DOUBLING CONDITIONS FOR HARMONIC MEASURE IN JOHN DOMAINS

HIROAKI AIKAWA AND KENTARO HIRATA

Abstract. We introduce new classes of domains, i.e., semi-uniform domains and inner semi-uniform domains. Both of them are intermediate between the class of John domains and the class of uniform domains. Under the capacity density condition, we show that the harmonic measure of a John domain D satisfies certain doubling conditions if and only if D is a semi-uniform domain or an inner semi-uniform domain.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, $\delta_D(x) = \operatorname{dist}(x, \partial D)$ and $x_0 \in D$. Let us recall some nonsmooth domains. By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \ldots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison. We write $B(x, R)$ and $S(x, R)$ for the open ball and the sphere of center at x and radius R, respectively.

We say that D is a *John domain* with John constant $c_J > 0$ and John center $x_0 \in D$ if each $x \in D$ can be joined to x_0 by a rectifiable curve $\gamma \subset D$ such that

$$\delta_D(y) \geq c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma, \tag{1}$$

where $\gamma(x, y)$ and $\ell(\gamma(x, y))$ stand for the subarc of γ connecting x and y and its length, respectively. In general, $0 < c_J < 1$. We say that D is a *uniform domain* if there exists a constant $A > 1$ such that each pair of points $x, y \in D$ can be joined by a rectifiable curve $\gamma \subset D$ such that $\ell(\gamma) \leq A|x - y|$ and

$$\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq A\delta_D(z) \quad \text{for all } z \in \gamma. \tag{2}$$

We call this curve γ a *cigar curve* connecting x and y. See [11, 12, 15]. If the complement of a uniform domain D satisfies the corkscrew condition, then D becomes an *NTA domain* ([13]). Observe that connectivity of a uniform domain can be extended from $x, y \in D$ to $x, y \in \overline{D}$. We introduce the following class of domains.

Definition 1. We say that D is a *semi-uniform domain* if every pair of points $x \in D$ and $y \in \partial D$ can be joined by a rectifiable curve γ such that $\gamma \setminus \{y\} \subset D$, $\ell(\gamma) \leq A|x - y|$ and (2) holds.

2000 *Mathematics Subject Classification.* 31B05, 31B25, 31C35.
Key words and phrases. John domain, semi-uniform domain, inner semi-uniform domain, harmonic measure, doubling condition, capacity density condition.
This work was supported in part by Grant-in-Aid for Scientific Research (B) (No. 15340046) Japan Society for the Promotion of Science.

A Denjoy domain is a typical semi-uniform domain which is not necessarily uniform. The relationships among above domains are summarized as

$$\text{NTA} \subsetneq \text{Uniform} \subsetneq \text{Semi-uniform} \subsetneq \text{John}. \tag{3}$$

Let $\omega(x, E, U)$ be the harmonic measure of the set E in an open set U evaluated at x. Jerison-Kenig [13] proved that harmonic measure of an NTA domain D satisfies the *strong doubling condition*: there is a constant $A_0 > 2$ such that

$$\omega(x, B(\xi, 2R) \cap \partial D, D) \le A\omega(x, B(\xi, R) \cap \partial D, D) \quad \text{for } x \in D \setminus B(\xi, A_0 R), \tag{4}$$

where $\xi \in \partial D$ and $R > 0$ small, say $R \le R_{SD}$. If (4) holds only for some fixed point $x = x_0$, we say that the harmonic measure of D satisfies the *doubling condition*. Obviously the strong doubling condition implies the doubling condition. Moreover, they showed that a bounded planar simply connected domain D is an NTA domain if and only if the harmonic measures both for D and $\overline{D}^c$ satisfy the doubling condition ([13, Theorem 2.7]). Kim and Langmeyer [14] gave the one-sided analogue; a bounded planar Jordan domain is a John domain if and only if the harmonic measure only for D satisfies the doubling condition. Their argument is based on complex analysis as well.

Balogh-Volberg [6, 7] showed a doubling condition similar to (4) in a planar uniformly John domain, or inner uniform domain (see Definition 2 below and the remarks before it). They also pointed out that there is a planar inner uniform domain for which (4) fails to hold. Indeed, let D be the complement of the line segments $[-1, 1]$ and $L_\theta = \{te^{-i\theta} : 0 \le t \le 1\}$ with $0 < \theta < \pi/2$. Let $B_1 = B(te^{-i\theta}, ct)$ and $B_2 = B(te^{-i\theta}, 2ct)$, where $\frac{1}{2}\sin\theta < c < \sin\theta$. Since $B_1 \cap [-1, 1] = \emptyset$ and $B_2 \cap [-1, 1] \ne \emptyset$, we have $\omega(x_0, B_1 \cap \partial D, D) \approx t^{\pi/(\pi-\theta)}$ and $\omega(x_0, B_2 \cap \partial D, D) \approx t$ as $t \to 0$. Hence $\omega(x_0, B_2 \cap \partial D, D)/\omega(x_0, B_1 \cap \partial D, D) \to \infty$. See Figure 1.

