

# TOWARD THE SIEGEL RING IN GENUS FOUR

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ABSTRACT. Runge gave the ring of genus three Siegel modular forms as a quotient ring,  $R_3/\langle J^{(3)} \rangle$ .  $R_3$  is the genus three ring of code polynomials and  $J^{(3)}$  is the difference of the weight enumerators for the  $e_8 \oplus e_8$  and  $d_{16}^+$  codes. Freitag and Oura gave a degree 24 relation,  $R_0^{(4)}$ , of the corresponding ideal in genus four;  $R_0^{(4)}$  is also a linear combination of weight enumerators. We take another step toward the ring of Siegel modular forms in genus four. We explain new techniques for computing with Siegel modular forms and actually compute six new relations, classifying all relations through degree 32. We show that the local codimension of any irreducible component defined by these known relations is at least 3 and that the true ideal of relations in genus four is not a complete intersection. Also, we explain how to generate an infinite set of relations by symmetrizing first order theta identities and give one example in degree 32. We give the generating function of  $R_5$  and use it to reprove results of Nebe [25] and Salvati Manni [37].

## §1. Introduction.

There is an important ring homomorphism,  $\text{Th}_2 : R_g \rightarrow M_g$ , from code polynomials to Siegel modular forms of genus  $g$ . The ring of code polynomials is defined as the  $H_g$ -invariant subring of  $\mathbb{C}[F_a : a \in \mathbb{F}_2^g]$  where the finite group  $H_g = \langle T_g, \{D_S\}_S \rangle \subseteq \text{GL}(2^g, \mathbb{C})$  is defined by  $(T_g)_{ab} = (\frac{1+i}{2})^g (-1)^{a \cdot b}$  and  $D_S = \text{diag}(i^{S[a]})$  for integral symmetric  $g \times g$  matrices  $S$ . The ring of Siegel modular forms is defined as follows: Let  $\mathcal{H}_g$  denote the Siegel upper half space of genus  $g$ . For  $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  and a function  $f : \mathcal{H}_g \rightarrow \mathbb{C}$ , we write

$$(f|_k \sigma)(\Omega) = \det(C\Omega + D)^{-k} f((A\Omega + B)(C\Omega + D)^{-1}).$$

We then have a right action of  $\text{Sp}_g(\mathbb{R})$  on such functions. Let  $\Gamma_g = \text{Sp}_g(\mathbb{Z})$  be the modular group of genus  $g$  and for any subgroup  $\Gamma$  of finite index in  $\Gamma_g$  let  $M^k(\Gamma)$  be the complex vector space of Siegel modular forms of weight  $k$  with respect to  $\Gamma$ . These are the holomorphic  $f : \mathcal{H}_g \rightarrow \mathbb{C}$  satisfying  $f|_k \sigma = f$  for all  $\sigma \in \Gamma$  and for which  $f|_k \sigma$  is bounded in domains of type  $\{\Omega \in \mathcal{H}_g : \text{Im}(\Omega) \geq Y_0\}$  for all  $\sigma \in \Gamma_g$  and any  $Y_0 > 0$ . We denote by  $M(\Gamma) = \bigoplus M^k(\Gamma)$  the graded ring of Siegel modular forms for  $\Gamma$ . Let  $S(\Gamma) = \bigoplus S^k(\Gamma)$  be the ideal of cusp forms, defined by  $S^k(\Gamma) = \{f \in M^k(\Gamma) : \forall \sigma \in \Gamma_g, \Phi(f|_k \sigma) = 0\}$  where  $\Phi$  is the standard Siegel operator [14]. In the case  $\Gamma = \Gamma_g$  we write  $M_g, M_g^k, S_g, S_g^k$  instead of  $M(\Gamma_g), M^k(\Gamma_g), S(\Gamma_g), S^k(\Gamma_g)$ , respectively. The map  $\text{Th}_2$  is defined by sending

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the variables  $F_a$  to the second order theta constants  $\theta_2[a]$ . In genus three, B. Runge [34] showed that the kernel of  $\text{Th}_2$  was generated by a degree 16 polynomial  $J^{(3)}$  related to the Schottky form. Thus B. Runge represented the complicated ring  $M_3$  as a quotient ring,  $R_3/\langle J^{(3)} \rangle$ .

E. Freitag and M. Oura [13] took a first step toward a similar goal in genus four by computing  $R_0^{(4)}$ , a polynomial relation of degree 24 among the second order theta constants. They were able to do this because M. Oura [26] had computed the generating function for  $R_4$ , because the dimension of  $M_4^{12}$  was known and because these Siegel forms were determined by their Fourier coefficients on root lattices.

In this article, we compute six additional relations, one of degree 28 and five of degree 32. These relations, along with  $R_0^{(4)}$ , linearly span all relations through degree 32. Let  $\mathbf{p}_4$  be the kernel of  $\text{Th}_2$  in genus four. It is natural to wonder how far our relations go toward defining the 10 dimensional variety  $Z(\mathbf{p}_4)$  inside the 15 dimensional projective space  $\mathbb{P}^{15}(\mathbb{C})$ . We show that any irreducible component of the algebraic set defined by our relations that contains  $Z(\mathbf{p}_4)$  has dimension at most 12. These new relations also show that  $\mathbf{p}_4$  is not a complete intersection.

The final section illustrates how to create many relations by using the known identities among the first order theta constants. We give one example of this in degree 32 and verify that it is a linear combination of our previous relations.

We also report progress on related topics. We give the generating function for  $R_5$ . Using this generating function we find all linear relations, in every genus, among the weight enumerators of the 9 Type II binary codes of length 24. This result may also be found in the forthcoming article [25] by G. Nebe where she also classifies length 32 by the elegant technique of using the neighbor-graph to define formal Hecke operators on code polynomials. Here we show that there is a unique genus six cusp code polynomial of degree 24 and that it defines a Siegel modular cusp form in  $S_6^{12}$ ; this Siegel modular cusp form is the coding theory analogue of the Siegel modular cusp form constructed from Niemeier lattices in [1]. This form has interesting Fourier coefficients and we give some in Table 9. We also use the generating function of  $R_5$  to help reprove a result of R. Salvati-Manni [37] that the map  $\text{Th}_2$  is not surjective for  $g \geq 5$ . The surjectivity of  $\text{Th}_2$  remains open only in  $g = 4$  and perhaps this adds more interest to computations in this genus.

Although this article is a natural continuation of [13], entirely different computational techniques are required because the relevant Siegel modular cusp forms are not determined by their Fourier coefficients on root lattices. The difficulty in computing the Fourier coefficients of images of  $\text{Th}_2$  lies in the multiplication of multivariable power series. Instead of computing Fourier coefficients directly, we use the Restriction technique [30][32] to specialize Siegel modular cusp forms to elliptic modular cusp forms with one variable power series. Let  $L$  be an integral rank  $g$  lattice and  $L^*$  the dual lattice of  $L$ . We denote by  $\ell = \exp(L^*/L)$  the exponent of the abelian group  $L^*/L$ . If  $M \in \text{GL}_g(\mathbb{R})$  is a basis for  $L = \mathbb{Z}^g M$  then  $s = MM'$  is a Gram matrix for  $L$ . A change of basis  $UM$  changes  $s$  to  $UsU'$  for some  $U \in \text{GL}_g(\mathbb{Z})$ . Let  $f$  be an element of  $M_g^k$ . We note that for any  $\tau \in \mathcal{H}_1$  that  $s\tau \in \mathcal{H}_g$  and that  $f(s\tau)$  is independent of the choice of basis  $M$  because  $f(UsU'\tau) = \det(U)^k f(s\tau) = f(s\tau)$  when  $k$  is even. We define  $\phi_L^* f : \mathcal{H}_1 \rightarrow \mathbb{C}$  by

$(\phi_L^* f)(\tau) = f(s\tau)$ . We use the following Theorem from [31].

**Theorem 1.1.** *Let  $L$  be an integral rank  $g$  lattice with  $\ell = \exp(L^*/L)$ . The map  $\phi_L^* : M_g \rightarrow M_1(\Gamma_0(\ell))$  is a graded ring homomorphism that multiplies weights by  $g$  and takes cusp forms to cusp forms.*

The homomorphism  $\phi_L^*$  has a transparent effect on Fourier expansions. We write  $\langle \Omega, T \rangle = \text{tr}(\Omega T)$ . If  $f(\Omega) = \sum_T a(T) e(\langle \Omega, T \rangle)$  then for  $q = e^{2\pi i \tau}$  we have

$$(\phi_L^* f)(\tau) = \sum_T a(T) e(\langle s\tau, T \rangle) = \sum_{j=0}^{\infty} q^j \left( \sum_{T: \langle s, T \rangle = j} a(T) \right).$$

These one variable power series are more easily handled. The general strategy is to replace the computation of Fourier coefficients of  $f$  by the specializations  $\phi_L^* f$  whenever possible.

The computations in this paper were done using C++, Fermat [11], GAP [15], Magma [20], and Mathematica.

## §2. Notation and Context

In this section we fix our notation and place our work in a broader context. For  $w \in \mathbb{C}$  write  $e(w) = e^{2\pi i w}$ . For  $\Omega \in \mathcal{H}_g$  and column vectors  $z \in \mathbb{C}^g, a, b \in \mathbb{R}^g$ , we define the theta function by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{m \in \mathbb{Z}^g} e \left( \frac{1}{2} (m+a)' \Omega (m+a) + (m+a)' (z+b) \right),$$

where  $X'$  denotes the transpose of  $X$ . The  $r$ -th order theta function is defined by  $\theta_r \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \theta \begin{bmatrix} a/r \\ b \end{bmatrix} (rz, r\Omega)$  and under this terminology the original definition becomes a first order theta function. The function  $\theta_r[a]$  defined by  $\theta_r[a](\Omega) = \theta_r \begin{bmatrix} a \\ 0 \end{bmatrix} (0, \Omega)$  is called an  $r$ -th order theta constant. We denote by  $\vec{\theta}_r = [\theta_r[a]]_{a \in \mathbb{Z}^g / r\mathbb{Z}^g}$  the vector of  $r$ -th order theta constants.

