Adaptive Dynamics for Interacting Markovian Processes

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Dynamics of information flow in adaptively interacting stochastic processes is studied. We give an extended form of game dynamics for Markovian processes and study its behavior to observe information flow through the system. Examples of the adaptive dynamics for two stochastic processes interacting through matching pennies game interaction are exhibited along with underlying causal structure.

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When studying the interaction and evolution of many stochastic processes that are endowed with the ability to adapt to their enviroment, a natural question arises: how does information flow though the system and, moreover, how can we measure or calculate this information flow? From the viewpoint of large networks of stochastic elements, flow of information in the network has been studied [1–4]. In general, mutual information is not a representative measure of information flow in adaptive dynamics as its causal structure forms a complex network, making the concept of information flow unclear. To address this problem, we give an extended form of game dynamics for interacting Markovian processes and investigate information flow quantitatively.

Suppose that N stochastic processes $X_1, \ldots, X_N$ are interacting with each other. At each time step τ, the unit n sends a symbol $s_n \in \{0, 1\}$ to the other units and receives at most $N-1$ symbols from the other units. We denote the global system state as $s = s_1 \cdots s_N$. The next symbol sent by the unit n, s_n', is dependent on the symbol received from the previous global state, s. Local transition probabilities for n-th unit are described as

$$x^{(n)}_{s_n'|s} = P(X_n(\tau{+}1) = s_n' | X_1(\tau) = s_1, \ldots, X_N(\tau) = s_N),$$

where $n = 1, \ldots, N$ and $x^{(n)}_{0|s} + x^{(n)}_{1|s} = 1$. The transition probabilities $(x^{(n)}_{0|s}, x^{(n)}_{1|s})$ is an element of a simplex denoted by $\Delta^{(n)}_s$.

We introduce a local adaptation process to change transition probabilities $x^{(n)}_{s_n'|s}$, assuming that adaptation is very slow compared with relaxation time of the global Markovian process. After the system reaches a stationary state, each unit independently changes its stochastic structure by changing its transition probabilities. Assuming strong connectivity of the global Markovian kernel, we study dynamics of transition probabilities in an ergodic subspace. This assumption corresponds to persistency of dynamics of transition probabilities $x^{(n)}_{s_n'|s}$ in the state space. Time evolution of $x^{(n)}_{s_n'|s}$ is driven by simple stochastic learning through interaction: reinforcements for transition probabilities of the unit n to send 0 and 1 in the previous global state s are given by the constants $a^{(n)}_{0|s}$ and $a^{(n)}_{1|s}$. The conditional expectation reinforcements $R^{(n)}_{s_n'|s}$ to chose each symbols s_n' given the previous state s are calculated with $a^{(n)}_{s_n'|s}$, $x_{s_n'|s}$, and the unique stationary distribution. For X_n, we give adaptive dynamics for probabilities of s_n' given s for $t \sim t + \Delta t$

$$x^{(n)}_{s_n'|s}(t+\Delta t) = \frac{x^{(n)}_{s_n'|s}(t) e^{\beta^{(n)} R^{(n)}_{s_n'|s}(t)}}{\sum_n^N x^{(n)}_{s_n'|s}(t) e^{\beta^{(n)} R^{(n)}_{s_n'|s}(t)}}, \qquad (1)$$

where $\beta^{(n)}$ is the learning rate for the unit n. Here Δt is much larger than the relaxation time of the global Markovian process. The continuous time model is given as

$$\frac{\dot{x}^{(n)}_{s_n'|s}(t)}{x^{(n)}_{s_n'|s}(t)} = \beta^{(n)} (R^{(n)}(t)_{s_n'|s} - R^n_{|s}(t)), \qquad (2)$$

for $n = 1, \ldots, N$, where $R^{(n)}_{|s} = \sum_{s_n'} x^{(n)}_{s_n'|s} R^{(n)}_{s_n'|s}$ is the conditional expectation of reinforcements over all possible symbols given the previous system state s. Intuitively, when $(R^{(n)}(t)_{s_n'|s} - R^n_{|s}(t))$ is positive, that is, the conditional expectation reinforcement for a symbol s_n' given s is greater than the average of the expectation reinforcement given s, the logarithmic derivative of $x^{(n)}_{s_n'|s}(t)$ increases, and when negative, it decreases. The learning rate, $\beta^{(n)}$, controls the time scales of the adaptive dynamics of each unit n. (See [5] for the derivation of this model.) Note that Eq. (2) represents adaptive dynamics with finite memories. Higher dimensional coupled ODEs are required for multiple Markovian process and PDEs for non-Markovian process with infinitely long memories.