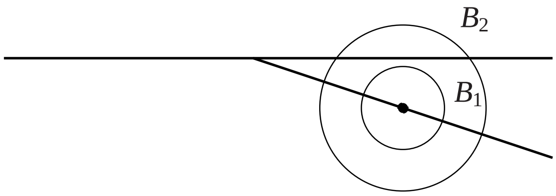

FIGURE 1. Harmonic measure fails to satisfy the doubling condition.

In this paper, we characterize John domains whose harmonic measure satisfies (4), the strong doubling condition. There is a John domain with polar boundary whose harmonic measure vanishes. For such domains any doubling conditions for harmonic measure is hopeless. To avoid such pathological domains, we assume the *capacity density condition* (abbreviated to CDC). See Section 3 for its definition. If $n = 2$, then the CDC coincides with the uniform perfectness of the boundary. Our main result is as follows.

Theorem 1. *Let D be a John domain with John constant c_J and suppose the CDC holds. Then the following are equivalent:*

(i) *D is a semi-uniform domain.*
(ii) *The harmonic measure of D satisfies the strong doubling condition, i.e.,* (4) *holds whenever $\xi \in \partial D$ and $R > 0$ is small.*

(iii) *For each* $\alpha > 1/c_J$, *there exist constants* $A > 1$ *and* $\tau > 0$ *depending only on* D *and* α *such that*

$$\omega(x, \partial D \cap B(\xi, R), D) \ge \frac{1}{A}\left(\frac{R}{R + |x - \xi|}\right)^{\tau} \quad \textit{for } |x - \xi| < \alpha\delta_D(x), \tag{5}$$

whenever $\xi \in \partial D$ *and* $R > 0$ *is small.*

Remark 1. The constant $1/c_J$ is a threshold; if α is less than c_J, then $\{x \in D : |x - \xi| < \alpha\delta_D(x)\}$ may be an empty set.

Next, we state a version of Theorem 1 with respect to the *inner diameter metric* $\rho_D(x, y)$ defined by

$$\rho_D(x, y) = \inf\{\mathrm{diam}(\gamma) : \ \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\},$$

where $\mathrm{diam}(\gamma)$ denotes the diameter of γ. If we replace $\mathrm{diam}(\gamma)$ by $\ell(\gamma)$ in the above definition, then we obtain the *inner length distance* $\lambda_D(x, y)$. Obviously $|x-y| \le \rho_D(x, y) \le \lambda_D(x, y)$. It turns out, however, that ρ_D and λ_D are comparable for a John domain (Väisälä [16, Theorem 3.4]). We say that D is an *inner uniform domain* or *uniformly John domain* if there exists a constant $A > 1$ such that every pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ with $\ell(\gamma) \le A\rho_D(x, y)$ and (2). See Balogh-Volberg [6, 7] and Bonk-Heinonen-Koskela [9]; actually, the latter use $\lambda_D(x, y)$ instead of $\rho_D(x, y)$ in the definition. However, ρ_D and λ_D are equivalent as noted above. For a John domain D, we can consider the completion D^* with respect to ρ_D ([4, Proposition 2.1]). Then $\partial^* D = D^* \setminus D$ is the ideal boundary of D with respect to ρ_D. Observe that connectivity of an inner uniform domain can be extended from $x, y \in D$ to $x, y \in D^*$. See [4, Lemma 2.1].

Definition 2. We say that D is an *inner semi-uniform domain* if every pair of points $x \in D$ and $y \in \partial^* D$ can be joined by a rectifiable curve γ such that $\gamma \setminus \{y\} \subset D$, $\ell(\gamma) \le A\rho_D(x, y)$ and (2) holds.

Let $\xi^* \in \partial^* D$. Then there are a point $\xi \in \partial D$ and a sequence $\{x_j\} \subset D$ converging to ξ with respect to the Euclidean metric as well as converging to ξ^* with respect to ρ_D. We say that ξ^* lies over ξ and define the projection π from D^* to $\overline{D}$ by $\pi(\xi^*) = \xi$ for $\xi^* \in \partial^* D$ and $\pi|_D = \mathrm{id}|_D$. Let $B_\rho(\xi, R)$ be the connected component of $B(\xi, R) \cap D$ from which ξ^* is accessible. We observe that $B_\rho(\xi, R)$ plays a role of a ball with center at ξ^* in the completion D^* ([4, Lemma 2.2]). Let $\Delta_\rho(\xi^*, R) = \{x^* \in \partial^* D : \rho_D(x^*, \xi^*) < R\}$. This is a surface ball with respect to ρ_D. Consider a version of (4) with respect to ρ_D: there is a constant $A_0 > 2$ such that

$$\omega(x, \Delta_\rho(\xi^*, 2R), D) \le A\omega(x, \Delta_\rho(\xi^*, R), D) \quad \text{for } x \in D \setminus B_\rho(\xi^*, A_0 R), \tag{6}$$

where $\xi^* \in \partial^* D$ and $R > 0$ small. We have the following.

Theorem 2. *Let* D *be a John domain with John constant* c_J *and suppose the CDC holds. Then the following are equivalent:*

(i) D *is an inner semi-uniform domain.*

(ii) (6) *holds whenever* $\xi^* \in \partial^* D$ *and* $R > 0$ *is small.*

(iii) *For each* $\alpha > 1/c_J$, *there exist constants* $A > 1$ *and* $\tau > 0$ *depending only on* D *and* α *such that*

$$\omega(x, \Delta_\rho(\xi^*, R), D) \ge \frac{1}{A}\left(\frac{R}{R + \rho_D(x, \xi^*)}\right)^{\tau} \quad \textit{for } \rho_D(x, \xi^*) < \alpha\delta_D(x),$$

whenever $\xi^* \in \partial^* D$ and $R > 0$ is small.

