

# Sums of Products of Kronecker's Double Series

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## Abstract

Closed expressions are obtained for sums of products of Kronecker's double series of the form  $\sum \binom{n}{j_1, \dots, j_N} B_{j_1}(x'_1, x_1; \tau) \cdots B_{j_N}(x'_N, x_N; \tau)$ , where the summation ranges over all nonnegative integers  $j_1, \dots, j_N$  with  $j_1 + \cdots + j_N = n$ . Corresponding results are derived for functions which are an elliptic analogue of the periodic Euler polynomials. As corollaries, we reproduce the formulas for sums of products of Bernoulli numbers, Bernoulli polynomials, Euler numbers, and Euler polynomials, which were given by K. Dilcher.

## 1 Introduction

The Bernoulli polynomials  $B_m(x)$  are defined by means of the following generating function:

$$\frac{\xi e^{x\xi}}{e^\xi - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} \xi^n.$$

The  $m$ -th Bernoulli number  $B_m$  is  $B_m(0)$ . A well-known identity for the Bernoulli numbers is

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1) B_{2n}, \quad (n \geq 2), \quad (1)$$

which was found by many authors, including Euler (for references, see, e.g., [SD]). This identity was generalized to formulas including sums of products of Bernoulli numbers of the forms

$$\sum \binom{2n}{2j_1, \dots, 2j_N} B_{2j_1} \cdots B_{2j_N}, \quad (N = 3, 4, 5, 6, 7). \quad (2)$$

Here the summation is extended over all positive integers  $j_1, \dots, j_N$  with  $j_1 + \dots + j_N = n$ , and  $\binom{2n}{2j_1, \dots, 2j_N} := \frac{(2n)!}{(2j_1)! \cdots (2j_N)!}$ . (see [SD] for  $N = 3$ , [San] for  $N = 5$ , and [Zha1] for  $3 \leq N \leq 7$ .) It should be noted that (2) can be written in terms of the Riemann zeta function via Euler's formula

$$\zeta(2m) = (-1)^{m-1} \frac{(2\pi)^{2m} B_{2m}}{2(2m)!}. \quad (3)$$

In 1996, K. Dilcher [Di] gave formulas for sums of products of Bernoulli numbers and polynomials which was the kind given in [SD], [San] and [Zha1]. His formulas include (2) for  $N \geq 2$  where the summation ranges over all nonnegative integers  $j_1, \dots, j_N$  with  $j_1 + \dots + j_N = n$ . He also produced corresponding results for sums of products of Euler numbers, Euler polynomials, and special values of the following zeta functions:

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}.$$

Recently Dilcher's result has been generalized by many people: I-C. Huang and S-Y. Huang [HH] have deduced some generalized formulas for sums of products of Bernoulli numbers and polynomials via a method called algebraic residues. J. Satoh [Sat] and T. Kim [Ki] have given formulas for sums of products of Carlitz's  $q$ -Bernoulli numbers. K-W. Chen and M. Eie [CE] and K-W. Chen [Ch] have produced formulas for sums of products of generalized Bernoulli numbers and polynomials by using special values of some zeta functions at nonpositive integers. T. Kim and C. Adiga [KA] have also obtained a relation between sums of products of generalized Bernoulli numbers and higher order generalized Bernoulli numbers.

The principal purpose of this paper is to establish sums of products of Kronecker's double series  $B_m(x', x; \tau)$  which are defined by means of the following generating function:

$$-\sum \frac{e e(n'x' + nx)}{-\xi + n'\tau + n} = \sum_{m=0}^{\infty} \frac{B_m(x', x; \tau)}{m!} (2\pi i)^m \xi^{m-1}, \quad (x', x \in \mathbb{R}).$$

Here  $\sum^e$  denotes Eisenstein summation [We]. So we have

$$B_m(x', x; \tau) = \begin{cases} 1 & (m = 0), \\ -\frac{m!}{(2\pi i)^m} \sum_{\substack{(n', n) \\ \neq (0, 0)}} \frac{e e(n'x' + nx)}{(n'\tau + n)^m} & (m \geq 1). \end{cases}$$

E.V. Ivashkevich *et al.* [III Sect.3.1] and K. Katayama [Ka] noted that these series can be considered as an elliptic generalization of the classical Bernoulli functions. The author [Ma] mentioned a relation between the generating function of Kronecker's double series and that of the (Debye) elliptic polylogarithms studied by A. Levin [Le] in order to enforce the validity of their elliptic generalization.

The formulas for sums of products of Kronecker's double series induce Dilcher's results for Bernoulli numbers and polynomials, which guarantees that the sums of products of Kronecker's double series are a natural generalization of the sums of products of Bernoulli numbers and polynomials. In addition these yield formulas for  $\eta(2m)$  which are slightly different from Dilcher's ones.

We also obtain corresponding results for functions which are an elliptic analogue of the periodic Euler polynomials defined by L. Carlitz [Ca]. (We call these functions *elliptic Euler functions* for short). These also produce the formulas for sums of products of Euler numbers, Euler polynomials and  $\lambda(2m)$ , which were given by Dilcher. (The formulas for  $\lambda(2m)$  are slightly different from Dilcher's ones too.)