Suppose that two biased coin tossing processes X and Y adaptively interact with each other. They produce a pair of symbols ij at each time step, where i and j are either heads (0) or tails (1). At the next time step, X send a symbol i' to Y based on the previous pair of symbols

ij, and vise versa. If there is a causal interaction with one step memory, the global stochastic process becomes a simple Markovian process. When X's and Y's behavior are causally separated, the whole system is a product of two biased coin tossing processes (case 10 in Fig. 1).

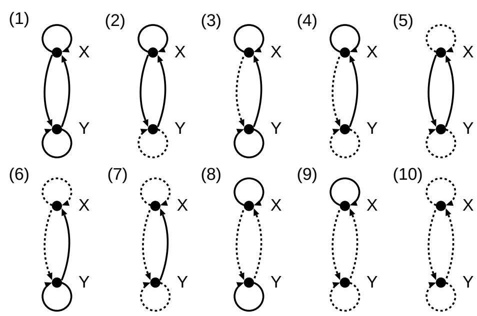

FIG. 1: Possible causal structure (case 1-10): $X \to Y$ indicates that Y receives symbols sent by X (information flow from X to Y). Dashed arrows indicate ignorance of received symbols (no information flow).

Considering Fig. 1, the extreme cases are 1 and 10. Case 1 corresponds to the situation, "each unit has one step memory of the previous global state s," and case 10 to, "no information of s." Local transition probabilities of X and Y are given as $(x_{i'|ij}) = P(X' = i'|X = i, Y = j)$, and $(y_{j'|ij}) = P(Y' = j'|X = i, Y = j)$, where $\sum_{i'} x_{i'|ij} = \sum_{j'} y_{j'|ij} = 1$. The global Markovian kernel is given with $(x_{i'|ij} y_{j'|ij})$ where $\sum_{i',j'} x_{i'|ij} y_{j'|ij} = 1$. When X and Y match heads (0) or tails (1) of coins (00 or 11), Y reinforces the choice, and when they don't (01 or 10), X reinforces the choice. This interaction is called the matching pennies game in game theory. The reinforcements are given by a bi-matrix

$$(A, B) = \left(\begin{bmatrix} -\epsilon_X & \epsilon_X \\ \epsilon_X & -\epsilon_X \end{bmatrix}, \begin{bmatrix} \epsilon_Y & -\epsilon_Y \\ -\epsilon_Y & \epsilon_Y \end{bmatrix} \right), \quad (3)$$

where $0 < \epsilon_X, \epsilon_Y < 1$. The intereaction matrices, $A = (a_{ij})$ and $B = (b_{ji})$, are the reinforcements for X and Y for the global state ij. The Nash equilibrium of the game (3) in terms of game theory is an uniformly random state $(1/2, 1/2)$. The conditional expectation reinforcements are given by $R^X_{i'|ij} = (A\mathbf{y}_{|ij})_{i'}$ and $R^Y_{j'|ij} = (B\mathbf{x}_{|ij})_{j'}$, where $\mathbf{x}_{|ij} = (x_{0|ij}, x_{1|ij})^T$, and $\mathbf{y}_{|ij} = (y_{0|ij}, y_{1|ij})^T$. Eq. (2) reduces to

$$\begin{aligned} \frac{\dot{x_{i'|ij}}}{x_{i'|ij}} &= \beta^X [(A\mathbf{y}_{|ij})_{i'} - \mathbf{x}_{|ij} \cdot A\mathbf{y}_{|ij}], \\ \frac{\dot{y_{j'|ij}}}{y_{j'|ij}} &= \beta^Y [(B\mathbf{x}_{|ij})_{j'} - \mathbf{y}_{|ij} \cdot B\mathbf{x}_{|ij}]. \end{aligned} \quad (4)$$

Eq. (4) corresponds to adaptive dynamics for an interacting Markovian processes in an 8-dimensional state space $\Pi_{i,j} \Delta^X_{ij} \times \Delta^Y_{ij}$, which is in the form of standard game dyamics. Similarly, for case 10, we have

$$\begin{aligned} \frac{\dot{x_{i'|**}}}{x_{i'|**}} &= \beta^X [(A\mathbf{y}_{|**})_{i'} - \mathbf{x}_{|**} \cdot A\mathbf{y}_{|**}], \\ \frac{\dot{y_{j'|**}}}{y_{j'|**}} &= \beta^Y [(B\mathbf{x}_{|**})_{j'} - \mathbf{y}_{|**} \cdot B\mathbf{x}_{|**}], \end{aligned} \quad (5)$$