By definition, a semi-uniform domain is an inner semi-uniform domain. The domain in Figure 1 is an inner semi-uniform domain and satisfies (6). Thus (3) is refined as follows:

$$\begin{array}{ccccc} & & \text{Inner uniform} & & \\ & \subsetneqq & & \subsetneqq & \\ \text{NTA} \subsetneqq \text{Uniform} & & & & \text{Inner semi-uniform} \subsetneqq \text{John.} \\ & \subsetneqq & & \subsetneqq & \\ & & \text{Semi-uniform} & & \end{array}$$

There is no direct relationship between the class of inner uniform domains and the class of semi-uniform domains. Theorem 2 and the above implications yield that (4) is a property stronger than (6). This is not straightforward from their definitions.

The plan of the present paper is as follows: In Section 2, some preliminary notions such as the quasihyperbolic metric and local reference points will be recalled. The relationship between the Green function and the harmonic measure will be extensively studied in Section 3. Theorem 1 will be proved in Section 4 based on the results in Section 3. Theorem 2 can be proved almost in the same manner. Necessary lemmas will be stated in the last section.

2. Preliminaries

We define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D. We observe that the shortest length of the Harnack chain connecting x and y is comparable to $k_D(x, y) + 1$. Therefore, the Harnack inequality yields that there is a constant $A > 1$ depending only on n such that

$$\exp(-A(k_D(x, y) + 1)) \le \frac{h(x)}{h(y)} \le \exp(A(k_D(x, y) + 1)) \tag{7}$$

for every positive harmonic function h on D. We say that D satisfies a quasihyperbolic boundary condition if

$$k_D(x, x_0) \le A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A \quad \text{for all } x \in D. \tag{8}$$

It is easy to see that a John domain satisfies the quasihyperbolic boundary condition (see [10, Lemma 3.11]). We have more precise estimate ([3, Proposition 2.1]).

Lemma A. *Let D be a John domain with John constant c_J. Then there exist a positive integer N and constants $R_D > 0$ and $A > 1$ depending only on D with the following property: for every $\xi \in \partial D$ and $0 < R < R_D$ there are N points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ such that $A^{-1}R \le \delta_D(y_i^R) \le R$ for $i = 1, \dots, N$ and*

$$\min_{i=1,\dots,N} \{k_{D_R}(x, y_i^R)\} \le A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap B(\xi, R/2),$$

where $D_R = D \cap B(\xi, 8R)$. Moreover, every $x \in D \cap B(\xi, R/2)$ can be connected to some y_i^R by a curve $\gamma \subset D_R$ with $\ell(\gamma(x, z)) \le A\delta_D(z)$ for all $z \in \gamma$.

If the conclusion of the above lemma holds, then we say that ξ has a *system of local reference points* $y_1^R, \dots, y_N^R$ of order N.

3. Green function and harmonic measure

We begin by recalling the capacity density condition (abbreviated to CDC).

Definition 3. By Cap we denote the logarithmic capacity if $n = 2$, and the Newtonian capacity if $n \geq 3$. We say that the CDC holds if there exist constants $A > 0$ and $R_D > 0$ such that

$$\operatorname{Cap}(B(\xi, R) \setminus D) \geq \begin{cases} AR & \text{if } n = 2, \\ AR^{n-2} & \text{if } n \geq 3, \end{cases}$$

whenever $\xi \in \partial D$ and $0 < R < R_D$.

It is well known that the CDC is equivalent to the uniformly Δ-regularity ([5]). Hence there is a positive constant β such that if $\xi \in \partial D$ and $0 < r < R$ are small, then

$$\sup_{D\cap B(\xi,r)} \omega(\cdot, D \cap S(\xi, R), D \cap B(\xi, R)) \leq A(r/R)^\beta, \tag{9}$$

so that there is a constant $A_1 > 1$ such that

$$\inf_{D\cap B(\xi,R/A_1)} \omega(\cdot, \partial D \cap B(\xi, R), D) \geq \frac{1}{2}. \tag{10}$$

Lemma 1. *Let $G(x, y)$ be the Green function for D with the CDC. Suppose $\delta_D(y) = R > 0$ is small. Then*

$$G(x, y) \approx R^{2-n} \quad \text{for } x \in S(y, R/2). \tag{11}$$

Moreover, there is a positive constant β such that

$$G(x, y) \leq AR^{2-n}\Big(\frac{\delta_D(x)}{R}\Big)^\beta \quad \text{for } x \in D \setminus B(y, R/2). \tag{12}$$

Proof. If $n \geq 3$, then the first assertion is obvious. The planar case will be given in Lemma 3. For the proof of (12) we may assume that $\delta_D(x) < R/4$. Let $x^* \in \partial D$ be a point such that $|x^* - x| = \delta_D(x) < R/4$. Then $|x^* - y| \geq \delta_D(y) = R$. Hence $B(x^*, R/2) \cap B(y, R/2) = \emptyset$, so that the maximum principle and (11) yield

$$G(x, y) \leq AR^{2-n}\omega(x, S(y, R/2), D\setminus B(y, R/2)) \leq AR^{2-n}\omega(x, D\cap S(x^*, R/2), D\cap B(x^*, R/2)).$$

