A singular perturbation problem with integral curvature bound

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Abstract

We consider a singular perturbation problem of Modica-Mortola functional as the thickness of diffused interface approaches to zero. We assume that sequence of functions have uniform energy and square-integral curvature bounds in two dimension. We show that the limit measure concentrate on one rectifiable set and has square integrable curvature.

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1 Introduction

The Modica-Mortola functional [12] has been used widely as an approximation of hypersurface area of diffused interface, both in static and time-dependent models and the functional often being coupled with other interacting fields. After a suitable normalization it is defined for scalar-valued function $u : U \subset \mathbf{R}^n \to \mathbf{R}$ by

$$E_\varepsilon(u) = \int_U \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\, dx, \tag{1.1}$$

where $W : \mathbf{R} \to [0, \infty)$ is a double-well potential with two equal minima at ± 1 and $\varepsilon > 0$ is a small parameter. In mathematical literature some of the first rigorous results are given by Modica [11], Sternberg [18] and others who proved that E_ε Γ-converges to the area functional as $\varepsilon \to 0$. Namely,

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consider a sequence of minimizers $\{u_\varepsilon\}$ of $E_\varepsilon(\cdot)$, $\varepsilon \to 0$, among functions with $\int_U u\,dx = m$ fixed. Here $-|U| < m < |U|$ and $|U|$ is the n-dimensional volume of U. One expects that u_ε is close to ± 1 for the most part of U and that it is advantageous to have as little transition region as possible. It is also straightforward to see that the transition region should have the thickness of order ε for E_ε to be of constant order with respect to ε. The aforementioned works show that there exist a converging subsequence and the limit u_0 such that $u_0 = \pm 1$ a.e. on U and u_0 minimizes the hypersurface area of $U \cap \partial\{u_0 = 1\}$ among such functions with equal integral value m. Such area-minimizing hypersurfaces are known to be smooth constant mean curvature hypersurfaces (CMC) if the dimension n of domain U is less than 8 and CMC with possible small singularities for $n \geq 8$ [7, 17]. The functional E_ε approximates the hypersurface area in the sense that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = 2\sigma \mathcal{H}^{n-1}(U \cap \partial\{u_0 = 1\}),$$

where

$$\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\,ds$$

and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. It is also proved [10] that the limit of Lagrange multipliers

$$\lambda_\varepsilon = -\varepsilon \Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon}$$

has the geometric meaning in that

$$\sigma H = \lambda_0,$$

where $\lambda_0 = \lim_{\varepsilon\to 0} \lambda_\varepsilon$ and H is the constant mean curvature of $U \cap \partial\{u_0 = 1\}$. It is of interest to study the limiting behavior of E_ε without the energy-minimizing properties in view of applications to various dynamical problems. In [8, 19, 20] motivated by the Cahn-Hilliard equation [4] they gave a geometric characterization of limit interfaces without minimizing property but with $W^{1,p}$ Sobolev norm control of

$$f_\varepsilon = -\varepsilon \Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon}, \tag{1.2}$$

where $p > \frac{n}{2}$, which corresponds to the chemical potential field in the framework of van der Waals-Cahn-Hilliard theory of phase transitions. The control of such quantity may be seen as an analogue of control of mean curvature field in view of above result by Luckhaus and Modica [10] and also Schätzle [16].

Recently there have been much interest in studying limit interface when we have a control of

$$\frac{1}{\varepsilon}\int_U |f_\varepsilon|^2\,dx, \tag{1.3}$$

in dimensions $n \le 3$ as $\varepsilon \to 0$ [3, 9, 14, 5]. If one makes the ansatz that the internal layer profile is the usual hyperbolic tangent shape, it is reasonable to relate this quantity to the L^2 norm of the mean curvature of interface. In general one expects as $\varepsilon \to 0$ that the limit interface should have L^2 mean curvature and that appropriately defined limit of f_ε should correspond to the mean curvature. For this problem Moser [14] showed for dimension $n \le 3$ with some technical monotonicity assumption that the limit interface is a rectifiable varifold [1] with L^2 mean curvature. Bellettini and Mugnai [3] considered the problem with radial symmetry assumption and showed that the quantity (1.3) converges to the L^2 norm of mean curvature for the limit interface as $\varepsilon \to 0$.

In this paper we extend the results of [3, 14] in that we make no assumptions on the sequence of functions $\{u_\varepsilon\}$ except for the uniform bounds on the energy (1.1) and L^2 norm of the chemical potential in the form (1.3), and conclude essentially the same results as in [3, 14] for the limit interface. Unfortunately we can prove the result only for $n = 2$. Here we state our main theorem. A few minor assumptions are made on the function W (see Sect. 2.1).

Theorem 1.1. *Suppose $U \subset \mathbf{R}^2$ is a bounded domain. Suppose a sequence of functions $\{u_{\varepsilon_i}\}_{i=1}^\infty \subset W^{2,2}(U)$ satisfies for $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\varepsilon_i = 0$*

$$\liminf_{i\to\infty} E_{\varepsilon_i}(u_{\varepsilon_i}) < \infty, \qquad \liminf_{i\to\infty} \frac{1}{\varepsilon_i}\int_U |f_{\varepsilon_i}|^2\,dx < \infty. \tag{1.4}$$

Define a sequence of Radon measures on U by

$$\mu_{\varepsilon_i}(\phi) = \int_U \phi\left(\frac{\varepsilon_i|\nabla u_{\varepsilon_i}|^2}{2} + \frac{W(u_{\varepsilon_i})}{\varepsilon_i}\right)dx$$

for $\phi \in C_c(U)$. By the weak star compactness of bounded measures there exists a subsequence (denoted by the same indices) $\{\mu_{\varepsilon_i}\}_{i=1}^\infty$ and the limit Radon measure μ on U. Then,

(i) $u_{\varepsilon_i} \to \pm 1$ locally uniformly on $U \setminus \mathrm{supp}\mu$.

(ii) *There exist a closed countably 1-rectifiable set* Σ *and* $\mathcal{H}^1$ *measurable function* θ *defined on* Σ *such that* $\mu = \theta \mathcal{H}^1 \lfloor_{\Sigma}$.

(iii) $\theta/(2\sigma)$ *is* $\mathcal{H}^1$ *a.e. integer-valued on* Σ.

(iv) *The generalized curvature* f *of* μ *satisfies*

$$\int_U |f|^2 \, d\mu \le \liminf_{i\to\infty} \frac{1}{\varepsilon_i} \int_U |f_{\varepsilon_i}|^2 \, dx.$$

For the definition of *rectifiable set* and *generalized (mean) curvature* see [1, 17]. The function f can be obtained as follows. Define any vector-valued limit measure of $\{f_{\varepsilon_i} \nabla u_{\varepsilon_i} \, dx\}_{i=1}^{\infty}$ as η. Note that the L^1 norms are uniformly bounded by

$$\int_U |f_{\varepsilon_i} \nabla u_{\varepsilon_i}| \, dx \le \left(\frac{1}{\varepsilon_i} \int_U |f_{\varepsilon_i}|^2 \, dx \right)^{1/2} \left(\varepsilon_i \int_U |\nabla u_{\varepsilon_i}|^2 \, dx \right)^{1/2}$$

and (1.4). Since η is absolutely continuous with respect to μ we define f as the Radon-Nikodym derivative $\frac{d\eta}{d\mu}$. We show f is indeed the generalized curvature of μ with property (iv).

Though it appears to us that it is not stated explicitly in the literature, any 1-dimensional integral varifold [1] with L^p $(p > 1)$ generalized curvature should have support consisting of finite number of $C^{1,1-\frac{1}{p}}$ curves possibly meeting at isolated junction points. The proof should follow more or less from stationary case studied by Allard and Almgren [2], where they proved that stationary integral varifold has support which are fininte number of lines with possible junction points.

The main point of the paper is to establish a properly scaled monotonicity formula for the energy density, which was also essential in [8, 19, 20]. There we assumed the Sobolev $W^{1,p}$ norm for some $p > \frac{n}{2}$ is bounded:

$$\liminf_{i\to\infty} ||f_{\varepsilon_i}||_{W^{1,p}(U)} < \infty.$$

Though we do not have any control of derivatives of f_{ε_i} in this paper, we find that we may still use many of the estimates in [8, 19, 20] if we regularize u_ε appropriately. More specifically, we consider the convolution of u_ε, $u_\varepsilon * \psi_{\varepsilon^{1+\beta}}$, where $\psi_{\varepsilon^{1+\beta}}$ is the usual mollifier scaled by $\varepsilon^{1+\beta}$, for a carefully chosen $\beta > 0$. The function still satisfies a similar equation while nonlinear term produces error terms. The regularization gives some control of derivatives of $f_\varepsilon * \psi_{\varepsilon^{1+\beta}}$, to which we apply estimates for the so-called discrepancy measure

$$\xi_\varepsilon(u_\varepsilon) = \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon} \tag{1.5}$$

obtained in [19].

After the main part of the paper is completed we were informed that Röger and Schätzle [15] obtained the similar results for $n \le 3$ using different estimates for the discrepancy measure. Since our method is different from theirs we believe that it should have an independent interest.

2 Assumptions and preliminaries

In the following we set up the assumptions, recall various definitions and the rectifiablity theorem due to Moser [13] which we use later in Section 4.

2.1 Assumptions

We assume that the double-well potential $W : \mathbf{R} \to [0,\infty)$ is a C^3 function satisfying the following assumptions;

(i) $W(1) = W(-1) = 0$,

(ii) there exists $\gamma \in (-1,1)$ such that $W' < 0$ on $(\gamma,1)$ and $W' > 0$ on $(-1,\gamma)$,

(iii) there exist $\alpha \in (0,1)$ and $\kappa > 0$ such that $W''(s) \ge \kappa$ for all $|s| \ge \alpha$.

Under the assumption (1.4) we may assume that there exist constants E_0 and a_1 such that

(A.1) $E_{\varepsilon_i}(u_{\varepsilon_i}) \le E_0$,

(A.2) $\frac{1}{\varepsilon_i}\int_U |f_{\varepsilon_i}|^2\, dx \le a_1$

for all $i = 1, 2, \cdots$.

By defining $\widetilde{u}_{\varepsilon_i}(x) = u_{\varepsilon_i}(\varepsilon_i x)$ and $\widetilde{f}_{\varepsilon_i}(x) = f_{\varepsilon_i}(\varepsilon_i x)$, (1.2), $(A.1)$ and $(A.2)$ are equivalent to

$$\varepsilon_i \widetilde{f}_{\varepsilon_i} = -\Delta \widetilde{u}_{\varepsilon_i} + W'(\widetilde{u}_{\varepsilon_i}) \tag{2.1}$$

and

$(\widetilde{A}.1)$ $\int_{U/\varepsilon_i} \frac{|\nabla \widetilde{u}_{\varepsilon_i}|^2}{2} + W(\widetilde{u}_{\varepsilon_i})\, dx \le \varepsilon_i^{-1} E_0$,

$(\widetilde{A}.2)$ $\int_{U/\varepsilon_i} |\widetilde{f}_{\varepsilon_i}|^2\, dx \le \varepsilon_i^{-1} a_1$.

Throughout this paper, different positive constants will be denoted by the same letter c. We write $c(s)$ when it is helpful to write out the dependence of c on s.

2.2 The generalized L^2 curvature functional

We find that it is convenient to work in the framework set out by Moser [14]. In the following we only need results for $n = 2$.

Let $\mathcal{F}$ be the set of all symmetric, positive semidefinite real $(n \times n)$-matrices. We write $\mathcal{M}(U)$ for the set of all pairs $M = (\mu, \nu)$ such that

(1) μ is Radon measure on U,

(2) ν is Radon measure on U with values in $\mathcal{F}$ and

(3) there exists a function $\Phi \in L^\infty(\mu, \mathcal{F})$ such that $\nu = \mu\lfloor_\Phi$.

In the following we set $M_{\varepsilon_i} = (\mu_{\varepsilon_i}, \nu_{\varepsilon_i})$ with

$$\mu_{\varepsilon_i} = \left(\frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} + \frac{W(u_{\varepsilon_i})}{\varepsilon_i} \right) dx, \qquad \nu_{\varepsilon_i}^{\alpha\beta} = \varepsilon_i \left(\frac{\partial u_{\varepsilon_i}}{\partial x^\alpha} \frac{\partial u_{\varepsilon_i}}{\partial x^\beta} \right) dx \tag{2.2}$$

and

$$\Phi_{\varepsilon_i}^{\alpha\beta} = \frac{\varepsilon_i \frac{\partial u_{\varepsilon_i}}{\partial x^\alpha} \frac{\partial u_{\varepsilon_i}}{\partial x^\beta}}{\frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} + \frac{W(u_{\varepsilon_i})}{\varepsilon_i}} \in L^\infty(\mu_{\varepsilon_i}, \mathcal{F}) \tag{2.3}$$

so that $\nu_{\varepsilon_i} = \mu_{\varepsilon_i}\lfloor_{\Phi_{\varepsilon_i}}$. Continuing with general framework, for $M = (\mu, \nu) \in \mathcal{M}(U)$ define the linear functional

$$\delta M(\phi) = \int_U (\text{div}\phi \; d\mu - \frac{\partial \phi^\alpha}{\partial x^\beta} \; d\nu^{\alpha\beta}) \tag{2.4}$$

on $C_c^1(U; \mathbf{R}^n)$. The usual summation convention is assumed. The functional δM is an analogue of the usual first variation [1] and it was introduced by Moser [13]. Using (1.2) and integration by parts, one verifies that

$$\delta M_{\varepsilon_i}(\phi) = - \int_U \phi^\alpha H_{\varepsilon_i}^\beta \; d\nu_{\varepsilon_i}^{\alpha\beta}, \tag{2.5}$$

where

$$H_{\varepsilon_i}^\beta = \frac{f_{\varepsilon_i}}{\varepsilon_i \frac{\partial u_{\varepsilon_i}}{\partial x^\beta}}. \tag{2.6}$$

Now define the *generalized L^2 curvature functional* $\mathcal{C}$ as the functional on $\mathcal{M}(U)$ by

$$\mathcal{C}(M) = \sup \left\{ (\delta M(\phi))^2 \;\middle|\; \phi \in C_c^1(U; \mathbf{R}^n), \; \int_U \phi^\alpha \phi^\beta \; d\nu^{\alpha\beta} \leq 1 \right\}. \tag{2.7}$$

$\mathcal{C}(M)$ corresponds to the usual L^2 norm square of mean curvature when $M = (\mu, \nu)$ is a pair of smooth objects, namely, μ is an $(n-1)$-Hausdorff measure restricted to a smooth $(n-1)$-dimensional submanifold Σ and $\Phi(x) = p(x) \otimes p(x)$, where $p(x)$ is the unit normal to the tangent space $T_x\Sigma$ at x. By (2.5)

$$(\delta M_{\varepsilon_i}(\phi))^2 \leq \int_U H^\alpha_{\varepsilon_i} H^\beta_{\varepsilon_i} \, d\nu^{\alpha\beta}_{\varepsilon_i} \int_U \phi^\alpha \phi^\beta \, d\nu^{\alpha\beta}_{\varepsilon_i}$$

so that (2.6), (2.7) and (A.2) show

$$\mathcal{C}(M_{\varepsilon_i}) \leq \int_U H^\alpha_{\varepsilon_i} H^\beta_{\varepsilon_i} d\nu^{\alpha\beta}_{\varepsilon_i} = \frac{1}{\varepsilon_i} \int_U |f_{\varepsilon_i}|^2 \, dx \leq a_1. \tag{2.8}$$

Write

$$\mathfrak{C}(U) = \{M \in \mathcal{M}(U) \,|\, \mathcal{C}(M) < \infty\}.$$

Important subclass of $\mathfrak{C}(U)$ we need is

$$\mathfrak{C}_1(U) = \{M \in \mathfrak{C}(U) \,|\, \mathrm{trace}\, \nu \leq \mu\},$$

which has the following rectifiability property:

Proposition 2.1. ([14, Propostion 2.2]) *If $M = (\mu, \nu) \in \mathfrak{C}_1(U)$, then the set*

$$\Sigma = \{x \in U : \theta(x) > 0\}$$

is 1-rectifiable. Moreover,

$$\mu = \theta \mathcal{H}^1 \lfloor_\Sigma.$$

If Φ is such that $\nu = \mu \lfloor_\Phi$, then $\Phi(x) = \mathrm{proj}^\perp_{T_x\Sigma}(x)$ for μ-almost every $x \in \Sigma$.