where $\mathbf{x}_{|**} = (x_{0|**}, x_{1|**})^T$, and $\mathbf{y}_{|**} = (y_{0|**}, y_{1|**})^T$. Here, the * indicates ignorance of received symbols. Eq. (5) is, again, standard game dynamics in a 2-dimensional state space $\Delta^X \times \Delta^Y$. It is known that the dynamics of Eq. (5) is Hamiltonian with a constant of motion $H = 1/\beta^X D(\mathbf{x}^*||\mathbf{x}) + 1/\beta^Y D(\mathbf{y}^*||\mathbf{y})$, where D is Kullback divergence, and where $(\mathbf{x}^*, \mathbf{y}^*)$ is the Nash equilibrium of the game (A, B). The dynamics are neutrally stable periodic orbits for all range of parameters ϵ_X, ϵ_Y [6, 7]. When the degree of freedom of the Hamiltonian systems is more than 2, and the bi-matrix (A, B) gives asymmetric cyclical interaction, the dynamics can be chaotic [5, 8, 9]. Summarizing, if all units have complete information of the previous global state s (case 1), or they are all causally separated with no information of s (case 10), we have a family of standard game dynamics given by Eqs. (4) and (5).

For intermediate cases $2 - 9$, showing in Fig. 1, where units have partial information of s, we have explicit stationary distribution terms in the adaptive dynamics. Assuming the process is ergodic, $0 < x_{i'|ij}, y_{j'|ij} < 1$, an unique stationary distribution $(p(i, j))$ exists. We denote the marginal stationary distributions $\mathbf{p}^X = (P(X = 0), P(X = 1))^T$, $\mathbf{p}^Y = (P(Y = 0), P(Y = 1))^T$. The conditional stationary distribution of i, given the previous state j, is denoted as $p(i|j) = p(i, j)/p(j)$, and those of j, given the previous state i, as $p(j|i) = p(i, j)/p(i)$.

For case 2, with $R^X_{i'|ij} = (A\mathbf{y}_{|i*})_{i'}$ and $R^Y_{j'|i*} = \sum_j p(j|i)(B\mathbf{x}_{|ij})_{j'}$, Eq. (2) reduces to

$$\begin{aligned} \frac{\dot{x_{i'|ij}}}{x_{i'|ij}} &= \beta^X [(A\mathbf{y}_{|i*})_{i'} - \mathbf{x}_{|ij} \cdot A\mathbf{y}_{|i*}], \quad (6) \\ \frac{\dot{y_{j'|i*}}}{y_{j'|i*}} &= \beta^Y [(\sum_j p(j|i) B\mathbf{x}_{|ij})_{j'} - \mathbf{y}_{|ij} \cdot (\sum_j p(j|i) B\mathbf{x}_{|ij})]. \end{aligned}$$

Similarly, for case 5, with $R^X_{i'|*j} = (A\mathbf{p}^X)_{i'}$ and $R^Y_{j'|i*} = (B\mathbf{p}^Y)_{j'}$, we obtain

$$\begin{aligned} \frac{\dot{x_{i'|*j}}}{x_{i'|*j}} &= \beta^X [(A\mathbf{p}^Y)_{i'} - \mathbf{x}_{|*j} \cdot A\mathbf{p}^Y], \\ \frac{\dot{y_{j'|i*}}}{y_{j'|i*}} &= \beta^Y [(B\mathbf{p}^X)_{j'} - \mathbf{y}_{|i*} \cdot B\mathbf{p}^X]. \end{aligned} \quad (7)$$

Note that $(p(i, j))$ are given as a function of $(x_{i'|ij})$ and $(y_{j'|ij})$, thus the equations of motion are in a closed form. For cases $2 - 9$, we have nonlinear couplings with a stationary distribution, which is in contrast to the quasilinear coupling of standard game dynamics. Eq. (6) - (7) are both in an extended form of standard game dynamics.