Hence we have (12) from (9). □

Lemma 2. *Let $G(x, y)$ be the Green function for D with the CDC. Suppose $\delta_D(y) = R > 0$ is small and $G(x, y) > A_2R^{2-n}$. Then there is a curve γ connecting x and y in D such that $\ell(\gamma) \leq AR$ and $\delta_D(z) \geq R/A$ for all $z \in \gamma$, where A depends only on D and A_2.*

Proof. Observe from the maximum principle that $\Omega = \{z \in D : G(z, y) > A_2R^{2-n}\}$ is a connected open set. If $n \geq 3$, then $G(z, y) \leq |z - y|^{2-n}$, so that $\operatorname{diam}\Omega \leq AR$. The planar case will be given in Lemma 3. Let γ be a curve connecting x and y in Ω. Lemma 1 says that

$$A_2R^{2-n} < G(z, y) \leq AR^{2-n}\Big(\frac{\delta_D(z)}{R}\Big)^\beta \quad \text{for } z \in \Omega \setminus B(y, \delta_D(y)/2).$$

Hence $\delta_D(z) \geq R/A$ for all $z \in \gamma$. Since $\operatorname{diam}\gamma \leq \operatorname{diam}\Omega \leq AR$, taking a polygonal curve, we may modify γ so that $\ell(\gamma) \leq AR$. The proof is complete. □

Lemma 3. *Let $n = 2$ and let $G(x, y)$ be the Green function for D with the CDC. Suppose $\delta_D(y) = R > 0$ is small. Then the following statements hold:*

(i) $G(x, y) \approx 1$ *for* $x \in S(y, R/2)$.
(ii) *Let* $\Omega = \{z \in D : G(z, y) > A_2\}$. *Then* $\operatorname{diam}\Omega \le AR$.

Proof. (i) Let $M_0 = \sup_{S(y,R/2)} G(\cdot, y)$. By the maximum principle $G(\cdot, y) \le M_0$ on $D \setminus B(y, R/2)$. Let $y^* \in \partial D$ be a point such that $|y^* - y| = \delta_D(y) = R$. By (9) we find a positive constant $\varepsilon_1 < 1/4$ such that $G(\cdot, y) \le M_0/2$ on $D \cap B(y^*, 2\varepsilon_1 R)$. Let y' be the point in $\overline{yy^*}$ with $|y' - y^*| = \varepsilon_1 R$. Then $G(\cdot, y) \le M_0/2$ on $B(y', \varepsilon_1 R)$. Cover the sphere $S(y, (1 - \varepsilon_1)R)$ with finitely many balls with the same radii $\varepsilon_1 R$. We may assume that $B(y', \varepsilon_1 R)$ appears in the covering, consecutive balls have an intersection with volume comparable to $(\varepsilon_1 R)^n$, and the number of balls is bounded by a constant depending only on ε_1 and the dimension n. Applying the mean value property of $G(\cdot, y)$, we can conclude $G(\cdot, y) \le (1 - c)M_0$ on $S(y, (1 - \varepsilon_1)R)$, and hence on $D \setminus B(y, (1 - \varepsilon_1)R)$ with $0 < c < 1$ independent of R and y (see [2, Proof of Lemma 2]). Let G_B be the Green function for $B = B(y, (1 - \varepsilon_1)R)$. Then

$$G_B(x, y) = G(x, y) - \widehat{R}^{D\setminus B}_{G(\cdot,y)}(x) \ge G(x, y) - (1 - c)M_0 \quad \text{for } x \in B,$$

where $\widehat{R}^{D\setminus B}_{G(\cdot,y)}$ is the regularized reduced function of $G(\cdot, y)$ relative to $D \setminus B$ in D. Take the supremum over $S(y, R/2)$ to obtain

$$A \ge M_0 - (1 - c)M_0 = cM_0.$$

Thus (i) follows, since $G(x, y) \ge G_{B(y,R)}(x, y) = \log 2$ for $x \in S(y, R/2)$.

(ii) For the proof it is sufficient to show the following claim: there is a positive constant λ such that if $\delta_D(y) \le 2|x - y|$ small, then

$$G(x, y) \le A\Big(\frac{\delta_D(y)}{|x - y|}\Big)^{\lambda}. \tag{13}$$

Let $|x - y| = L$ be sufficiently small. The first named author ([2, Lemma 1]) showed the uniform perfectness of ∂D. Hence we find a constant $b \ge 2$ and an increasing sequence $\delta_D(y) = R = R_1 < R_2 < \cdots < R_{k-1} < L \le R_k$ such that $S(y, R_j) \cap \partial D \ne \emptyset$ and that $2 \le R_j/R_{j-1} \le b$ for $j = 1, \dots, k$. Here $R_0 = \delta_D(y)/2$. Let $u = G(\cdot, y)$ in D and let $u = 0$ in $\mathbb{R}^n \setminus D$. Then u is a nonnegative subharmonic function in $\mathbb{R}^n \setminus \{y\}$. We employ an argument similar to (i). Cover the sphere $S(y, R_j)$ with finitely many balls with the same radii $\varepsilon_1 R_j$. We find $y'' \in S(y, R_j) \cap \partial D$. We may assume that $B(y'', \varepsilon_1 R_j)$ appears in the covering, consecutive balls have an intersection with volume comparable to $(\varepsilon_1 R_j)^n$, and the number of balls is bounded by a constant depending only on ε_1 and the dimension n. Moreover, observe that these balls lie outside $B(y, R_{j-1})$. Applying the mean value property of u, we obtain