Here $\theta(x)$ is the 1-dimensional density of μ:

$$\theta(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{2r},$$

where $B_r(x) = \{y \in \mathbf{R}^2; |y - x| < r\}$.

The main task in this paper is to show that $\mathrm{trace}\, \nu \leq \mu$ holds for the limit measure pair (μ, ν) of $\{M_{\varepsilon_i}\}_{i=1}^\infty$ and $\theta(x) \geq c > 0$ uniformly on $\mathrm{supp}\mu$, which follows once we establish the monotonicity formula in Section 3.

3 Monotonicity formula

The main task of this section is to prove Theorem 3.10 which gives the uniform lower bound of the scaled energy. It is the crucial ingredient for the application of the rectifiability theorem, Proposition 2.1. In the following we drop the index i for simplicity.

Lemma 3.1. ([14, Lemma 2.2]) *Define μ_ε, ν_ε and $\xi_\varepsilon(u_\varepsilon)$ as in (2.2) and (1.5). For any $\delta > 0$ and for $B_s(x_0) \subset B_r(x_0) \subset U$,*

$$\begin{aligned}(1+\delta)\frac{1}{r}\mu_\varepsilon(B_r(x_0)) - (1-\delta)\frac{1}{s}\mu_\varepsilon(B_s(x_0))& \\ \geq -(\frac{2}{\delta}+1)a_1 r - \int_s^r \frac{1}{\rho^2}\int_{B_\rho(x_0)} \xi_\varepsilon(u_\varepsilon)\, dx d\rho.&\end{aligned} \tag{3.1}$$

Proof. We may assume $x_0 = 0$ by a suitable translation. Write $B_r = B_r(0)$. By (2.4) and (2.5), for any $\phi \in C_c^1(U; \mathbf{R}^2)$,

$$\int \left(\mathrm{div}\phi\, d\mu_\varepsilon - \frac{\partial \phi^\alpha}{\partial x^\beta}\, d\nu_\varepsilon^{\alpha\beta}\right) + \int \phi^\alpha H_\varepsilon^\beta\, d\nu_\varepsilon^{\alpha\beta} = 0. \tag{3.2}$$

Suppose $h \in C_c^\infty(\mathbf{R})$ satisfies $h(s) = 0$ for $s \in [1,\infty)$. By substituting $\phi(x) = xh(\frac{|x|}{\rho})$ into (3.2), we obtain

$$\begin{aligned}2\int h\, d\mu_\varepsilon - \int h \text{ trace } d\nu_\varepsilon + \frac{1}{\rho}\int |x| h'\, d\mu_\varepsilon& \\ - \int \left(\frac{x^\alpha x^\beta}{\rho|x|}h' - h x^\alpha H_\varepsilon^\beta\right)\, d\nu_\varepsilon^{\alpha\beta} = 0.&\end{aligned} \tag{3.3}$$

By the definitions of μ_ε and $\nu_\varepsilon^{\alpha\beta}$ we obtain

$$\begin{aligned}&2\int h\, d\mu_\varepsilon - \int h\text{trace } d\nu_\varepsilon \\ &= \int h\, d\mu_\varepsilon + \int h\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\right)\, dx - \int h\varepsilon|\nabla u_\varepsilon|^2\, dx \\ &= \int h\, d\mu_\varepsilon - \int \xi_\varepsilon(u_\varepsilon) h\, dx.\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and multiplying both sides by $-\frac{1}{\rho^2}$, we obtain

$$\begin{aligned}&\frac{d}{d\rho}\left(\frac{1}{\rho}\int h\, d\mu_\varepsilon + \frac{1}{\rho}\int h x^\alpha H_\varepsilon^\beta\, d\nu_\varepsilon^{\alpha\beta}\right) \\ &\quad = -\frac{1}{\rho^3}\int \left(h'\frac{x^\alpha x^\beta}{|x|} + |x| h' x^\alpha H_\varepsilon^\beta\right)\, d\nu_\varepsilon^{\alpha\beta} - \frac{1}{\rho^2}\int h\xi_\varepsilon(u_\varepsilon)\, dx.\end{aligned} \tag{3.5}$$

Integrating over (s,r), we obtain

$$\begin{aligned}
&\frac{1}{r}\int h\,d\mu_\varepsilon + \frac{1}{r}\int hx^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta} - \frac{1}{s}\int h\,d\mu_\varepsilon - \frac{1}{s}\int hx^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta} \\
&\quad = \int\left(\int_s^r \frac{1}{\rho}\frac{d}{d\rho}h\,d\rho\right)\left(\frac{x^\alpha x^\beta}{|x|^2} + x^\alpha H_\varepsilon^\beta\right)d\nu_\varepsilon^{\alpha\beta} - \int_s^r \frac{1}{\rho^2}\int h\xi_\varepsilon(u_\varepsilon)\,dxd\rho.
\end{aligned} \tag{3.6}$$

Let $\{h_k\}_{k=1}^\infty \subset C_c^\infty(\mathbf{R})$ be a sequence of approximate functions for the characteristic function of $(-\infty,1)$ and use $h = h_k$ in (3.6). Since $\int_s^r \frac{1}{\rho}\frac{d}{d\rho}h_k(\frac{|x|}{\rho})d\rho$ converges to $\frac{1}{|x|}$ for $|x| \in [s,r)$ and otherwise 0, we obtain

$$\begin{aligned}
&\frac{1}{r}\mu_\varepsilon(B_r) + \frac{1}{r}\int_{B_r} x^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta} - \frac{1}{s}\mu_\varepsilon(B_s) - \frac{1}{s}\int_{B_s} x^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta} \\
&= \int_{B_r\setminus B_s}\left(\frac{x^\alpha x^\beta}{|x|^3} + \frac{x^\alpha H_\varepsilon^\beta}{|x|}\right)d\nu_\varepsilon^{\alpha\beta} - \int_s^r \frac{1}{\rho^2}\int_{B_\rho}\xi_\varepsilon(u_\varepsilon)\,dxd\rho.
\end{aligned} \tag{3.7}$$

By $\Phi_\varepsilon^{\alpha\beta}\mu_\varepsilon = \nu_\varepsilon^{\alpha\beta}$ and (2.8), we obtain for any $\delta > 0$

$$\begin{aligned}
\left|\frac{1}{\rho}\int_{B_\rho} x^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta}\right| &\le \frac{1}{\rho}\left(\int_{B_\rho} x^\alpha x^\beta \Phi^{\alpha\beta}\,d\mu_\varepsilon\right)^{\frac{1}{2}}\left(\int_{B_\rho} H_\varepsilon^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta}\right)^{\frac{1}{2}} \\
&\le \frac{1}{\rho}\left(\int_{B_\rho}\varepsilon|x|^2|\nabla u_\varepsilon|^2\,dx\right)^{\frac{1}{2}} a_1^{\frac{1}{2}} \\
&\le \frac{\delta}{\rho}\mu_\varepsilon(B_\rho) + \frac{a_1\rho}{\delta}
\end{aligned} \tag{3.8}$$

and

$$\left|\int_{B_r\setminus B_s}\frac{x^\alpha H_\varepsilon^\beta}{|x|}\,d\nu_\varepsilon^{\alpha\beta}\right| \le \int_{B_r\setminus B_s}\frac{x^\alpha x^\beta}{|x|^3}\,d\nu_\varepsilon^{\alpha\beta} + r\int_{B_r\setminus B_s} H_\varepsilon^\alpha H_\varepsilon^\beta\,d\nu_\varepsilon^{\alpha\beta}. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7), we proved (3.1). □

Later we use the following L^∞ estimates of u_ε away from 1.

Lemma 3.2. *For each open set $V \subset\subset U$, there exist constants c_1 depending only on W, $\mathrm{dist}(\partial U, V)$, κ, E_0 and a_1 such that*

$$\sup_V |u_\varepsilon| \le 1 + c_1\varepsilon^{\frac{1}{2}} \tag{3.10}$$

for $\varepsilon \le 1$.

Proof. Let $x \in V$ be arbitrary and set $r = \frac{1}{2}\text{dist}(\partial U, x)$. Write $B_r = B_r(x)$. First we show

$$\|(u_\varepsilon - 1)_+\|_{L^2(B_{\frac{r}{2}})} \leq c\varepsilon^{\frac{3}{2}}, \tag{3.11}$$

where $(u_\varepsilon - 1)_+ = \max\{(u_\varepsilon - 1), 0\}$. Let $\phi \in C_c^\infty(B_r)$ be a smooth function with $\phi = 1$ on $B_{\frac{r}{2}}$. Multiplying (1.2) by $(u_\varepsilon - 1)_+\phi^2$ and by integration by parts we obtain

$$\begin{aligned}\int_{\{u_\varepsilon>1\}} &\varepsilon|\nabla u_\varepsilon|^2\phi^2 + 2\varepsilon(u_\varepsilon - 1)_+\phi\nabla u_\varepsilon \cdot \nabla\phi \\ &+ \frac{W'(u_\varepsilon)}{\varepsilon}(u_\varepsilon - 1)_+\phi^2\, dx = \int_{\{u_\varepsilon>1\}} f_\varepsilon(u_\varepsilon - 1)_+\phi^2\, dx.\end{aligned} \tag{3.12}$$

By applying Cauchy's inequality to (3.12), for any $\delta > 0$

$$\begin{aligned}&\int_{\{u_\varepsilon>1\}} \varepsilon|\nabla u_\varepsilon|^2\phi^2 + \frac{W'(u_\varepsilon)}{\varepsilon}(u_\varepsilon - 1)_+\phi^2\, dx \\ &\leq \int_{\{u_\varepsilon>1\}} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\phi^2 + 2\varepsilon[(u_\varepsilon - 1)_+]^2|\nabla\phi|^2 + \frac{\varepsilon}{4\delta}f_\varepsilon^2\phi^2 + \frac{\delta}{\varepsilon}[(u_\varepsilon - 1)_+]^2\phi^2\, dx.\end{aligned} \tag{3.13}$$

By the assumption on W,

$$W'(u_\varepsilon) \geq \kappa(u_\varepsilon - 1)_+ \tag{3.14}$$

and

$$\begin{aligned}\int_{\{u_\varepsilon>1\}} 2\varepsilon[(u_\varepsilon - 1)_+]^2|\nabla\phi|^2\, dx &\leq c(\kappa)(\sup|\nabla\phi|^2)\varepsilon\int_{B_r\cap\{u_\varepsilon>1\}} W(u_\varepsilon)\, dx \\ &\leq c(\kappa, r)E_0\varepsilon^2.\end{aligned} \tag{3.15}$$

By (3.13-15) and $(A.2)$,

$$\frac{\varepsilon}{2}\int_{\{u_\varepsilon>1\}} \phi^2|\nabla u_\varepsilon|^2\, dx + (\frac{\kappa}{\varepsilon} - \frac{\delta}{\varepsilon})\int_{\{u_\varepsilon>1\}} \phi^2[(u_\varepsilon - 1)_+]^2\, dx \leq \left\{cE_0 + \frac{a_1}{4\delta}\right\}\varepsilon^2.$$

Since δ is arbitrary, we choose $\delta = \frac{\kappa}{2}$. Thus we have

$$\int_{B_{\frac{r}{2}}} [(u_\varepsilon - 1)_+]^2\, dx \leq c(a_1, \kappa, E_0, r)\varepsilon^3. \tag{3.16}$$

With a suitable choice of constant, (3.16) shows (3.11). Next we consider $\widetilde{u}(x) = u(\varepsilon x)$ and $\widetilde{f}(x) = f(\varepsilon x)$ which satisfy (2.1). Set $h(s) = s_+$ and let $\{h_k\}_{k=1}^{\infty} \subset C^3(\mathbf{R})$ be a sequence of approximate functions for h with $h_k(s) = 0$ for $s \in (-\infty, 0]$ and $h_k' \geq 0$, $h_k'' \geq 0$. Consider the functions $h_k \circ (\widetilde{u}_\varepsilon(x) - 1)$. Using (2.1),

$$\begin{aligned}\Delta h_k \circ (\widetilde{u}_\varepsilon - 1) &= h_k'' \circ (\widetilde{u}_\varepsilon - 1)|\nabla \widetilde{u}_\varepsilon|^2 + h_k' \circ (\widetilde{u}_\varepsilon - 1)\Delta \widetilde{u}_\varepsilon \geq h_k' \circ (\widetilde{u}_\varepsilon - 1)\Delta \widetilde{u}_\varepsilon \\ &\geq h_k' \circ (\widetilde{u}_\varepsilon - 1)(W'(\widetilde{u}_\varepsilon) - \varepsilon \widetilde{f}_\varepsilon) \geq -h_k' \circ (\widetilde{u}_\varepsilon - 1)\varepsilon \widetilde{f}_\varepsilon.\end{aligned} \tag{3.17}$$