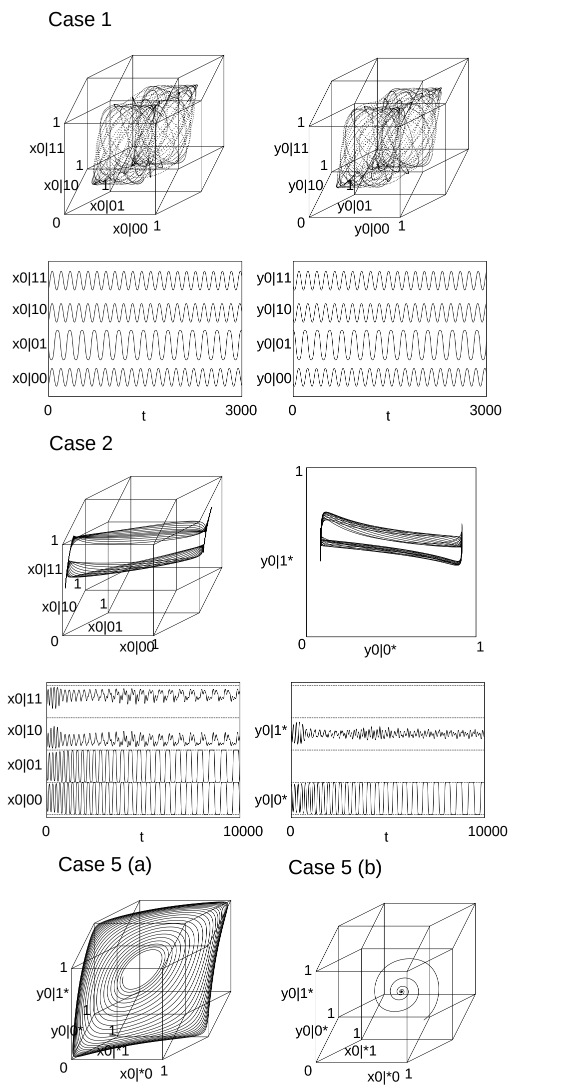

FIG. 2: (Top) Case 1: Neutrally stable quasi-periodic tori. (Middle) Case 2: A combination of quasi-periodic tori and transients to a heteroclinic cycle. (Bottom) Case 5: (a) Transients to a heteroclinic cycle which consists of vertex saddles $(x_{0|*0}, x_{0|*1}, y_{0|0*}, y_{0|1*}) = (0,0,0,0)$, $(1,1,0,0)$, $(0,0,1,1)$, $(1,1,1,1)$ and (b) convergence to one of infinitely many neutrally stable fixed points which gives an uniform stationary distribution $p(i,j) = \frac{1}{4}$; (in this case, converging to $(x_{0|*0}, x_{0|*1}, y_{0|0*}, y_{0|1*}) = (0.539057, 0.460943, 0.671772, 0.328228)$) are attracting sets.

Let us now consider several examples. In examples, where the parameters are fixed to $\beta^X = \beta^Y$ and $\epsilon_X = \epsilon_Y = 0.5$, we have four types of dynamics: (1) neutrally stable periodic motion of Markovian kernel, (2) convergence to a fixed Markovian kernel that gives a uniform stationary distribution, (3) sharp switching among almost deterministic Markovian kernel, (4) a combination of (1)-(3). In contrast to the matching pennies game dynamics which shows only neutrally stable periodic orbits, we obtain new types of dynamics naturally given by the Markovian structure.

Case 1 (Eq. (4)): Neutrally stable quasi-periodic tori are observed. They are simply a product of periodic orbits in the matching pennies game dynamics. The dynamics of Eq. (5) is embedded in a subspace in the state space, given by $x_{i'|00} = x_{i'|01} = x_{i'|10} = x_{i'|11}$ and $y_{j'|00} = y_{j'|01} = y_{j'|10} = y_{j'|11}$.

Case 2 (Eq. (6)): A combination of the dynamics of Eq. (4), quasi-periodic tori, and the dynamics of Eq. (7), transients to a heteroclinic cycle, are observed (Fig. 2, middle). One of the infinitely many attracting periodic orbits corresponding to periodic orbits in Eqs. (4) is selected depending on initial conditions.

Case 5 (Eq. (7)): Bi-stable dynamics is observed. A manifold which gives uniform stationary distribution $p(i,j) = 1/4$, is an attracting set. Fixed points on this attracting manifold are all neutrally stable. Heteroclinic cycles which consists of several vertex saddles are also attracting sets. Depending on initial conditions, either convergence to one of the fixed points on the attracting manifold or transients to one of the heteroclinic cycles are observed (Fig. 2, bottom).

In standard game dynamics which describes causally separated stochastic process, information flow is always 0. By using the Markovian extention of game dynamics, we can now quantify bi-directional information flow between stochastic units. Eq. (9) gives conditional mutual information of Y and X' given Y and X', which is a measure of stochastic dependence of X' and Y (sometime called transfer entropy, see [10–13]). Recently, a new measure of information flow which describes deviation of two random variables from causal dependence, is formulated by Ay and Polani [4]. Information flow from Y to X', given X and Y, is defined by Eq. (9) as a measure of causal dependence.