$$M_j = \sup_{\mathbb{R}^n\setminus B(y,R_j)} u = \sup_{S(y,R_j)} u \le (1 - c)M_{j-1} \le (1 - c)^j M_0$$

for $j = 1, 2, \dots, k$. Since $L \le R_k \le b^k R_0$, it follows that

$$M_k \le \exp(k \log(1 - c) + \log M_0) \le \exp\Big(\log M_0 + \frac{\log(1 - c)}{\log b} \log \frac{L}{R_0}\Big) = M_0\Big(\frac{R}{2L}\Big)^{\lambda}$$

with $\lambda = -\log(1 - c)/\log b$. Thus (13) follows. $\square$

Lemma 4. *Let D be a John domain with the CDC. Let $\xi \in \partial D$ have a system of local reference points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ of order N for $0 < R < R_D$. Then*

$$R^{n-2} \sum_{i=1}^{N} G(x, y_i^R) \le A\omega(x, \partial D \cap B(\xi, 2A_1 R), D) \quad \text{for } x \in D \setminus B(\xi, 2R), \tag{14}$$

where A depends only on D and A_1 is the constant in (10).

Proof. The maximum principle and (11) give

$$R^{n-2} \sum_{i=1}^{N} G(x, y_i^R) \approx 1 \quad \text{for } x \in \bigcup_i S(y_i^R, \delta_D(y_i^R)/2).$$

Since $\bigcup_i S(y_i^R, \delta_D(y_i^R)/2) \subset D \cap B(\xi, 2R)$, it follows from (10) that

$$\omega(x, \partial D \cap B(\xi, 2A_1 R), D) \approx 1 \quad \text{for } x \in \bigcup_i S(y_i^R, \delta_D(y_i^R)/2).$$

The maximum principle completes the proof. □

The following is an estimate opposite to Lemma 4.

Lemma 5. *Let D be a John domain. Let $\xi \in \partial D$ have a system of local reference points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ of order N for $0 < R < R_D$. Then*

$$\omega(x, \partial D \cap B(\xi, R/8), D) \le AR^{n-2} \sum_{i=1}^{N} G(x, y_i^R) \quad \text{for } x \in D \setminus B(\xi, R/4), \tag{15}$$

where A depends only on D.

Proof. For $0 < r < \delta_D(x_0)/2$ let $U(r) = \{x \in D : \delta_D(x) < r\}$. Then each point $x \in U(r)$ can be connected to x_0 by a curve such that (1) holds. Hence, $B(x, A_3 r) \setminus U(r)$ includes a ball with radius r, provided A_3 is large. This implies that

$$\omega(x, U(r) \cap S(x, A_3 r), U(r) \cap B(x, A_3 r)) \le 1 - \varepsilon_0 \quad \text{for } x \in U(r)$$

with $0 < \varepsilon_0 < 1$ depending only on A_3 and the dimension. Let $R \ge r$ and repeat this argument with the maximum principle. Then

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \le A \exp\Big(- A' \frac{R}{r} \Big) \quad \text{for } x \in U(r) \tag{16}$$

for some $A' > 0$. See [1, Lemma 1] for details.

Let $0 < R < R_D$. For each $x \in D \cap B(\xi, R/2)$ there is a local reference point $y(x) \in \{y_1^R, \dots, y_N^R\}$ such that

$$k_D(x, y(x)) \le A \log \frac{R}{\delta_D(x)} + A$$

by Lemma A. Let $y'(x) \in S(y(x), \delta_D(y(x))/2)$. Observe that $k_{D\setminus\{y(x)\}}(x, y'(x)) \le A \log(R/\delta_D(x)) + A$. Letting $u(x) = R^{n-2} \sum_{i=1}^{N} G(x, y_i^R)$, we obtain from (7) and (11) that

$$u(x) \ge A\Big(\frac{\delta_D(x)}{R}\Big)^{\lambda} \quad \text{for } x \in D \cap B(\xi, R/2)$$

with some $\lambda > 0$ depending only on D.