By the standard elliptic estimate (cf.[6, Theorem 8.17.]) applied to (3.17), we obtain for any $B_1 \subset U/\varepsilon$

$$\sup_{B_{\frac{1}{2}}} h_k \circ (\widetilde{u}_\varepsilon - 1) \leq c(\|h_k \circ (\widetilde{u}_\varepsilon - 1)\|_{L^2(B_1)} + \|h_k' \circ (\widetilde{u}_\varepsilon - 1)\varepsilon \widetilde{f}_\varepsilon\|_{L^2(B_1)}).$$

By taking the limit $k \to \infty$ we obtain

$$\sup_{B_{\frac{1}{2}}} (\widetilde{u}_\varepsilon - 1)_+ \leq c(\|(\widetilde{u}_\varepsilon - 1)_+\|_{L^2(B_1)} + \|\varepsilon \widetilde{f}_\varepsilon\|_{L^2(B_1)}).$$

We have $\|(\widetilde{u}_\varepsilon - 1)_+\|_{L^2(B_1)} \leq c\varepsilon^{\frac{1}{2}}$ from (3.11) and $\|\varepsilon \widetilde{f}_\varepsilon\|_{L^2(B_1)} \leq a_1^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}$ by $(\widetilde{A}.2)$. Thus

$$\sup_{B_{\frac{1}{2}}} (\widetilde{u}_\varepsilon - 1)_+ \leq c\varepsilon^{\frac{1}{2}}.$$

With a suitable choice of c_1 we have

$$\sup_{V} (u_\varepsilon - 1)_+ \leq c_1 \varepsilon^{\frac{1}{2}}.$$

Repeating the same arguement, we obtain $\sup_V |(u + 1)_-| \leq c_1 \varepsilon^{\frac{1}{2}}$, which proves (3.10). □

Remark 3.3. *In the following we often use the fact that the scaled function* $\widetilde{u_\varepsilon}$ *has uniform* $C^{0,\gamma}$ *estimate for any* $0 < \gamma < 1$. *This follows from (2.1) and the standard* $W^{2,2}$ *estimate*

$$\|\widetilde{u}_\varepsilon\|_{W^{2,2}(B_1)} \leq c(\|\widetilde{u}_\varepsilon\|_{L^2(B_1)} + \|W'(\widetilde{u}_\varepsilon)\|_{L^2(B_1)} + \|\varepsilon \widetilde{f}_\varepsilon\|_{L^2(B_1)})$$

as well as the Sobolev inequality (recalling $n = 2$*)*

$$\|\widetilde{u}_\varepsilon\|_{C^{0,\gamma}(B_1)} \leq c(\gamma)\|\widetilde{u}_\varepsilon\|_{W^{2,2}(B_1)}.$$

Remark 3.4. *In the remaining part of the paper we fix constants as follows. First fix*

$$0 < \beta_2 < \frac{1}{2}, \qquad 0 < \gamma < 1.$$

Choose $\beta_1 > 0$ *to be small so that*

$$0 < \frac{1}{2} - \beta_2 - \beta_1, \qquad 0 < \gamma\beta_2 - 2\beta_1$$

hold. Fix β_0 *so that*

$$0 < \beta_0 \le \beta_1$$

and define

$$\iota = \max\left\{\beta_1 + \beta_2 + \frac{1}{2}, 1 - \gamma\beta_2 + 2\beta_1, 1 - \beta_0\right\}.$$

We note that $0 < \iota < 1$ *by above choice. Finally fix* β_3 *so that*

$$\iota < \beta_3 < 1.$$

We next quote the following from [19], which holds for W with the properties stated in Sec. 2.1.

Lemma 3.5. ([19, Lemma 3.5]) *There exist constants* $\varepsilon_1 > 0$ *and* $c_2 > 0$ *depending only on* β_0, β_1, c_3 *and* W *with the following properties. Suppose (A)* $\widetilde{v}_\varepsilon \in C^3(B_{\varepsilon^{-\beta_1}})$, $g \in C^1(B_{\varepsilon^{-\beta_1}})$ *and* $\varepsilon \le \varepsilon_1$ *satisfy*

$$-\Delta\widetilde{v}_\varepsilon + W'(\widetilde{v}_\varepsilon) = \varepsilon g$$

on $B_{\varepsilon^{-\beta_1}}$ *and*
(B) $\sup_{B_{\varepsilon^{-\beta_1}}} |\widetilde{v}_\varepsilon| \le 1 + \varepsilon^{\beta_0}$, $\sup_{B_{\varepsilon^{-\beta_1}}} \left(\frac{|\nabla\widetilde{v}_\varepsilon|^2}{2} - W(\widetilde{v}_\varepsilon)\right) \le c_3$.
Then

$$\sup_{B_{\frac{1}{2}\varepsilon^{-\beta_1}}} \left(\frac{|\nabla\widetilde{v}_\varepsilon|^2}{2} - W(\widetilde{v}_\varepsilon)\right) \le c_2(\varepsilon^{1-\beta_1}\|g\|_{W^{1,2}(B_{\varepsilon^{-\beta_1}})} + \varepsilon^{\beta_0}).$$

The above estimate is derived via the Alexsandroff-Bakelman-Pucci estimate and it is essential that W is a double well potential.

Let us consider the estimate of the discrepancy measure $\xi_\varepsilon(u_\varepsilon) = (\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} - \frac{W'(u_\varepsilon)}{\varepsilon})$. Since we have no control of the derivative of f_ε, we cannot apply Lemma 3.5 to u_ε directly. Thus we consider the regularization of u_ε. Let $\psi \in C_0^\infty(\mathbf{R}^2)$ be a positive radial symmetric function with $\mathrm{supp}\psi \subset B_1(0)$ and $\int_{\mathbf{R}^2} \psi(x)dx = 1$. For $\varepsilon > 0$, set $\psi_{\varepsilon^{1+\beta_2}}(x) = \frac{1}{\varepsilon^{2(1+\beta_2)}}\psi(\frac{x}{\varepsilon^{1+\beta_2}})$ where β_2 is chosen in Remark 3.4. Next proposition follows from applying Lemma 3.5 to $u_\varepsilon * \psi_{\varepsilon^{1+\beta_2}}$.

Proposition 3.6. *Define*

$$v_\varepsilon = u_\varepsilon * \psi_{\varepsilon^{1+\beta_2}}. \tag{3.18}$$

For $V \subset\subset U$ *there exist a constant* $0 < \varepsilon_2 < \varepsilon_1$ *and* c_4 *depending only on* W, $\mathrm{dist}(V, \partial U)$, E_0 *and* a_1 *satisfying that*

$$\sup_V \xi_\varepsilon(v_\varepsilon) \le c_4 \varepsilon^{-\iota} \tag{3.19}$$

if $\varepsilon < \varepsilon_2$. *Here* $\xi_\varepsilon(v_\varepsilon) = \left(\varepsilon \frac{|\nabla v_\varepsilon|^2}{2} - \frac{W(v_\varepsilon)}{\varepsilon}\right)$.

Proof. By scaling $\widetilde{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x)$ and $\widetilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, (3.18) is

$$\widetilde{v}_\varepsilon = \widetilde{u}_\varepsilon * \psi_{\varepsilon^{\beta_2}}.$$

Since $\widetilde{u}_\varepsilon$ satisfies (2.1), $\widetilde{v}_\varepsilon$ satisfies

$$-\Delta \widetilde{v}_\varepsilon + W'(\widetilde{v}_\varepsilon) = \varepsilon g \tag{3.20}$$

with

$$g = \widetilde{f}_\varepsilon * \psi_{\varepsilon^{\beta_2}} + \frac{W'(\widetilde{v}_\varepsilon)}{\varepsilon} - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} * \psi_{\varepsilon^{\beta_2}}.$$

In the following we apply Lemma 3.5 thus we need to estimate $W^{1,2}$-norm of g on $B_{\varepsilon^{-\beta_1}} \subset \frac{V}{\varepsilon}$. First we consider the estimate of $\|g\|_{L^2(B_{\varepsilon^{-\beta_1}})}$. By inserting the term $\pm\frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon}$ we obtain

$$\|g\|_{L^2(B_{\varepsilon^{-\beta_1}})} \le \|\widetilde{f}_\varepsilon * \psi_{\varepsilon^{\beta_2}}\|_{L^2(B_{\varepsilon^{-\beta_1}})} + \|\frac{W'(\widetilde{v}_\varepsilon)}{\varepsilon} - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon}\|_{L^2(B_{\varepsilon^{-\beta_1}})}$$

$$+\|\frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} * \psi_{\varepsilon^{\beta_2}}\|_{L^2(B_{\varepsilon^{-\beta_1}})} := (I_1) + (II_1) + (III_1). \tag{3.21}$$

By $(\widetilde{A}.2)$,

$$(I_1) \le \|\widetilde{f}_\varepsilon\|_{L^2(B_{2\varepsilon^{-\beta_1}})} \|\psi_{\varepsilon^{\beta_2}}\|_{L^1(B_{\varepsilon^{\beta_2}})} \le a_1^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}. \tag{3.22}$$

By $C^{0,\gamma}$ estimate for $\widetilde{u}_\varepsilon$ we have

$$\sup_{B_{\varepsilon^{-\beta_1}}} |\widetilde{v}_\varepsilon - \widetilde{u}_\varepsilon| \le c\varepsilon^{\gamma\beta_2}.$$

Thus with this sup bound

$$(II_1) \le \frac{1}{\varepsilon}\left\{\int_{B_{\varepsilon^{-\beta_1}}} (\sup|W''|)^2 |\widetilde{v}_\varepsilon - \widetilde{u}_\varepsilon|^2\,dx\right\}^{\frac{1}{2}} \le c\varepsilon^{\gamma\beta_2 - \beta_1 - 1}. \tag{3.23}$$

Similarly with

$$|W'(\widetilde{u}_\varepsilon) - W'(\widetilde{u}_\varepsilon) * \psi_{\varepsilon^{\beta_2}}| \leq c\varepsilon^{\gamma\beta_2},$$

$$(III_1) \leq \frac{1}{\varepsilon}\left\{\int_{B_{\varepsilon^{-\beta_1}}} |W'(\widetilde{u}_\varepsilon) - W'(\widetilde{u}_\varepsilon) * \psi_{\varepsilon^{\beta_2}}|^2\,dx\right\}^{\frac{1}{2}} \leq c\varepsilon^{\gamma\beta_2-\beta_1-1}. \quad (3.24)$$

Next, we estimate $\|\nabla g\|_{L^2(B_{\varepsilon^{-\beta_1}})}$. By inserting the terms $\pm\nabla\frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon}$,

$$\|\nabla g\|_{L^2(B_{\varepsilon^{-\beta_1}})} \leq \|\nabla(\widetilde{f}_\varepsilon * \psi_{\varepsilon^{\beta_2}})\|_{L^2(B_{\varepsilon^{-\beta_1}})} + \|\nabla(\frac{W'(\widetilde{v}_\varepsilon)}{\varepsilon} - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon})\|_{L^2(B_{\varepsilon^{-\beta_1}})}$$

$$+\|\nabla(\frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} - \frac{W'(\widetilde{u}_\varepsilon)}{\varepsilon} * \psi_{\varepsilon^{\beta_2}})\|_{L^2(B_{\varepsilon^{-\beta_1}})} := (I_2) + (II_2) + (III_2). \quad (3.25)$$

By $(\widetilde{A}.2)$,

$$(I_2) \leq c\varepsilon^{-\beta_2-\frac{1}{2}}. \quad (3.26)$$

For (II_2) we have

$$|\nabla(W'(\widetilde{v}_\varepsilon) - W'(\widetilde{u}_\varepsilon))| \leq \sup|W''||\nabla\widetilde{v}_\varepsilon - \nabla\widetilde{u}_\varepsilon| + c(\sup|W'''|)\varepsilon^{\gamma\beta_2}|\nabla\widetilde{u}_\varepsilon|. \quad (3.27)$$

Since

$$|\nabla\widetilde{u}_\varepsilon(x-y) - \nabla\widetilde{u}_\varepsilon(x)| \leq \int_0^1 |\nabla^2\widetilde{u}_\varepsilon(x-ty)||y|\,dt,$$

we estimate the L^2-norm of the first term of (3.27) as

$$\begin{aligned}\int_{B_{\varepsilon^{-\beta_1}}} |\nabla\widetilde{v}_\varepsilon - \nabla\widetilde{u}_\varepsilon|^2\,dx &\leq \int_{B_{\varepsilon^{-\beta_1}}}\int_{B_{\varepsilon^{\beta_2}}} |\nabla\widetilde{u}_\varepsilon(x-y) - \nabla\widetilde{u}_\varepsilon(x)|^2\psi_{\varepsilon^{\beta_2}}(y)\,dydx \\ &\leq \int_{B_{\varepsilon^{-\beta_1}}}\int_{B_{\varepsilon^{\beta_2}}} \psi_{\varepsilon^{\beta_2}}(y)\ |y|^2\int_0^1 |\nabla^2\widetilde{u}_\varepsilon(x-ty)|^2\,dtdydx \\ &\leq \varepsilon^{2\beta_2}\int_{B_{2\varepsilon^{-\beta_1}}} |\nabla^2\widetilde{u}_\varepsilon|^2\,dx.\end{aligned} \quad (3.28)$$

By the $W^{2,2}$ estimate of $\widetilde{u}_\varepsilon$ in Remark 3.3 we have

$$\int_{B_{2\varepsilon^{-\beta_1}}} |\nabla\widetilde{u}_\varepsilon|^2 + |\nabla^2\widetilde{u}_\varepsilon|^2\,dx \leq c\varepsilon^{-2\beta_1}. \quad (3.29)$$

Substituting (3.29) into (3.28), we obtain

$$\int_{B_{\varepsilon^{-\beta_1}}} |\nabla\widetilde{v}_\varepsilon - \nabla\widetilde{u}_\varepsilon|^2\,dx \leq c\varepsilon^{2\beta_2-2\beta_1}. \quad (3.30)$$

As the L^2-norm of the second term of (3.27) can be estimated by (3.29), we obtain

$$(II_2) \leq c(\varepsilon^{\beta_2-\beta_1-1} + \varepsilon^{\gamma\beta_2-\beta_1-1}) \leq c\varepsilon^{\gamma\beta_2-\beta_1-1}. \tag{3.31}$$