The paper is organized as follows: In Section 2 we obtain formulas for sums of products of Kronecker's double series, and produce Corresponding results for sums of products of Bernoulli numbers, Bernoulli polynomials and  $\eta(2m)$ . Section 3 deals with formulas for sums of products of elliptic Euler functions and related Dilcher's results.

## 2 Kronecker's double series

In this section we give formulas for sums of products of Kronecker's double series. As corollaries, we reproduce Dilcher's formulas for sums of products of Bernoulli numbers and polynomials, and produce formulas for special values of the zeta function  $\eta(s)$  which are slightly different from Dilcher's ones.

We begin with introducing another expression of the generating function of Kronecker's double series  $B_m(x', x; \tau)$ . Let  $\tau$  be in the upper half-plane. In what follows we shall use the following notations:  $e(x) := e^{2\pi ix}$ ,  $q := e(\tau)$ , and Jacobi's theta function

$$\theta(x; \tau) := \sum_{m \in \mathbb{Z}} e\left(\frac{1}{2}\left(m + \frac{1}{2}\right)^2 \tau + \left(m + \frac{1}{2}\right)\left(x + \frac{1}{2}\right)\right).$$

We define the function  $F(x, \xi; \tau)$  as follows:

$$F(x, \xi; \tau) = \frac{\theta'(0; \tau)\theta(x + \xi; \tau)}{\theta(x; \tau)\theta(\xi; \tau)}, \quad (x, \xi \in \mathbb{C} \setminus \mathbb{Z} + \tau\mathbb{Z}),$$

where  $\theta'(x; \tau) = \frac{\partial}{\partial x}\theta(x; \tau)$ . For fixed  $x \in \mathbb{C} \setminus \mathbb{Z} + \tau\mathbb{Z}$ , the function  $F(x, \xi; \tau)$  with respect to  $\xi$  is meromorphic with only simple poles on the lattice  $\mathbb{Z} + \tau\mathbb{Z}$ . In addition, it satisfies the following properties (see, e.g., [Jor, We]):

$$F(x, \xi + 1; \tau) = F(x, \xi; \tau), \quad F(x, \xi + \tau; \tau) = e(-x)F(x, \xi; \tau). \quad (4)$$

Set  $\underline{F}(x', x; \xi; \tau) := e(x\xi)F(-x' + x\tau, \xi; \tau)$ . It is necessary to suppose that  $x' \notin \mathbb{Z}$  or  $x \notin \mathbb{Z}$  if  $x', x$  are real numbers because  $F(0, 0; \tau)$  becomes infinity. When  $x'$  and  $x$  are real numbers with  $-1 < x < 0$ , Kronecker proved the following equation [We].

$$\underline{F}(x', x; \xi; \tau) = -\sum^e \frac{e(n'x' + nx)}{-\xi + n'\tau + n},$$

where  $\sum^e$  denotes Eisenstein summation [We], i.e.,  $\sum^e = \lim_{N' \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{n'=-N'}^{N'} \sum_{n=-N}^N$ .

So Kronecker's double series  $B_m(x', x; \tau)$  are expressed as

$$\underline{F}(x', x; \xi; \tau) = \sum_{n=0}^{\infty} \frac{B_n(x', x; \tau)}{n!} (2\pi i)^n \xi^{n-1}. \quad (5)$$

We note that  $B_m(x', x; \tau)$  have the following periodicity by (4):

$$B_m(x' + 1, x; \tau) = B_m(x', x + 1; \tau) = B_m(x', x; \tau). \quad (6)$$

Let us introduce the function  $\underline{F}^{(m)}(x', x; \xi; \tau)$ :

$$\underline{F}^{(m)}(x', x; \xi; \tau) := \frac{1}{(2\pi i)^m} \left( \frac{\partial}{\partial \xi} \right)^m \underline{F}(x', x; \xi; \tau), \quad (m \geq 0), \quad (7)$$

especially  $\underline{F}^{(0)}(x', x; \xi; \tau) = \underline{F}(x', x; \xi; \tau)$ . We see from (4) that  $\underline{F}^{(m)}(x', x; \xi; \tau)$  satisfy the following periodicity:

$$\begin{aligned} \underline{F}^{(m)}(x', x; \xi + 1; \tau) &= e(x)\underline{F}^{(m)}(x', x; \xi; \tau), \\ \underline{F}^{(m)}(x', x; \xi + \tau; \tau) &= e(x')\underline{F}^{(m)}(x', x; \xi; \tau). \end{aligned} \quad (8)$$

They also have the following expression by (5):

$$\underline{F}^{(m)}(x', x; \xi; \tau) = \frac{(-1)^m m!}{(2\pi i)^m \xi^{m+1}} + \sum_{n=0}^{\infty} \frac{B_{n+m+1}(x', x; \tau)}{(n+m+1)n!} (2\pi i)^{n+1} \xi^n. \quad (9)$$

