Finite Dimensional Semisimple Q-Algebras

Takahiko Nakazi* and Takanori Yamamoto*

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan.

Abstract A Q-algebra can be represented as an operator algebra on an infinite dimensional Hilbert space. However we don't know whether a finite n-dimensional Q-algebra can be represented on a Hilbert space of dimension n except $n = 1, 2$. It is known that a two dimensional Q-algebra is just a two dimensional commutative operator algebra on a two dimensional Hilbert space. In this paper we study a finite n-dimensional semisimple Q-algebra on a finite n-dimensional Hilbert space. In particular we describe a three dimensional Q-algebra of the disc algebra on a three dimensional Hilbert space. Our studies are related to the Pick interpolation problem for a uniform algebra.

AMS Classification: 46J05; 46J10; 47A30; 47B38

Keywords: commutative Banach algebra; semisimple Q-algebra; three dimension; norm; Pick interpolation

Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: nakazi@math.sci.hokudai.ac.jp

Takanori Yamamoto
Department of Mathematics
Hokkai-Gakuen University
Sapporo 062-8605, Japan
E-mail: yamamoto@elsa.hokkai-s-u.ac.jp

1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X. If I is a closed ideal of A, then the quotient algebra A/I is a commutative Banach algebra with unit. In this paper, if a Banach algebra $\mathcal{B}$ is isometrically isomorphic to A/I, then $\mathcal{B}$ is called a Q-algebra. (F. Bonsall and J. Duncan called $\mathcal{B}$ an IQ-algebra.(cf. [1], p.270)) B. Cole (cf. [1], p.272) showed that any Q-algebra is an operator algebra on a Hilbert space H, that is, there exists an isometric isomorphism to an operator algebra on H. Let μ be a probability measure on X and $H^2(\mu)$ the closure of A in $L^2(\mu)$. $H^2(\mu) \cap I^{\perp}$ denotes the annihilator of I in $H^2(\mu)$. Let P be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^{\perp}$ and for any $f \in A$ put

$$S_f^{\mu}\phi = P(f\phi), \quad (\phi \in H^2(\mu) \cap I^{\perp}).$$

Then $S_{f+k}^{\mu} = S_f^{\mu}$ for k in I and $\|S_f^{\mu}\| \leq \|f+I\|$. S^{μ} is the map of A/I on operators on $H^2(\mu)\cap I^{\perp}$ which sends $f + I \to S_f^{\mu}$ for each f in A. Hence S^{μ} is a contractive homomorphism from A into $B(H^2(\mu) \cap I^{\perp})$ where $B(H^2(\mu) \cap I^{\perp})$ is the set of all bounded linear operators on $H^2(\mu) \cap I^{\perp}$. The kernel of S^{μ} contains I. Then we say that S^{μ} gives a contractive representation of A/I into $B(H^2(\mu) \cap I^{\perp})$. If $\|S_f^{\mu}\| = \|f + I\|, (f \in A)$ then $\ker S^{\mu} = I$ and we say that S^{μ} gives an isometric representation of A/I on $H^2(\mu) \cap I^{\perp}$.

Problem 1. Prove that any finite n-dimensional Q-algebra can be represented on a Hilbert space of finite dimension n.

If S^{μ} is isometric then we solve Problem 1. In fact, T. Nakazi and K. Takahashi (cf. [9]) solved Problem 1 for $n = 2$ in this way. It seems to be unknown for $n \geq 3$.

Problem 2. Describe a finite n-dimensional Q-algebra in finite n-dimensional commutative operator algebras with unit on a Hilbert space of finite dimension n.

Problem 2 is clear for $n = 1$ and it was proved by S.W. Drury (cf. [4]) and T. Nakazi (cf. [8]) that a 2-dimensional commutative operator algebra with unit on a Hilbert space is just a Q-algebra. J. Holbrook (cf. [6]) proved that von Neumann's inequality

$$\|p(T)\| \leq \|p\|_{\infty}$$

can fail for some polynomials p in 3 variables, where $T = (T_1, T_2, T_3)$ is a triple of commuting contractions on $\mathbf{C}^4$, and T_1, T_2, T_3 are simultaneously diagonalizable. Then we can construct a 4-dimensional commutative matrix algebra with unit on $\mathbf{C}^4$, which is not a Q-algebra. If $n \geq 4$, then this implies that the set of all n-dimensional Q-algebra A/I is smaller than the set of all set of all n-dimensional commutative oparator algebras with unit on an n-dimensional Hilbert space. If $n = 3$, then Problem 2 has not been solved yet. In this paper, we concentrate on a semisimple commutative Banach algebra and we study Problem 2. In Section 2, we will prove several general results of finite dimensional semisimple Q-algebras that will be used in the latter sections. In Section 3, we will study arbitrary n-dimensional semisimple Q-algebras for $n = 2, 3$. In Section 4, we will study the isometric representation of A/I. In Section 5, we will describe completely 3-dimensional semisimple Q-algebras of the disc algebra in 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

2. Semisimple and commutative matrix algebra

In this section, we study 3-dimensional semisimple commutative operator algebras on a 3-dimensional Hilbert space. In particular, we study when two such operator algebras are isometric or unitary equivalent. S. McCullough and V. Paulsen (cf. [7], Proposition 2.2) proved the similar result of Proposition 2.3. We use Lemma 2.1 to prove Proposition 2.2 and Proposition 2.3.