For every positive integer  $r$ , we denote by  $\Gamma_g(r, 2r)$  the subgroup of  $\Gamma_g$  of elements  $\sigma$  satisfying

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \pmod{r}, \quad (B)_0 \equiv (C)_0 \equiv 0 \pmod{2r},$$

where  $(X)_0$  means to take the vector determined by the diagonal coefficients of a square matrix  $X$ . If we drop the second condition, we get the principal congruence subgroup  $\Gamma_g(r)$  of level  $r$ . We define  $r^g$  variables  $F_a$  for  $a \in \mathbb{Z}^g / r\mathbb{Z}^g$ . Let  $\mathbb{C}[F_a : a \in \mathbb{Z}^g / r\mathbb{Z}^g]$  be the polynomial ring in these variables and  $\mathbb{C}[F_a : a \in \mathbb{Z}^g / r\mathbb{Z}^g]^{(2)}$  the subring of even degree. For even  $r$ , there is a map

$$\text{Th}_r : \mathbb{C}[F_a : a \in \mathbb{Z}^g / r\mathbb{Z}^g]^{(2)} \rightarrow M(\Gamma_g(r, 2r))$$

induced by sending  $F$  to  $\vec{\theta}_r$  and the ring  $M(\Gamma_g(r, 2r))$  is the integral closure of  $\text{Im Th}_r$  inside its quotient field when  $r$  is divisible by 4. This theorem is called the ‘fundamental lemma’ of Igusa. Runge proved the same conclusion in the case  $r = 2$  for the map

$$\text{Th}_2 : \mathbb{C}[F_a : a \in \mathbb{F}_2^{g(2)}] \rightarrow M(\Gamma_g^*(2, 4))$$

where  $\Gamma_g^*(2, 4)$  is a subgroup of  $\Gamma_g(2, 4)$  of index two [33]. Similar results hold for larger subgroups by the process of taking invariant subrings.

We know that the group  $\Gamma_g$  is generated by the elements  $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$  and  $t(S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$  for integral symmetric  $S$ . For these elements, we have

$$\vec{\theta}_2|_{\frac{1}{2}} t(S) = D_S \vec{\theta}_2 \text{ and } \vec{\theta}_2|_{\frac{1}{2}} J = \pm T_g \vec{\theta}_2.$$

The  $\pm 1$  comes from the choice of square root. The group  $H_g = \langle T_g, \{D_S\}_S \rangle \subseteq \text{GL}(2^g, \mathbb{C})$  is finite and the representation  $\phi : \Gamma_g \rightarrow H_g / \pm 1$  defined by  $\vec{\theta}_2|_{\sigma} = \pm \phi(\sigma) \vec{\theta}_2$  defines  $\Gamma_g^*(2, 4)$  as its kernel. We see that an  $H_g$ -invariant polynomial goes to a level one Siegel form under the map  $\text{Th}_2$ . Let  $R_g$  be the  $H_g$ -invariant subring of  $\mathbb{C}[F_a : a \in \mathbb{F}_2^{g(2)}]$  and  $R_g^m$  the vector space of  $H_g$ -invariant homogeneous polynomials of degree  $m$ . Let the exact sequence

$$0 \rightarrow \mathfrak{p}_g \rightarrow R_g \xrightarrow{\text{Th}_2} M_g$$

define a prime ideal  $\mathfrak{p}_g$  of height  $2^g - \binom{g+1}{2} - 1$ . For any ideal  $I \subseteq \mathbb{C}[F_a : a \in \mathbb{F}_2^g]$  let  $Z(I) = \{[F] \in \mathbb{P}^{2^g-1} : \forall P \in I, P(F) = 0\}$ . The Zariski closure of  $\{\vec{\theta}_2(\Omega) : \Omega \in \mathcal{H}_g\}$  in  $\mathbb{P}^{2^g-1}$  is  $Z(\mathfrak{p}_g)$ .

Examples of Siegel modular forms are given by the Siegel theta series  $\vartheta_{\Lambda}^{(g)} : \mathcal{H}_g \rightarrow \mathbb{C}$  of a rank  $n$  lattice  $\Lambda \subseteq \mathbb{R}^n$ :

$$\vartheta_{\Lambda}^{(g)}(\Omega) = \sum_{\lambda \in \Lambda^g} e\left(\frac{1}{2} \text{tr}(\lambda \Omega \lambda')\right).$$

We will write  $\vartheta_{\Lambda}$  when the dependence on  $g$  is clear. The function  $\vartheta_{\Lambda}$  depends only upon the isometry class of the lattice. If  $\Lambda$  is even unimodular then we have  $\vartheta_{\Lambda} \in M_g^{n/2}$ . Useful properties are  $\vartheta_{\Lambda_1 \oplus \Lambda_2} = \vartheta_{\Lambda_1} \vartheta_{\Lambda_2}$  and  $\Phi \vartheta_{\Lambda}^{(g)} = \vartheta_{\Lambda}^{(g-1)}$ . Examples of  $H_g$ -invariant polynomials are given by the weight enumerators  $W_C^{(g)} \in R_g^m$  of a Type II code  $C \subseteq \mathbb{F}_2^m$  of length  $m$ :

$$W_C^{(g)}(F) = \sum_{x \in C^g} \prod_{i=1}^m F_{\text{row}_i(x)},$$

in which each element of  $C$  is expressed as a column vector and  $\text{row}_i(x)$  denotes the  $i$ -th row vector of  $x$ . A Type II code means a binary self-dual doubly-even code. Again, we write  $W_C$  unless the dependence on  $g$  is noteworthy. The weight enumerator has properties analogous to those of the theta series:  $W_{C_1 \oplus C_2} = W_{C_1} W_{C_2}$  and also  $\tilde{\Phi} W_C^{(g)} = W_C^{(g-1)}$  for  $\tilde{\Phi} : R_g \rightarrow R_{g-1}$  defined by

$$\tilde{\Phi}(F_a) = \begin{cases} F_b, & \text{if } \exists b \in \mathbb{F}_2^{g-1} : a = b0, \\ 0, & \text{if } \exists b \in \mathbb{F}_2^{g-1} : a = b1. \end{cases}$$

It is known that  $\text{Th}_2 W_C = \vartheta_{\Lambda(C)}$  for the lattice  $\Lambda(C) = \frac{1}{\sqrt{2}}\{x \in \mathbb{Z}^m : x \bmod 2 \in C\}$ , see [6]. In fact, we have the following commutative diagram:

$$\begin{array}{ccc} R_g^{2k} & \xrightarrow{\text{Th}_2} & M_g^k \\ \tilde{\Phi} \downarrow & & \downarrow \Phi \\ R_{g-1}^{2k} & \xrightarrow{\text{Th}_2} & M_{g-1}^k \quad . \end{array}$$

For  $\mathbb{F} = \mathbb{Z}, \mathbb{Q}$  or  $\mathbb{R}$  we let  $V_g(\mathbb{F})$  denote the  $g$ -by- $g$  symmetric matrices with coefficients in  $\mathbb{F}$  and let  $\mathcal{P}_g(\mathbb{F}) \subseteq V_g(\mathbb{F})$  denote the positive definite ones. Sometimes we use the same notation for an integral lattice in  $\mathbb{R}^g$  and one of its Gram matrices in  $\mathcal{P}_g(\mathbb{Z})$ .

We now place our work in a broader context. There are at least four reasons to study polynomial relations among the second order theta constants. These constants occur in the theory of Prym varieties and relations among them have a natural place in the Schottky-Jung theory where the search for such relations originated. Second, relations are elements of the kernel of the map

$$\text{Th}_2 : R_g \rightarrow M_g.$$

Third, when these relations can be lifted from genus  $g$  to genus  $g+1$  they provide examples of Siegel modular cusp forms in genus  $g+1$ . Finally, they are test cases to address Mumford's question [23, pp. 220-221]: are all relations among theta constants consequences of the Riemann theta relation and its generalizations?

We briefly summarize the previous results in these areas. The Schottky-Jung identities take a relation and produce a modular form of one higher genus that vanishes on the Jacobian locus. This form, however, is not necessarily automorphic with respect to the full modular group. There are no relations among the second order theta constants in genera one and two. In his 1972 paper, *Prym Varieties I*, D. Mumford commented that simple relations among the second order theta constants do not seem to be known. Based on the existence of the Schottky form in  $S_4^8$  he pointed out that the essentially unique relation in  $g=3$  appears to be of degree 16. The traditional theory circumnavigated these relations by reformulating the Schottky-Jung identities in terms of first order theta constants and applying them to the known relations among the first order theta constants. It is coding theory that gave the first direct method for displaying relations among the second order theta constants.

