}} \int_{t_k}^{t_{k+1}} h dt,
\end{aligned}$$

the last line following from Young's inequality. Taking the sum and applying Lemma 1.3 on each of  $(a, m_1^-)$ ,  $(m_1^+, m_2^-)$ ,  $\dots$ ,  $(m_R^+, b)$ , we conclude by (3.70) that

$$\begin{aligned}
& \frac{(\tilde{c} - \gamma)^2}{\tilde{c}} \int_a^b (\dot{g})^2 dt \\
& \leq (1 + \delta) \lim_{\varepsilon \rightarrow 0} \int_a^b \int_0^1 m_\varepsilon dx dt + \frac{1 + \delta^{-1}}{\tilde{c}} \int_a^b h dt.
\end{aligned}$$

Letting  $\gamma \rightarrow 0$  gives precisely (3.22).  $\square$

### 3.4.2 Proof of Lemma 3.2

*Proof.* By symmetry, it suffices to prove (3.26) for  $j = 1$ , i.e., to show

$$\dot{g}_1 \in L^2(\{\text{Mult} \leq J\}). \quad (3.71)$$

By definition, (3.71) holds if for every  $\phi \in C_0^1(\{\text{Mult} \leq J\})$ ,

$$- \int_{\{\text{Mult} \leq J\}} g_1 \dot{\phi} dt \leq C \int_{\{\text{Mult} \leq J\}} \phi^2 dt.$$

We have:

$$\begin{aligned}
& - \int_{\{\text{Mult} \leq J\}} g_1 \dot{\phi} dt \\
& = - \int_{\{\text{Mult} = J\}} g_1 \dot{\phi} dt - \int_{\{\text{Mult} < J\}} g_1 \dot{\phi} dt \\
& = - \int_{\{\text{Mult} = J\}} \sum_{j=1}^J \frac{g_j}{J} \dot{\phi} dt - \int_{\{\text{Mult} < J\}} g_1 \dot{\phi} dt \\
& = - \int_{\{\text{Mult} \leq J\}} \sum_{j=1}^J \frac{g_j}{J} \dot{\phi} dt + \int_{\{\text{Mult} < J\}} \sum_{j=1}^J \frac{g_j}{J} \dot{\phi} dt - \int_{\{\text{Mult} < J\}} g_1 \dot{\phi} dt \\
& = - \int_{\{\text{Mult} \leq J\}} \sum_{j=1}^J \frac{g_j}{J} \dot{\phi} dt + \int_{\{\text{Mult} < J\}} \left( -g_1 \frac{J-1}{J} + \sum_{j=2}^J \frac{g_j}{J} \right) \dot{\phi} dt.
\end{aligned}$$

Notice that  $\phi$  does not necessarily vanish on the boundary of  $\{\text{Mult} < J\}$  (we only know that it is compactly supported on  $\{\text{Mult} \leq J\}$ ), but we can nonetheless integrate by parts in the second term since

$$\left(-g_1 \frac{J-1}{J} + \sum_{j=2}^J \frac{g_j}{J}\right) = 0 \quad \text{on} \quad \{\text{Mult} = J\}.$$

Thus, we obtain:

$$\begin{aligned} & - \int_{\{\text{Mult} \leq J\}} g_1 \dot{\phi} dt \\ &= - \int_{\{\text{Mult} \leq J\}} \sum_{j=1}^J \frac{g_j}{J} \dot{\phi} dt - \int_{\{\text{Mult} < J\}} \left(-\dot{g}_1 \frac{J-1}{J} + \sum_{j=2}^J \frac{\dot{g}_j}{J}\right) \phi dt \\ &\stackrel{(3.24), (3.25)}{\leq} C_1 \int_{\{\text{Mult} \leq J\}} \phi^2 dt + C_2 \int_{\{\text{Mult} < J\}} \phi^2 dt \\ &\leq (C_1 + C_2) \int_{\{\text{Mult} \leq J\}} \phi^2 dt. \end{aligned}$$

□

### 3.4.3 Proof of Lemma 3.4

*Proof.* Without loss, we may assume

$$\sup_{(a,b)} g_2 \neq \sup_{(a,b)} g_m, \quad \inf_{(a,b)} g_1 \neq \inf_{(a,b)} g_m.$$

Let

$$t_{**} := \operatorname{argmax}_{[a,b]} g_2.$$

By (3.48), there exists a point  $t_m \in (a, b)$  such that

$$g_1(t_m) = g_2(t_m) = g_m(t_m) \quad \text{and} \quad g_1 \neq g_2 \quad \text{in between } t_m \text{ and } t_{**}.$$

We have:

$$\begin{aligned} \sup_{(a,b)} g_2 - \sup_{(a,b)} g_m &\leq |g_2(t_{**}) - g_m(t_m)| \\ &= |g_2(t_{**}) - g_2(t_m)| \\ &= \left| \int_{t_m}^{t_{**}} (\dot{g}_2) dt \right| \\ &\leq \left( (b-a) \int_{(a,b) \setminus \{g_1=g_2\}} (\dot{g}_2)^2 dt \right)^{1/2}. \end{aligned} \quad (3.72)$$

Similarly,

$$\inf_{(a,b)} g_m - \inf_{(a,b)} g_1 \leq \left( (b-a) \int_{(a,b) \setminus \{g_1=g_2\}} (\dot{g}_1)^2 dt \right)^{1/2}. \quad (3.73)$$

Thus,

$$\begin{aligned} & \sup_{(a,b)} g_2 - \inf_{(a,b)} g_1 \\ &= \sup_{(a,b)} g_2 - \sup_{(a,b)} g_m + \sup_{(a,b)} g_m - \inf_{(a,b)} g_m + \inf_{(a,b)} g_m - \inf_{(a,b)} g_1 \\ &\stackrel{(3.72),(3.73)}{\leq} \left( (b-a) \int_{(a,b) \setminus \{g_1=g_2\}} (\dot{g}_2)^2 dt \right)^{1/2} \\ &\quad + \left( (b-a) \int_{(a,b) \setminus \{g_1=g_2\}} (\dot{g}_1)^2 dt \right)^{1/2} + \text{osc}_{(a,b)} g_m. \end{aligned}$$

□

### 3.4.4 Proof of Lemma 3.5

*Proof.* The proof is nearly the same as for Lemma 3.4. Without loss, we assume

$$\sup_{(a,b)} g_J \neq \sup_{(a,b)} g_m, \quad \inf_{(a,b)} g_1 \neq \inf_{(a,b)} g_m.$$

Let

$$t_{**} := \operatorname{argmax}_{[a,b]} g_J.$$

By (3.64), there exists a point  $t_m \in (a, b)$  such that

$$\begin{aligned} g_1(t_m) &= \dots = g_J(t_m) = g_m(t_m) \quad \text{and} \\ \nexists t \text{ in between } t_m \text{ and } t_{**} \text{ such that } g_1(t) &= \dots = g_J(t). \end{aligned}$$

We have:

$$\begin{aligned} \sup_{(a,b)} g_J - \sup_{(a,b)} g_m &\leq |g_J(t_{**}) - g_m(t_m)| \\ &= |g_J(t_{**}) - g_J(t_m)| \\ &= \left| \int_{t_m}^{t_{**}} (\dot{g}_J) dt \right| \\ &\leq \left( (b-a) \int_{(a,b) \setminus \{g_1=g_J\}} (\dot{g}_J)^2 dt \right)^{1/2}. \end{aligned} \quad (3.74)$$

As in the proof of Lemma 3.4, combining (3.74) with the similar argument for  $g_1$  leads to (3.65). □

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