Lemma 2.1. *Let $n \geq 2$ and let H be an n-dimensional Hilbert space which is spanned by $k_1, k_2, ..., k_n$. Let*

$$\psi_1 = \frac{k_1}{\|k_1\|}, \quad \psi_j = \frac{k_j - \sum_{i=1}^{j-1}\langle k_j, \psi_i\rangle\psi_i}{\|k_j - \sum_{i=1}^{j-1}\langle k_j, \psi_i\rangle\psi_i\|} \ (2 \leq j \leq n).$$

Then $\{\psi_1, \ \dots \ , \psi_n\}$ is an orthonormal basis for H. Let $P_1, ..., P_n$ be the idempotent operators on H such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For $1 \leq m \leq n$, let $a_{ij}^{(m)} = \langle P_m\psi_j, \psi_i\rangle$, $(1 \leq i, j \leq n)$. Then $P_m = (a_{ij}^{(m)})_{1\leq i,j\leq n}$ is an $n \times n$ matrix such that

$$P_1 = \begin{pmatrix} B_1 \\ O \end{pmatrix}, \ \dots \ , \ P_m = \begin{pmatrix} O & B_m \\ O & O \end{pmatrix}, \ \dots \ , \ P_n = \begin{pmatrix} O & B_n \end{pmatrix},$$

where B_m is an $m \times (n - m + 1)$ matrix such that

$$B_1 = \begin{pmatrix} 1 & . & . & . & a_{1n}^{(1)} \end{pmatrix}, \ \dots \ , \ B_m = \begin{pmatrix} a_{1m}^{(m)} & . & . & . & a_{1n}^{(m)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & . & . & . & a_{mn}^{(m)} \end{pmatrix}, \ \dots \ , \ B_n = \begin{pmatrix} a_{1n}^{(n)} \\ . \\ . \\ . \\ 1 \end{pmatrix}.$$

Then $a_{mm}^{(m)} = 1$, and for $m \geq 2$,

$$a_{im}^{(m)} = \frac{\langle k_m, \psi_i\rangle}{\|k_m - \sum_{h=1}^{m-1}\langle k_m, \psi_h\rangle\psi_h\|},$$

and for $m + 1 \leq j \leq n$,

$$a_{ij}^{(m)} = \frac{-\sum_{h=m}^{j-1}\langle k_j, \psi_h\rangle a_{ih}^{(m)}}{\|k_j - \sum_{h=1}^{j-1}\langle k_j, \psi_h\rangle\psi_h\|}.$$

Since this lemma is proved by elementary calculations, the proof is omitted. It is well known that any n-dimensional semisimple commutative Banach algebra with unit I is spanned by commuting idempotents $P_1, ..., P_n$ satisfying $P_1 + ... + P_n = I$.

Proposition 2.2. *In Lemma 2.1, for* $1 \le m \le n$, $\mathrm{rank} P_m = 1$, *and* $\mathcal{B} = \mathrm{span}\{P_1, ..., P_n\}$ *is an* n*-dimensional semisimple commutative operator algebra with unit on* H. *Then* $n \times n$ *matrix* $(a_{ij}^{(m)})$ *for* P_m *with respect to* $\{\psi_1, ..., \psi_n\}$ *is* $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$, *and*

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & a_{12}^{(1)} & . & . & . & . & a_{1n}^{(1)} \\ 0 & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & 0 \end{pmatrix}, \quad P_2 = (a_{ij}^{(2)}) = \begin{pmatrix} 0 & a_{12}^{(2)} & a_{13}^{(2)} & . & . & . & a_{1n}^{(2)} \\ 0 & 1 & a_{23}^{(2)} & . & . & . & a_{2n}^{(2)} \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \end{pmatrix},$$

$$......, \quad P_n = (a_{ij}^{(n)}) = \begin{pmatrix} 0 & . & . & . & . & 0 & a_{1n}^{(n)} \\ 0 & . & . & . & . & 0 & a_{2n}^{(n)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & a_{n-1n}^{(n)} \\ 0 & . & . & . & . & 0 & 1 \end{pmatrix}.$$

In Lemma 2.1, $a_{ij}^{(m)}$ *is written using* $k_1, ..., k_n$ *and* $\psi_1, ..., \psi_n$.

Proof. By the assumption of Lemma 2.1, $P_i k_i = k_i$ and $P_i k_j = 0$ if $i \ne j$. Hence $\mathrm{rank} P_m = 1$. If $i \ne j$, then $P_i P_j k_m = \delta_{jm} P_i k_j = 0$, $(1 \le m \le n)$. Since $H = \mathrm{span}\{k_1, ..., k_n\}$, this implies that $P_i P_j = 0$ if $i \ne j$. Hence $\mathcal{B}$ is commutative. Since $P_i^2 k_m = \delta_{im} P_i k_m = P_i k_m$, $(1 \le m \le n)$, it follows that $P_i^2 = P_i$. Hence $\mathcal{B}$ is semisimple and n-dimensional. Since $(P_1 + ... + P_n) k_m = P_m k_m = k_m$, $(1 \le m \le n)$, it follows that $P_1 + ... + P_n = I$. Hence $\mathcal{B}$ has a unit I. This completes the proof.