W. Duke [6] and B. Runge [33] both noticed that

$$J^{(g)} = W_{e_8 \oplus e_8}^{(g)} - W_{d_{16}^+}^{(g)} \in R_g^{16}$$

is zero in  $g=2$  but nonzero in  $g=3$ . Witt's conjecture, as proved by Igusa and Kneser, tells us that  $\text{Th}_2 J^{(3)} = 0$  so that  $J^{(3)}$  is the degree 16 relation alluded to by D. Mumford. B. Runge was the first to systematically investigate the process of taking invariant subrings for higher genus and showed that  $J^{(3)}$  is essentially the only relation in  $g=3$  by proving

that  $R_3/\langle J^{(3)} \rangle \cong M_3$ . In  $g = 4$  the only relation in degree 24,  $R_0^{(4)}$ , was found by E. Freitag and M. Oura, who showed

$$R_0^{(g)} = 3W_{C_1}^{(g)} + 20W_{C_2}^{(g)} - 75W_{C_3}^{(g)} + 96W_{C_4}^{(g)} - 55W_{C_6}^{(g)} + 12W_{C_7}^{(g)} - W_{C_9}^{(g)}.$$

Here, the  $C_i$  are the 9 Type II codes of length 24 as indexed in [13], see our Table 1. From  $\tilde{\Phi}W_C^{(g)} = W_C^{(g-1)}$  we see that  $\text{Th}_2 J^{(4)}$  is a cusp form; in fact, by a result of Igusa, it is the Schottky form  $J_8 \in S_4^8$  [17] up to a constant. In a similar vein, E. Freitag and M. Oura showed that  $\text{Th}_2 R_0^{(5)}$  is a nontrivial cusp form in  $S_5^{12}$ . Modular forms of higher genus are large objects from a computational point of view. Some artifice is always needed to identify the cusp forms and this technique of lifting relations is a good one.

**Table 1.**

Type II codes of length 24 ( $k = 12$ ) and their root systems.

$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$
$d_{12}^2$	$d_{10}e_7^2$	$d_8^3$	$d_6^4$	$d_{24}$	$d_4^6$	$a_1^{24}$	$d_{16}e_8$	$e_8^3$

Finally, B. Runge gave a construction of  $J^{(3)}$  that is independent of coding theory, compare [12]. He constructed  $J^{(3)}$  from a quartic relation on the first order theta constants that is indeed a consequence of the classical Riemann theta relations. The direct verification of the  $H_3$ -invariance of  $J^{(3)}$  constructed in this way is a computation not so very different from one given by Schottky in 1877, showing that some relations on the second order theta constants have always been implicit in the Schottky-Jung literature [38, pp. 345-348]. We do not address the Schottky-Jung theory here but refer the reader to the recent treatment by B. Runge [36] and note that five new Schottky relations of weight 16 and level  $\Gamma(2)$  could be generated from the relations we give here.

We now place the new contributions of this article in this context. E. Freitag and M. Oura knew the generating function for  $R_4$  and the dimension of  $S_4^{12}$ . Subsequently, C. Poor and D. Yuen computed  $\dim M_4^k$  for  $k = 10, 14, 16$  and gave bases [32]. Possessing bases for both  $R_g^{2k}$  and  $M_g^k$  allows the computation of all the polynomial relations among the second order theta constants of degree  $2k$ .

**Table 2.**

$k$	0	2	4	6	8	10	12	14	16	18	20
$\dim R_4^{2k}$	1	0	1	1	2	3	7	7	19	27	52
$\dim M_4^k$	1	0	1	1	2	3	6	6	14	?	?

In  $g = 4$  we give a basis for all relations of degree 32. In Table 3 this basis is denoted by  $R_1^{(4)}, \dots, R_5^{(4)}$ , and is expressed as linear combinations of a chosen basis of 19 weight enumerators. We use the numbering scheme of the 85 Type II codes of length 32 from [3], [4]; this particular choice of a basis of 19 weight enumerators was arbitrary.

**Table 3.**  
Relations in  $g = 4$  and  $k = 16$

Basis	Root system	Coefficients of Relations				
		$R_1^{(4)}$	$R_2^{(4)}$	$R_3^{(4)}$	$R_4^{(4)}$	$R_5^{(4)}$
$\mathcal{C}_1$	$d_{32}$	0	-5	-63	-944	-64
$\mathcal{C}_2$	$d_{24}e_8$	-3	110	990	11597	-2041
$\mathcal{C}_3$	$d_{20}d_{12}$	0	-616	-2016	-22624	15400
$\mathcal{C}_4$	$d_{18}e_7^2$	0	880	-960	-6080	-880
$\mathcal{C}_5$	$d_{16}^2$	0	121	957	1252	14559
$\mathcal{C}_6$	$d_{16}e_8^2$	153	-121	-2610	3269	42186
$\mathcal{C}_7$	$d_{16}d_8^2$	0	-385	2520	23660	-38465
$\mathcal{C}_{10}$	$d_{12}^2e_8$	-420	0	2520	-10500	50540
$\mathcal{C}_{11}$	$d_{12}^2d_8$	0	0	0	0	-117600
$\mathcal{C}_{16}$	$d_{10}^3a_1^2$	0	0	0	0	-225792
$\mathcal{C}_{18}$	$d_{10}e_8e_7^2$	640	0	1280	-4480	3200
$\mathcal{C}_{23}$	$d_{10}a_1^{22}$	0	16	-576	-5696	-26128
$\mathcal{C}_{24}$	$e_8^4$	-83	0	498	-1931	-63675
$\mathcal{C}_{25}$	$d_8^3e_8$	-315	0	-1890	7245	-21315
$\mathcal{C}_{27}$	$d_4^6e_8$	28	0	280	-1092	16716
$\mathcal{C}_{29}$	$d_8^4$	0	0	-1890	22260	-47985
$\mathcal{C}_{44}$	$d_4e_7^4$	0	0	0	0	409600
$\mathcal{C}_{67}$	$d_4^8$	0	0	960	-37440	-29760
$\mathcal{C}_{82}$	$a_1^{32}$	0	0	0	21504	21504

We also give a generator  $\mathcal{R}_1$  for the one dimensional space of relations in degree 28. Since  $k \equiv 2 \pmod{4}$  in this case, we do no better than to store  $\mathcal{R}_1$  as a large polynomial [40]. We prove that the map

$$\text{Th}_2 : R_4^{2k} \rightarrow M_4^k$$

is onto for  $k \leq 16$ ,  $k \equiv 0 \pmod{2}$ , and show that  $R_0^{(4)}$  is the  $H_4$ -invariant relation of lowest degree in  $g = 4$ . Freitag and Oura's result may be viewed as placing the 10 dimensional variety  $Z(\mathbf{p}_4)$  inside the 14 dimensional hypersurface  $Z(R_0^{(4)})$ ; in these terms our new

relations of degree 32 cut the dimension down to 12.

**Theorem 4.2.** *Any irreducible component of the algebraic set  $Z(R_1^{(4)}, \dots, R_5^{(4)})$  that contains  $Z(\mathfrak{p}_4)$  has dimension at most 12.*

The hope, of course, is that someday generators of  $\mathfrak{p}_4$  will be known and that  $M_4$  can be realized as  $R_4/\mathfrak{p}_4$ . This paper is another step toward that distant goal. It will not be easy to compute sufficiently many relations nor to prove that the ideal they generate is prime nor that the quotient ring is integrally closed in its quotient field, if in fact it is. These tasks are all the more difficult because  $\mathfrak{p}_4$  is not a complete intersection, a fact that we can prove from the relations we have already found.

We also use relations to construct cusp forms. We show that  $\text{Th}_2 \text{Span}(R_1^{(5)}, \dots, R_5^{(5)}) \subseteq S_5^{16}$  is 5 dimensional. This subspace depends upon which weight enumerators are chosen for a basis but illustrates the technique of lifting relations. Finally, we construct a  $H_4$ -invariant relation of degree 32 among the second order theta constants by beginning with first order quartic theta identities. Thus, at least one of the relations of degree 32 in  $\mathfrak{p}_4$  is a consequence of Riemann's theta relation.

### §3. Theta relations of degree 32 in genus 4.

In this section we compute the kernel of the map  $\text{Th}_2 : R_4^{32} \rightarrow M_4^{16}$ . It is known [35] that  $R_g^{2k}$  is spanned by weight enumerators when  $k$  is divisible by 4. There are 85 classes of Type II codes of length 32 and we follow the indexing of the Catalogue of Lattices, [3], [4]. Since  $\dim R_4^{32} = 19$  there exists a basis of 19 linearly independent weight enumerators, see Table 3. The computation of weight enumerators is eased by the following considerations: not all monomials may occur in a weight enumerator, a supported monomial must be invariant under the subgroup  $\langle D_S \rangle$  to occur and such monomials are called *admissible*. The condition on the multi-index  $I$  for the admissibility of  $F^I$  is:

$$\forall S \in V_n(\mathbb{Z}), \quad \sum_{a \in \mathbb{F}_2^g} I_a S[a] \equiv 0 \pmod{4}.$$

Many admissible monomials necessarily have the same coefficient in a weight enumerator. The permutation subgroup of  $H_g$  is  $H_g \cap S_{2^g} = \text{AGL}(g)$  and admissible monomials are broken down into  $\text{AGL}(g)$ -orbits. A useful alternative description of  $\text{AGL}(g)$  is as  $\{\sigma \in S_{2^g} : \exists U \in \text{GL}_g(\mathbb{Z}), b \in \mathbb{F}_2^g : \sigma(F_a) = F_{aU+b}\}$ .

In order to specify  $W_C$  we need only give the coefficient for each  $\text{AGL}(g)$ -orbit of admissible monomials. For  $g = 4$ ,  $k = 16$ , there are 51 463 749 admissible monomials and 1083  $\text{AGL}(4)$ -classes of admissible monomials. The coefficients for each of the 85 weight enumerators on each class of admissible monomials may be viewed at [27]. There one will also find the 66 nonbasis weight enumerators expressed in terms of the 19 chosen basis elements.  $M_4^{16}$  satisfies the exact sequence

$$0 \rightarrow S_4^{16} \rightarrow M_4^{16} \xrightarrow{\Phi} M_3^{16} \rightarrow 0$$



and since  $\dim M_3^{16} = 7$  [39] we see that  $\dim M_4^{16} = 14$  follows from  $\dim S_4^{16} = 7$ , a result proved in [32]. Cusp forms in  $S_4^{16}$  are not determined solely by their Fourier coefficients on root lattices, as is seen by the example  $J_8^2 \neq 0$ , and so linear relations among theta series of lattices may not be deduced solely from the root systems of the lattices. A determining set of Fourier coefficients  $T$  is given by  $2T = D_4, A_4, A_3A_1, A_2^2, A_2A_1^2, A_1^4$  and  $2D_4$ . These results from [32] were proven by the technique of restriction to modular curves.