$$
\begin{aligned}
& I(Y : X'|X,Y) \\
={}& \sum_{i',i,j} p(i',i,j) \log \frac{p(i'|i,j)}{\sum_j p(j|i) p(i'|i,j)} \\
={}& -\sum_{i',i} p(i) (\sum_j p(j|i) x_{i'|ij}) \log(\sum_j p(j|i) x_{i'|ij}) \\
& + \sum_{i,j} p(i,j) [\sum_{i'} x_{i'|ij} \log(x_{i'|ij})]. \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
& I(Y \to \ X'|X,Y) \\
={}& \sum_{i',i,j} p(i)p(j)p(i'|i,j) \log \frac{p(i'|i,j)}{\sum_j p(j) p(i'|i,j)} \\
={}& -\sum_{i',i} p(i) (\sum_j p(j) x_{i'|ij}) \log(\sum_j p(j) x_{i'|ij}) \\
& + \sum_{i,j} p(i)p(j) [\sum_{i'} x_{i'|ij} \log(x_{i'|ij})]. \qquad (9)
\end{aligned}
$$

In the case that Y is a fixed information source, $(y_{0|00}, y_{0|01}, y_{0|10}, y_{0|11}) = (1,0,1,0)$, the dynamics (4) with $\beta^Y = 0$ monotonically converges to an optimal $(x_{0|00}, x_{0|01}, x_{0|10}, x_{0|11}) = (0, 1, 0, 1)$. The system state s is either 00 or 11 and X is always rewarded. In this case,

$$\begin{aligned} I(X : Y'|X,Y) &= 0, \quad I(X \to Y'|X,Y) = 0, \\ I(Y : X'|X,Y) &= 0, \quad I(Y \to X'|X,Y) = \log 2. \end{aligned} \tag{10}$$

There is information flow from Y to X because X receives symbols sent by Y and extracts information from Y's behavior. Thus, X is not stochastically dependent on Y but, is causally dependent on Y. The above measure defined by (9) clearly captures this property. Thus, intuitively, we can say that $I(Y \to X'|X,Y)$ is a more appropriate measure of the information flow.

As shown in Fig. 3, we observe (case 1) aperiodic, (case 2) periodic switching among aperiodic, and (case 5) stationary information flow. In general, information flow vanishes when the system state is on a manifold M_0 defined by $x_{i'|ij} = \sum_j p(j) x_{i'|ij}$ and $y_{j'|ij} = \sum_i p(i) y_{j'|ij}$. Information flow is maximized to $\log 2$ when the system state is on a manifold M_1 defined by the set of points which have maximal distance from M_0. Case 5 with bistability between a fixed point and heteloclinic cycle gives us a clear example of stationary information flow. Between the manifold M_0 and M_1 we have dynamic flow of information such as those in case 1 and 2 in Fig. 3. Through adaptation, dynamic information flow emerges by keeping rewards as large as possible at each moment, and because of the complex game interaction and underlying causal structure.

The above is an extention of game dynamics for interacting Markovian processes. If all units have complete information of the previous global state s, or they are all causally separated with no information of s, we have a family of standard game dynamics. For intermediate cases with partial information of s, we have explicit stationary distribution terms in the equations of motion. The presented examples show new types of phenomena in contrast to standard game dynamics. Dynamics of information flow between two units is discussed based on underlying causal structure. When units are ternary information sources, the presented game dynamics shows chaotic behavior even in the simplest case Eqs. (5) [5, 8, 9]. Studying adaptive dynamics for N units with heterogeneous game interaction, and with various types of causal networks is left for a future work. Rigorous information theoretic analysis of the presented adaptive dynamics will be covered more elsewhere. The relationship between global and individual reward structure and information flow among units would give us new insights in game theory. Applications to ecological and social dynamics, econophysics, and studies on learning in game are all straightforward.

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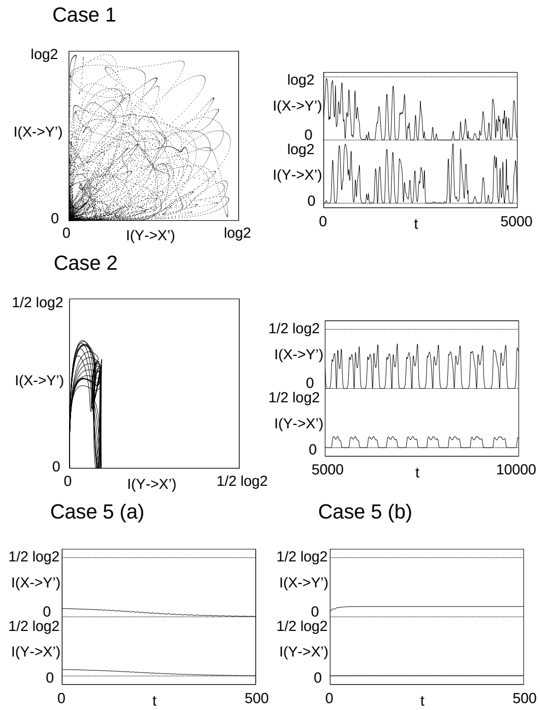