Let  $N$  be a positive integer and  $n$  a nonnegative integer. We set  $\mathbf{x}_i = (x'_i, x_i)$  for  $i = 1, \dots, N$ . Our aim in this section is to evaluate the sum

$$S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) := \sum_{\substack{j_1, \dots, j_N \geq 0 \\ j_1 + \dots + j_N = n}} \binom{n}{j_1, \dots, j_N} B_{j_1}(x'_1, x_1; \tau) \cdots B_{j_N}(x'_N, x_N; \tau), \quad (10)$$

where  $\binom{n}{j_1, \dots, j_N} := \frac{n!}{j_1! \cdots j_N!}$  is the multinomial coefficient. The generating function of  $S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N)$  is

$$\xi^N \prod_{i=1}^N \underline{F}(x'_i, x_i; \xi; \tau) = \sum_{n=0}^{\infty} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) \frac{(2\pi i \xi)^n}{n!}. \quad (11)$$

To produce our result, we need the following lemma:

**LEMMA 2.1.** *For any  $i = 1, \dots, N$ , let  $x'_i$  and  $x_i$  be real numbers with  $x'_i \notin \mathbb{Z}$ . Set*

$$\mathbf{x}_i = (x'_i, x_i) \quad (i = 1, \dots, N), \quad (y', y) = (x'_1 + \cdots + x'_N, x_1 + \cdots + x_N).$$

*Suppose that  $y' \notin \mathbb{Z}$ . Then we have*

$$\begin{aligned} \frac{(N-1)!}{(2\pi i)^{N-1}} \prod_{i=1}^N \underline{F}(x'_i, x_i; \xi; \tau) &= (-1)^{N-1} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \\ &\quad \times S_N^\tau(m; \mathbf{x}_1, \dots, \mathbf{x}_N) \underline{F}^{(N-1-m)}(y', y; \xi; \tau). \end{aligned} \quad (12)$$

*Proof.* Set  $G(\xi) :=$  (left-hand side of (12))  $-$  (right-hand side of (12)). We see from (8) that

$$G(\xi + 1) = e(y)G(\xi), \quad G(\xi + \tau) = e(y')G(\xi). \quad (13)$$

Using Liouville's theorem, we will show that  $G(\xi) = 0$ . Let  $\xi$  be a complex number near the origin. It follows from (9) that

$$\underline{F}^{(m)}(x', x; \xi; \tau) = \frac{(-1)^m m!}{(2\pi i)^m \xi^{m+1}} + O(1),$$

where  $O$  denotes the Landau symbol. Thus

$$\begin{aligned} G(\xi) &= \frac{(N-1)!}{(2\pi i)^{N-1} \xi^N} \sum_{n=0}^{\infty} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) \frac{(2\pi i \xi)^n}{n!} \\ &\quad - \sum_{m=0}^{N-1} \binom{N-1}{m} S_N^\tau(m; \mathbf{x}_1, \dots, \mathbf{x}_N) \frac{(N-1-m)!}{(2\pi i)^{N-1-m} \xi^{N-m}} + O(1) \\ &= O(1). \end{aligned}$$

So the function  $G(\xi)$  is holomorphic at  $\xi = 0$ . Since the function  $F(x, \xi; \tau)$  with respect to  $\xi$  is meromorphic with only simple poles on the lattice  $\mathbb{Z} + \tau\mathbb{Z}$ , the possible poles of  $G(\xi)$  are on  $\mathbb{Z} + \tau\mathbb{Z}$ . These together with (13) imply that  $G(\xi)$  is a holomorphic function. On the other hand one sees that  $|e(y')| = |e(y)| = 1$  and  $e(y') \neq 1$  since  $y', y$  are real numbers and  $y' \notin \mathbb{Z}$ . So it follows by (13) that  $G(\xi)$  is a bounded function. By Liouville's theorem, one can obtain  $G(\xi) = 0$ . This completes the proof.  $\square$

**THEOREM 2.1** (Sums of products of Kronecker's double series). *Let  $n$  be an integer with  $n \geq N$ . For any  $i = 1, \dots, N$ , let  $x'_i$  and  $x_i$  be real numbers with  $x'_i \notin \mathbb{Z}$ . Set*

$$\mathbf{x}_i = (x'_i, x_i) \quad (i = 1, \dots, N), \quad (y', y) = (x'_1 + \dots + x'_N, x_1 + \dots + x_N).$$

Suppose that  $y' \notin \mathbb{Z}$ . Then we have

$$\begin{aligned} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) &= (-1)^{N-1} N \binom{n}{N} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \\ &\quad \times S_N^\tau(m; \mathbf{x}_1, \dots, \mathbf{x}_N) \frac{B_{n-m}(y', y; \tau)}{n-m}. \end{aligned} \quad (14)$$

*Proof.* Comparing the coefficient of  $\xi^{n-N}$  in (12) together with (9) and (11) induces (14).  $\square$

Let us reproduce the sums of products of Bernoulli numbers and polynomials given by Dilcher. In analogy to (10), we denote

$$S_N(n; x_1, \dots, x_N) := \sum_{\substack{j_1, \dots, j_N \geq 0 \\ (j_1 + \dots + j_N = n)}} \binom{n}{j_1, \dots, j_N} B_{j_1}(x_1) \cdots B_{j_N}(x_N),$$

$$S_N(n) := \begin{cases} \sum_{\substack{j_1, \dots, j_N \geq 0 \\ (2j_1 + \dots + 2j_N = n)}} \binom{n}{2j_1, \dots, 2j_N} B_{2j_1} \cdots B_{2j_N} & (n : \text{even}), \\ 0 & (n : \text{odd}). \end{cases}$$

We remark that  $S_N(2n)$  in this paper corresponds to  $S_N(n)$  in [Di, Section 2].