To find the kernel of  $\text{Th}_2$  on  $R_4^{32}$  we write

$$\text{Th}_2 \left( \sum_C^{\text{nineteen}} \alpha_C W_C \right) = \sum_C^{\text{nineteen}} \alpha_C \vartheta_{\Lambda(C)}.$$

We cannot hope to store each  $\vartheta_{\Lambda(C)}$  as large pieces of power series in ten variables and so some specialization technique is required. We can compute initial pieces of  $\sum_C^{19} \alpha_C \phi_L^* \vartheta_{\Lambda(C)}$  however, because  $\phi_L^* \vartheta_{\Lambda(C)}$  is a one variable  $q$ -expansion.

Let  $\pi_N : M_1^K(\Gamma_0(\ell)) \rightarrow \mathbb{C}^N$  be the truncation of the Fourier series to  $O(q^N)$ . We then have maps

$$M_g^k \xrightarrow{\phi_L^*} M_1^{gk}(\Gamma_0(\ell)) \xrightarrow{\pi_N} \mathbb{C}^N.$$

We will combine these for  $L = D_4, A_4$  and consider:

$$M_4^k \xrightarrow{\phi_{D_4}^* \oplus \phi_{A_4}^*} M_1^{4k}(\Gamma_0(2)) \oplus M_1^{4k}(\Gamma_0(5)) \xrightarrow{\pi_{N_1} \oplus \pi_{N_2}} \mathbb{C}^{N_1+N_2}.$$

Calling the composition above  $\beta$ , the next Lemma will apply to the sequence

$$R_4^{32} \xrightarrow{\text{Th}_2} M_4^{16} \xrightarrow{\beta} \mathbb{C}^{N_1+N_2}.$$

**Lemma 3.1.** *Let  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  be linear maps of finite dimensional vector spaces. If  $\dim B = \dim(\beta \circ \alpha)(A)$  then  $\beta$  injects,  $\alpha$  surjects and the kernel of  $\alpha$  is the kernel of  $\beta \circ \alpha$ .*

*Proof.* From the sequence  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  we have  $\dim(\beta \circ \alpha)(A) \leq \dim \beta(B) \leq \dim B$ . From the hypothesis  $\dim B = \dim(\beta \circ \alpha)(A)$  we have  $\dim(\beta \circ \alpha)(A) = \dim \beta(B) = \dim B$ . Finite dimensionality implies that  $\beta$  injects and hence that  $\ker(\beta \circ \alpha) = \ker(\alpha)$ . Finite dimensionality also implies that  $(\beta \circ \alpha)(A) = \beta(B)$  so that we have  $\alpha(A) = B$ .  $\square$

The following theorem is analogous to Proposition 1.1 and Theorem 1.3 from [13].

**Theorem 3.2.** *The 19 weight enumerators  $W_i$  given in Table 3 are a basis of  $R_4^{32}$ . The kernel of  $\text{Th}_2$  on  $R_4^{32}$  has dimension 5 and a basis is given in Table 3.*

*Proof.* Let  $\beta = (\pi_N \oplus \pi_N)(\phi_{D_4}^* \oplus \phi_{A_4}^*)$ . By the Lemma, if we can find an  $N$  for which  $\dim(\beta \circ \text{Th}_2)(R_4^{32}) = \dim M_4^{16} = 14$ , we will have proven that  $\beta$  injects,  $\text{Th}_2$  surjects and  $\ker(\text{Th}_2) = \ker(\beta \circ \text{Th}_2)$ . We ran our programs with  $N = 10$ . Each  $\beta(\text{Th}_2(W_i))$  is a pair of  $q$ -polynomials of degree at most 9 and we observed that  $\dim(\beta \circ \text{Th}_2)(R_4^{32}) = 14$ . By

the Lemma  $\beta$  injects and we may calculate the linear relations among the  $\text{Th}_2 W_i^{(4)}$  by finding the linear relations among the  $\beta \text{Th}_2 W_i^{(4)}$ . The computations of

$$(\pi_N \phi_L^* \text{Th}_2 W_i)(\tau) = (\pi_N \phi_L^* W_i(\theta_2[a]))(\tau) = \pi_N W_i(\theta_2[a](s\tau)) = \pi_N W_i(\pi_N \theta_2[a](s\tau))$$

were performed by specializing each second order theta constant  $\theta_2[a](\Omega)$  to  $\pi_N \theta_2[a](s\tau)$ , computing *each* of the 51 million specialized admissible monomials, and evaluating the specialized weight enumerator  $\pi_N W_i(\pi_N \theta_2[a](s\tau))$  as a linear combination of these specialized admissible monomials. Table 4 gives the values of  $\beta \text{Th}_2 W_i^{(4)}$  and the coefficients for a basis of linear dependence relations may be computed to be as in Table 3. In order to show that the chosen 19 weight enumerators are linearly independent it now suffices to show that the relations  $R_1^{(4)}, \dots, R_5^{(4)}$  are linearly independent. Table 5 accomplishes this by giving the coefficients for the  $R_i^{(4)}$  on five admissible monomials as a rank 5 matrix.  $\square$

Table 4.

$C$	$\pi_{10}\phi_{D_4}^* \text{Th}_2 W_C^{(4)}$
$C_1$	$1 + 23808q^2 + 3809280q^3 + 790712064q^4 + 101677301760q^5 + 11406195194880q^6 + 1030558541365248q^7 + 79617119299305216q^8 + 5257235129594216448q^9$
$C_2$	$1 + 16128q^2 + 1769472q^3 + 293011200q^4 + 29554900992q^5 + 3005136589824q^6 + 259509205204992q^7 + 21023497963208448q^8 + 1528586760809349120q^9$
$C_3$	$1 + 12288q^2 + 1044480q^3 + 149554944q^4 + 13139558400q^5 + 1229633863680q^6 + 102942760132608q^7 + 8653855475187456q^8 + 680225510739836928q^9$
$C_4$	$1 + 10368q^2 + 755712q^3 + 100949760q^4 + 8072110080q^5 + 724799632896q^6 + 59253973438464q^7 + 5116063423627008q^8 + 422389215603572736q^9$
$C_5$	$1 + 11520q^2 + 860160q^3 + 116248320q^4 + 9904619520q^5 + 1005942881280q^6 + 87289853460480q^7 + 7123155471302400q^8 + 534936846962196480q^9$
$C_6$	$1 + 11520q^2 + 860160q^3 + 121409280q^4 + 9739468800q^5 + 1037899545600q^6 + 88543017123840q^7 + 7279369018801920q^8 + 541722447773368320q^9$
$C_7$	$1 + 8448q^2 + 516096q^3 + 63631104q^4 + 4616847360q^5 + 390827879424q^6 + 31170012954624q^7 + 2779434833084160q^8 + 244380666976075776q^9$
$C_{10}$	$1 + 9216q^2 + 552960q^3 + 71550720q^4 + 5212717056q^5 + 524387856384q^6 + 43451333959680q^7 + 3680842807459584q^8 + 292567115403264000q^9$
$C_{11}$	$1 + 7680q^2 + 380928q^3 + 44824320q^4 + 3192471552q^5 + 301273098240q^6 + 24874195034112q^7 + 2138196457012992q^8 + 177848376370102272q^9$
$C_{16}$	$1 + 6528q^2 + 276480q^3 + 31037184q^4 + 2092523520q^5 + 193214292480q^6 + 15723304562688q^7 + 1373338788216576q^8 + 118521925880659968q^9$
$C_{18}$	$1 + 8064q^2 + 436224q^3 + 55736064q^4 + 3708297216q^5 + 360342759936q^6 + 28782893666304q^7 + 2523852810600192q^8 + 210812170505207808q^9$
$C_{23}$	$1 + 2688q^2 + 92160q^3 + 8427264q^4 + 335093760q^5 + 27014008320q^6 + 1756971307008q^7 + 195264403259136q^8 + 22853207967449088q^9$
$C_{24}$	$1 + 11520q^2 + 860160q^3 + 126570240q^4 + 9574318080q^5 + 1069856209920q^6 + 89796180787200q^7 + 7462218074423040q^8 + 546803376064757760q^9$
$C_{25}$	$1 + 6912q^2 + 344064q^3 + 43147008q^4 + 2617049088q^5 + 233865421824q^6 + 17889543880704q^7 + 1642833974492928q^8 + 149087792867966976q^9$
$C_{27}$	$1 + 4608q^2 + 233472q^3 + 27866880q^4 + 1199849472q^5 + 82356639744q^6 + 5294415716352q^7 + 602572018007808q^8 + 71712860211609600q^9$
$C_{29}$	$1 + 5376q^2 + 172032q^3 + 19664640q^4 + 1266843648q^5 + 116772707328q^6 + 9399206952960q^7 + 835043893313280q^8 + 73032506428293120q^9$
$C_{44}$	$1 + 6336q^2 + 261120q^3 + 30083328q^4 + 1923084288q^5 + 183286071552q^6 + 14614907295744q^7 + 1297633329245952q^8 + 111242267100880896q^9$
$C_{67}$	$1 + 2304q^2 + 24576q^3 + 4697856q^4 + 248610816q^5 + 20125584384q^6 + 1549271678976q^7 + 148779019337472q^8 + 14926421059239936q^9$
$C_{82}$	$1 + 768q^2 + 2191104q^4 + 60948480q^5 + 6206653440q^6 + 459649056768q^7 + 45528070684416q^8 + 4856441173966848q^9$

Table 4 (continued).