FIG. 3: (Top) Case 1: Aperiodic information flow. (Middle) Case 2: Periodic switching among aperiodic information flow. (Bottom) Case 5: (a) stationary information flow $I(Y \to X' : X, Y) = I(X \to Y' : X, Y) = 0$. (b) stationary information flow $I(Y \to X' : X, Y) = 0.00305395$ and $I(Y \to X' : X, Y) = 0.06023025$.

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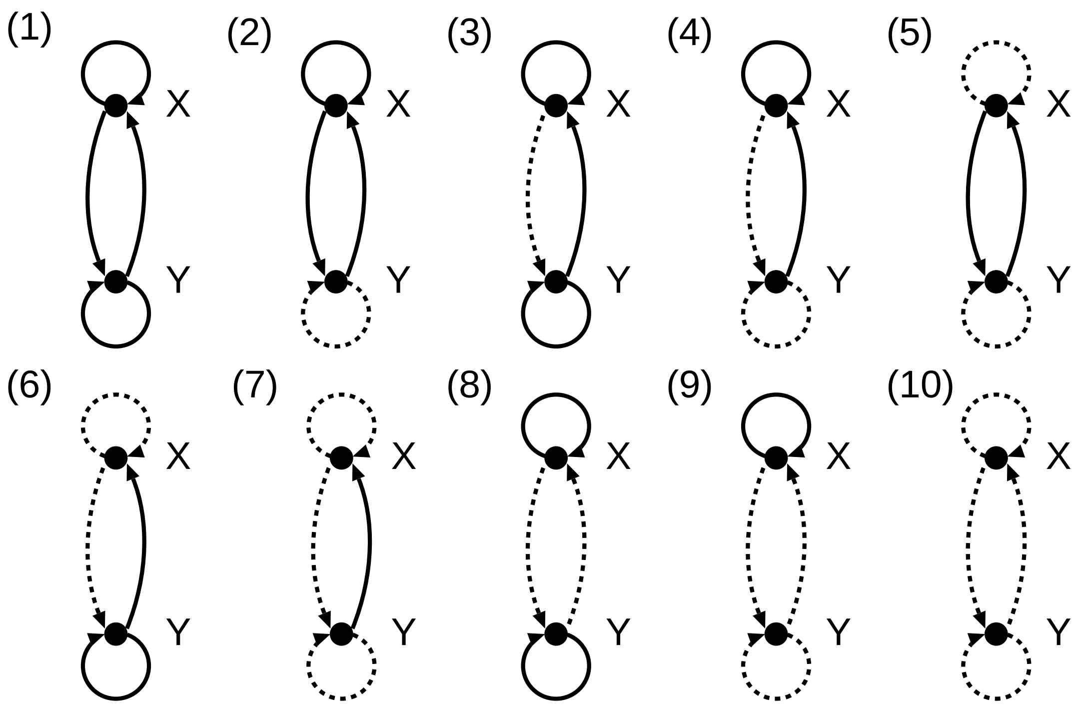