Proposition 2.3. *Let* H *be a 3-dimensional Hilbert space which is spanned by* k_1, k_2, k_3. *Let* $\langle \cdot, \cdot \rangle$ *denote the inner product, and let* $|| \cdot ||$ *denote the norm of* H.

$$\psi_1 = \frac{k_1}{||k_1||}, \quad \psi_2 = \frac{k_2 - \langle k_2, \psi_1 \rangle \psi_1}{||k_2 - \langle k_2, \psi_1 \rangle \psi_1||}, \quad \psi_3 = \frac{k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2}{||k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2||}.$$

Then ψ_1, ψ_2, ψ_3 *is an orthonormal basis in* H. *Let* P_i *be the idempotent operator on* H *such that* $P_i k_i = k_i$, $P_i k_j = 0$ *if* $i \ne j$. *For* $m = 1, 2, 3$, *the* 3×3 *matrix* $(a_{ij}^{(m)})$ *for* P_m *with respect to* $\{\psi_1, \psi_2, \psi_3\}$ *is* $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$. *Then*

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = (a_{ij}^{(2)}) = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = (a_{ij}^{(3)}) = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{||k_1||^2 ||k_2||^2 - |\langle k_1, k_2 \rangle|^2}}, \quad y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{||k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2||},$$

$$z = \frac{-\langle k_3, \psi_2 \rangle}{||k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2||}.$$

Proof. By Proposition 2.2, there exist x, y such that

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 2.1,

$$x = a_{12}^{(1)} = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}},$$

and

$$y = a_{13}^{(1)} = \frac{-\sum_{h=1}^{2} \langle k_3, \psi_h \rangle a_{1h}^{(1)}}{\|k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h\|} = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

By Proposition 2.2, there exist z, w such that

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & w \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & -w-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

because $P_1 + P_2 + P_3 = I$. By Lemma 2.1,

$$z = a_{23}^{(2)} = \frac{-\sum_{h=2}^{2} \langle k_3, \psi_h \rangle a_{2h}^{(2)}}{\|k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h\|} = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

By Lemma 2.1,

$$a_{12}^{(1)} = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}} = -a_{12}^{(2)}.$$

Hence

$$w = a_{13}^{(2)} = \frac{-\sum_{h=2}^{2} \langle k_3, \psi_h \rangle a_{1h}^{(2)}}{\|k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h\|} = \frac{-\langle k_3, \psi_2 \rangle a_{12}^{(2)}}{\|k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h\|} = z a_{12}^{(2)} = -z a_{12}^{(1)} = -xz.$$

This completes the proof.

Theorem 2.4. *Let P_1, P_2, P_3 be idempotent operators defined in Proposition 2.3. Let H' be a 3-dimensional Hilbert space. Let $\mathcal{B}'$ be a 3-dimensional semisimple commutative operator algebra on H'. Then, there are idempotent operators Q_1, Q_2, Q_3 on H', an orthonormal basis $\psi_1', \psi_2', \psi_3'$ in H' and complex numbers x_0, y_0, z_0 such that $\mathcal{B}' = \mathrm{span}\{Q_1, Q_2, Q_3\}$ and, as matrices relative to $\psi_1', \psi_2', \psi_3'$,*

$$Q_1 = \begin{pmatrix} 1 & x_0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -x_0 & -x_0 z_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & x_0 z_0 - y_0 \\ 0 & 0 & -z_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let τ be the map of $\mathcal{B}$ on $\mathcal{B}'$ such that

$$\tau(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3) = \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3, \quad (\lambda_1, \lambda_2, \lambda_3 \in \mathbf{C}).$$

(1) *τ is isometric if and only if*

$$\begin{aligned}
&|x|^2+|y|^2=|x_0|^2+|y_0|^2,\\
&(1+|x|^2)(1+|z|^2)=(1+|x_0|^2)(1+|z_0|^2),\\
&|x|^2+xz\overline{y}=|x_0|^2+x_0z_0\overline{y_0}.
\end{aligned}$$

(2) *τ is induced by a unitary map from H to H' if and only if there are complex numbers u_1,u_2,u_3 such that*

$$|u_1|=|u_2|=|u_3|=1,\quad u_1x=u_2x_0,\quad u_1y=u_3y_0,\quad u_2z=u_3z_0.$$

Then $|x|=|x_0|,\quad |y|=|y_0|,\quad |z|=|z_0|,\quad xz\overline{y}=x_0z_0\overline{y_0}.$

Proof. (1) By the theorem of B. Cole and J. Wermer (cf. [3]), τ is isometric if and only if, writing tr for trace,

$$\operatorname{tr}(P_i^*P_j)=\operatorname{tr}(Q_i^*Q_j),\quad (1\le i,j\le 3).$$

If τ is isometric, then

$$\begin{aligned}
&1+|x|^2+|y|^2=\operatorname{tr}(P_1^*P_1)=\operatorname{tr}(Q_1^*Q_1)=1+|x_0|^2+|y_0|^2,\\
&(1+|x|^2)(1+|z|^2)=\operatorname{tr}(P_2^*P_2)=\operatorname{tr}(Q_2^*Q_2)=(1+|x_0|^2)(1+|z_0|^2),\\
&|x|^2+xz\overline{y}=\operatorname{tr}(P_1^*P_2)=\operatorname{tr}(Q_1^*Q_2)=|x_0|^2+x_0z_0\overline{y_0}.
\end{aligned}$$