$C$	$\pi_{10}\phi_{A_4}^* \text{Th}_2 W_C^{(4)}$
$C_1$	$1 + 19840q^2 + 2380800q^3 + 379152960q^4 + 42324910080q^5 + 3939620169600q^6 + 303368206049280q^7 + 19850777207081280q^8 + 1118726703849308160q^9$
$C_2$	$1 + 13440q^2 + 1105920q^3 + 147026880q^4 + 13154365440q^5 + 1091273170560q^6 + 79543588700160q^7 + 5273655205688640q^8 + 318089474385960960q^9$
$C_3$	$1 + 10240q^2 + 652800q^3 + 78610560q^4 + 6076078080q^5 + 469744704000q^6 + 33042318151680q^7 + 2219974040834880q^8 + 140915798842306560q^9$
$C_4$	$1 + 8640q^2 + 472320q^3 + 54056160q^4 + 3826621440q^5 + 284769424320q^6 + 19627937556480q^7 + 1333644711750720q^8 + 87388365564364800q^9$
$C_5$	$1 + 9600q^2 + 537600q^3 + 63489600q^4 + 4780154880q^5 + 384780566400q^6 + 27602426880000q^7 + 1826055288187200q^8 + 114516377854771200q^9$
$C_6$	$1 + 9600q^2 + 537600q^3 + 63489600q^4 + 4769832960q^5 + 385167638400q^6 + 27546430464000q^7 + 1825199342971200q^8 + 114380380107571200q^9$
$C_7$	$1 + 7040q^2 + 322560q^3 + 35077440q^4 + 2244280320q^5 + 160418190720q^6 + 10835316940800q^7 + 748588213903680q^8 + 50906074555269120q^9$
$C_{10}$	$1 + 7680q^2 + 345600q^3 + 39000960q^4 + 2632919040q^5 + 203691264000q^6 + 14139428474880q^7 + 950215898744640q^8 + 62147328282316800q^9$
$C_{11}$	$1 + 6400q^2 + 238080q^3 + 26269440q^4 + 1645393920q^5 + 124615054080q^6 + 8518657536000q^7 + 579657496814400q^8 + 39024647361146880q^9$
$C_{16}$	$1 + 5440q^2 + 172800q^3 + 18771360q^4 + 1109038080q^5 + 82360123200q^6 + 5568463226880q^7 + 385014688687680q^8 + 26550412870133760q^9$
$C_{18}$	$1 + 6720q^2 + 272640q^3 + 30108960q^4 + 1901537280q^5 + 141847103040q^6 + 9637484267520q^7 + 657190750420800q^8 + 44340877336657920q^9$
$C_{23}$	$1 + 2240q^2 + 57600q^3 + 5144160q^4 + 180034560q^5 + 14470968000q^6 + 834416302080q^7 + 71960575318080q^8 + 5534043961128960q^9$
$C_{24}$	$1 + 9600q^2 + 537600q^3 + 63489600q^4 + 4759511040q^5 + 385554710400q^6 + 27490434048000q^7 + 1824343397755200q^8 + 114244382360371200q^9$
$C_{25}$	$1 + 5760q^2 + 215040q^3 + 23083200q^4 + 1342771200q^5 + 94976991360q^6 + 6303360614400q^7 + 439799651876160q^8 + 30847127104757760q^9$
$C_{27}$	$1 + 3840q^2 + 145920q^3 + 13524480q^4 + 590469120q^5 + 36791036160q^6 + 2209341173760q^7 + 178980560735040q^8 + 13461265132892160q^9$
$C_{29}$	$1 + 4480q^2 + 107520q^3 + 12563520q^4 + 699310080q^5 + 52294396800q^6 + 3479643217920q^7 + 243009857935680q^8 + 17173493256683520q^9$
$C_{44}$	$1 + 5280q^2 + 163200q^3 + 17773200q^4 + 1033574400q^5 + 76674648480q^6 + 5149476817920q^7 + 357587516393280q^8 + 24804993334179840q^9$
$C_{67}$	$1 + 1920q^2 + 15360q^3 + 3504960q^4 + 150405120q^5 + 10960920960q^6 + 718905876480q^7 + 55014006047040q^8 + 4295551658803200q^9$
$C_{82}$	$1 + 640q^2 + 1786560q^4 + 38092800q^5 + 3980342400q^6 + 256196935680q^7 + 21531393065280q^8 + 1764817337794560q^9$

**Table 5.**  
Some coefficients of  $R_i^{(4)}$  on monomials.

monomial	$R_1^{(4)}$	$R_2^{(4)}$	$R_3^{(4)}$	$R_4^{(4)}$	$R_5^{(4)}$
$F_{0010}^{12} F_{0011}^{12} F_{0111}^4 F_{1101}^4$	-8515584	-11708928	-4257792	140507136	-10930816512
$F_{0001}^8 F_{1001}^8 F_{1010}^8 F_{1100}^8$	-50577408	-149022720	73801728	2894782464	-84402984960
$F_{0010}^4 F_{0011}^4 F_{0100}^8 F_{0111}^{12} F_{1101}^4$	-44556288	-217383936	-107433984	-9086128128	21135614976
$F_{0000}^8 F_{0001}^4 F_{0010}^4 F_{0101}^8 F_{1001}^8$	-101154816	-1170892800	-665247744	-48748879872	21050910720
$F_{0011}^6 F_{0100}^4 F_{0111}^2 F_{1000}^8 F_{1001}^6 F_{1101}^6$	-283594752	-2552901120	-2497517568	-67229374464	-633092866560

Using  $\beta = (\pi_{11} \oplus \pi_{11} \oplus \pi_{11} \oplus \pi_{11}) (\phi_{D_4}^* \oplus \phi_{A_4}^* \oplus (\phi_{A_3}^* \circ \Phi) \oplus (\phi_{A_2}^* \circ \Phi^2))$ , we ran our programs again and obtained the same answer. This served as a consistency check in so far as generic errors will increase the dimension of the image. As a further check we have computed

$$32 W_{e_8} R_0^{(4)} = R_1^{(4)}.$$

We emphasize that by showing  $\beta$  injects the restriction technique has replaced the usual technique of computing Fourier coefficients. Whereas a certain set of 14 Fourier coefficients would have been required to determine an  $f \in M_4^{16}$  it turned out that 2 specializations to order 10 sufficed.

Next we lift our 5 relations to cusp forms in  $S_5^{16}$  and show that these 5 cusp forms are linearly independent. Let  $R_1^{(5)}, \dots, R_5^{(5)}$  be the linear combinations of the  $W_{C_i}^{(5)}$  given by the coefficients in Table 3. We have  $\text{Th}_2 R_i^{(5)} \in S_5^{16}$  and we test for linear independence of the  $\text{Th}_2(R_i^{(5)})$  by computing a few Fourier coefficients. Although this is in general computationally difficult, note that when  $R$  is a root lattice the coefficient  $a(R; \vartheta_L)$  is easier to compute because it depends only on the first shell of  $L$ , compare [8], [9] and the PARI program of Borchers [1](cf. [19]). For  $x^{(5)} = \sum_i c_i W_{C_i}^{(5)}$  we have

$$a(R, \text{Th}_2(x^{(5)})) = \sum_i c_i a(R, \vartheta_{L_i}^{(5)}).$$

This is tractable since we know the root systems  $L_i = \Lambda(C_i)$ . Table 3 gives these root systems. Table 6 gives the normalized values of  $a(R, \text{Th}_2(R_j^{(5)}))$  for  $j = 1, 2, 3, 4, 5$  and root lattices  $R$ .

**Table 6.**The normalized Fourier coefficients of  $\text{Th}_2(R_j^{(5)})$ ,  $j = 1, 2, 3, 4, 5$ .

Root lattice	normalizing factor f	$\text{Th}_2(R_j^{(5)})/f$				
		$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$D_5$	123863040	-8	-11	-4	132	-10269
$D_4A_1$	1981808640	-24	55	72	7468	-70615
$A_5$	82575360	48	-33	-1404	-47724	86609
$A_4A_1$	220200960	18	-1859	-18756	-530642	390869
$A_3A_2$	82575360	-1224	-17579	-110916	-2482648	-1732685
$A_3A_1^2$	110100480	-4536	-57937	-1035756	-23203924	-11310215
$A_2^2A_1$	3963617280	-198	-3245	-60828	-1303426	-579993
$A_2A_1^3$	3963617280	-1320	-43703	-758460	-16266348	-4930881
$A_1^5$	21139292160	-2376	-90904	-1701441	-35156776	-13893854

**Theorem 3.3.**  $\text{Span}\left(\text{Th}_2(R_j^{(5)})\right) \subseteq S_5^{16}$  is 5 dimensional.

*Proof.* The 9-by-5 matrix in Table 6 has rank 5.  $\square$

#### §4. Dimensions of Components

The 5-by-16 Jacobian matrix in the next Lemma appears to have rank exactly 3. This is another consistency check in that generic errors would give rank 5. Augmenting the Jacobian with the relation  $\mathcal{R}_1$  does not seem to increase the rank.