To give Dilcher's results, we need the following proposition and lemma:

**PROPOSITION 2.1.** *Let  $x$  be a real number and  $x'$  a complex number with  $x' \notin \mathbb{Z}$ . The  $m$ -th Bernoulli function  $\tilde{B}_m(x)$  is defined by  $\tilde{B}_m(x) := B_m(\{x\})$ , where  $\{x\}$  denotes the fractional part of  $x$ . Then we have*

$$\lim_{\tau \rightarrow i\infty} B_m(x', x; \tau) = \begin{cases} \frac{1 + e(x')}{2 - e(x')} & (m = 1, x \in \mathbb{Z}), \\ \tilde{B}_m(x) & (\text{otherwise}). \end{cases} \quad (15)$$

*Proof.* See, e.g., [Ma, Proposition 2.1] for the proof.  $\square$

**LEMMA 2.2.**

(i) *Let  $x_1, \dots, x_N$  be real numbers and  $x'_1, \dots, x'_N$  complex numbers with  $x'_1, \dots, x'_N \notin \mathbb{Z}$ . Set  $\mathbf{x}_i = (x'_i, x_i)$  for  $i = 1, \dots, N$ . If  $0 \leq x_1, \dots, x_N < 1$ , then we have*

$$\lim_{x' \rightarrow -i\infty} \lim_{\tau \rightarrow i\infty} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) = S_N(n; x_1, \dots, x_N). \quad (16)$$

(ii) *Set  $\mathbf{x}_i = (1/2, 0)$  for  $i = 1, \dots, N$ . Then we have*

$$\lim_{\tau \rightarrow i\infty} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) = S_N(n). \quad (17)$$

(iii) *Set  $\mathbf{x}_i = (1/2, 0)$  for  $i = 1, \dots, N - 1$  and  $\mathbf{x}_N = (x'_N, 0)$ . Then we have*

$$\left[ \text{the coefficient of } x_n^0 (= 1) \text{ of } \lim_{\tau \rightarrow i\infty} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) \right] = S_N(n). \quad (18)$$

*Proof.* If  $0 \leq x < 1$ , then it follows from (15) that

$$\lim_{x' \rightarrow -i\infty} \lim_{\tau \rightarrow i\infty} B_m(x', x; \tau) = B_m(x)$$

because  $B_1 = -1/2$ . This induces (16). Since  $B_{2m+1} = 0$  ( $m \geq 1$ ) and  $\lim_{\tau \rightarrow i\infty} B_1(1/2, 0; \tau) = 0$ , we can deduce (17). We will show (18). One sees from (15) that

$$\lim_{\tau \rightarrow i\infty} S_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) = S_N(n) + \frac{1}{2} \frac{1 + e(x'_N)}{1 - e(x'_N)} \sum_{\substack{j_1, \dots, j_{N-1} \geq 0 \\ (j_1 + \dots + j_{N-1} = n-1)}} \binom{n}{j_1, \dots, j_{N-1}, 1} B_{j_1} \cdots B_{j_{N-1}}.$$

We obtain (18) because  $\frac{1}{2} \frac{1 + e(x'_N)}{1 - e(x'_N)}$  is an odd function.  $\square$

The higher-order Bernoulli polynomials  $B_m^{(N)}(y)$  are defined by the following generating function (see, e.g., [Nör, p.145]):

$$\frac{\xi^N e^{y\xi}}{(e^\xi - 1)^N} = \sum_{n=0}^{\infty} \frac{B_n^{(N)}(y)}{n!} \xi^n. \quad (19)$$

Thus  $B_m^{(N)}(y) = S_N(m; x_1, \dots, x_N)$  when  $y = x_1 + \dots + x_N$ . K. Dilcher [Di] obtained the following two identities:

**THEOREM 2.2** (Sums of products of Bernoulli polynomials). *Let  $x_1, \dots, x_N, y$  be complex numbers with  $y = x_1 + \dots + x_N$ . Then for  $n \geq N$  we have*

$$S_N(n; x_1, \dots, x_N) = (-1)^{N-1} N \binom{n}{N} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m B_m^{(N)}(y) \frac{B_{n-m}(y)}{n-m}. \quad (20)$$

*Proof.* It is sufficient to show (20) when  $0 \leq x_1, \dots, x_N, y < 1$  by analyticity of  $B_m(x)$ . This is derived from (14) and (16).  $\square$

**REMARK 2.1.** The above theorem corresponds to [Di, Lemma 4]. One can easily obtain [Di, Theorem 3] from the theorem by using [Di, (3,7)].

**REMARK 2.2.** Eq.(20) with  $x_1 = \dots = x_N = 0$  was proved by H.S. Vandiver [Van, Eq.(142)].



**THEOREM 2.3** (Sums of products of Bernoulli numbers). *For  $2n > N$  we have*

$$S_N(2n) = (-1)^{N-1} N \binom{2n}{N} \sum_{m=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1}{2m} S_N(2m) \frac{B_{2n-2m}}{2n-2m}, \quad (21)$$

where  $\lfloor x \rfloor$  denotes the greatest integer not exceeding  $x$ .