Now let us employ a modified version of the box argument (cf. [8] and [1, Lemma 2]). Let $D_j = \{x \in D : \exp(-2^{j+1}) \le u(x) < \exp(-2^j)\}$ and $U_j = \{x \in D : u(x) < \exp(-2^j)\}$. Then we see that

$$U_j \cap B(\xi, R/2) \subset \Big\{x \in D : \delta_D(x) < AR \exp\Big(-\frac{2^j}{\lambda}\Big)\Big\}. \tag{17}$$

Define sequences R_j, r_j and ρ_j by $R_0 = 3R/8$, $r_0 = R/8$ and

$$\rho_j = \frac{3}{4\pi^2}\frac{R}{j^2}, \quad R_j = \frac{3}{8}R - \sum_{k=1}^{j}\rho_k, \quad r_j = \frac{R}{8} + \sum_{k=1}^{j}\rho_k$$

for $j \ge 1$. We observe

$$\frac{R}{8} = r_0 < r_1 < \cdots < \frac{R}{4} < \cdots < R_1 < R_0 = \frac{3}{8}R. \tag{18}$$

Let $A(\xi, r, R) = B(\xi, R) \setminus \overline{B(\xi, r)}$ be the annulus with center at ξ and radii r and R. Since $R_{j-1} - R_j = r_j - r_{j-1} = \rho_j$, it follows that if $x \in A(\xi, r_j, R_j)$, then $B(x, \rho_j) \subset A(\xi, r_{j-1}, R_{j-1})$. See Figure 2.

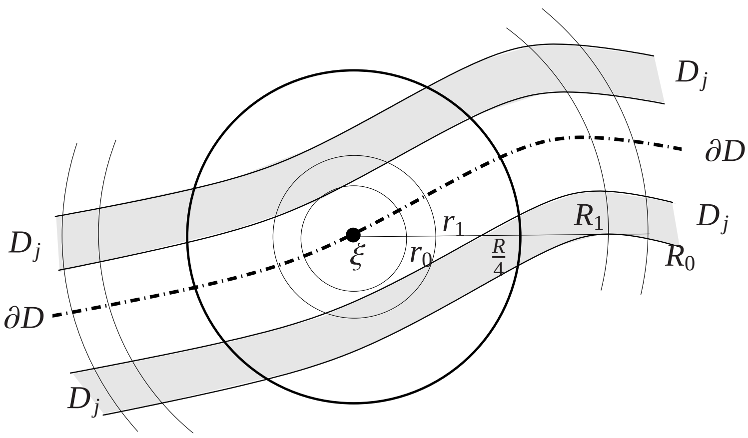

Figure 2. A box argument for annuli.

The maximum principle, (16) and (17) give

$$\begin{aligned} &\omega(x, U_j \cap \partial A(\xi, r_{j-1}, R_{j-1}), U_j \cap A(\xi, r_{j-1}, R_{j-1})) \\ &\le \omega(x, U_j \cap S(x, \rho_j), U_j \cap B(x, \rho_j)) \le A \exp\Big(-Aj^{-2}\exp\Big(\frac{2^j}{\lambda}\Big)\Big) \end{aligned} \tag{19}$$

for $x \in U_j \cap A(\xi, r_j, R_j)$. Let $\omega_0 = \omega(\cdot, \partial D \cap B(\xi, R/8), D)$ and put

$$d_j = \begin{cases} \displaystyle\sup_{x \in D_j \cap A(\xi, r_j, R_j)} \frac{\omega_0(x)}{u(x)} & \text{if } D_j \cap A(\xi, r_j, R_j) \neq \emptyset, \\ 0 & \text{if } D_j \cap A(\xi, r_j, R_j) = \emptyset. \end{cases}$$

By (18) it is sufficient to show that d_j is bounded by a constant independent of R and j. Apply the maximum principle to $U_j \cap A(\xi, r_{j-1}, R_{j-1})$ to obtain

$$\omega_0(x) \le \omega(x, U_j \cap \partial A(\xi, r_{j-1}, R_{j-1}), U_j \cap A(\xi, r_{j-1}, R_{j-1})) + d_{j-1}u(x).$$

Divide the both sides by $u(x)$ and take the supremum over $D_j \cap A(\xi, r_j, R_j)$. Then (19) yields

$$d_j \le A \exp\left(2^{j+1} - Aj^{-2}\exp(2^j/\lambda)\right) + d_{j-1}.$$

Since $\sum_j \exp\left(2^{j+1} - Aj^{-2}\exp(2^j/\lambda)\right) < \infty$, we obtain $\sup_{j\ge 0} d_j < \infty$. Thus (15) follows from the maximum principle. □

4. Proof of Theorem 1

Proof of Theorem 1. (i) $\Longrightarrow$ (ii). Suppose first D is a semi-uniform domain. Let $\xi \in \partial D$ and let $R > 0$ be sufficiently small. Then by Lemma 5 and scaling we find a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, 16R)$ such that

$$\omega(x, \partial D \cap B(\xi, 2R), D) \le AR^{n-2}\sum_{i=1}^{N} G(x, y_i) \quad \text{for } x \in D \setminus B(\xi, 4R).$$

Let $\{y_1^*, \dots, y_N^*\} \subset D \cap S(\xi, R/2A_1)$ be a system of local reference points. Lemma 4 implies that

$$R^{n-2}\sum_{i=1}^{N} G(x, y_i^*) \le A\omega(x, \partial D \cap B(\xi, R), D) \quad \text{for } x \in D \setminus B(\xi, R/A_1).$$

By the semi-uniformity, each y_i is connected to ξ by a cigar curve γ_i. Let $y_i' \in \gamma_i \cap S(\xi, R/4A_1)$. Observe $k_D(y_i', y_j^*) \le A$ for some j. Since $k_D(y_i, y_j^*) \le k_D(y_i, y_i') + k_D(y_i', y_j^*) \le A$ and $y_i, y_j^*, y_i' \in D \cap \overline{B(\xi, 16R)}$, it follows that

$$G(x, y_i) \approx G(x, y_j^*) \quad \text{for } x \in D \setminus B(\xi, 32R),$$

so that

$$\omega(x, \partial D \cap B(\xi, 2R), D) \le A\omega(x, \partial D \cap B(\xi, R), D) \quad \text{for } x \in D \setminus B(\xi, 32R).$$

Hence (4) follows with $A_0 = 32$.