Figure 1

Case 1

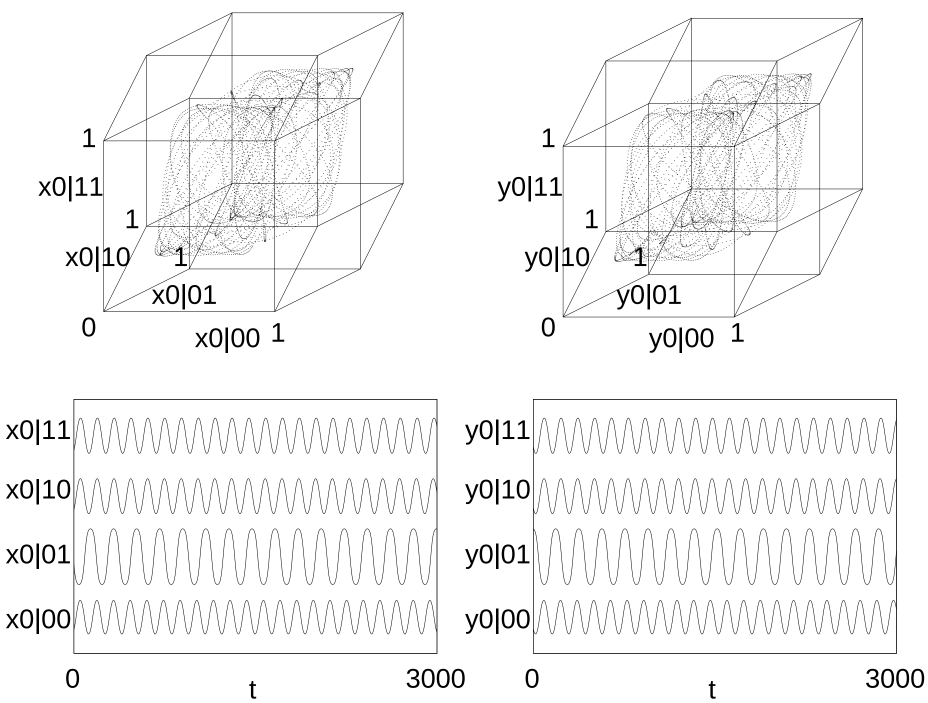


Figure 2a

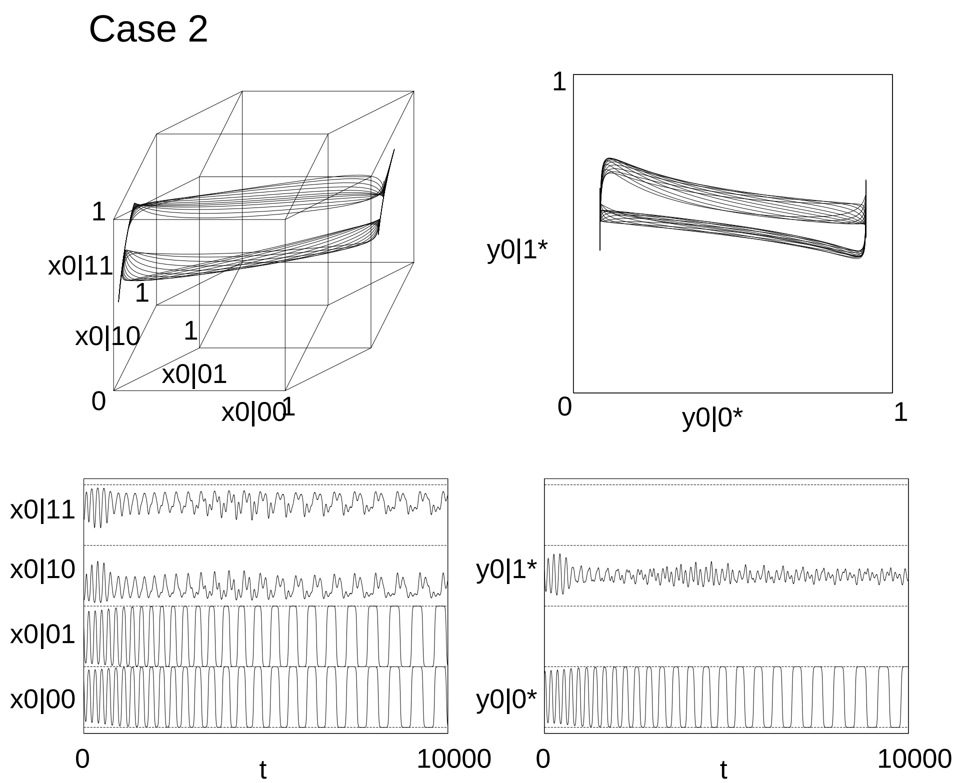

Case 2
1
x0|11
1
x0|10
1
x0|01
0
x0|00
1
1
y0|1*
0
y0|0*
1
x0|11
x0|10
x0|01
x0|00
0
t
10000
y0|1*
y0|0*
0
t
10000