**Lemma 4.1.** Let  $H = \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 2 \\ 1 & 1 & 2 & 6 \end{pmatrix}$ . Let  $j = 1, \dots, 5$  and  $a \in \mathbb{F}_2^4$ . The following

5-by-16 Jacobian matrix has rank at least 3 for  $\tau$  in an open dense set of  $\mathcal{H}_1$ :

$$\left( \frac{\partial R_j^{(4)}}{\partial F_a} \right)_{F_a := \theta_2[a](H\tau)}$$

*Proof.* Each 3-by-3 minor is a holomorphic function in  $q = e(\tau)$  and as such either vanishes identically or is nonzero on an open dense set. Here is the  $q$ -expansion of one 3-by-3 minor,

evaluated by restriction:

$$\begin{aligned} & \left| \begin{array}{ccc} \frac{\partial R_1^{(4)}}{\partial F_{0000}} & \frac{\partial R_1^{(4)}}{\partial F_{0001}} & \frac{\partial R_1^{(4)}}{\partial F_{0010}} \\ \frac{\partial R_2^{(4)}}{\partial F_{0000}} & \frac{\partial R_2^{(4)}}{\partial F_{0001}} & \frac{\partial R_2^{(4)}}{\partial F_{0010}} \\ \frac{\partial R_3^{(4)}}{\partial F_{0000}} & \frac{\partial R_3^{(4)}}{\partial F_{0001}} & \frac{\partial R_3^{(4)}}{\partial F_{0010}} \end{array} \right|_{F_a := \theta_2[a](H\tau)} \\ & = -13\,969\,614\,962\,861\,087\,124\,029\,561\,634\,816\,000q^{117} + \dots \quad \square \end{aligned}$$

**Theorem 4.2.** *Any irreducible component of the algebraic set  $Z(R_1^{(4)}, \dots, R_5^{(4)})$  that contains  $Z(\mathbf{p}_4)$  has dimension at most 12.*

*Proof.* Let  $\mathbf{q}$  be an irreducible component of  $Z(R_1^{(4)}, \dots, R_5^{(4)})$  that contains  $Z(\mathbf{p}_4)$ . For any  $\tau \in \mathcal{H}_1$  we have  $P = [\vec{\theta}_2(H\tau)] \in Z(\mathbf{p}_4) \subseteq \mathbf{q}$ . First of all we have  $\dim \mathbf{q} = \min_{Q \in \mathbf{q}} \dim T_Q(\mathbf{q})$ , see [22, pg. 24]. Second, we may pass back and forth between the definitions of the tangent spaces on  $\mathbf{q}$  and on an affine piece of  $\mathbf{q}$  to assert that if  $I(\mathbf{q})$  has generators  $g_1, \dots, g_\ell$  then  $\dim T_Q(\mathbf{q}) = 15 - \text{rank} \left( \frac{\partial g_i}{\partial F_a} \right)_{1 \leq i \leq \ell, a \in \mathbb{F}_2^4}$ , see [22, pp. 3, 24]. Since  $R_1^{(4)}, \dots, R_5^{(4)} \in I(\mathbf{q})$ , we may use Lemma 4.1 for generic  $\tau$  to deduce that

$$\dim \mathbf{q} \leq \dim T_P(\mathbf{q}) \leq 15 - \text{rank} \left( \frac{\partial R_j^{(4)}}{\partial F_a} \right)_{1 \leq j \leq 5, a \in \mathbb{F}_2^4} (P) \leq 15 - 3 = 12. \quad \square$$

### §5. Weights $k$ congruent to 2 modulo 4

A different approach is used for weights congruent to 2 modulo 4. Weight  $k = 2$  is simple because both  $R_g^4$  and  $M_g^2$  are trivial. In order to find a basis of  $H_g$ -invariant polynomials in the other cases we write an element from  $R_g^{2k}$  as:

$$P(c, F) = \sum_{\substack{\text{AGL}(g)\text{-classes } [N] \\ \text{of degree } 2k \text{ admissible monomials}}} c_N \sum_{M \in [N]} M.$$

This is the most general form of a  $\langle D_S \rangle$ -invariant polynomial of degree  $2k$ . In order to show  $H_g$  invariance it suffices to solve for  $c_N$  such that  $P(c, F) = P(c, T_g F)$ . It is not computationally feasible to expand out  $P(c, F) = P(c, T_g F)$  and to solve for the  $c_N$  so we again proceed by specialization. Let  $\mathbf{f}_0 \in \mathbb{Z}^{2^g}$  be a choice of  $2^g$  integers, preferably small integers like 0 or  $\pm 1$ . If  $P(c, F)$  is  $T_g$ -invariant then  $P(c, \mathbf{f}_0) = P(c, T_g \mathbf{f}_0)$ . In the case of  $g = 4$  and  $k = 14$  when  $\dim R_4^{28} = 7$  there are 394  $\text{AGL}$ -classes and a total of 11 005 344 admissible monomials. We continue specializing and wait for the linear equations

$$(5.1) \quad P(c, \mathbf{f}_i) = P(c, T_g \mathbf{f}_i)$$

in the variables  $c_N$  to have rank  $394 - 7 = 387$ . After about 350 integral specializations the integral coefficients of the  $c_N$  in (5.1) became uncomfortably large. The size of the coefficients can be reduced by specializing the  $F_a$  to polynomials with integer coefficients instead but this makes the specializations of admissible monomials harder to compute. A happy medium was found in specializing to small bivariate polynomials and after using 300 of these the rank was computed to be indeed 387. The  $\bar{c}_N$  in the null space of these linear equations are the only possibilities for  $H_4$ -invariant polynomials in  $R_4^{28}$ . Since the dimension of  $R_4^{28}$  is known to be 7 this proves that the  $P(\bar{c}, F)$  are  $H_4$ -invariant. Knowledge of the dimension of  $R_4^{28}$  saves us from having to directly show that any specific polynomial is  $T_g$ -invariant, a computation that would not have been feasible. For  $k = 6, 10, 14$  the spaces  $R_4^{2k}$  have dimensions 1, 3 and 7, respectively. Bases for  $R_4^{2k}$  in these weights can be viewed at [40]. In all three cases the restriction technique of section 3 shows that  $\text{Th}_2$  is onto. We may minimally use  $\pi_N \circ \phi_{D_4}^*$  with  $N = 1, 5, 8$ , for  $k = 6, 10, 14$ , respectively, but we ran  $N = 11$  for purposes of consistency checking. There are no relations in  $k = 6$  or 10 but  $k = 14$  has one relation  $\mathcal{R}_1$  which may be viewed at [40]. If we normalize the code polynomial  $E_6 \in R_3^{12}$  so that  $\text{Th}_2 E_6$  is the Eisenstein series  $E_6 \in M_3^6$  then we have

$$\tilde{\Phi}\mathcal{R}_1 = 1344 E_6 J^{(3)}.$$

**Theorem 5.2.** *The map  $\text{Th}_2 : R_4^{2k} \rightarrow M_4^k$  is surjective for  $k \leq 16$ ,  $k \equiv 0 \pmod{2}$ .*

**Theorem 5.3.**  *$R_4^{28}$  has a basis of 7 polynomials and the kernel of  $\text{Th}_2$  on  $R_4^{28}$  has a basis of one element,  $\mathcal{R}_1$ . These are as given at [40].*

**Corollary 5.4.** *The prime ideal  $\mathfrak{p}_4$  is not generated by 5 elements.*

*Proof.* We have one relation only in degrees 24 and 28 and five relations in degree 32. Since there is one invariant polynomial in degree 8 and none in degree 4, our assertion follows.  $\square$

## §6. Weight enumerators of degree 24.

We are now in a position to classify the linear relations, for every  $g$ , among the weight enumerators of the 9 Type II codes of length 24. Compare Nebe [25] for a less computational proof. This is analogous to the classification of the linear relations among the theta series of even unimodular lattices begun by Witt, Igusa, Kneser, Erokhin and Nebe-Venkov. Let  $V$  be the  $\mathbb{Q}$ -vector space spanned by classes of Type II codes of length  $m$ . Let  $\text{WE}_g : V \rightarrow R_g^m$  be the linear map that sends a code  $C$  to its weight enumerator  $W_C^{(g)}$ . For  $g = 0$  we send each code to 1. Let  $V_g = \ker(\text{WE}_g)$ , the space of relations among the weight enumerators in genus  $g$ , so that we have a decreasing filtration

$$V = V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{m-1} = 0.$$

Before proceeding, we explain the notation of Tables 7 and 8. The codes  $C_i$  are those from Table 1,  $\tau_i$  is the kissing number of  $C_i$ , and  $\text{coeff}(\mu, W_{C_i})$  is the coefficient of the monomial  $\mu$  in the polynomial  $W_{C_i}$ . The results for  $g \leq 4$  are due to [13].



The first result we give here is the generating function of  $R_5$ .

**Theorem 6.1.** *The generating function of  $R_5$  is given by*

$$\begin{aligned} \sum_{m=0}^{\infty} (\dim R_5^m) t^m &= 1 + t^8 + 2t^{16} + 2t^{20} + 8t^{24} + 8t^{28} + 34t^{32} + 60t^{36} \\ &\quad + 203t^{40} + 553t^{44} + 2063t^{48} + 7359t^{52} + 30811t^{56} \\ &\quad + 127416t^{60} + 541644t^{64} + 2235677t^{68} + 8966371t^{72} + \dots \\ &= \frac{N}{D}. \end{aligned}$$

The complete forms of  $N$  and  $D$  can be seen at [27].

Since the calculation is carried out by the same way as in [26], we omit the description of the proof. Once we know that  $\dim R_5^{24} = 8$ , it is enough to examine 8 appropriate monomials to investigate  $R_5^{24}$  and to show that  $7F^{(5)} + 99G^{(5)} = 0$ . To double-check this result we classified all 457  $AGL(5)$ -orbits of admissible monomials of degree 24 in genus 5. These computations allow us to give a basis of  $R_5^{24}$  and linear relations among the weight enumerators of Type II codes of length 24.

**Table 7.**

Relations among weight enumerators of Type II codes of length 24.