For (III_2),

$$\begin{aligned} &|\nabla(W'(\widetilde{u}_\varepsilon(x))) - \nabla(W'(\widetilde{u}_\varepsilon(x)) * \psi_{\varepsilon^{\beta_2}})|^2 \\ &\quad \leq \int_{B_{\varepsilon^{\beta_2}}} |\nabla(W'(\widetilde{u}_\varepsilon(x))) - \nabla(W'(\widetilde{u}_\varepsilon(x-y)))|^2 \psi_{\varepsilon^{\beta_2}}(y)dy \\ &\quad \leq 2\int_{B_{\varepsilon^{\beta_2}}} (\sup|W''|)^2 |\nabla\widetilde{u}_\varepsilon(x) - \nabla\widetilde{u}_\varepsilon(x-y)|^2 \psi_{\varepsilon^{\beta_2}}(y)dy \\ &\qquad + 2\int_{B_{\varepsilon^{\beta_2}}} (c\sup|W'''|)^2 \varepsilon^{2\beta_2\gamma} |\nabla\widetilde{u}_\varepsilon(x)|^2 \psi_{\varepsilon^{\beta_2}}(y)\, dy. \end{aligned} \tag{3.32}$$

In the same way as we obtained (3.31) we have from (3.32)

$$\begin{aligned} (III_2) &\leq c\varepsilon^{-1}\left\{ \int_{B_{\varepsilon^{-\beta_1}}} \int_{B_{\varepsilon^{\beta_2}}} |\nabla\widetilde{u}_\varepsilon(x) - \nabla\widetilde{u}_\varepsilon(x-y)|^2 \psi_{\varepsilon^{\beta_2}}(y)\, dydx \right. \\ &\qquad \left. + \int_{\varepsilon^{-\beta_1}} \varepsilon^{2\beta_2\gamma} |\nabla\widetilde{u}_\varepsilon(x)|^2\, dx \right\}^{\frac{1}{2}} \\ &\leq c(\varepsilon^{\beta_2-\beta_1-1} + \varepsilon^{\gamma\beta_2-\beta_1-1}) \leq c\varepsilon^{\gamma\beta_2-\beta_1-1}. \end{aligned} \tag{3.33}$$

Estimates (3.21), (3.22), (3.23), (3.24), (3.25), (3.26), (3.31), (3.33) show

$$\|g\|_{W^{1,2}(B_{\varepsilon^{-\beta_1}})} \leq c(\varepsilon^{-\beta_2-\frac{1}{2}} + \varepsilon^{\gamma\beta_2-\beta_1-1}). \tag{3.34}$$

Before we apply Lemma 3.5 we need to have a uniform estimate for $\|\widetilde{v}_\varepsilon\|_{C^1}$. By the choice of β_1, β_2 and γ we have

$$-\beta_2 + \frac{1}{2} > 0 \text{ and } \gamma\beta_2 - \beta_1 > 0.$$

so in particular we have $\|\varepsilon g\|_{W^{1,2}(B_{\varepsilon^{-\beta_1}})} \leq c$. Since $\widetilde{v}_\varepsilon$ satisfies (3.20), by $W^{3,2}$-estimate of the elliptic PDE and Sobolev's inequality, we obtain a uniform C^1 estimate for $\widetilde{v}_\varepsilon$ and thus we have ε-independent c_3 such that

$$\sup_{B_{\varepsilon^{-\beta_1}}} \left(\frac{|\nabla\widetilde{v}_\varepsilon|^2}{2} - W(\widetilde{v}_\varepsilon) \right) \leq c_3.$$

Now we are ready to use Lemma 3.5 to conclude with (3.34) that

$$\sup_{B_{\frac{1}{2}\varepsilon^{-\beta_1}}} \left(\frac{|\nabla\widetilde{v}_\varepsilon|^2}{2} - W(\widetilde{v}_\varepsilon) \right) \leq c(\varepsilon^{\frac{1}{2}-\beta_1-\beta_2} + \varepsilon^{\gamma\beta_2-2\beta_1} + \varepsilon^{\beta_0}).$$

By scaling,

$$\sup_V \left(\frac{\varepsilon |\nabla v_\varepsilon|^2}{2} - \frac{W(v_\varepsilon)}{\varepsilon} \right) \le c(\varepsilon^{-\frac{1}{2}-\beta_1-\beta_2} + \varepsilon^{\gamma\beta_2 - 2\beta_1 - 1} + \varepsilon^{\beta_0 - 1}).$$

We defined ι so that the right-hand side is bounded by $c\varepsilon^{-\iota}$ thus we proved (3.19) with an appropriate choice of constant c_4. □

Proposition 3.7. *There exist constants $c_5 > 0$ and $\varepsilon_2 > 0$ depending only on W, $\mathrm{dist}(V, \partial U)$, E_0 and a_1 such that for $\varepsilon \le \varepsilon_2$ and $B_r \subset V$*

$$\int_{B_r} \{\xi_\varepsilon(u_\varepsilon)\}_+ \, dx \le \int_{B_r} \{\xi_\varepsilon(v_\varepsilon)\}_+ \, dx + c_5 \varepsilon^{\frac{\beta_2}{2}} \{r + \mu_\varepsilon(B_r)\}.$$

Proof. Since $|\nabla u_\varepsilon| \le |\nabla u_\varepsilon - \nabla v_\varepsilon| + |\nabla v_\varepsilon|$, by considering the square of both sides and Cauchy's inequality,

$$\begin{aligned} |\nabla u_\varepsilon|^2 &\le |\nabla v_\varepsilon|^2 + |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + 2|\nabla u_\varepsilon - \nabla v_\varepsilon||\nabla v_\varepsilon| \\ &\le |\nabla v_\varepsilon|^2 + (1 + \varepsilon^{-\beta_2})|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + \varepsilon^{\beta_2}|\nabla v_\varepsilon|^2. \end{aligned} \tag{3.35}$$

Similarly

$$|\nabla v_\varepsilon|^2 \le |\nabla u_\varepsilon|^2 + (1 + \varepsilon^{-\beta_2})|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + \varepsilon^{\beta_2}|\nabla u_\varepsilon|^2. \tag{3.36}$$

By substituting (3.36) to (3.35) we obtain

$$|\nabla u_\varepsilon|^2 \le |\nabla v_\varepsilon|^2 + c\varepsilon^{-\beta_2}|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + c\varepsilon^{\beta_2}|\nabla u_\varepsilon|^2. \tag{3.37}$$

Using (3.37) we obtain

$$\begin{aligned} \{\xi_\varepsilon(u_\varepsilon)\}_+ \le \{\xi_\varepsilon(v_\varepsilon)\}_+ &+ \frac{c}{2}\varepsilon^{1-\beta_2}|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \\ &+ \frac{|W(v_\varepsilon) - W(u_\varepsilon)|}{\varepsilon} + c\varepsilon^{\beta_2}\left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right). \end{aligned} \tag{3.38}$$

Integrating (3.38) on B_r for $\varepsilon \le r$,

$$\begin{aligned} \int_{B_r} \{\xi_\varepsilon(u_\varepsilon)\}_+ \, dx \le \int_{B_r} \{\xi_\varepsilon(v_\varepsilon)\}_+ \, dx + c\varepsilon^{-\beta_2} \int_{B_r} \varepsilon|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \, dx \\ + \int_{B_r} \frac{|W(v_\varepsilon) - W(u_\varepsilon)|}{\varepsilon} \, dx + c\varepsilon^{\beta_2} \int_{B_r} (\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}) \, dx. \end{aligned} \tag{3.39}$$

We estimate each term of (3.39). As we proved in (3.28),

$$c\varepsilon^{-\beta_2} \int_{B_r} \varepsilon|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \, dx \le c\varepsilon^{3+\beta_2} \int_{B_{r+\varepsilon^{1+\beta_2}}} |\nabla^2 u_\varepsilon|^2 \, dx. \tag{3.40}$$

We estimate L^2-norm of $\nabla^2 u_\varepsilon$ in $B_{r+\varepsilon^{1+\beta_2}}$. Write $\tilde r = r + \varepsilon^{1+\beta_2}$. By scaling $B_{\tilde r}$ to B_1 by $\tilde x = x/\tilde r$ and applying $W^{2,2}$ estimate to $\Delta u_\varepsilon = \frac{\tilde r^2 W'(u_\varepsilon)}{\varepsilon^2} - \frac{\tilde r^2 f_\varepsilon}{\varepsilon}$, we obtain after scaling back

$$\int_{B_{\tilde r}} |\nabla^2 u_\varepsilon|^2\,dx \le \frac{1}{\tilde r^4}\int_{B_{\tilde r}} |u_\varepsilon|^2\,dx + \frac{1}{\varepsilon^4}\int_{B_{\tilde r}} (W'(u_\varepsilon))^2\,dx + \frac{1}{\varepsilon^2}\int_{B_{\tilde r}} f_\varepsilon^2\,dx. \tag{3.41}$$

For the second term of (3.41), we split the integral to B_r and $B_{\tilde r}\setminus B_r$. Since $|W'(u_\varepsilon)|^2 \le cW(u_\varepsilon)$ and $|B_{\tilde r}\setminus B_r| \le c\varepsilon^{1+\beta_2} r$, we obtain

$$\int_{B_{\tilde r}} |W'(u_\varepsilon)|^2\,dx \le \varepsilon\int_{B_r} \frac{W(u_\varepsilon)}{\varepsilon}\,dx + c\int_{B_{\tilde r}\setminus B_r} dx \le \varepsilon(\mu_\varepsilon(B_r) + r\varepsilon^{\beta_2}). \tag{3.42}$$

By Lemma 3.2 for the first term and by (3.42) and (A.2), (3.41) is estimated as

$$\int_{B_{\tilde r}} |\nabla^2 u_\varepsilon|^2\,dx \le \frac{c}{r^2} + \frac{c}{\varepsilon^3}(\mu_\varepsilon(B_r) + r\varepsilon^{\beta_2}) + \frac{a_1}{\varepsilon}. \tag{3.43}$$

By (3.40) and (3.43),

$$\begin{aligned} c\varepsilon^{-\beta_2}\int_{B_r} \varepsilon|\nabla u_\varepsilon - \nabla v_\varepsilon|^2\,dx &\le c\left\{\frac{\varepsilon^{\beta_2+3}}{r^2} + r\varepsilon^{2\beta_2} + \varepsilon^{\beta_2+2} + \varepsilon^{\beta_2}\mu_\varepsilon(B_r)\right\} \\ &\le c\left\{r\varepsilon^{\beta_2} + \varepsilon^{\beta_2}\mu_\varepsilon(B_r)\right\} \end{aligned} \tag{3.44}$$

since $r \ge \varepsilon$. For the estimate of the third term of (3.39),

$$\begin{aligned} |W(v_\varepsilon) - W(u_\varepsilon)| &\le |v_\varepsilon - u_\varepsilon||W'(u_\varepsilon)| + \frac{\sup|W''|}{2}|v_\varepsilon - u_\varepsilon|^2 \\ &\le c\varepsilon^{-\frac{\beta_2}{2}}|v_\varepsilon - u_\varepsilon|^2 + c\varepsilon^{\frac{\beta_2}{2}}|W'(u_\varepsilon)|^2. \end{aligned} \tag{3.45}$$

Since $|W'(u_\varepsilon)|^2 \le cW(u_\varepsilon)$,

$$\begin{aligned} \int_{B_r} \frac{|W(v_\varepsilon) - W(u_\varepsilon)|}{\varepsilon}\,dx &\le c\varepsilon^{-1-\frac{\beta_2}{2}}\int_{B_r} |v_\varepsilon - u_\varepsilon|^2\,dx + c\varepsilon^{\frac{\beta_2}{2}}\int_{B_r} \frac{W(u_\varepsilon)}{\varepsilon}\,dx \\ &\le c\varepsilon^{-1-\frac{\beta_2}{2}}\int_{B_r} |v_\varepsilon - u_\varepsilon|^2\,dx + c\varepsilon^{\frac{\beta_2}{2}}\mu_\varepsilon(B_r). \end{aligned} \tag{3.46}$$

For the right-hand side of (3.46)

$$\int_{B_r} |u_\varepsilon - v_\varepsilon|^2\,dx \le \int_{B_r\setminus B_{r-\varepsilon^{1+\beta_2}}} |u_\varepsilon - v_\varepsilon|^2\,dx + \int_{B_{r-\varepsilon^{1+\beta_2}}} |u_\varepsilon - v_\varepsilon|^2\,dx$$
$$\le c\left(r\varepsilon^{1+\beta_2} + \varepsilon^{2+2\beta_2}\int_{B_r}|\nabla u_\varepsilon|^2\,dx\right) \le c\left(r\varepsilon^{1+\beta_2} + \varepsilon^{1+2\beta_2}\mu_\varepsilon(B_r)\right). \tag{3.47}$$

By substituting (3.47) to (3.46) we obtain

$$\begin{aligned}\int_{B_r}\frac{|W(v_\varepsilon)-W(u_\varepsilon)|}{\varepsilon}\,dx &\le r\varepsilon^{\frac{\beta_2}{2}} + c\varepsilon^{\frac{3\beta_2}{2}}\mu_\varepsilon(B_r) + c\varepsilon^{\frac{\beta_2}{2}}\mu_\varepsilon(B_r)\\ &\le r\varepsilon^{\frac{\beta_2}{2}} + c\varepsilon^{\frac{\beta_2}{2}}\mu_\varepsilon(B_r).\end{aligned} \tag{3.48}$$

The claim of the proposition follows from (3.39), (3.44), (3.48). □

Next we estimate the lower bound of the energy density ratio for 'small' scale, namely, for $\varepsilon \le r \le t_1\varepsilon^\iota$ with small t_1 independent of ε.