Conversely, if three equalities in (1) hold, then

$$\begin{aligned}
&\operatorname{tr}(P_1^*P_1)=1+|x|^2+|y|^2=1+|x_0|^2+|y_0|^2=\operatorname{tr}(Q_1^*Q_1),\\
&\operatorname{tr}(P_2^*P_2)=(1+|x|^2)(1+|z|^2)=(1+|x_0|^2)(1+|z_0|^2)=\operatorname{tr}(Q_2^*Q_2),\\
&\operatorname{tr}(P_1^*P_2)=|x|^2+xz\overline{y}=|x_0|^2+x_0z_0\overline{y_0}=\operatorname{tr}(Q_1^*Q_2),\\
&\operatorname{tr}(P_2^*P_3)=\overline{xz}(y-xz)-|z|^2=\overline{x_0z_0}(y_0-x_0z_0)-|z_0|^2=\operatorname{tr}(Q_2^*Q_3),\\
&\operatorname{tr}(P_3^*P_1)=y(\overline{xz-y})=y_0(\overline{x_0z_0-y_0})=\operatorname{tr}(Q_3^*Q_1),\\
&\operatorname{tr}(P_3^*P_3)=1+|z|^2+|xz-y|^2=1+|z_0|^2+|x_0z_0-y_0|^2=\operatorname{tr}(Q_3^*Q_3).
\end{aligned}$$

(2) Suppose τ is induced by a unitary map $U=(u_{ij})$, $(1\le i,j\le 3)$ from H to H'. Since $UP_1=Q_1U$, it follows that $u_{21}=u_{31}=0$. Since $UP_2=Q_2U$, it follows that $u_{32}=0$. Hence U is an upper triangular matrix. Since the columns of U are pairwise orthogonal, U is a diagonal matrix. Hence there are complex numbers u_1,u_2,u_3 such that u_1,u_2,u_3 are diagonal element of U, and $|u_1|=|u_2|=|u_3|=1$. Since $UP_1=Q_1U$, it follows that $u_1x=u_2x_0$, $u_1y=u_3y_0$. S-ince $UP_2=Q_2U$, it follows that $u_2z=u_3z_0$. The converse is also true. This completes the proof.

Example 2.5. Let $\mathcal{B}_0=\operatorname{span}\{P_1,P_2,P_3\}$, where

$$P_1=\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix},\quad P_2=\begin{pmatrix}0&-1&-1\\0&1&1\\0&0&0\end{pmatrix},\quad P_3=\begin{pmatrix}0&0&0\\0&0&-1\\0&0&1\end{pmatrix},$$

and let $\mathcal{B}_1 = \text{span}\{P_1^*, P_2^*, P_3^*\}$. This is an example which is established by W. Wogen (cf. [3]). He proved that $\mathcal{B}_0$ and $\mathcal{B}_1$ are isometrically isomorphic, and not unitarily equivalent. There is another example as the following. Let $\mathcal{B}_2 = \text{span}\{Q_1, Q_2, Q_3\}$, where

$$Q_1 = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2/3} \\ 0 & 1 & 1/\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & \sqrt{2/3} \\ 0 & 0 & -1/\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{B}_0$ and $\mathcal{B}_2$ are 3-dimensional commutative operator algebras with unit. By the calculation,

$$\begin{aligned} &|x|^2 + |y|^2 = |x_0|^2 + |y_0|^2 = 2, \\ &(1 + |x|^2)(1 + |z|^2) = (1 + |x_0|^2)(1 + |z_0|^2) = 4, \\ &|x|^2 + xz\overline{y} = |x_0|^2 + x_0 z_0 \overline{y_0} = 2. \end{aligned}$$

By (1) of Theorem 2.4, this implies that $\mathcal{B}_0$ and $\mathcal{B}_2$ are isometrically isomorphic. By (2) of Theorem 2.4, $\mathcal{B}_0$ and $\mathcal{B}_2$ are not unitarily equivalent.

3. One to one representation

In this section, we assume that A/I is n-dimensional and semisimple. Hence there exist $\tau_1, ..., \tau_n$ in the maximal ideal space $M(A)$ of A such that $\tau_i \neq \tau_j$ $(i \neq j)$ and $I = \cap_{j=1}^n \ker \tau_j$. S^μ gives a contractive representation of A/I into $B(H^2(\mu) \cap I^\perp)$ and $\dim H^2(\mu) \cap I^\perp \leq \dim A/I = n$. We study when S^μ is one to one from A/I to $B(H^2(\mu) \cap I^\perp)$. It is clear that S^μ is one to one if and only if $\dim H^2(\mu) \cap I^\perp = \dim A/I$. For $1 \leq j \leq n$, there exist $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \text{span}\{f_1 + I, ..., f_n + I\}$. The following two quantities are important to study S^μ. For $1 \leq j \leq n$,

$$\rho_j = \sup\{|\tau_j(f)| \ ; \ f \in \cap_{l \neq j} \ker \tau_l, \ \|f\| \leq 1\}$$

and

$$\rho_j(\mu) = \sup\{|\tau_j(f)| \ ; \ f \in \cap_{l \neq j} \ker \tau_l, \ \|f\|_\mu \leq 1\},$$

where $\|f\|$ denotes the supnorm of f in A and $\|f\|_\mu = \langle f, f \rangle_\mu^{1/2} = \left(\int |f|^2 d\mu\right)^{1/2}$. Then it is easy to see that

$$\|f_j + I\| = \frac{1}{\rho_j}, \quad \|f_j + I\|_\mu = \frac{1}{\rho_j(\mu)},$$

and

$$\|f_j + I\| \geq \|S_{f_j}^\mu\| \geq \|f_j + I\|_\mu.$$

If S^μ is one to one then τ_j has a bounded extension to $H^2(\mu)$. In fact, if S^μ is one to one then $\dim H^2(\mu) \cap I^\perp = n$ and so $\dim H^2(\mu) \cap (\ker \tau_j)^\perp = 1$ for $1 \leq j \leq n$. Then for $1 \leq j \leq n$, there exists $k_j \in H^2(\mu)$ such that

$$\tau_j(f) = \langle f, k_j \rangle_\mu = \int_X f \overline{k_j} d\mu, \ (f \in A).$$