*Proof.* Suppose that  $N$  is odd. Set  $\mathbf{x}_i = (1/2, 0)$  ( $i = 1, \dots, N$ ) and  $(y', y) = (N/2, 0)$ . It follows from  $2n > N$  that

$$\lim_{\tau \rightarrow i\infty} B_{2n-m}(y', y; \tau) = B_{2n-m} = 0, \quad (1 \leq m \leq N-1, m : \text{odd}).$$

This together with (14) and (17) induces (21). Next suppose that  $N$  is even. Set

$$\mathbf{x}_i = \left(\frac{1}{2}, 0\right) \quad (i = 1, \dots, N-1), \quad \mathbf{x}_N = (x'_N, 0), \quad (y', y) = \left(\frac{N-1}{2} + x'_N, 0\right).$$

By virtue of (14) and  $2n - m \geq 2$  ( $m = 0, \dots, N-1$ ), we have

$$\begin{aligned} & \left[ \text{the coefficient of } x_n^{j_0} (= 1) \text{ of } \lim_{\tau \rightarrow i\infty} S_N^\tau(2n; \mathbf{x}_1, \dots, \mathbf{x}_N) \right] \\ &= (-1)^{N-1} N \binom{2n}{N} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \frac{B_{2n-m}}{2n-m} \\ & \quad \times \left[ \text{the coefficient of } x_n^{j_0} (= 1) \text{ of } \lim_{\tau \rightarrow i\infty} S_N^\tau(m; \mathbf{x}_1, \dots, \mathbf{x}_N) \right] \end{aligned}$$

This together with (18) induces (21).  $\square$

**REMARK 2.3.** We can not derive directly Theorem 2.3 from Theorem 2.2. In general  $S_N(2n) \neq S_N(2n; 0, \dots, 0)$  since  $B_1(0) = B_1 = -1/2 \neq 0$ .

**REMARK 2.4.**  $S_N(2m)$  are expressed as the numbers  $b_m^{(N)}$  defined in [Di] (see [Di, Theorem 2]). So Theorem 2.3 corresponds to [Di, Theorem 1].

As motivated by the work [SD], Dilcher also deal with formulas for sums of products of special values of the following zeta function:

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

We here produce formulas for  $\eta(2m)$  which are slightly different from [Di, (3.12)].

**THEOREM 2.4** (Sums of products of  $\eta(2m)$ ). *For  $2n \geq N$  we have*

$$\sum_{\substack{j_1 + \dots + j_N \geq 0 \\ (j_1 + \dots + j_N = n)}} \eta(2j_1) \cdots \eta(2j_N) = \frac{(-1)^{n+N-1} (2\pi)^{2n}}{2^N (N-1)! (2n-N)!} \\ \times \sum_{m=0}^{[(N-1)/2]} \binom{N-1}{2m} B_{2m}^{(N)} \left(\frac{N}{2}\right) \frac{\tilde{B}_{2n-2m}(\frac{N}{2})}{2n-2m}, \quad (22)$$

with the convention  $\eta(0) = 1/2$ .

*Proof.* It follows from (19) that

$$\sum_{n=0}^{\infty} \frac{B_n^{(N)}(\frac{N}{2})}{n!} \xi^n = \frac{\xi^N e^{\frac{N\xi}{2}}}{(e^\xi - 1)^N} = \frac{\xi^N}{(e^{\xi/2} - e^{-\xi/2})^N}, \quad (23)$$

and from this we see that  $B_m^{(N)}(\frac{N}{2}) = 0$  for odd  $m$ . Set  $\mathbf{x}_i = (x'_i, 1/2)$  for any  $i = 1, \dots, N$ . By virtue of (14), (16) and (23), one obtains

$$S_N(2n; \frac{1}{2}, \dots, \frac{1}{2}) = (-1)^{N-1} N \binom{2n}{N} \sum_{m=0}^{[(N-1)/2]} \binom{N-1}{2m} B_{2m}^{(N)} \left(\frac{N}{2}\right) \frac{\tilde{B}_{2n-2m}(\frac{N}{2})}{2n-2m}. \quad (24)$$

On the other hand, we can easily see that, for  $m > 1$ ,

$$\eta(m) = (1 - 2^{1-m}) \zeta(m).$$

Using Euler's formula (3) and the identity  $B_m(1/2) = -(1 - 2^{1-m})B_m$  (see, e.g., [Nör, p22, (19)]), we obtain

$$B_{2m}(\frac{1}{2}) = (-1)^m \frac{2(2m)!}{(2\pi)^{2m}} \eta(2m). \quad (25)$$

We note that (25) with  $m = 0$  also holds since  $\eta(0) = 1/2$ . Eqs.(24) and (25) yield (22).  $\square$

**REMARK 2.5.** The difference between (22) and [Di, (3.12)] is the Bernoulli function  $\tilde{B}_{2n-2m}(N/2)$  by virtue of [Di, (3.7)]; In [Di, (3.12)] the Bernoulli polynomials  $B_{2n-2m}(N/2)$  appeared because Dilcher used the sums of products of Bernoulli polynomials (Theorem 2.2) for his results. The right-hand

side of (22) can be written in terms of  $\eta(2m)$  or  $\zeta(2m)$  since

$$\tilde{B}_{2n-2m}\left(\frac{N}{2}\right) = \begin{cases} (-1)^{n-m-1} \frac{2(2n-2m)!}{(2\pi)^{2n-2m}} \zeta(2n-2m) & (N : \text{even}), \\ (-1)^{n-m} \frac{2(2n-2m)!}{(2\pi)^{2n-2m}} \eta(2n-2m) & (N : \text{odd}). \end{cases}$$

It seems that (22) is better than [Di, (3.12)] for writing sums of products of  $\eta(2m)$  in terms of  $\eta(2m)$  or  $\zeta(2m)$  since we have to use the difference equation  $B_m(x+1) - B_m(x) = mx^{m-1}$  for it in [Di, (3.12)].