(ii) $\Longrightarrow$ (iii). Suppose $\xi \in \partial D$ and $R > 0$ is small and $|x - \xi| < \alpha\delta_D(x)$. It is easy to see from (10) that (5) holds for $|x - \xi| \le R/A_1$. Now let $r = |x - \xi| > R/A_1$. Suppose first $A_0 r > R_{SD}$ with R_{SD} for (4). Take $y \in D \cap S(\xi, R/A_1)$ with $\delta_D(y) \ge R/A$. Then $k_D(x, y) \le A\log(1/R) + A$, so that (7) and (10) give

$$\omega(x, \partial D \cap B(\xi, R), D) \ge \frac{1}{A}R^{\tau}\omega(y, \partial D \cap B(\xi, R), D) \ge \frac{1}{2A}R^{\tau}$$

with some $\tau > 0$ depending only on D and α. Since $R + |x - \xi| \ge R_{SD}/A_0$, we obtain (5). Suppose next $A_0 r \le R_{SD}$. We find a local reference point $y_i \in D \cap S(\xi, A_0 A_1 r)$ such that

$$k_D(x, y_i) \le A(D, \alpha). \tag{20}$$

Note that $R < A_1 r$. Applying (4) with y_i in please of x repeatedly, we obtain

$$\omega(y_i, \partial D \cap B(\xi, A_1 r), D) \le A\left(\frac{r}{R}\right)^{\tau}\omega(y_i, \partial D \cap B(\xi, R), D),$$

where A and τ depend only on A_1 and the doubling constant. Therefore (7) and (20) give

$$\omega(x, \partial D \cap B(\xi, A_1 r), D) \le A\left(\frac{r}{R}\right)^{\tau}\omega(x, \partial D \cap B(\xi, R), D).$$

Since $\omega(x, \partial D \cap B(\xi, A_1 r), D) \geq 1/2$ by (10), we obtain (5) as

$$\Big(\frac{R}{r}\Big)^{\tau} \geq \Big(\frac{R}{R + |x - \xi|}\Big)^{\tau}.$$

(iii) $\Longrightarrow$ (i). Let $x \in D$ and $\xi \in \partial D$. We may assume that $|x - \xi| = R$ is small. Then by Lemma A and scaling we find a system of local reference points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ and $y_1^{2R}, \dots, y_N^{2R} \in D \cap S(\xi, 2R)$. We claim that every y_i^{2R} can be connected to some y_j^R by a curve γ with $\ell(\gamma) \leq AR$ and $\delta_D(z) \geq R/A$ for all $z \in \gamma$. By (iii) and Lemma 5,

$$\frac{1}{A} \leq \omega(y_i^{2R}, \partial D \cap B(\xi, R/8), D) \leq AR^{n-2} \sum_{j=1}^{N} G(y_i^{2R}, y_j^R).$$

Hence there is y_j^R such that $G(y_i^{2R}, y_j^R) \geq AR^{2-n}$. Lemma 2 gives a curve γ connecting y_i^{2R} to y_j^R in D such that $\ell(\gamma) \leq AR$ and $\delta_D(z) \geq R/A$ for all $z \in \gamma$. Thus the claim follows.

Now the proof is easy. By Lemma A we find a point y_i^{2R} which can be connected to x by a cigar curve with length bounded by AR. The claim gives a point y_j^R which can be connected to y_i^{2R} by a cigar curve with length bounded by AR. See Figure 3.

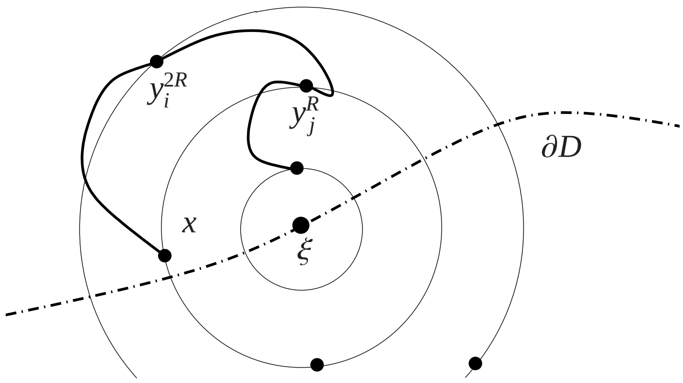

FIGURE 3. A cigar curve connecting x to ξ.