Proposition 3.1. *There exists a one to one contractive representation S^μ of A/I.*

Proof. Since $\tau_j \in M(A)$, there exists a positive representing measure m_j of τ_j on X. Let $\mu = \sum_{j=1}^n m_j/n$. Then

$$|\tau_j(f)| = |\int_X f dm_j| \leq n(\int_X |f|^2 d\mu)^{1/2} = n\|f\|_\mu, \ (f \in A).$$

Hence τ_j has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$ and $\tilde{\tau}_j \neq \tilde{\tau}_i$ $(j \neq i)$. Since $H^2(\mu) \cap I^\perp = \cap_{j=1}^n \ker \tilde{\tau}_j$, $\dim H^2(\mu) \cap I^\perp = n$. Hence $H^2(\mu) \cap I^\perp = \text{span}\{k_1, ..., k_n\}$. Suppose $S_f^\mu = 0$, then $\overline{\tau_j(f)} k_j = (S_f^\mu)^* k_j = 0$. Hence $\tau_j(f) = 0$, $(1 \leq j \leq n)$. Hence $f \in \cap_j \ker \tau_j = I$. This implies that S^μ is one to one from A/I to $B(H^2(\mu) \cap I^\perp)$. This completes the proof.

Theorem 3.2. *Suppose that S^μ is a one to one contractive representation of A/I. Let k_j be a function in $H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_\mu$ $(f \in A)$, for $1 \leq j \leq n$. Then*
(1) $H^2(\mu) \cap I^\perp = \text{span}\{k_1, ..., k_n\}$ and $H^2(\mu) \cap I_j^\perp = \text{span}\{k_j\}$ where $I_j = \ker \tau_j$.
(2) If $m_j = \|k_j\|_\mu^{-2} |k_j|^2 d\mu$ and $m = \sum_{j=1}^n m_j/n$, then m_j is a representing measure for τ_j for each $1 \leq j \leq n$, and we may assume that μ is absolutely continuous with respect to m.
(3) $\|S_{f_j}^\mu\| = \|k_j\|_\mu \|f_j + I\|_\mu$ for $1 \leq j \leq n$.

Proof. (1) Since S^μ is one to one, τ_j has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$. In fact, if S^μ is one to one then $\dim H^2(\mu) \cap I^\perp = n$ and so $\dim H^2(\mu) \cap (\ker \tau_j)^\perp = 1$ for $1 \leq j \leq n$. Then there exists $k_j \in H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_\mu$ $(f \in A)$. If $g \in I$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^\perp$ for each j. Since $k_1, ..., k_n$ are linearly independent, $\{k_1, ..., k_n\}$ is a basis of $H^2(\mu) \cap I^\perp$. If $g \in I_j$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I_j^\perp$ for each j. Hence k_j is a basis of $H^2(\mu) \cap I_j^\perp$.
(2) For $1 \leq j \leq n$,

$$\int_X f dm_j = \int_X f \frac{|k_j|^2}{\|k_j\|_\mu^2} d\mu = \frac{\langle f k_j, k_j \rangle_\mu}{\|k_j\|_\mu^2} = \frac{\tilde{\tau}_j(f k_j)}{\|k_j\|_\mu^2} = \frac{\tau_j(f)\tilde{\tau}_j(k_j)}{\|k_j\|_\mu^2} = \tau_j(f), \ (f \in A).$$

Hence m_j is a representing measure for τ_j. Let $\mu = \mu^a + \mu^s$ be a Lebesgue decomposition by m. Then $H^2(\mu^a) \cap I^\perp = H^2(\mu) \cap I^\perp$ and so $H^2(\mu^s) \cap I^\perp = \{0\}$ where μ^a and μ^s are divided by their total masses. Hence $S_f^\mu = S_f^{\mu^a} \oplus S_f^{\mu^s} = S_f^{\mu^a} \oplus 0$ and so $\|S_f^\mu\| = \|S_f^{\mu^a}\|$ for $f \in A$.
(3) Since $\text{rank}(S_{f_j}^\mu)^* = 1$, there exists $x_j \in H^2(\mu) \cap I^\perp$ such that $(S_{f_j}^\mu)^* \phi = \langle \phi, x_j \rangle k_j = (k_j \otimes x_j)\phi$, $(\phi \in H^2(\mu) \cap I^\perp)$. Then $\|S_{f_j}^\mu\| = \|(S_{f_j}^\mu)^*\| = \|k_j \otimes x_j\| = \|k_j\|_\mu \|x_j\|_\mu$. Let P be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^\perp$. Then

$$\langle P f_j, \phi \rangle = \langle S_{f_j}^\mu 1, \phi \rangle = \langle 1, (S_{f_j}^\mu)^* \phi \rangle = \langle x_j, \phi \rangle \langle 1, k_j \rangle = \langle x_j, \phi \rangle, \quad (\phi \in H^2(\mu) \cap I^\perp),$$

because $\langle 1, k_j \rangle = 1$. Hence $P f_j = x_j$. Hence

$$\|f_j + I\|_\mu = \|P f_j\|_\mu = \|x_j\|_\mu.$$

Hence $\|S_{f_j}^\mu\| = \|k_j\|_\mu \|f_j + I\|_\mu$. This completes the proof.