### 3 Elliptic Euler functions

We give formulas for sums of products of *elliptic Euler functions* which are an elliptic analogue of the periodic Euler polynomials defined by L. Carlitz [Ca]. In complete analogy to the method of Section 2, we obtain results concerning Euler numbers, Euler polynomials, and special values of the zeta function  $\lambda(s)$ .

The Euler polynomials  $E_m(x)$  are defined by means of the following generating function:

$$\frac{2e^{x\xi}}{e^\xi + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} \xi^n. \quad (26)$$

They satisfy the following (see, e.g., [Nör, p24–26]):

$$E_m(1) = (-1)^m E_m(0), \quad E_{2m}(0) = 0 \quad (m > 0). \quad (27)$$

In the course of study of a generalization of multiplication formulas (distribution property) for Bernoulli and Euler polynomials, L. Carlitz [Ca] introduced the *periodic Euler polynomials*  $\tilde{E}_m(x)$ :

$$\tilde{E}_m(x) := E_m(x) \quad (0 \leq x < 1), \quad \tilde{E}_m(x+1) := -E_m(x),$$

namely  $\tilde{E}_m(x) = (-1)^{[x]} E_m(\{x\})$  ( $x \in \mathbb{R}$ ).

**REMARK 3.1.**  $\tilde{E}_0(x)$  is a discontinuous function because  $E_0(x) = 1$ . The others are continuous functions by virtue of (27).

The elliptic Euler functions  $E_m(x', x; \tau)$  are defined by means of the following generating function:

$$-2e\left(-\frac{x}{2}\right)\underline{F}(x', x; \xi + \frac{1}{2}; \tau) = \sum_{n=0}^{\infty} \frac{E_n(x', x; \tau)}{n!} (2\pi i)^{n+1} \xi^n. \quad (28)$$

Since  $-2\left(-\frac{x}{2}\right)\underline{F}(x', x; \xi + \frac{1}{2}; \tau) = -2e(\xi x)F(-x' + x\tau, \xi + \frac{1}{2}; \tau)$ , it is seen from (4) that

$$E_m(x', x + 1; \tau) = -E_m(x', x; \tau). \quad (29)$$

The elliptic Euler functions degenerate into the periodic Euler polynomials:

**PROPOSITION 3.1.** *Let  $x$  be a real number and  $x'$  a complex number with  $x' \notin \mathbb{Z}$ . Then we have*

$$\lim_{\tau \rightarrow i\infty} E_m(x', x; \tau) = \begin{cases} (-1)^{[x]} \frac{e(x') + 1}{e(x') - 1} & (m = 0, x \in \mathbb{Z}), \\ \tilde{E}_m(x) & (\text{otherwise}). \end{cases} \quad (30)$$

*Proof.* We have  $E_m(x', x; \tau) = (-1)^{[x]} E_m(x', \{x\}; \tau)$  by (29). So it is sufficient to show (30) when  $0 \leq x < 1$ . Suppose that  $0 \leq x < 1$ . The function  $F(x, \xi; \tau)$  has the following expression [We]:

$$F(x, \xi; \tau) = 2\pi i \left[ \sum_{j=1}^{\infty} \frac{q^j}{e(x) - q^j} e(-j\xi) - \sum_{j=1}^{\infty} \frac{q^j}{e(-x) - q^j} e(j\xi) + \frac{1}{e(x) - 1} + \frac{1}{e(\xi) - 1} + 1 \right], \quad (|\operatorname{Im} x|, |\operatorname{Im} \xi| < \operatorname{Im} \tau).$$

After direct calculation we can get by (28) that

$$E_m(x', x; \tau) = -2 \left[ \sum_{j=1}^{\infty} (x - j)^m \frac{(-1)^j q^j}{e(-x' + x\tau) - q^j} - \sum_{j=1}^{\infty} (x + j)^m \frac{(-1)^j q^j}{e(x' - x\tau) - q^j} + x^m \frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1} \right] + E_m(x).$$

Since  $\lim_{\tau \rightarrow i\infty} e(x\tau)q^j = \lim_{\tau \rightarrow i\infty} e(-x\tau)q^j = 0$  ( $j \in \mathbb{Z}_{\geq 1}$ ), and

$$\lim_{\tau \rightarrow i\infty} x^m \frac{e(-x' + x\tau)}{e(-x' + x\tau) - 1} = \begin{cases} \frac{1}{1 - e(x')} & (m = 0, x = 0), \\ 0 & (\text{otherwise}), \end{cases}$$

one obtains (30). □

Set  $\mathbf{x}_i = (x'_i, x_i)$  for  $i = 1, \dots, N$ . Our aim in this section is to evaluate the sum

$$T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) := \sum_{\substack{j_1, \dots, j_N \geq 0 \\ (j_1 + \dots + j_N = n)}} \binom{n}{j_1, \dots, j_N} E_{j_1}(x'_1, x_1; \tau) \cdots E_{j_N}(x'_N, x_N; \tau). \quad (31)$$