$g$	$V_g = \ker(\text{WE}_g)$	$\dim R_g^{24}$
0	$\sum_{i=1}^9 \alpha_i W_{C_i}$ where $\sum \alpha_i = 0$ .	1
1	$\sum_{i=1}^9 \alpha_i W_{C_i}$ where $\sum \alpha_i = 0$ and $\sum \alpha_i \tau_i = 0$ .	2
2	$\sum_{i=1}^9 \alpha_i W_{C_i}$ where $\sum \alpha_i = 0$ , $\sum \alpha_i \tau_i = 0$ and $\sum \alpha_i \tau_i^2 = 0$ .	3
3	$\sum_{i=1}^9 \alpha_i W_{C_i}$ where $\sum \alpha_i = 0$ , $\sum \alpha_i \tau_i = 0$ , $\sum \alpha_i \tau_i^2 = 0$ , $\sum \alpha_i \tau_i^3 = 0$ , and $\sum \alpha_i \beta_i = 0$ where $\beta_i = \text{coeff}(F_{000}^{16} \prod_{a \in \mathbb{F}_2^3} F_a, W_{C_i})$ .	5
4	span of $F = W_{C_5} - 66W_{C_1} + 495W_{C_3} - 880W_{C_4} + 594W_{C_6} - 144W_{C_7}$ , $G = W_{C_9} + 14W_{C_1} - 70W_{C_3} + 112W_{C_4} - 70W_{C_6} + 16W_{C_7} - 3W_{C_8}$ .	7
5	span of $7F + 99G$	8
$g \geq 6$	0	9

**Table 8.**

Constants for Table 7.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$
$\tau_i/24$	22	18	14	10	46	6	2	30	30
$\beta_i/336$	0	1	0	0	0	0	0	4	12

In  $g = 6$ , there is a unique cusp form  $\text{Th}_2(7F + 99G) \in S_6^{12}$  obtained at the end of this process. This cusp form has interesting coefficients, see Table 9. If you mutilpy by 26 159 874 048, you will get a correct number. For instance the Fourier coefficient of  $\text{Th}_2(7F^{(6)} + 99G^{(6)})$  for the  $A_6$  lattice is  $10 \times 261\,598\,740\,48 = 261\,598\,740\,480$ .

**Table 9.**Normalized Fourier Coefficients of  $\text{Th}_2(7F^{(6)} + 99G^{(6)})$ 

$A_6$	$A_5A_1$	$A_4A_2$	$A_4A_1^2$	$A_3^2$	$A_3A_2A_1$	$A_3A_1^3$	$A_2^3$
10	120	180	1600	160	2880	21120	5760

**Table 9(continued).**

$A_2^2A_1^2$	$A_2A_1^4$	$A_1^6$	$D_6$	$D_5A_1$	$D_4A_2$	$D_4A_1^2$	$E_6$
40320	276480	1735680	0	0	0	0	0

We have thus obtained the following Theorems.

**Theorem 6.2.** *The linear combination  $\text{Th}_2(7F^{(6)} + 99G^{(6)})$  is a non-trivial cusp form of weight 12 in genus 6.*

**Theorem 6.3.** *The weight enumerators of the 9 Type II codes of length 24 are linearly independent if and only if  $g \geq 6$ . All linear relations among them are given in Tables 7 and 8.*

The Fourier coefficients of  $\text{Th}_2(7F + 99G)$  vanish on the more complicated root lattices as indicated by Igusa in [18, pg. 105]. Can the image of  $\text{Th}_2$  be nicely characterized in terms of the vanishing of Fourier coefficients?

We conclude this section by reproving a result of Salvati-Manni [37], which states that  $\text{Th}_2$  is not surjective if  $g \geq 5$ . We recall that  $\dim M_5^6 = 1$  from [7]. Since we have  $\dim R_5^{12} = 0$  by Theorem 6.1, we know that  $\text{Th}_2$  is not surjective in genus 5. For  $g \geq 6$ , we know that  $\dim M_g^{12} \geq 11$ , see [24], and the fact  $\dim M_g^{12} > 9$  implies that  $\text{Th}_2$  is not surjective. We have thus shown the assertion.

### §7. Construction of a relation from first order theta identities.

As for theta functions, we have so far been mainly concerned with the second order theta functions. In this section, we shall construct a homogeneous polynomial in the first order theta constants which vanishes identically as a Siegel modular form in genus 4. The relation we obtain by converting to second order theta constants will be expressed as a linear combination of our 5 relations  $R_1^{(4)}, \dots, R_5^{(4)}$ .

The half-integral characteristics of a first order theta function may be identified with  $\mathbb{F}_2^{2g}$ . We define an action of  $\text{Sp}_g(\mathbb{F}_2)$  on  $\mathbb{F}_2^{2g}$  to mimick the action of  $\text{Sp}_g(\mathbb{Z})$  on the half-integral characteristics of the first order theta function. In fact we put

$$\zeta \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}' \zeta + \begin{pmatrix} (A'C)_0 \\ (B'D)_0 \end{pmatrix}$$

for  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{F}_2)$ ,  $\zeta \in \mathbb{F}_2^{2g}$ . When trying to bring some order into the maze of theta identities it is natural to ask whether two sequences of theta characteristics are in the same  $\mathrm{Sp}_g(\mathbb{F}_2)$ -orbit. Frobenius found a complete set of invariants for this action, [16, pg. 212].

For  $\zeta = \begin{bmatrix} a \\ b \end{bmatrix}$ ,  $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{F}_2^{2g}$ , we put

$$e_*(\zeta) = (-1)^{a \cdot b},$$

$$e(\zeta_1, \zeta_2, \zeta_3) = e_*(\zeta_1)e_*(\zeta_2)e_*(\zeta_3)e_*(\zeta_1 + \zeta_2 + \zeta_3).$$

The Theorem of Frobenius can be stated as follows.

**Theorem 7.1.** *Two sequences of characteristics,  $(\zeta_1, \dots, \zeta_m)$  and  $(\xi_1, \dots, \xi_m)$ , are in the same  $\mathrm{Sp}_g(\mathbb{F}_2)$ -orbit if and only if sending  $\zeta_i \mapsto \xi_i$  preserves*

- (1) *all linear relations with an even number of summands,*
- (2) *all  $e_*$  values and*
- (3) *all  $e$  values.*

We can restore linearity to the peculiar affine action of  $\mathrm{Sp}_g(\mathbb{F}_2)$  on characteristics  $\mathbb{F}_2^{2g}$  in the following way: In  $\mathbb{F}_2^{2g} \times \mathbb{F}_2$  let the characteristics  $\zeta \in \mathbb{F}_2^{2g}$  that we have been considering be identified with  $(\zeta, 1) \in \mathbb{F}_2^{2g} \times \mathbb{F}_2$ . Let the coset  $(P, 0) \in \mathbb{F}_2^{2g} \times \mathbb{F}_2$  be identified with  $\mathbb{F}_2^{2g}$  by calling  $P \in \mathbb{F}_2^{2g}$  a *period*. Then we let  $\mathrm{Sp}_g(\mathbb{F}_2)$  act on periods  $\mathbb{F}_2^{2g}$  by  $P \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}' P$  for  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{F}_2)$ ,  $P \in \mathbb{F}_2^{2g}$ . If we note that the sum of two characteristics is a period, we see that the action of  $\mathrm{Sp}_g(\mathbb{F}_2)$  on the  $P \oplus \zeta \in \mathbb{F}_2^{2g} \times \mathbb{F}_2$  is linear. The action of  $\mathrm{Sp}_g(\mathbb{F}_2)$  on periods is characterized by linear dependencies and by the invariant:

$$e_2(P_1, P_2) = e_*(P_1)e_*(P_2)e_*(P_1 + P_2).$$

A characteristic  $\zeta$  is called *even* or *odd* as  $e_*(\zeta)$  is  $+1$  or  $-1$ . A triple  $(\zeta_1, \zeta_2, \zeta_3)$  is called *syzygetic* or *azygetic* as  $e(\zeta_1, \zeta_2, \zeta_3)$  is  $+1$  or  $-1$ . A sequence of characteristics is called *syzygetic* or *azygetic* if every triple in the sequence is. A set of sets of characteristics is called *syzygetic* or *azygetic* if every sequence extracted by selecting one characteristic from each set is. Periods are called *syzygetic* or *azygetic* as  $e_2(P_1, P_2)$  is  $+1$  or  $-1$ . Consider a typical first order theta identity in  $g = 4$  closely studied in [10, pp. 25, 217].

$$\begin{aligned} & \theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & - \theta \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & - \theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \theta \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \theta \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \theta \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & - \theta \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \theta \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \theta \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \theta \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 0. \end{aligned}$$

Let  $G$  be the first row in the above identity;  $G$  is a syzygetic period group of rank 2. The first column is an azygetic characteristic 4-sequence  $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$  with the  $\zeta_i + G$  all even. Thus the identity has the form

$$\sum_{i=1}^4 \pm \prod_{\zeta \in \zeta_i + G} \theta[\zeta] = 0.$$

The Theorem of Frobenius and the transformation law of the theta function show that there is such an identity for *any* syzygetic group  $G$  of rank 2 and azygetic 4-sequence  $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$  with  $\zeta_i + G$  all even. Denote an unordered set of this nature as  $\Xi = \{\zeta_1 + G, \zeta_2 + G, \zeta_3 + G, \zeta_4 + G\}$ . Following [12] we use quadratic polynomials  $Q[\begin{smallmatrix} a \\ b \end{smallmatrix}]$  so that  $\text{Th}_2 Q[\begin{smallmatrix} a \\ b \end{smallmatrix}] = \theta[\begin{smallmatrix} a \\ b \end{smallmatrix}]^2$ : we define  $Q[\begin{smallmatrix} a \\ b \end{smallmatrix}] = \sum_{\alpha \in \mathbb{F}_2^g} (-1)^{\alpha \cdot b} F_\alpha F_{\alpha+a}$ , modeled after  $\theta[\begin{smallmatrix} a \\ b \end{smallmatrix}]^2 = \sum_{\alpha \in \mathbb{F}_2^g} (-1)^{\alpha \cdot b} \theta_2[\alpha] \theta_2[\alpha + a]$ . The action of the generators of  $H_g$  [33] is:

$$Q[\begin{smallmatrix} a \\ b \end{smallmatrix}]|T_g = i^{g-2a \cdot b} Q[\begin{smallmatrix} b \\ a \end{smallmatrix}]; \quad Q[\begin{smallmatrix} a \\ b \end{smallmatrix}]|D_S = i^{-a \cdot (S a + 2S_0)} Q[\begin{smallmatrix} a \\ b + S a + S_0 \end{smallmatrix}].$$