Let $G(\tau)$ denote the Gleason part of τ. If $G(\tau_i) = G(\tau_j)$, then we write $\tau_i \sim \tau_j$.

Proposition 3.3. *Suppose that* $\dim H^2(\mu)\cap I^\perp = n$, $I^1 = \cap_{j\in N^1}\ker\tau_j$, $I^2 = \cap_{j\in N^2}\ker\tau_j$, $N^1\cap N^2 = \emptyset$ *and* $N^1\cup N^2 = \{1,2,...,n\}$. *Let* $\sharp N^j$ *denote the number of elements in* N^j. *If* $\tau_j \not\sim \tau_k$ *whenever* $j\in N^1$ *and* $k\in N^2$, *then* $H^2(\mu) = H^2(\mu^1)\oplus H^2(\mu^2)$, $H^2(\mu)\cap I^\perp = (H^2(\mu^1)\cap(I^1)^\perp)\oplus(H^2(\mu^2)\cap(I^2)^\perp)$, $S^\mu_\phi = S^{\mu^1}_\phi\oplus S^{\mu^2}_\phi$ *and* $\dim H^2(\mu^j)\cap(I^j)^\perp = \sharp N^j$, *where* $\mu = \frac{\mu^1+\mu^2}{2}$, $\mu^1\perp\mu^2$ *and* μ^j *is a probability measure for* $j = 1,2$.

Proof. By (1) of Theorem 3.2, $H^2(\mu)\cap I^\perp = \text{span}\{k_1,...,k_n\}$. We may assume that $N^1 = \{1,2,...,l\}$ and $N^2 = \{l+1,...,n\}$. By (2) of Theorem 3.2, $m_j = \|k_j\|_\mu^{-2}|k_j|^2 d\mu$ is a representing measure for τ_j for each $1\le j\le n$. Put $\lambda^1 = \frac{1}{l}\sum_{j=1}^l m_j$ and $\lambda^2 = \frac{1}{n-l}\sum_{j=l+1}^n m_j$ then $\lambda^1\perp\lambda^2$ by definitions of N^1 and N_2. Let $\mu = \mu_0^1+\mu_0^2$ be a Lebesgue decomposition with respect to λ^1 such that $\mu_0^1\ll\lambda^1$ and $\mu_0^2\perp\lambda^1$. Put $\mu^1 = \mu_0^1/\|\mu_0^1\|$ and $\mu^2 = \mu_0^2/\|\mu_0^2\|$. This completes the proof.

4. Isometric representation

.

In this section, we assume that A/I is n-dimensional and semisimple. Hence there exist $\tau_1,...,\tau_n$ in the maximal ideal space $M(A)$ of A such that $\tau_i\ne\tau_j$ $(i\ne j)$ and $I = \cap_{j=1}^n\ker\tau_j$. For $1\le j\le n$, there exist $f_j\in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then f_j+I is idempotent in A/I and $A/I = \text{span}\{f_1+I,...,f_n+I\}$. If S^μ is an isometric representation of A/I, then $\|S^\mu_{f_j}\| = \|f_j+I\|$ for $1\le j\le n$. By (3) of Theorem 3.2, this implies that $\|f_j+I\| = \|k_j\|_\mu\|f_j+I\|_\mu$. Hence, if S^μ is an isometric representation of A/I, then $\|k_j\|_\mu = \|f_j+I\|/\|f_j+I\|_\mu$ for $1\le j\le n$. Is the converse of this statement true? If $n=2$, then the answer will be given in Proposition 4.4.

Theorem 4.1. *Suppose that* $G(\tau_i)\cap G(\tau_j)\cap G(\tau_l) = \emptyset$ *if* i,j *and* l *are different from each other. Then there exists an isometric representation* S^μ *of* A/I.