The generating function of  $T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N)$  is

$$\frac{1}{(2\pi i)^N} \prod_{i=1}^N \left( -2e\left(-\frac{x_i}{2}\right) \underline{F}(x'_i, x_i; \xi + \frac{1}{2}; \tau) \right) = \sum_{n=0}^{\infty} T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) \frac{(2\pi i \xi)^n}{n!}. \quad (32)$$

**THEOREM 3.1** (Sums of products of elliptic Euler functions). *Let  $n$  be a nonnegative integer. For any  $i = 1, \dots, N$ , let  $x'_i$  and  $x_i$  be real numbers with  $x'_i \notin \mathbb{Z}$ . Set*

$$\mathbf{x}_i = (x'_i, x_i) \quad (i = 1, \dots, N), \quad (y', y) = (x'_1 + \dots + x'_N, x_1 + \dots + x_N).$$

*Suppose that  $y' \notin \mathbb{Z}$ . Then we have*

$$T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{2^{N-1}}{(N-1)!} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \times S_N^\tau(m; \mathbf{x}_1, \dots, \mathbf{x}_N) E_{n+N-1-m}(y', y; \tau). \quad (33)$$

*Proof.* It follows from (12) that

$$\begin{aligned} & \frac{(N-1)!}{(2\pi i)^{N-1}} \prod_{i=1}^N \left( -2e\left(-\frac{x_i}{2}\right) \underline{F}(x'_i, x_i; \xi + \frac{1}{2}; \tau) \right) = \\ & 2^{N-1} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m S_N^\tau(m; \mathbf{x}_1, \dots, \mathbf{x}_N) \left( -2e\left(-\frac{y}{2}\right) \underline{F}^{(N-1-m)}(y', y; \xi + \frac{1}{2}; \tau) \right). \end{aligned} \quad (34)$$

It is seen from (7) and (28) that

$$\begin{aligned} -2e\left(-\frac{y}{2}\right) \underline{F}^{(m)}(y', y; \xi + \frac{1}{2}; \tau) &= \frac{1}{(2\pi i)^m} \left( \frac{\partial}{\partial \xi} \right)^m \left( -2e\left(-\frac{y}{2}\right) \underline{F}(y', y; \xi + \frac{1}{2}; \tau) \right) \\ &= \sum_{n=0}^{\infty} \frac{E_{n+m}(y', y; \tau)}{n!} (2\pi i)^{n+1} \xi^n. \end{aligned}$$

Comparing the coefficient of  $\xi^n$  in (34), one gets (33).  $\square$

In complete analogy to the method of Section 2 one can reproduce Dilcher's formulas for sums of products of Euler polynomials and special values of the zeta function  $\lambda(s)$ :

$$\lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}.$$

We denote

$$T_N(n; x_1, \dots, x_N) := \sum_{\substack{j_1, \dots, j_N \geq 0 \\ (j_1 + \dots + j_N = n)}} \binom{n}{j_1, \dots, j_N} E_{j_1}(x_1) \cdots E_{j_N}(x_N),$$

$$\tilde{T}_N(n) := \sum_{\substack{j_1, \dots, j_N \geq 1 \\ (j_1 + \dots + j_N = n)}} \binom{n}{j_1, \dots, j_N} E_{j_1}(0) \cdots E_{j_N}(0).$$

We remark that the second summation is extended over all positive integers  $j_1, \dots, j_N$  with  $j_1 + \dots + j_N = n$ , namely  $T_N(n; 0, \dots, 0) \neq \tilde{T}_N(n)$  in general.  $\tilde{T}_N(n)$  is the same as that in the proof of [Di, Theorem 7].

**LEMMA 3.1.**

(i) Let  $x_1, \dots, x_N$  be real numbers and  $x'_1, \dots, x'_N$  complex numbers with  $x'_1, \dots, x'_N \notin \mathbb{Z}$ . Set  $\mathbf{x}_i = (x'_i, x_i)$  for  $i = 1, \dots, N$ . If  $0 \leq x_1, \dots, x_N < 1$ , then we have

$$\lim_{x' \rightarrow -i\infty} \lim_{\tau \rightarrow i\infty} T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) = T_N(n; x_1, \dots, x_N). \quad (35)$$

(ii) Set  $\mathbf{x}_i = (1/2, 0)$  for  $i = 1, \dots, N$ . Then we have

$$\lim_{\tau \rightarrow i\infty} T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) = \tilde{T}_N(n). \quad (36)$$

(iii) Set  $\mathbf{x}_i = (1/2, 0)$  for  $i = 1, \dots, N-1$  and  $\mathbf{x}_N = (x'_N, 0)$ . Then we have

$$\left[ \text{the coefficient of } x_n^0 (= 1) \text{ of } \lim_{\tau \rightarrow i\infty} T_N^\tau(n; \mathbf{x}_1, \dots, \mathbf{x}_N) \right] = \tilde{T}_N(n). \quad (37)$$