Theorem 3.8. *There exist constants $c_6 > 0$, $t_1 > 0$ and $\varepsilon_3 > 0$ such that if $\varepsilon \le r \le t_1\varepsilon^\iota$, $B_r(x_0) \subset V$, $|u_\varepsilon(x_0)| < \alpha$ and $0 < \varepsilon \le \varepsilon_3$, then*

$$c_6 \le \frac{1}{r}\mu_\varepsilon(B_r(x_0)).$$

Proof. We may assume $x_0 = 0$. For $\widetilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$,

$$\frac{1}{\varepsilon}\mu_\varepsilon(B_\varepsilon) = \frac{1}{\varepsilon}\int_{B_\varepsilon}\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon}\,dx = \int_{B_1}\frac{|\nabla\widetilde{u}_\varepsilon|^2}{2} + W(\widetilde{u}_\varepsilon)\,dx.$$

Since $|\widetilde{u}_\varepsilon(0)| < \alpha$ and $\widetilde{u}_\varepsilon \in C^{0,\gamma}$ there exists a constant $c_7 > 0$ satisfying $W(\widetilde{u}_\varepsilon) > c_7$ on B_{c_7} and thus

$$\frac{1}{\varepsilon}\mu_\varepsilon(B_\varepsilon) \ge \int_{B_{c_7}} W(\widetilde{u}_\varepsilon)\,dx \ge c_7\mathcal{L}^2(B_{c_7}) = c_8. \tag{3.49}$$

Let $t_1 > 0$ be a constant to be determined shortly. We claim that $\mu_\varepsilon(B_r)/r \ge c_8/2$ for $\varepsilon \le r \le t_1\varepsilon^\iota$. To derive a contradiction assume that there exists a constant r_1 with $\varepsilon \le r_1 \le t_1\varepsilon^\iota$ satisfying

$$\frac{1}{r_1}\mu_\varepsilon(B_{r_1}) = \frac{c_8}{2}.$$

By continuity of $\frac{1}{r}\mu_\varepsilon(B_r)$ with respect to r, there exists r_0 with $\varepsilon \le r_0 < r_1$ satisfying $\frac{1}{r_0}\mu_\varepsilon(B_{r_0}) = c_8$ and $\frac{c_8}{2} \le \frac{1}{r}\mu_\varepsilon(B_r) \le c_8$ for $r_0 \le r \le r_1$. By Lemma 3.1, Proposition 3.6 and 3.7, for $\varepsilon \le s \le r \le t_1\varepsilon^\iota$,

$$\begin{aligned}&(1+\delta)\frac{1}{r}\mu_\varepsilon(B_r) - (1-\delta)\frac{1}{s}\mu_\varepsilon(B_s)\\ &\quad \ge -(\frac{2}{\delta}+1)a_1 r - \int_s^r c_4\varepsilon^{-\iota}dr - \int_s^r \frac{c_5\varepsilon^{\frac{\beta_2}{2}}}{r}\{1+\frac{1}{r}\mu_\varepsilon(B_r)\}dr.\end{aligned} \tag{3.50}$$

Using (3.50) with $s = r_0$ and $r = r_1$ as well as $r_1 \le t_1\varepsilon^\iota$ we obtain

$$\frac{c_8}{2}(3\delta - 1) \ge -(\frac{2}{\delta}+1)a_1 t_1\varepsilon^\iota - c_4 t_1 - c_5\varepsilon^{\frac{\beta_2}{2}}(1+c_8)\log(\frac{t_1\varepsilon^\iota}{\varepsilon}). \tag{3.51}$$

Set $\delta = \frac{1}{6}$ so the left-hand side of (3.51) is equal to $-\frac{c_8}{4}$. Choose t_1 small so the right-hand side of (3.51) is greater than $-\frac{c_8}{4}$ for sufficiently small ε. This leads to a contradiction. We set $c_6 = c_8/2$. □

Next we estimate the discrepancy $\xi_\varepsilon(v_\varepsilon)$ for 'large' r, namely, for $t_1\varepsilon^\iota \le r$. The proof is a suitable modification of [19, Prop.3.5].

Proposition 3.9. *Set* $\beta_4 = \min\{2-2\beta_2,\, 2\beta_2,\, \beta_3 - \iota\} > 0$. *There exist constants* $c_9 > 0$ *and* $\varepsilon_4 > 0$ *such that, if* $B_r \subset V$, $t_1\varepsilon^\iota \le r \le 1$ *and* $0 < \varepsilon \le \varepsilon_4$

$$\int_{B_r}\{\xi_\varepsilon(v_\varepsilon)\}_+\,dx \le c_9\left(r\varepsilon^{\beta_3-\iota} + \varepsilon^{2-2\beta_2} + \varepsilon^{\beta_4}\mu_\varepsilon(B_r)\right). \tag{3.52}$$

Proof. We estimate the integral on three domains,

$$\mathcal{A} = \{x \in B_r \setminus B_{r-\varepsilon^{\beta_3}}\}, \quad \mathcal{B} = \{x \in B_{r-\varepsilon^{\beta_3}} \,|\, \mathrm{dist}(\{|u_\varepsilon| \le \alpha\}, x) < \varepsilon^{\beta_3}\},$$

$$\mathcal{C} = \{x \in B_{r-\varepsilon^{\beta_3}} \,|\, \mathrm{dist}(\{|u_\varepsilon| \le \alpha\}, x) \ge \varepsilon^{\beta_3}\}.$$

Case 1. (estimate on $\mathcal{A}$)
By Proposition 3.6 and $\mathcal{L}^2(\mathcal{A}) \le cr\varepsilon^{\beta_3}$,

$$\int_{\mathcal{A}}\{\xi_\varepsilon(v_\varepsilon)\}_+\,dx \le c_5\varepsilon^{-\iota}\mathcal{L}^2(\mathcal{A}) \le cr\varepsilon^{\beta_3-\iota}. \tag{3.53}$$

Case 2. (estimate on $\mathcal{B}$)
We first estimate $\mathcal{L}^2(\mathcal{B})$. We apply Vitali's covering lemma to the family of balls $\{B_{\varepsilon^{\beta_3}(x)}\}_{x\in\{|u_\varepsilon|\le\alpha\}\cap\mathcal{B}}$ (which covers $\mathcal{B}$), so that $\{B_{\varepsilon^{\beta_3}(x_i)}\}_{i=1}^N$ is a

pairwise disjoint subset of the family and so that $\mathcal{B} \subset \cup_{i=1}^{N} B_{5\varepsilon^{\beta_3}}(x_i)$. Then we have

$$\mathcal{L}^2(\mathcal{B}) \le c(5\varepsilon^{\beta_3})^2 N = cN\varepsilon^{2\beta_3}. \tag{3.54}$$

Since $\iota < \beta_3$ (Remark 3.4), $\varepsilon^{\beta_3} < t_1\varepsilon^{\iota}$ for all sufficiently small ε. Thus by Theorem 3.8

$$c_6\varepsilon^{\beta_3} \le \mu_\varepsilon(B_{\varepsilon^{\beta_3}}(x_i))$$

holds for each $i = 1, ..., N$. Since they are pairwise disjoint, summing over i we have

$$Nc_6\varepsilon^{\beta_3} \le \mu_\varepsilon(B_r) \tag{3.55}$$

and (3.54) and (3.55) show that

$$\mathcal{L}^2(\mathcal{B}) \le c\varepsilon^{\beta_3}\mu_\varepsilon(B_r). \tag{3.56}$$

Finally, with Proposition 3.6 and (3.56)

$$\int_{\mathcal{B}} \{\xi_\varepsilon(v_\varepsilon)\}_+ \, dx \le c_4\varepsilon^{-\iota}\mathcal{L}^2(B) \le c\varepsilon^{\beta_3-\iota}\mu_\varepsilon(B_r). \tag{3.57}$$

<u>Case 3.</u> (estimate on $\mathcal{C}$)
We define a Lipschitz function ρ as follows;

$$\rho(x) = \min\{1, 2\varepsilon^{-\beta_3}\mathrm{dist}(\{|x| \ge r - \varepsilon^{\beta_3}/2\} \cup \{|u_\varepsilon| \le \alpha\}, x)\}.$$

ρ is 0 on the set $\{|x| \ge r - \varepsilon^{\beta_3}/2\} \cup \{|u_\varepsilon| \le \alpha\}$, 1 on $\mathcal{C}$ and $|\nabla\rho| \le 2\varepsilon^{-\beta_3}$. Using this ρ, we estimate $\frac{1}{2}\varepsilon|\nabla v_\varepsilon|^2$. By (3.18) and (3.20), v_ε (without scaling) satisfies $-\varepsilon\Delta v_\varepsilon + \frac{W'(v_\varepsilon)}{\varepsilon} = g$ where $g = f_\varepsilon * \psi_{\varepsilon^{1+\beta_2}} + \frac{W'(v_\varepsilon)}{\varepsilon} - \frac{W'(u_\varepsilon)}{\varepsilon} * \psi_{\varepsilon^{1+\beta_2}}$. Differentiating this equation with respect to the k-th variable, multiplying it by $D_k v_\varepsilon \rho^2$ and integrating on B_r, we have

$$\int (\varepsilon\Delta D_k v_\varepsilon) D_k v_\varepsilon \rho^2 \, dx = \int (\frac{W''(v_\varepsilon)}{\varepsilon} D_k v_\varepsilon - D_k g) D_k v_\varepsilon \rho^2 \, dx. \tag{3.58}$$

By integrating by parts, the left-hand side of (3.58) is

$$\int (\varepsilon\Delta D_k v_\varepsilon) D_k v_\varepsilon \rho^2 \, dx = -\varepsilon \int |\nabla^2 v_\varepsilon|^2 \rho^2 \, dx - 2\varepsilon \int D_{ik} v_\varepsilon D_k v_\varepsilon \rho D_i \rho \, dx. \tag{3.59}$$

Since $W'' \geq \kappa$ on $\{|u_\varepsilon| \geq \alpha\}$ and $\frac{\kappa}{2} \geq \sup|W'''|\varepsilon^{\gamma\beta_2}$ for sufficiently small ε, by Cauchy's inequality, the right-hand side of (3.58) is

$$\begin{aligned}
&\int (\frac{W''(v_\varepsilon)}{\varepsilon} D_k v_\varepsilon - D_k g) D_k v_\varepsilon \rho^2 \, dx \\
&\geq \int \frac{W''(u_\varepsilon)}{\varepsilon} |\nabla v_\varepsilon|^2 \rho^2 - \frac{|W''(v_\varepsilon) - W''(u_\varepsilon)|}{\varepsilon} |\nabla v_\varepsilon|^2 \rho^2 - |\nabla g||\nabla v_\varepsilon| \rho^2 \, dx \\
&\geq \int \frac{\kappa |\nabla v_\varepsilon|^2}{\varepsilon} \rho^2 - \sup|W'''|\varepsilon^{\gamma\beta_2} \frac{|\nabla v_\varepsilon|^2}{\varepsilon} \rho^2 \, dx - \int |\nabla g||\nabla v_\varepsilon| \rho^2 \, dx \\
&\geq \int \frac{\kappa |\nabla v_\varepsilon|^2}{2\varepsilon} \rho^2 \, dx - \int (\frac{\varepsilon}{\kappa} |\nabla g|^2 \rho^2 + \frac{\kappa}{4\varepsilon} |\nabla v_\varepsilon|^2 \rho^2) \, dx.
\end{aligned} \tag{3.60}$$

By (3.58), (3.59) and (3.60), we obtain

$$\begin{aligned}
&\int \frac{\kappa |\nabla v_\varepsilon|^2}{4\varepsilon} \rho^2 \, dx + \frac{\varepsilon}{2} \int |\nabla^2 v_\varepsilon|^2 \rho^2 \, dx \\
&\leq 2\varepsilon \int |\nabla v_\varepsilon|^2 |\nabla \rho|^2 \, dx + \frac{\varepsilon}{\kappa} \int |\nabla g|^2 \rho^2 \, dx.
\end{aligned} \tag{3.61}$$

We estimate the L^2-norm of ∇g in a similar manner as in the proof of Proposition 3.6 and 3.7. Now the scale is different from Proposition 3.6 and 3.7. By inserting the term $\pm \frac{\nabla W'(u_\varepsilon)}{\varepsilon}$ like (3.25),

$$\begin{aligned}
\int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla g|^2 \, dx \leq &\int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla (f_\varepsilon * \psi_{1+\varepsilon^{\beta_2}})|^2 \, dx \\
&+ \int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla (\frac{W'(v_\varepsilon)}{\varepsilon} - \frac{W'(u_\varepsilon)}{\varepsilon})|^2 \, dx \\
&+ \int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla (\frac{W'(u_\varepsilon)}{\varepsilon} - \frac{W'(u_\varepsilon)}{\varepsilon} * \psi_{1+\varepsilon^{\beta_2}})|^2 \, dx \\
:= &(I_3) + (II_3) + (III_3).
\end{aligned} \tag{3.62}$$

By $(A.2)$ like (3.26),

$$(I_3) \leq c\varepsilon^{-2\beta_2 - 1}. \tag{3.63}$$

For (II_3) we have

$$|\nabla (W'(v_\varepsilon) - W'(u_\varepsilon))| \leq \sup|W''||\nabla v_\varepsilon - \nabla u_\varepsilon| + c(\sup|W'''|)\varepsilon^{\gamma\beta_2} |\nabla u_\varepsilon|. \tag{3.64}$$

By the similar calculation to (3.28),

$$\int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2\, dx \le \varepsilon^{2\beta_2+2} \int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}+\varepsilon^{1+\beta_2}}} |\nabla^2 u_\varepsilon|^2\, dx. \tag{3.65}$$

Write $\tilde{r} = r - \frac{\varepsilon^{\beta_3}}{2} + \varepsilon^{1+\beta_2}$. We estimate $\int_{B_{\tilde{r}}} |\nabla^2 u_\varepsilon|^2\, dx$ by the same way as the proof of Proposition 3.7. By scaling $B_{\tilde{r}}$ to B_1 by $\tilde{x} = \frac{x}{\tilde{r}}$ and apply the $W^{2,2}$ estimate to $\Delta u_\varepsilon = \frac{\tilde{r}^2 W'(u_\varepsilon)}{\varepsilon^2} - \frac{\tilde{r}^2 f_\varepsilon}{\varepsilon}$, we obtain after scaling back,

$$\int_{B_{\tilde{r}}} |\nabla^2 u_\varepsilon|^2\, dx \le \frac{1}{\tilde{r}^4} \int_{B_{\tilde{r}}} |u_\varepsilon|^2\, dx + \frac{1}{\varepsilon^4} \int_{B_{\tilde{r}}} (W'(u_\varepsilon))^2\, dx + \frac{1}{\varepsilon^2} \int_{B_{\tilde{r}}} f_\varepsilon^2\, dx. \tag{3.66}$$

Since $|W'(u_\varepsilon)|^2 \le cW(u_\varepsilon)$ and $B_{\tilde{r}} \subset B_r$,

$$\frac{1}{\varepsilon^4} \int_{B_{\tilde{r}}} |W'(u_\varepsilon)|^2\, dx \le \frac{1}{\varepsilon^3} \int_{B_{\tilde{r}}} \frac{cW(u_\varepsilon)}{\varepsilon}\, dx \le \frac{c}{\varepsilon^3} \mu_\varepsilon(B_r) \tag{3.67}$$