Repeat the claim again. We find a point $y_k^{R/2}$ which can be connected to y_j^R by a cigar curve with length bounded by $AR/2$. Thus we can construct a cigar curve connecting points as follows:

$$x \to y_i^{2R} \to y_j^R \to y_k^{R/2} \to \cdots \to \xi.$$

The length of the curve is bounded by AR. Thus D is a semi-uniform domain. □

5. PROOF OF THEOREM 2

Replacing Lemmas A, 4 and 5 by the following three lemmas, we can prove Theorem 2 almost in the same way as for Theorem 1. The details are left to the reader. Recall π is the natural projection from D^* to $\overline{D}$. Let $\xi^* \in \partial^* D$, $\xi = \pi(\xi^*)$ and $S_\rho(\xi^*, R) = \{x \in D : \rho_D(x, \xi^*) = R\}$. Observe that $S_\rho(\xi^*, R) \subset S(\xi, R)$, that $B_\rho(\xi^*, R)$ is the connected component of $B(\xi, R) \cap D$ from which ξ^* is accessible, and that the boundary of $B_\rho(\xi^*, R)$ is included in $S_\rho(\xi^*, R) \cup \partial D$. The following lemma corresponds to Lemma A.

Lemma 6. *Let D be a John domain with John constant c_J. Then there exist a positive integer M and constants $R_D > 0$ and $A > 1$ depending only on D with the following property: for every $\xi^* \in \partial^* D$ and $0 < R < R_D$ there are M points $y_1^R, \dots, y_M^R \in S_\rho(\xi^*, R)$ such that $A^{-1}R \le \delta_D(y_i^R) \le R$ for $i = 1, \dots, M$ and*

$$\min_{i=1,\dots,M} \{k_{B_\rho(\xi^*,8R)}(x, y_i^R)\} \le A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in B_\rho(\xi^*, R/2).$$

Moreover, every $x \in B_\rho(\xi^*, R/2)$ can be connected to some y_i^R by a curve $\gamma \subset B_\rho(\xi^*, 8R)$ with $\ell(\gamma(x,z)) \le A\delta_D(z)$ for all $z \in \gamma$.

If the conclusion of the above lemma holds, then we say that $\xi^* \in \partial^* D$ has a *system of inner local reference points* $y_1^R, \dots, y_M^R$ of order M. We emphasize that inner local reference points $y_1^R, \dots, y_M^R$ lie on $S_\rho(\xi^*, R)$ and that $M \le N$ in general. The following two lemmas replace Lemmas 4 and 5.

Lemma 7. *Let D be a John domain with the CDC. Let $\xi^* \in \partial^* D$ have a system of inner local reference points $y_1^R, \dots, y_M^R \in S_\rho(\xi^*, R)$ of order M. Then*

$$R^{n-2} \sum_{i=1}^{M} G(x, y_i^R) \le A\omega(x, \Delta_\rho(\xi^*, 2A_1 R), D) \quad \text{for } x \in D \setminus B_\rho(\xi^*, 2R),$$

where A depends only on D.

Lemma 8. *Let D be a John domain. Let $\xi^* \in \partial^* D$ have a system of inner local reference points $y_1^R, \dots, y_M^R \in S_\rho(\xi^*, R)$ of order M. Then*

$$\omega(x, \Delta_\rho(\xi^*, R/8), D) \le AR^{n-2} \sum_{i=1}^{M} G(x, y_i^R) \quad \text{for } x \in D \setminus B_\rho(\xi^*, R/4),$$

where A depends only on D.

References

[1] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan **53** (2001), no. 1, 119–145.

[2] ———, *Hölder continuity of the Dirichlet solution for a general domain*, Bull. London Math. Soc. **34** (2002), no. 6, 691–702.

[3] H. Aikawa, K. Hirata, and T. Lundh, *Martin boundary points of a John domain and unions of convex sets*, J. Math. Soc. Japan **58** (2006), no. 1, 247–274.

[4] H. Aikawa, T. Lundh, and T. Mizutani, *Martin boundary of a fractal domain*, Potential Anal. **18** (2003), no. 4, 311–357.

[5] A. Ancona, *On strong barriers and an inequality of Hardy for domains in* $\mathbf{R}^n$, J. London Math. Soc. (2) **34** (1986), no. 2, 274–290.

[6] Z. Balogh and A. Volberg, *Boundary Harnack principle for separated semihyperbolic repellers, harmonic measure applications*, Rev. Mat. Iberoamericana **12** (1996), no. 2, 299–336.

[7] ———, *Geometric localization, uniformly John property and separated semihyperbolic dynamics*, Ark. Mat. **34** (1996), no. 1, 21–49.

[8] R. F. Bass and K. Burdzy, *A boundary Harnack principle in twisted Hölder domains*, Ann. of Math. (2) **134** (1991), no. 2, 253–276.

[9] M. Bonk, J. Heinonen, and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, Astérisque (2001), no. 270, viii+99.

[10] F. W. Gehring and O. Martio, *Lipschitz classes and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 203–219.

[11] ———, *Quasiextremal distance domains and extension of quasiconformal mappings*, J. Analyse Math. **45** (1985), 181–206.
[12] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74 (1980).
[13] D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147.
[14] K. Kim and N. Langmeyer, *Harmonic measure and hyperbolic distance in John disks*, Math. Scand. **83** (1998), no. 2, 283–299.
[15] J. Väisälä, *Uniform domains*, Tohoku Math. J. (2) **40** (1988), no. 1, 101–118.
[16] ———, *Relatively and inner uniform domains*, Conform. Geom. Dyn. **2** (1998), 56–88 (electronic).

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: `aik@math.sci.hokudai.ac.jp`

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: `hirata@math.sci.hokudai.ac.jp`