Proof. By Proposition 3.3, if $G(\tau_j) = \{\tau_j\}$, for all $1\le j\le n$, then there exists an isometric representation S^{μ^j} of A/I_j where $I_j = \ker\tau_j$, and $\mu^i\perp\mu^j$. If $\mu = (\mu^1+...+\mu^n)/n$, then $H^2(\mu)\cap I^\perp = (H^2(\mu^1)\cap I^\perp)\oplus...\oplus(H^2(\mu^n)\cap I^\perp)$ and $S^\mu_f = S^{\mu^1}_f\oplus...\oplus S^{\mu^n}_f$ $(f\in A)$. Therefore, the theorem is proved in the case when $G(\tau_j) = \{\tau_j\}$, for all $1\le j\le n$. It is sufficient to prove the theorem when $\tau_i\sim\tau_j$ for some $i,j(i\ne j)$. Suppose $\tau_{2k-1}\sim\tau_{2k}, (1\le k\le n_0)$ and $G(\tau_l) = \{\tau_l\}, (2n_0+1\le l\le n)$ for some n_0. Since $G(\tau_i)\cap G(\tau_j)\cap G(\tau_l) = \emptyset$, it follows that $\dim A/I_{ij} = 2$ where $I_{ij} = I_i\cap I_j = \ker\tau_i\cap\ker\tau_j$. By Corollary 1 in [9], there is a probability measure μ^{ij} such that $\|S^{\mu^{ij}}_f\| = \|f+I_{ij}\|$ for all $f\in A$. By Proposition 3.3, there are probability measures $\mu^{2k-1,2k}, (1\le k\le n_0)$ and $\mu^l, (2n_0+1\le l\le n)$ such that $\mu = (\mu^{12}+\mu^{34}+...+\mu^{2n_0-1,2n_0}+\mu^{2n_0+1}+...+\mu^n)/(n-n_0)$, $H^2(\mu)\cap I^\perp = (H^2(\mu^{12})\cap I^\perp_{12})\oplus...\oplus(H^2(\mu^{2n_0-1,2n_0})\cap I^\perp_{2n_0-1,2n_0})\oplus(H^2(\mu^{2n_0+1})\cap I^\perp_{2n_0+1})\oplus...\oplus(H^2(\mu^n)\cap I^\perp_n)$, $S^\mu_f = S^{\mu^{12}}_f\oplus...\oplus S^{\mu^{2n_0-1,2n_0}}_f\oplus S^{\mu^{2n_0+1}}_f\oplus...\oplus S^{\mu^n}_f$ Hence S^μ is an isometric representation of A/I

where $I=(\cap_{k=1}^{n_0} I_{2k-1,2k})\cap(\cap_{l=2n_0+1}^{n} I_l)$. This completes the proof.

For example, we consider when $n=3$ and $\tau_1\sim\tau_2\not\sim\tau_3$. Let $I_{12}=I_1\cap I_2=\ker\tau_1\cap\ker\tau_2$. Then $\dim A/I_{12}=2$. By Corollary 1 in [9], there is a probability measure μ^{12} such that $\|S_f^{\mu^{12}}\|=\|f+I_{12}\|$ for all $f\in A$. Let S^{μ^3} be the isometric representation of A/I_3 where $I_3=\ker\tau_3$. Let $\mu=(\mu^{12}+\mu^3)/2$. Then $\mu^{12}\perp\mu^3$, $H^2(\mu)\cap I^\perp=(H^2(\mu^{12})\cap I_{12}^\perp)\oplus(H^2(\mu^3)\cap I_3^\perp)$, $S_f^\mu=S_f^{\mu^{12}}\oplus S_f^{\mu^3}$, $\quad(f\in A)$, $(S_f^{\mu^{12}})^*k_j=\overline{\tau_j(f)}k_j$, $(j=1,2)$, and $(S_f^{\mu^3})^*k_3=\overline{\tau_3(f)}k_3$. Hence

$$\|S_f^\mu\|=\max(\|S_f^{\mu^{12}}\|,\|S_f^{\mu^3}\|)=\max(\|f+I_{12}\|,|\tau_3(f)|)=\sup_{\nu\in(A/I)^*,\|\nu\|\le1}|\int_X fd\nu|=\|f+I\|.$$

Hence S^μ is an isometric representation of A/I where $I=I_{12}\cap I_3$. By the theorem of T. Nakazi (cf. [8]), $\|f+I_{12}\|$ can be written using $\rho_1=\sup\{|\tau_1(f)|\ ;\ f\in\ker\tau_2,\ \|f\|\le1\}$.

Corollary 4.2. *Let A be a uniform algebra and $I=\cap_{j=1}^n\ker\tau_j$ and $\tau_i\not\sim\tau_j(i\neq j)$. Then there exists an isometric representation S^μ of A/I, and $\|f+I\|=\max(|\tau_1(f)|,...,|\tau_n(f)|)$.*

Proof. Since $\tau_i\not\sim\tau_j$ $(i\neq j)$, there exist probability measures $\mu^1,...,\mu^n$ such that $\mu=(\mu^1+...+\mu^n)/n$, $\mu^i\perp\mu^j$ $(i\neq j)$, $H^2(\mu)\cap I^\perp=(H^2(\mu^1)\cap I^\perp)\oplus...\oplus(H^2(\mu^n)\cap I^\perp)$, $S_f^\mu=S_f^{\mu^1}\oplus...\oplus S_f^{\mu^n}$. Since $(S_f^{\mu^j})^*k_j=\overline{\tau_j(f)}k_j$, and $(S_f^{\mu^j})^*$ is a rank 1 operator on $H^2(\mu)\cap(\ker\tau_j)^\perp=$ span $\{k_j\}$, it follows that $\|S_f^{\mu^j}\|=\|(S_f^{\mu^j})^*\|=|\tau_j(f)|$. Then

$$\|S_f^\mu\|=\max(\|S_f^{\mu^1}\|,...,\|S_f^{\mu^n}\|)=\max(|\tau_1(f)|,...,|\tau_n(f)|)=\sup_{\nu\in(A/I)^*,\|\nu\|\le1}|\int_X fd\nu|=\|f+I\|.$$

This completes the proof.