Thus by (3.66), (3.67) and $(A.2)$, we obtain

$$\int_{B_{\tilde{r}}} |\nabla^2 u_\varepsilon|^2\, dx \le c(a_1)(\varepsilon^{-1} + \varepsilon^{-3} \mu_\varepsilon(B_r)). \tag{3.68}$$

By (3.65) and (3.68),

$$\int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla u_\varepsilon - \nabla v_\varepsilon|^2\, dx \le c(\varepsilon^{2\beta_2+1} + \varepsilon^{2\beta_2-1} \mu_\varepsilon(B_r)). \tag{3.69}$$

As the L^2 norm of the second term of (3.64) can be estimated by $\mu_\varepsilon(B_r)$, we obtain

$$(II_3) = \int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla(\frac{W'(u_\varepsilon)}{\varepsilon} - \frac{W'(v_\varepsilon)}{\varepsilon})|^2\, dx \le c(\varepsilon^{2\beta_2-1} + \varepsilon^{2\beta_2-3} \mu_\varepsilon(B_r)). \tag{3.70}$$

For (III_3),

$$\begin{aligned} &|\nabla(W'(u_\varepsilon(x))) - \nabla(W'(u_\varepsilon(x)) * \psi_{\varepsilon^{1+\beta_2}})|^2 \\ &\quad \le \int_{B_{\varepsilon^{1+\beta_2}}} |\nabla(W'(u_\varepsilon(x))) - \nabla(W'(u_\varepsilon(x-y)))|^2 \psi_{\varepsilon^{1+\beta_2}}(y)dy \\ &\quad \le 2\int_{B_{\varepsilon^{1+\beta_2}}} (\sup|W''|)^2 |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2 \psi_{\varepsilon^{1+\beta_2}}(y)dy \\ &\qquad + 2\int_{B_{\varepsilon^{1+\beta_2}}} (c\sup|W'''|)^2 \varepsilon^{2\beta_2\gamma} |\nabla u_\varepsilon(x)|^2 \psi_{\varepsilon^{1+\beta_2}}(y)\, dy. \end{aligned} \tag{3.71}$$

In the same way as we obtained (3.69) we have

$$
\begin{aligned}
(III_3) &\le c(\varepsilon^{2\beta_2-1} + \varepsilon^{2\beta_2-3}\mu_\varepsilon(B_r) + \varepsilon^{2\beta_2\gamma-3}\mu_\varepsilon(B_r)) \\
&\le c(\varepsilon^{2\beta_2-1} + \varepsilon^{2\beta_2-3}\mu_\varepsilon(B_r)).
\end{aligned}
\tag{3.72}
$$

Estimates (3.62), (3.63), (3.70) and (3.72) show

$$
\int_{B_{r-\frac{\varepsilon^{\beta_3}}{2}}} |\nabla g|^2\,dx \le c(\varepsilon^{-2\beta_2-1} + \varepsilon^{2\beta_2-3}\mu_\varepsilon(B_r)). \tag{3.73}
$$

With $|\nabla\rho| \le 2\varepsilon^{-\beta_3}$, (3.61) and (3.73), we obtain

$$
\int_{\mathcal{C}} \frac{\kappa|\nabla v_\varepsilon|^2}{4\varepsilon}\,dx \le c\varepsilon^{1-2\beta_3}\int_{B_{r-\varepsilon^{\beta_3}/2}} |\nabla v_\varepsilon|^2\,dx + \frac{c}{\kappa}(\varepsilon^{-2\beta_2} + \varepsilon^{2\beta_2-2}\mu_\varepsilon(B_r)).
$$

Since $\int_{B_{r-\varepsilon^{\beta_3}/2}} \frac{\varepsilon|\nabla v_\varepsilon|^2}{2}\,dx \le \int_{B_r} \frac{\varepsilon|\nabla u_\varepsilon|^2}{2}\,dx \le \mu_\varepsilon(B_r)$, multiplying above by $2\varepsilon^2\kappa^{-1}$, we have

$$
\int_{\mathcal{C}} \{\xi_\varepsilon(v_\varepsilon)\}_+\,dx \le \int_{\mathcal{C}} \frac{\varepsilon|\nabla v_\varepsilon|^2}{2}\,dx \le c(\varepsilon^{2-2\beta_3}\mu_\varepsilon(B_r) + \varepsilon^{2\beta_2}\mu_\varepsilon(B_r) + \varepsilon^{2-2\beta_2}). \tag{3.74}
$$

Combining (3.53), (3.57) and (3.74), and recalling the definition of β_4, we obtain the desired estimate. □

Next, we obtain the lower bound of the energy density ratio for $t_1\varepsilon^\iota \le r \le t_2$.

Theorem 3.10. *There exist constants $c_{10} > 0$, $t_2 > 0$ and $\varepsilon_5 > 0$ such that if $B_r(x_0) \subset V$, $|u_\varepsilon(x_0)| < \alpha$ and $\varepsilon \le \varepsilon_5$, then for $t_1\varepsilon^\iota \le r \le t_2$,*

$$
c_{10} \le \frac{1}{r}\mu_\varepsilon(B_r(x_0)).
$$

Proof. By Theorem 3.8, $c_6 \le \frac{1}{r}\mu_\varepsilon(B_r)$ with $r = t_1\varepsilon^\iota$. The proof of the claim is similar to that of Theorem 3.8. Let $t_2 > 0$ be a constant to be determined shortly. We claim that $\mu_\varepsilon(B_r)/r \ge c_6/2$ for $t_1\varepsilon^\iota \le r \le t_2$. To derive a contradiction assume that there exists a constant r_3 with $t_1\varepsilon^\iota \le r_3 \le t_2$ satisfying

$$
\frac{1}{r_3}\mu_\varepsilon(B_{r_3}) = \frac{c_6}{2}.
$$

By continuity of $\frac{1}{r}\mu_\varepsilon(B_r)$ with respect to r, there exists r_2 with $t_1\varepsilon^\iota \le r_2 < r_3$ satisfying $\frac{1}{r_2}\mu_\varepsilon(B_{r_2}) = c_6$ and $\frac{c_6}{2} \le \frac{1}{r}\mu_\varepsilon(B_r) \le c_6$ for $r_2 \le r \le r_3$. By Lemma 3.1, Proposition 3.7 and 3.9, for $t_1\varepsilon^\iota \le s \le r \le t_2$,

$$\begin{aligned}(1+\delta)\frac{1}{r}\mu_\varepsilon(B_r) - (1-\delta)\frac{1}{s}\mu_\varepsilon(B_s)\\ \ge -(\frac{2}{\delta}+1)a_1 r - \int_s^r \rho^{-2}\{c_9(\rho\varepsilon^{\beta_3-\iota} + \varepsilon^{2-2\beta_2} + \varepsilon^{\beta_4}\mu_\varepsilon(B_\rho))\\ + c_5\varepsilon^{\frac{\beta_2}{2}}(\rho + \mu_\varepsilon(B_\rho))\}\,d\rho.\end{aligned} \tag{3.75}$$

Using (3.75) with $s = r_2$ and $r = r_3$ as well as $r_3 \le t_2$ we obtain

$$\begin{aligned}\frac{c_6}{2}(3\delta - 1) \ge &- (\frac{2}{\delta}+1)a_1 t_2 - c_9\{\varepsilon^{\beta_3-\iota}\log(\frac{r_3}{r_2}) + \varepsilon^{2-2\beta_2-2\iota}\frac{t_2}{t_1^2}\\ &+ c_6\varepsilon^{\beta_4}\log(\frac{r_3}{r_2})\} - c_5\varepsilon^{\frac{\beta_2}{2}}\log(\frac{r_3}{r_2})(1+c_6).\end{aligned} \tag{3.76}$$

Set $\delta = \frac{1}{6}$ so the left-hand side of (3.76) is equal to $-\frac{c_6}{4}$. Choose t_2 small so the right-hand side of (3.76) is greater than $-\frac{c_6}{4}$ for sufficiently small ε. This leads to a contradiction. We set $c_{10} = c_6/2$. ☐

Similarly, we can also show the upper bound of the energy density ratio.

Proposition 3.11. *There exist constants $c_{11} > 0$ and $\varepsilon_6 > 0$ such that if $B_r(x_0) \subset V$, $|u_\varepsilon(x_0)| < \alpha$ and $\varepsilon \le \varepsilon_6$, then for $\varepsilon \le r \le t_2$,*

$$\frac{1}{r}\mu_\varepsilon(B_r(x_0)) \le c_{11}.$$

Proof. By $(A.1)$, we obtain $\frac{1}{t_2}\mu_\varepsilon(B_{t_2}) \le \frac{E_0}{t_2}$ where t_2 is the same constant as Theorem 3.10. By using Proposition 3.7 and 3.9 for discrepancy term in (3.1), and by the similar proof to Theorem 3.8 and 3.10, we obtain the upper bound. ☐

Similarly, as the consequence of Proposition 3.7 and 3.9, by Proposition 3.11, we can establish the following monotonicity estimates.

Theorem 3.12. *For $B_s(x_0) \subset B_r(x_0) \subset V$, $\varepsilon < s < r < t_2$,*

$$\begin{aligned}(1+\delta)\frac{1}{r}\mu_\varepsilon(B_r(x_0)) - (1-\delta)\frac{1}{s}\mu_\varepsilon(B_s(x_0))\\ \ge -(\frac{2}{\delta}+1)a_1 r + \int_s^r \frac{1}{\rho^2}\int_{B_\rho(x_0)}\left(\frac{W(u_\varepsilon)}{\varepsilon} - \frac{\varepsilon|\nabla u_\varepsilon|^2}{2}\right)_+ dxd\rho - K(\varepsilon).\end{aligned} \tag{3.77}$$

Here, $K(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$.

4 Rectifiability of limit interface

In this section we show that the support of the limit measure μ is a 1-rectifiable set and that it has generalized L^2 curvature expressed as the Radon-Nikodym derivative as we described after Theorem 1.1. Define the (signed) vector-valued measure $\nu^{\alpha\beta}$ on U

$$\nu^{\alpha\beta}(\phi) = \lim_{i\to\infty} \int (\varepsilon_i \frac{\partial u_{\varepsilon_i}}{\partial x^\alpha} \frac{\partial u_{\varepsilon_i}}{\partial x^\beta}) \phi \, dx$$

for $\phi \in C_c(U)$.

Theorem 4.1. *There exist constants $0 < D_1 \le D_2 < \infty$ and $t_2 > 0$ which depend only on E_0, a_1, $\mathrm{dist}(V, \partial U)$ and W such that*

$$D_1 r \le \mu(B_r(x)) \le D_2 r$$

for all $0 < r < t_2$ and $x \in \operatorname{supp} \mu$ with $B_r(x) \subset V$.

Proof. The existence of D_2 follows immediately from Proposition 3.11. We show the existence of D_1. Let $x_0 \in \operatorname{supp} \mu$. We claim that on passing to a subsequence, there exist $\{x_i\}_{i=1}^\infty \subset V$ such that $u_{\varepsilon_i}(x_i) \in [-\alpha, \alpha]$ and $x_i \to x_0$ as $i \to \infty$. We show the claim by contradiction. Suppose there exists $s > 0$ satisfying $B_s(x_0) \subset V$ and $B_s(x_0) \cap \{|u_{\varepsilon_i}| \le \alpha\} = \emptyset$ for all sufficiently large i. Suppose $u_{\varepsilon_i} > \alpha$ without loss of generality. Let $\phi \in C_c^1(B_s(x_0))$ be a function satisfying $\phi = 1$ on $B_{\frac{s}{2}}(x_0)$. Multiplying $\phi^2(u_{\varepsilon_i} - 1)$ to (1.2) and using Cauchy's inequality and (A.2), we obtain

$$\begin{aligned} &\int_{B_{\frac{s}{2}}(x_0)} \frac{W'(u_{\varepsilon_i})}{\varepsilon_i}(u_{\varepsilon_i} - 1) + c\varepsilon_i |\nabla u_{\varepsilon_i}|^2 \, dx \\ &\le a_1^{1/2} \left(\int \varepsilon_i \phi^2 (u_{\varepsilon_i} - 1)^2 \, dx \right)^{1/2} + c\varepsilon_i \sup |\nabla \phi|^2 \int_{B_s(x_0)} (u_{\varepsilon_i} - 1)^2 \, dx. \end{aligned} \tag{4.1}$$

Since $(u-1)W'(u) \ge cW(u)$ for $2 > u > \alpha$ for some $c > 0$, (4.1) shows $\lim_{i\to\infty} \mu_{\varepsilon_i}(B_{\frac{s}{2}}(x_0)) = 0$, which is a contradiction to $x_0 \in \operatorname{supp}\mu$. Thus for $r \le t_2$ Theorem 3.10 shows

$$\frac{1}{r}\mu(B_r(x)) \ge \lim_{i\to\infty} \frac{1}{r}\mu_{\varepsilon_i}(B_r(x)) \ge \lim_{i\to\infty} \frac{1}{r}\mu_{\varepsilon_i}(B_{\frac{r}{2}}(x_i)) \ge c_{10}/2.$$

We set $D_2 = c_{10}/2$. $\square$

From the proof of Theorem 4.1, next proposition follows.

Proposition 4.2. *$u_{\varepsilon_i} \to +1$ or $u_{\varepsilon_i} \to -1$ uniformly on each compact subset of $U \backslash \operatorname{supp}\mu$ and $\operatorname{supp}\|\partial\{u_0 = 1\}\| \subset \operatorname{supp}\mu$, where $u_0 = \lim_{i\to\infty} u_{\varepsilon_i}$ and $\|\partial\{u_0 = 1\}\|$ is a measure on U defined by $\|\partial\{u_0 = 1\}\|(U) = \int_U |Du_0|$. (For the details of the measure $\int_U |Du_0|$, see [7].)*

The proof of next proposition is similar to [19, Proposition 4.3] but we include it for the convinence of the reader.