Figure 2b

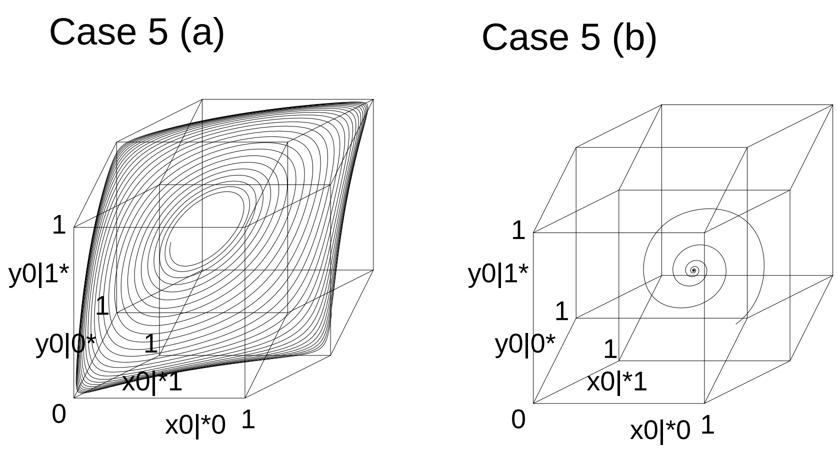


Figure 2c

Case 1

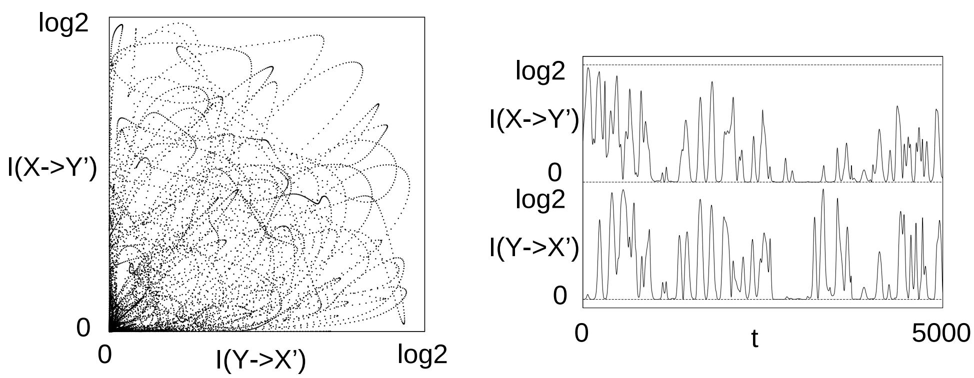


Figure 3a

Case 2

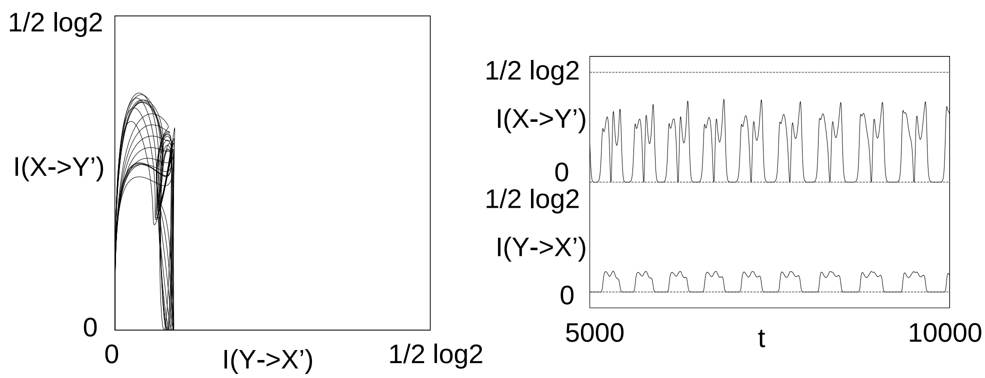
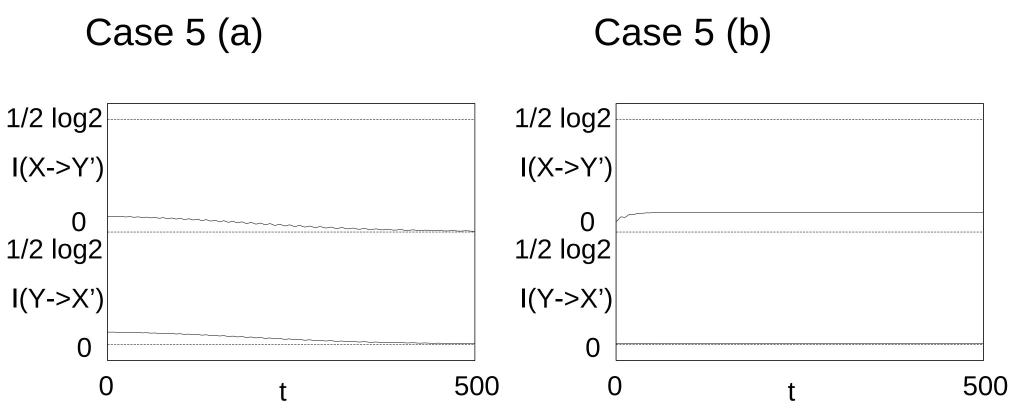


Figure 3b



Case 5 (a)
Case 5 (b)
1/2 log2
I(X->Y')
0
1/2 log2
I(Y->X')
0
0
t
500
1/2 log2
I(X->Y')
0
1/2 log2
I(Y->X')
0
0
t
500

Figure 3c