*Proof.* We can prove this lemma as the same method of Lemma 2.2, so omit the proof.  $\square$

Dilcher gave the following two formulas:

**THEOREM 3.2** (Sums of products of Euler polynomials). *Let  $x_1, \dots, x_N, y$  be complex numbers with  $y = x_1 + \dots + x_N$ . Then for  $n \geq N$  we have*

$$T_N(n; x_1, \dots, x_N) = \frac{2^{N-1}}{(N-1)!} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \times B_m^{(N)}(y) E_{n+N-1-m}(y). \quad (38)$$

*Proof.* It is sufficient to show (38) when  $0 \leq x_1, \dots, x_N, y < 1$  by analyticity of  $E_m(x)$ . This is derived from (33) and (35).  $\square$

**REMARK 3.2.** The above theorem corresponds to [Di, Lemma 5]. One can easily deduce [Di, Theorem 5] from the theorem.

**THEOREM 3.3** (Sums of products of  $\lambda(2m)$ ). *For  $n \geq N$  we have*

$$\sum_{\substack{j_1 + \dots + j_N \geq 1 \\ (j_1 + \dots + j_N = n)}} \lambda(2j_1) \cdots \lambda(2j_N) = \frac{2^{1-N}}{(2n-N)!(N-1)!} \times \sum_{m=0}^{[(N-1)/2]} \binom{N-1}{2m} (-1)^m \pi^{2m} (2n-2m-1)! S_N(2m) \lambda(2n-2m). \quad (39)$$

*Proof.* By [Di, (4.16)] we have

$$E_{2m-1}(0) = (-1)^m \frac{2^{2(2m-1)}!}{\pi^{2m}} \lambda(2m). \quad (40)$$

In a similarly way of the proof of Theorem 2.3, one can derive

$$\tilde{T}_N(n) = \frac{2^{N-1}}{(N-1)!} \sum_{m=0}^{[(N-1)/2]} \binom{N-1}{2m} S_N(2m) E_{n+N-1-2m}(0) \quad (41)$$

from Lemma 3.1. One can also see from (40) that, for  $n \geq N$ ,

$$\tilde{T}_N(n) = (-1)^{\frac{n+N}{2}} \frac{n! 2^{2N}}{\pi^{n+N}} \sum_{\substack{j_1 + \dots + j_N \geq 1 \\ (2j_1 + \dots + 2j_N = n+N)}} \lambda(2j_1) \cdots \lambda(2j_N),$$

namely

$$\tilde{T}_N(2n-N) = (-1)^n \frac{(2n-N)! 2^{2N}}{\pi^{2n}} \sum_{\substack{j_1 + \dots + j_N \geq 1 \\ (j_1 + \dots + j_N = n)}} \lambda(2j_1) \cdots \lambda(2j_N). \quad (42)$$

Eqs.(41) and (42) induce (39).  $\square$

**REMARK 3.3.** By virtue of [Di, Theorem 2.(b)] and [Di, Lemma 3] , we obtain

$$S(2n) = (N - 2n - 1)!(2n)!N \sum_{i=0}^{2n} \binom{N-1}{i} \frac{s(N-i, N-2n)}{2^i(N-i)!},$$

where  $s(n, k)$  denotes the Stirling numbers of the first kind. By using this and Theorem 3.3, we can get (4.18) in [Di, Theorem 7].

The Euler numbers are defined by means of the following generating functions:

$$\frac{2}{e^\xi + e^{-\xi}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \xi^n.$$

Thus the generating function is an even function. So it follows from (26) that

$$E_{2m}\left(\frac{1}{2}\right) = 2^{-2m} E_{2m}, \quad E_{2m+1} = 0. \quad (43)$$

Finally we produce formulas for Euler numbers which are slightly different from [Di, (4.9)]. The difference is the Euler functions  $\tilde{E}_{2n+N-1-2m}(N/2)$  as in Theorem 2.4

**THEOREM 3.4** (Sums of products of Euler numbers). *For  $2n \geq N$  we have*

$$\begin{aligned} & \sum_{\substack{j_1+\dots+j_N \geq 0 \\ (j_1+\dots+j_N=n)}} \binom{2n}{2j_1, \dots, 2j_N} E_{j_1} \cdots E_{j_N} \\ &= \frac{2^{2n+N-1}}{(N-1)!} \sum_{m=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1}{2m} B_{2m}^{(N)}\left(\frac{N}{2}\right) \tilde{E}_{2n+N-1-2m}\left(\frac{N}{2}\right). \end{aligned} \quad (44)$$

*Proof.* Set  $\mathbf{x}_i = (x'_i, 1/2)$  for any  $i = 1, \dots, N$ . By virtue of (30), (33) and (35), one gets

$$\begin{aligned} T_N(2n; \frac{1}{2}, \dots, \frac{1}{2}) &= \frac{2^{N-1}}{(N-1)!} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \\ &\quad \times B_m^{(N)}\left(\frac{N}{2}\right) \tilde{E}_{2n+N-1-m}\left(\frac{N}{2}\right). \end{aligned} \quad (45)$$

This together with (43) induces (44).  $\square$



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