Proposition 4.3.

$$\lim_{i\to\infty} \int_V |\xi_{\varepsilon_i}(u_{\varepsilon_i})| \, dx = 0. \tag{4.2}$$

Proof. Let $|\xi|$ be a Radon measure defined as the limit of $|\xi_{\varepsilon_i}(u_{\varepsilon_i})|$. We need to prove that $|\xi| = 0$. First we show

$$\liminf_{r\to 0} \frac{1}{r} |\xi|(B_r(x)) = 0 \tag{4.3}$$

for all $x \in \operatorname{supp}|\xi|$ by contradiction. Thus we assume that there exist $x_0 \in \operatorname{supp}|\xi|$, $R > 0$ and $b > 0$ with $|\xi|(B_r(x_0)) \geq br$ for $0 < r < R$. Fix δ (e.g. $\delta = 1/2$) and fix $r_1 = \min\{R, t_2\}$ and

$$r_2 = r_1 \exp[-\frac{4}{b}\{(\frac{2}{\delta} + 1)a_1 r_1 + 4D_2\}]. \tag{4.4}$$

By Theorem 4.1 and the definition of $|\xi|$, we may choose large enough i such that $t_1 \varepsilon_i^{\iota} \leq r_2$ and

$$\frac{1}{\tau} \int_{B_\tau(x_0)} \frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} + \frac{W(u_{\varepsilon_i})}{\varepsilon_i} \, dx \leq 2D_2, \qquad \frac{1}{\tau} \int_{B_\tau(x_0)} |\xi_{\varepsilon_i}(u_{\varepsilon_i})| \, dx \geq \frac{b}{2}$$

for all $r_2 \leq \tau \leq r_1$. By Propositon 3.7 and 3.9 we have for $r_2 \leq \tau \leq r_1$

$$\frac{1}{\tau} \int_{B_\tau(x_0)} \{\xi_{\varepsilon_i}(u_{\varepsilon_i})\}_+ \, dx \leq o(1)$$

as $i \to \infty$. Thus for all large i and $r_2 \leq \tau \leq r_1$ we have

$$\begin{aligned} &\frac{1}{\tau} \int_{B_\tau(x_0)} \left(\frac{W(u_{\varepsilon_i})}{\varepsilon_i} - \frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} \right)_+ dx \\ &\quad \geq \frac{1}{\tau} \int_{B_\tau(x_0)} |\xi_{\varepsilon_i}(u_{\varepsilon_i})| \, dx - \frac{1}{\tau} \int_{B_\tau(x_0)} \{\xi_{\varepsilon_i}(u_{\varepsilon_i})\}_+ \, dx \geq \frac{b}{4}. \end{aligned} \tag{4.5}$$

By Theorem 3.11 with $s = r_2$ and $r = r_1$ and using (4.5) we obtain

$$(1+\delta)2D_2 \geq -(\frac{2}{\delta}+1)a_1 r_1 + \frac{b}{4}\log\frac{r_1}{r_2}. \tag{4.6}$$

By (4.4), the right-hand side of (4.6) is estimated from below by $4D_2$. This is a contradiction. The claim with Theorem 4.1 shows

$$\liminf_{r\to 0}\frac{|\xi|(B_r(x))}{\mu(B_r(x))} \leq \liminf_{r\to 0}\frac{|\xi|(B_r(x))}{D_1 r} = 0$$

for all $x \in \mathrm{supp}|\xi|$. A standard result in measure theory then shows that $|\xi| = 0$. □

Next we show that the limit measure μ has a well-defined curvature.

Theorem 4.4. *The support of μ is a 1-rectifiable set. Moreover, η defined as the vector-valued limit measure of $\{f_{\varepsilon_i}\nabla u_{\varepsilon_i}\}_i^\infty$ in* Sect.1 *is absolutely continuous with respect to μ. $f = \frac{d\eta}{d\mu} \in L^2(\mu)$ is the generalized curvature of μ and satisfies*

$$\int |f|^2\,d\mu \leq \liminf_{i\to\infty}\frac{1}{\varepsilon_i}\int |f_{\varepsilon_i}|^2\,dx. \tag{4.7}$$

Proof. We consider the rectifiability of $\mathrm{supp}\,\mu$ first. By Proposition 4.3 and by recalling the definitions (2.2), we have $|\mu_\varepsilon - \mathrm{trace}\,\nu_\varepsilon^{\alpha\beta}| = |\xi_\varepsilon(u_\varepsilon)| \to 0$ in $L^1_{loc}(U)$. This shows

$$\mathrm{trace}\,\nu = \mu \tag{4.8}$$

in the limit. The lower density bound (Theorem 4.1), (4.8) and the rectifiability theorem (Propostion 2.1) show that $\mathrm{supp}\,\mu$ is a 1-rectifiable set and

$$\Phi(x) = \mathrm{proj}^{\perp}_{T_x(\mathrm{supp}\,\mu)}(x), \tag{4.9}$$

where $\Phi = \frac{d\nu}{d\mu} \in L^\infty(\mu,\mathcal{F})$. The fact that η is absolutely continuous with respect to μ follows from

$$\left(\int \phi\,d|\eta|\right)^2 \leq \liminf_{i\to\infty}\frac{1}{\varepsilon_i}\int |f_{\varepsilon_i}|^2\,dx \lim_{i\to\infty}\int \phi^2\varepsilon_i|\nabla u_{\varepsilon_i}|^2\,dx \leq a_1\int \phi^2 d\,\mu \tag{4.10}$$

for $\phi \in C_c(U)$, where we used (A.2) and (4.2). Moreover, by taking supremum of the left-hand side over ϕ with $\int \phi^2\,d\mu \leq 1$, (4.10) shows (4.7). To show that f is the curvature of μ, (2.4-5) gives

$$\int \mathrm{div}\phi\,d\mu_{\varepsilon_i} - \int \frac{\partial\phi^\alpha}{\partial x^\beta}\,d\nu_{\varepsilon_i}^{\alpha\beta} = -\int \phi\cdot\nabla u_{\varepsilon_i} f_{\varepsilon_i}\,dx \tag{4.11}$$

for $\phi \in C^1_c(U,\mathbf{R}^2)$. The limit of (4.11) gives

$$\int \mathrm{div}\phi \, d\mu - \int \frac{\partial \phi^\alpha}{\partial x^\beta}\, d\nu^{\alpha\beta} = -\int \phi \cdot f \, d\mu. \tag{4.12}$$

By (4.9), the left-hand side is $\int \mathrm{div}_{T_x(\mathrm{supp}\mu)}\phi \, d\mu$, where the integrand is the divergence restricted to the tangent line which exists a.e. on $\mathrm{supp}\,\mu$. The relation (4.12) shows that f is the generalized (mean) curvature in the sense of varifold [1]. □

5 Integrality of the limit interface

The remaining part of the paper concerns (iii) of Theorem 1.1, namely, we need to prove that the densities of the measure μ are integer multiple of 2σ for a.e. on $\mathrm{supp}\,\mu$. The proof is very similar to [8, Section 5] though one needs to modify the argument as in [15, Section 5]. Thus we write the outline and often omit the details.

Lemma 5.1. *Suppose $B_2 \subset V$. Given $s > 0$ there exist constants $b > 0$ and $\varepsilon_9 > 0$ depending only on a_1, E_0, W and s such that*

$$\int_{B_1\cap\{|u_\varepsilon|\geq 1-b\}} \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) dx \leq s \tag{5.1}$$

if $\varepsilon \leq \varepsilon_9$.

Proof. The estimate for $\int_{B_1\cap\{|u_\varepsilon|\geq 1-b\}} \frac{W(u_\varepsilon)}{\varepsilon}\, dx$ can be obtained by the same argument as in [20, Prop. 4.5]. To estimate the gradient term, one shows that replacing u_ε by v_ε causes a small error, which can be estimated as in Section 3. One then uses (3.19) to show that the gradient term is also small. □

We define $T : \mathbf{R}^2 \to \mathbf{R}$ by $T(x_1,x_2) = x_1$ and $T^\perp : \mathbf{R}^2 \to \mathbf{R}$ by $T^\perp(x) = x_2$. Also we define $n = (n_1,n_2) = \frac{\nabla u}{|\nabla u|}$ where $|\nabla u| \neq 0$ and $n = (0,0)$ where $|\nabla u| = 0$.

Lemma 5.2. *Suppose*

(1) $N \geq 1$ is an integer, Y is a subset of $\mathbf{R}^2$, $0 < R < \infty$, $1 < M < \infty$, $0 < a < \infty$, $0 < \varepsilon < 1$, $0 < \eta < 1$, $0 < E_0 < \infty$ and $-\infty \leq t_1 < t_2 \leq \infty$.

(2) Y *has no more than* $N+1$ *elements,* $T(x)=0$ *and* $t_1+a<T^{\perp}(x)<t_2+a$ *for all* $x\in Y$ *and* $|x-\tilde{x}|>3a$ *for any distinct* $x,\tilde{x}\in Y$.

(3) $(M+1)\mathrm{diam}Y<R$, *and denote* $\widetilde{R}=M\mathrm{diam}Y$.

(4) *On* $\{y\in\mathbf{R}^2\,|\,\mathrm{dist}(y,Y)<R\}$, u_ε *and* f_ε *satisfy* (1.2), $\int_{B_R(x)}|f_\varepsilon|^2\,dx\le\eta\varepsilon$ *and* $\int_a^R\frac{1}{r^2}\int_{B_r(x)}\{\xi_\varepsilon(u_\varepsilon)\}_+\,dxdr\le\eta$ *for each* $x\in Y$.

(5) *For each* $x\in Y$,

$$\int_0^R\frac{d\tau}{\tau^2}\int_{B_\tau(x)\cap\{y_2=t_j\}}|e_\varepsilon(u_\varepsilon)(y_2-x_2)-\varepsilon(y-x)\cdot\nabla u_\varepsilon D_2u_\varepsilon|\,d\mathcal{H}^1(y)\le\eta \tag{5.2}$$

for $j=1,2$. *Here* $e_\varepsilon(u_\varepsilon)=\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{W(u_\varepsilon)}{\varepsilon}$.

(6) *For each* $x\in Y$ *and* $a\le r\le R$,

$$\int_{B_r(x)}|\xi_\varepsilon(u_\varepsilon)|+(1-(n_2)^2)\varepsilon|\nabla u_\varepsilon|^2\,dy\le\eta r \tag{5.3}$$

and

$$\int_{B_\tau(x)}\varepsilon|\nabla u_\varepsilon|^2\,dy\le E_0r. \tag{5.4}$$

Then the following hold:

(A) *There exists* $t_3\in(t_1,t_2)$ *such that* $|T^{\perp}(x)-t_3|\ge a$ *and*

$$\begin{aligned}&\int_0^{\widetilde{R}}\frac{d\tau}{\tau^2}\int_{B_\tau(x)\cap\{y_2=t_3\}}|e_\varepsilon(u_\varepsilon)(y_2-x_2)-\varepsilon(y-x)\cdot\nabla u_\varepsilon D_2u_\varepsilon|\,d\mathcal{H}^1(y)\\&\le 3(N+1)NM(\eta+E_0^{\frac{1}{2}}\eta^{\frac{1}{2}})\end{aligned} \tag{5.5}$$

holds for each $x\in Y$.

(B) *Define* $Y_1=Y\cap\{x\,|\,t_1<T^{\perp}(x)<t_3\}$, $Y_2=Y\cap\{x\,|\,t_3<T^{\perp}(x)<t_2\}$,
$S_0=\{x\,|\,t_1<T^{\perp}(x)<t_2$ *and* $\mathrm{dist}(Y,x)<R\}$,
$S_1=\{x\,|\,t_1<T^{\perp}(x)<t_3$ *and* $\mathrm{dist}(Y_1,x)<\widetilde{R}\}$,
$S_2=\{x\,|\,t_3<T^{\perp}(x)<t_2$ *and* $\mathrm{dist}(Y_2,x)<\widetilde{R}\}$,
Then Y_1 *and* Y_2 *are non-empty and for all* $0<\delta<1$

$$\begin{aligned}&\frac{1}{\widetilde{R}}\{\mu_\varepsilon(S_1)+\mu_\varepsilon(S_2)\}\\&\le(1+\frac{1}{M})(\frac{1}{1-\delta})\{(1+\delta)\frac{1}{R}\mu_\varepsilon(S_0)+(\frac{2}{\delta}+1)\eta R+3\eta\}.\end{aligned} \tag{5.6}$$

Proof. Set $S = \{t_1 < T^{\perp}(x) < t_2\}$. We establish the monotonicity formula restricted on S. Let $\rho(y) : \mathbf{R}^2 \to \mathbf{R}$ be a smooth approximation to the charactaristic function of S. For $x \in Y$ we may assume $x = 0$ by a suitable translation. Let ζ be a smooth approximation to the characteristic function of $(-\infty, 1)$. We substitute $\phi = y\zeta(\frac{|y|}{r})\rho(y)$ into (3.2) and multiply the result by $-\frac{1}{r^2}$. After letting $\zeta \to \chi_{(-\infty,1)}$ and $\rho \to \chi_S$ and by similarly proceeding as in Lemma 3.1, we obtain for $0 < \delta < 1$

$$\begin{aligned}(1-\delta)\frac{1}{s}\mu_\varepsilon(B_s \cap S) &\le (1+\delta)\frac{1}{R}\mu_\varepsilon(B_R \cap S) + (\frac{2}{\delta}+1)\eta R \\ &+ \int_s^R \frac{1}{r^2}\int_{B_r\cap S}\{\xi_\varepsilon(u_\varepsilon)\}_+ \, dxdr \\ &+ \int_s^R \frac{1}{r^2}\int_{B_r\cap \partial S}|y_2 e_\varepsilon(u_\varepsilon) - \varepsilon D_2 u(y\cdot Du)| \, d\mathcal{H}^1(y)dr.\end{aligned} \tag{5.7}$$

By *(4)* and *(5)* applied to (5.7) we obtain

$$(1-\delta)\frac{1}{s}\mu_\varepsilon(B_s \cap S) \le (1+\delta)\frac{1}{R}\mu_\varepsilon(B_R \cap S) + (\frac{2}{\delta}+1)\eta R + 3\eta. \tag{5.8}$$

By the definition of $\widetilde{R}$, $S_1 \cup S_2 \subset B_{(\widetilde{R}+\mathrm{diam}Y)} \cap S \subset S_0$. Thus by (5.8) we obtain

$$\begin{aligned}\frac{1}{\widetilde{R}}\{\mu_\varepsilon(S_1) + \mu_\varepsilon(S_2)\} &\le \frac{1}{\widetilde{R}}\mu_\varepsilon(B_{(\widetilde{R}+\mathrm{diam}Y)} \cap S) \\ &\le \frac{1}{\widetilde{R}}(\frac{\widetilde{R}+\mathrm{diam}Y}{1-\delta})\{(1+\delta)\frac{1}{R}\mu_\varepsilon(S_0) + (\frac{2}{\delta}+1)\eta R + 3\eta\}.\end{aligned} \tag{5.9}$$

Since $\widetilde{R} = M\mathrm{diam}Y$ we obtain *(B)*. The proof of *(A)* is similar to [8, Lemma 5.4]. ☐

Next lemma can be proved by using Lemma 5.2 inductively.