We put  $r_i(\Xi) = \prod_{\zeta \in \zeta_i + G}^{\text{four}} Q[\zeta]$ . We define a homogeneous polynomial Norm of degree four in four variables  $x_1, \dots, x_4$ :

$$\begin{aligned} \text{Norm}(x_1, x_2, x_3, x_4) &= \prod^{\text{eight}} (\sqrt{x_1} \pm \sqrt{x_2} \pm \sqrt{x_3} \pm \sqrt{x_4}) \\ &= \sum_i^{\text{four}} x_i^4 - 4 \sum_{i \neq j}^{\text{twelve}} x_i x_j^3 + 6 \sum_{i < j}^{\text{six}} x_i^2 x_j^2 + 4 \sum_{i < j, i \neq k, j \neq k}^{\text{twelve}} x_i x_j x_k^2 - 40 x_1 x_2 x_3 x_4 \end{aligned}$$

and define  $\text{Norm}(\Xi) = \text{Norm}(r_1(\Xi), \dots, r_4(\Xi)) \in \mathbb{Z}[F_a : a \in \mathbb{F}_2^4]^{32}$ . We write  $G = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$  and  $\gamma_j = \begin{bmatrix} s_j \\ t_j \end{bmatrix}$  so that we can define  $\mu(G) = \sum_{j=1}^4 s_j s'_j$ . To pass between the groups  $H_g$  and  $\text{Sp}_g(\mathbb{F}_2)$  we use the interesting exact sequence [35] (*cf.* [2])

$$0 \rightarrow N_g \rightarrow H_g \xrightarrow{\psi} \text{Sp}_g(\mathbb{F}_2) \rightarrow 0.$$

Here we only need to know that  $\psi$  exists and that the following composite is the identity:

$$H_g/N_g \xrightarrow{\psi} \text{Sp}_g(\mathbb{F}_2) \cong \Gamma_g/\Gamma_g(2) \xrightarrow{\phi} H_g/N_g.$$

**Theorem 7.2.** *Let  $g = 4$ .*

- (1) *There are 5 355 syzygetic period 2-groups  $G$ . For each fixed  $G$  there are 10 distinct  $\zeta + G$  such that  $\zeta \in \mathbb{F}_2^{2g}$  and the  $\zeta + G$  are all even. For each fixed  $G$  there are 15 azygetic  $\Xi = \{\zeta_1 + G, \zeta_2 + G, \zeta_3 + G, \zeta_4 + G\}$  such that the  $\zeta_i + G$  are all even. There are thus  $15 \cdot 5\,355 = 80\,325$  distinct  $\Xi$ .*
- (2) *We have  $r_i(\Xi)|T_4 = r_i(\Xi|J)$  and  $r_i(\Xi)|D_S = i^{-\langle S, \mu(G) \rangle} r_i(\Xi|t(S))$ .*
- (3) *We have  $\text{Norm}(\Xi)|\sigma = \text{Norm}(\Xi|\psi(\sigma))$  and  $\{\text{Norm}(\Xi)\}$  is a set of 80 325 distinct polynomials in  $\mathbb{Z}[F_a : a \in \mathbb{F}_2^g]^{32}$ .*
- (4) *Let  $\text{BigNorm} = \sum_{\Xi}^{80\,325} \text{Norm}(\Xi)$ . The polynomial  $\text{BigNorm}$  is nontrivial and  $H_4$ -invariant. We have  $\text{Th}_2 \text{BigNorm} = 0$ .*

*Proof.* Write  $G = (0, p_1, p_2, p_1 + p_2)$  as a list. There are  $255 = 2^8 - 1$  nontrivial periods to choose for  $p_1$ . Including  $\{0, p_1\}$ , half of the periods  $p$  make  $(p_1, p)$  syzygetic so there are  $126 = 2^7 - 2$  choices for  $p_2$ . The last period is then determined and we divide by  $3!$  to count sets instead of lists:

$$255 \cdot 126/3! = 5\,355.$$

It is long known [10, pg. 40] that the set of  $g$  characteristics with uniform parity modulo a syzygetic period group of rank  $r$  behaves like a set of  $g - r$  characteristics. Thus there are 10 distinct  $\zeta + G$  such that the  $\zeta + G$  are all even because there are 10 even characteristics in  $g = 2$ . There are 15 azygetic 4-sets in  $g = 2$  so there are 15  $\Xi$  for any given syzygetic 2-group  $G$ . This is, in fact, Theorem 19 in [10, pg. 54].

For even characteristics  $\zeta = \begin{bmatrix} a \\ b \end{bmatrix}$  we have

$$Q[\zeta]|T_4 = Q\begin{bmatrix} a \\ b \end{bmatrix}|T_4 = i^{4-2a \cdot b} Q\begin{bmatrix} b \\ a \end{bmatrix} = Q\begin{bmatrix} b \\ a \end{bmatrix} = Q[\zeta|J].$$

Looking at each factor, we have  $r_i(\Xi)|T_4 = r_i(\Xi|J)$ . For generators  $D_S$  we have

$$Q[\zeta]|D_S = Q\begin{bmatrix} a \\ b \end{bmatrix}|D_S = i^{-a \cdot (Sa+2S_0)} Q\begin{bmatrix} a \\ b + Sa + S_0 \end{bmatrix} = i^{-a \cdot (Sa+2S_0)} Q[\zeta|t(S)].$$

Therefore for  $G = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$  and  $\gamma_j = \begin{bmatrix} s_j \\ t_j \end{bmatrix}$  we have

$$\begin{aligned} r_i(\Xi)|D_S &= \prod_j Q[\zeta_i + \gamma_j]|D_S = \prod_j i^{-(a_i+s_j) \cdot (S(a_i+s_j)+2S_0)} Q[(\zeta_i + \gamma_j)|t(S)] \\ &= \left(i^{-4S[a_i]}\right) \left(i^{-2a'_i S(\sum s_j)}\right) \left(i^{-\sum_j S[s_j]}\right) \left(i^{-2S_0 \cdot \sum(a_i+s_j)}\right) \prod_j Q[(\zeta_i + \gamma_j)|t(S)] \\ &= i^{-\sum_j S[s_j]} r_i(\Xi|t(S)) = i^{-\langle S, \mu(G) \rangle} r_i(\Xi|t(S)). \end{aligned}$$

We see that each  $r_i(\Xi)$  transforms by the same factor independently of  $i$ . This factor  $i^{-\langle S, \mu(G) \rangle} = \pm 1$  because  $\mu(G)$  is an even form. As a polynomial of even degree in the  $r_i$ ,  $\text{Norm}(\Xi)$  satisfies  $\text{Norm}(\Xi)|D_S = \text{Norm}(\Xi|t(S))$ . The  $H_4$ -symmetrization of  $\text{Norm}(\Xi)$ ,  $\sum_{\sigma \in H_4} \text{Norm}(\Xi)|\sigma$ , is a multiple of  $\text{BigNorm}$  because the stabilizers of  $\Xi$  in  $H_4$  all have the same order by the Theorem of Frobenius. This shows that  $\text{BigNorm}$  is  $H_4$ -invariant. A computation using the specialization  $F = (1, 7, -1, 0, 3, 2, -2, 5, 2, 5, 1, 2, -3, 1, 5, 3)$  shows that the  $\text{Norm}(\Xi)$  are all distinct and that  $\text{BigNorm}$  is nontrivial.

Finally, we show that  $\text{Th}_2 \text{BigNorm} = 0$ . Actually,  $\text{Th}_2 \text{Norm}(\Xi) = 0$  for each  $\Xi$ . One of the 8 factors  $\text{Th}_2 \left( \sqrt{r_1(\Xi)} \pm \sqrt{r_2(\Xi)} \pm \sqrt{r_3(\Xi)} \pm \sqrt{r_4(\Xi)} \right)$  is an identity

$$\sum_{i=1}^4 \pm \prod_{\zeta \in \zeta_i + G}^{\text{four}} \theta[\zeta] = 0. \quad \square$$

The invariant polynomial BigNorm should be a linear combination of  $R_1, \dots, R_5$  and it is:

$$\begin{aligned} 573\,102\,233\,555 \text{ BigNorm} = & \\ & 151\,595\,494\,160 R_1^{(4)} \\ & -292\,362\,643\,392 R_2^{(4)} \\ & +82\,765\,857\,152 R_3^{(4)} \\ & +5\,300\,722\,416 R_4^{(4)} \\ & +230\,972\,544 R_5^{(4)}. \end{aligned}$$

If  $p(x_1, x_2, x_3, x_4)$  is any homogeneous symmetric polynomial of even degree in four variables then  $\sum_{\Xi}^{80\,325} p(r_1(\Xi), r_2(\Xi), r_3(\Xi), r_4(\Xi)) \text{Norm}(\Xi)$  is a relation in  $\mathbf{p}_4$ . In this way we can construct many more relations. We tried to derive the Freitag-Oura relation,  $R_0^{(4)}$ , by a variant of this scheme but the symmetrization gave zero.

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