Lemma 5.3. *Corresponding to each* R, E_0, s *and* N *such that* $0 < R < \infty$, $0 < E < \infty$, $0 < s < 1$ *and* N *is a positive integer, there exists* $\eta > 0$ *with the following property.*
Assume the following.

(1) $Y \subset \mathbf{R}^2$ *has no more than* $N+1$ *elements,* $T(y) = 0$ *for all* $y \in Y$, $a > 0$, $|y - z| > 3a$ *for all* $y, z \in Y$ *and* $\mathrm{diam}Y \le \eta R$.

(2) *On* $\{x \in \mathbf{R}^2 \,|\, \mathrm{dist}(x, Y) < R\}$, u_ε *and* f_ε *satisfies* (1.2), $\int |f_\varepsilon|^2 \, dx \le \eta\varepsilon$ *and*

$$\int_a^R \frac{dr}{r^2}\int_{B_r(x)}\{\xi_\varepsilon(u_\varepsilon)\}_+ \, dy \le \eta R \quad \textit{for each } x \in Y.$$

(3) For each $y \in Y$ and $a \le r \le R$,

$$\int_{B_r(y)} |\xi_\varepsilon(u_\varepsilon)| + (1-(\nu_2)^2)\varepsilon|\nabla u_\varepsilon|^2 dy \le \eta r,$$

$$\int_{B_r(x)} \varepsilon|\nabla u_\varepsilon|^2\, dy \le E_0 r.$$

Then we have

$$\sum_{y\in Y} \frac{1}{a}\mu_\varepsilon(B_a(y)) \le s + \frac{1+s}{R}\mu_\varepsilon(\{x \,|\, \mathrm{dist}(Y,x) < R\}). \tag{5.10}$$

The next Lemma 5.4 is identical to [15, Lemma 5.5] .

Lemma 5.4. *Given $0 < s < 1$ and $0 < b < 1$, there exist $0 < \eta < 1$ and $1 < L < \infty$, depending on W, with the following property. Let $0 < \varepsilon < 1$. Suppose u_ε and f_ε satisfies* (1.2) *on $B_{4\varepsilon L}(0)$, with $\int |f|^2\, dx \le \varepsilon\eta$, $|u(0)| \le 1 - b$ and*

$$\int_{B_{4\varepsilon L}(0)} (|\xi_\varepsilon(u_\varepsilon)| + (1-(n_2)^2)\varepsilon|\nabla u_\varepsilon|^2)\, dy \le \eta(4\varepsilon L).$$

Then we have

$$|u_\varepsilon(0, x_2)| \ge 1 - \frac{b}{2} \text{ for } L\varepsilon \le |x_2| \le 3L\varepsilon, \tag{5.11}$$

$$\left|\frac{1}{2L\varepsilon}\mu_\varepsilon(B_{L\varepsilon}(0)) - 2\sigma\right| \le s \tag{5.12}$$

and

$$\left|\int_{-L\varepsilon}^{L\varepsilon} \frac{W(u(0,x_2))}{\varepsilon}\, dx_2 - \sigma\right| \le s. \tag{5.13}$$

Proof. We rescale the domain by ε for convenience. Let $q : \mathbf{R}^2 \to (-1, 1)$ be the unique solution of the ODE

$$\begin{aligned} &q'(t) = \{2W(q(t))\}^{\frac{1}{2}} \text{ for } t \in \mathbf{R} \\ &q(0) = u(0). \end{aligned} \tag{5.14}$$

We note that

$$\int_{-\infty}^{\infty} \frac{|q'(t)|^2}{2}\, dt = \int_{-\infty}^{\infty} \{\frac{W(q(t))}{2}\}^{\frac{1}{2}} q'(t)\, dt = \int_{-1}^{1} \{\frac{W(s)}{2}\}^{\frac{1}{2}} ds = \sigma. \tag{5.15}$$

We also identify q on $\mathbf{R}^2$ by $q(x_1, x_2) = q(x_2)$. Let b and s be given. For large L, we have

$$\left| \frac{1}{2L} \int_{B_L} \frac{|\nabla q|^2}{2} + W(q)\, dx - 2\sigma \right| \leq \frac{s}{8}, \quad \left| \int_{-L}^{L} W(q(t))\, dt - \sigma \right| \leq \frac{s}{8}, \quad (5.16)$$

and

$$|q(t)| \geq 1 - \frac{b}{4} \text{ for } L \leq t \leq 3L. \quad (5.17)$$

We show the claim of the theorem by contradiction. Assume that there exists a sequence $\{\eta_i\}_{i=1}^{\infty} \subset \mathbf{R}$ and $\{\widetilde{u}_i\}$ with $\eta_i \to 0$ as $i \to \infty$ satisfying for $0 < L < \infty$,

$$|\widetilde{u}_i(0)| \leq 1 - b, \qquad \int_{B_L} |\varepsilon_i \widetilde{f}_i|^2\, dx \leq \eta_i \varepsilon_i, \quad (5.18)$$

$$\int_{B_L} |\xi(\widetilde{u}_i)| + (1 - (n_2)^2)|\nabla \widetilde{u}_i|^2\, dx \leq 4\eta_i L \quad (5.19)$$

but one of the following fails,

$$\left| \frac{1}{2L} \int_{B_L} \frac{|\nabla \widetilde{u}_i|^2}{2} + W(\widetilde{u}_i)\, dx - 2\sigma \right| \geq s, \quad \left| \int_{-L}^{L} W(\widetilde{u}_i(0, x^2) dx^2 - \sigma \right| \geq s \quad (5.20)$$

or there exist x_2 with $L \leq |x_2| \leq 3L$ satisfying

$$|\widetilde{u}_i(0, x_2)| \leq 1 - \frac{b}{2}. \quad (5.21)$$

By $W^{2,2}$ bound there exists a subsequence of $\{\widetilde{u}_i\}$ (denoted by the same notation) converging weakly to $\widetilde{u}_\infty \in W^{2.2}$. By (5.18) $\widetilde{u}_\infty$ satisfies

$$-\Delta \widetilde{u}_\infty + W'(\widetilde{u}_\infty) = 0. \quad (5.22)$$

By (5.19) we also have $\int_{B_L} \frac{|D_1 \widetilde{u}_\infty|^2}{2}\, dx = 0$ and $|D_2 \widetilde{u}_\infty|^2 = 2W(\widetilde{u}_\infty)$. As we may assume $D_2 \widetilde{u}_\infty > 0$, we obtain $D_2 \widetilde{u}_\infty = \{2W(\widetilde{u}_\infty)\}^{\frac{1}{2}}$. Thus, $\widetilde{u}_\infty = q$. Since $\widetilde{u}_\infty$ satisfies (5.15-17), and the convergence is strong in $W^{1,p}$ for any $1 \leq p < \infty$, we obtain a contradiction to (5.20-21). □

The proof of Theorem 5.5 proceeds just like [15, Prop. 5.2].

Theorem 5.5. *The density of the limit measure μ is an integer multiple of σ for $\mathcal{H}^1$ a.e. on* suppμ.

Proof. By the rectifiability of $\operatorname{supp}\mu$ and the lower density bound μ has an approximate tangent line for $\mathcal{H}^1$ a.e. on $\operatorname{supp}\mu$. Fix such a point and choose coordinates so that the point is the origin and the approximate tangent line is $P = \{x = (x_1, x_2) \,|\, x_2 = 0\}$. We consider the scaling $\widetilde{u}_{\varepsilon_i}(x) = u_{\varepsilon_i}(r_i x)$ and $\widetilde{f}_{\varepsilon_i}(x) = f_{\varepsilon_i}(r_i x)$ with $r_i \to 0$. Let $\widetilde{\varepsilon}_i = \frac{\varepsilon_i}{r_i}$. $\widetilde{u}_{\varepsilon_i}$ satisfies

$$-\widetilde{\varepsilon}_i \Delta \widetilde{u}_{\varepsilon_i} + \frac{W'(\widetilde{u}_{\varepsilon_i})}{\widetilde{\varepsilon}_i} = r_i \widetilde{f}_i.$$

Define a sequence of measures $\mu_{\widetilde{\varepsilon}_i}^{r_i}$ by $\frac{\widetilde{\varepsilon}_i |\nabla \widetilde{u}_i|^2}{2} + \frac{W(\widetilde{u}_i)}{\widetilde{\varepsilon}_i}\, dx$. By the rectifiability of $\operatorname{supp}\mu$ we may choose a suitable subsequence (by choosing smaller ε_i if necessary)

$$\lim_{i\to\infty} \mu_{\widetilde{\varepsilon}_i}^{r_i}(B_1(0)) = \theta \int_{P\cap B_1} d\mathcal{H}^1 = 2\theta. \tag{5.23}$$

Here, θ is density of μ, that is,

$$\theta = \lim_{r\to 0} \frac{1}{2r}\mu(B_r(0)). \tag{5.24}$$

Write $\widetilde{u}_{\varepsilon_i} = u_{\varepsilon_i}$ and $\mu_{\widetilde{\varepsilon}_i}^{r_i} = \mu_{\varepsilon_i}$. Since $\nu^{11} = 0$, we obtain

$$\lim_{i\to\infty} \int_{B_3(0)} (1-n_2^2) \frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2}\, dx = \lim_{i\to\infty} \int_{B_3(0)} \frac{\varepsilon_i (D_1 u_{\varepsilon_i})^2}{2}\, dx = \int_{B_3(0)} d\nu^{11} = 0. \tag{5.25}$$

Suppose N is the smallest positive integer greater than $\frac{\theta}{2\sigma}$. Fix an arbitrary small $s > 0$. By Lemma 5.1 we may choose $b > 0$ so that

$$\int_{B_3(0)\cap\{|u_{\varepsilon_i}|\geq 1-b\}} \left(\frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} + \frac{W(u_{\varepsilon_i})}{\varepsilon_i} \right) dx \leq s \tag{5.26}$$

for sufficiently large i. With those s, b and $R = 1$, we choose η and L via Lemma 5.3 and 5.4. For large i we define

$$G_i = B_2(0) \cap \{|u_{\varepsilon_i}| \leq 1 - b\} \cap \left\{ x \;\middle|\; \int_{B_r(x)} |\xi_\varepsilon(u_{\varepsilon_i})| + (1-(n_2)^2)\varepsilon_i |\nabla u_{\varepsilon_i}|^2\, dx \leq \eta r \text{ for all } 4\varepsilon_i L \leq r \leq 1 \right\}. \tag{5.27}$$

By Besicovich's covering theorem and monotonicity formula,

$$\mu_{\varepsilon_i}(B_2(0) \cap \{|u_{\varepsilon_i}| \leq 1-b\} \setminus G_i) + \mathcal{L}^1(T(B_2(0) \cap \{|u_{\varepsilon_i}| \leq 1-b\} \setminus G_i)) \leq \frac{c}{\eta} \int_{B_3(0)} |\xi_\varepsilon(u_{\varepsilon_i})| + (1-n_2^2)\varepsilon_i |\nabla u_{\varepsilon_i}|^2 \tag{5.28}$$

which goes to 0 as $i \to \infty$ by (4.2) and (5.25). For any $x = (x_1, 0) \in B_1(0) \cap P$ define $Y = \{x_1\} \times \cup_{k=1}^{m} \{s_k\} \subset T^{-1}(x) \cap G_i$ with $s_1 < s_2 < \cdots < s_m$ where m is the largest integer so the each element of Y is separated by at least $3L\varepsilon_i$. We prove that Y does not contain more than $N-1$ elements. First note that all the assumptions for applying Lemma 5.3 and 5.4 are satisfied . Since

$$\sup_{x \in B_1(0) \cap P} \frac{1}{2} \int_{B_1(x)} (\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i})\, dy \leq \theta + s \tag{5.29}$$

for large i, Y having more than $N-1$ elements would imply that

$$2\sigma N \leq s(N+1) + (1+s)(\theta + s) \tag{5.30}$$

by Lemma 5.4. This would be a contradiction to $\frac{\theta}{2\sigma} < N$ for sufficiently small s depending only on N. Finally

$$\begin{aligned} 2\theta = \lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0)) &= \lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0) \cap \{|u_{\varepsilon_i}| \leq 1-b\} \cap G_i) \\ &+ \lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0) \cap \{|u_{\varepsilon_i}| \leq 1-b\} \setminus G_i) + \lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0) \cap \{|u_{\varepsilon_i}| \geq 1-b\}). \end{aligned} \tag{5.31}$$

Note that $m \leq N-1$. As $T^{-1}(x) \cap G_i \subset \{x_1\} \times \cup_{k=1}^{m} (s_k - L\varepsilon_i, s_k + L\varepsilon_i)$, by Lemma 5.4, we obtain

$$\begin{aligned} &\lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0) \cap \{|u_{\varepsilon_i}| \leq 1-b\} \cap G_i) \\ &\leq \lim_{i\to\infty} \left\{ \int_{B_1(0)\cap G_i} |\xi_{\varepsilon_i}(u_{\varepsilon_i})|\, dy + 2\int_{B_1(0)\cap P} \int_{T^{-1}(x)\cap G_i} \frac{W(u_{\varepsilon_i})}{\varepsilon_i}\, d\mathcal{H}^1 dy \right\} \\ &\leq \lim_{i\to\infty} \left\{ \int_{B_1(0)\cap G_i} |\xi_{\varepsilon_i}(u_{\varepsilon_i})|\, dy + 2\sum_{k=1}^{N-1} \int_{B_1(0)\cap T(G_i)\cap P} \int_{s_k - L\varepsilon_i}^{s_k + L\varepsilon_i} \frac{W(u_{\varepsilon_i})}{\varepsilon_i}\, dx_2 dx_1 \right\} \\ &\leq s + 4(\sigma + s)(N-1), \end{aligned} \tag{5.32}$$

where we note that $\mathcal{H}^1(B_1) = 2$. By (5.26) and (5.28), we obtain

$$\lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0) \cap \{|u_{\varepsilon_i}| \leq 1-b\} \setminus G_i) + \lim_{i\to\infty} \mu_{\varepsilon_i}(B_1(0) \cap \{|u_{\varepsilon_i}| \geq 1-b\}) \leq s. \tag{5.33}$$

Since $s > 0$ is arbitrary, $2\theta \leq 4\sigma(N-1)$. By the assumption of N we obtain $\theta = 2\sigma(N-1)$. This shows that the density at this point is integer multiple of 2σ. □

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