Corollary 4.3. *Let A be a uniform algebra and $I=\cap_{j=1}^n\ker\tau_j$ and $\tau_i\not\sim\tau_j(i\neq j)$. Suppose that S^μ is an isometric representation of A/I. Then,*
(1) $\mu=\sum_{j=1}^n\mu^j$, $\mu^i\perp\mu^j$ $(i\neq j)$, $\mu^j\ll m^j$ *where μ^j is a positive measure and m^j is some representing measure for τ_j.*
(2) $S_f^\mu=\sum_{j=1}^n\oplus S_f^{\mu^j}$ $(f\in A)$ *where μ^j is divided by its total variation and S^{μ^j} is an isometric representation of A/I_j, where $I_j=\ker\tau_j$.*
(3) S_f^μ *is an isometric representation of a diagonal $n\times n$ matrix for any f in A.*

Proof. By the proof of (2) of Theorem 3.2 and Theorem 4.1, (1), (2) and (3) holds.

If A/I is 2-dimensional and semisimple, then there exist τ_1,τ_2 in $M(A)$ such that $\tau_1\neq\tau_2$ and $I=\ker\tau_1\cap\ker\tau_2$. For $j=1,2$, there exists $f_j\in A$ such that $\tau_i(f_j)=\delta_{ij}$. Then f_j+I is idempotent in A/I and $A/I=\text{span}\{f_1+I,f_2+I\}$. If $n=2$, then

$$\rho_1=\sup\{|\tau_1(f)|\ ;\ f\in\ker\tau_2,\ \|f\|\le1\},$$

$$\rho_1(\mu) = \sup\{|\tau_1(f)| \ ; \ f \in \ker \tau_2, \ \|f\|_\mu \le 1\}$$

where $\|f\|$ denotes the supnorm of f in A and $\|f\|_\mu = \langle f, f\rangle_\mu = (\int |f|^2 d\mu)^{1/2}$. Then ρ_1 is a Gleason distance between τ_1 and τ_2, and $\|f_1 + I\| = 1/\rho_1$, $\quad \|f_1+I\|_\mu = 1/\rho_1(\mu)$. The following proposition is essentially known (cf. Lemma 3 of [9]).

Proposition 4.4. *If A/I is 2-dimensional and semisimple, then the following conditions are equivalent.*
(1) *S^μ is an isometric representation of A/I.*
(2) $\|k_1\|_\mu = \rho_1(\mu)/\rho_1$.
(3) $\|k_1\|_\mu = \|f_1 + I\|/\|f_1 + I\|_\mu$.

Proof. By Theorem 3.2, (1) implies (3). By the above remark, (2) is equivalent to (3). It is sufficient to show that (3) implies (1). By Theorem 3.2, if (3) holds, then $\|S^\mu_{f_1}\| = \|f_1 + I\|$. By the above remark, this implies $\|S^\mu_{f_1}\| = 1/\rho_1$. By the theorem of T. Nakazi (cf. [8]), if $I = \{f \in A \ ; \ \tau_1(f) = \tau_2(f) = 0\}$, then

$$\begin{aligned} \|f + I\| \ &= \ \sqrt{\left|\frac{\tau_1(f) - \tau_2(f)}{2}\right|^2 \left(\frac{1}{\rho_1^2} - 1\right) + \left(\frac{|\tau_1(f)| + |\tau_2(f)|}{2}\right)^2} \\ &+ \ \sqrt{\left|\frac{\tau_1(f) - \tau_2(f)}{2}\right|^2 \left(\frac{1}{\rho_1^2} - 1\right) + \left(\frac{|\tau_1(f)| - |\tau_2(f)|}{2}\right)^2}. \end{aligned}$$

Since $\|S^\mu_{f_1}\| = 1/\rho_1$, it follows from the theorem of I. Feldman, N. Krupnik and A. Markus (cf. [5]) that

$$\|f + I\| = \|\tau_1(f)S^\mu_{f_1} + \tau_2(f)S^\mu_{f_2}\| = \|S^\mu_f\|.$$

This completes the proof.

T. Nakazi and K. Takahashi [9] proved that there exists an isometric representation of A/I in the case when $\dim A/I = 2$. The following theorem gives a concrete matrix representation of A/I.

Theorem 4.5. *Suppose A/I is 3-dimensional and semisimple. If $\tau_1 \sim \tau_2 \not\sim \tau_3$ and S^μ is an isometric representation of A/I, then A/I is isometric to $\{S^\mu_f \ ; \ f \in A\} =$ span$\{S^\mu_{f_1}, S^\mu_{f_2}, S^\mu_{f_3}\}$, $S^\mu_f = \tau_1(f)S^\mu_{f_1} + \tau_2(f)S^\mu_{f_2} + \tau_3(f)S^\mu_{f_3}$, and*

$$(S^\mu_{f_1})^* = \begin{pmatrix} 1 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S^\mu_{f_2})^* = \begin{pmatrix} 0 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S^\mu_{f_3})^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$x = \frac{-\langle k_2, k_1\rangle_\mu}{\sqrt{\|k_1\|^2_\mu \|k_2\|^2_\mu - |\langle k_1, k_2\rangle_\mu|^2}}.$$

Proof. This follows from Lemma 2.1 and Theorem 4.1.

If $\mathcal{B} \subset B(H)$ and $\dim H = 3$, then

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

It follows from a 2-dimensional case that if $y = z = 0$, then $\mathcal{B}$ is a Q-algebra.

If the following condition (1) implies (2) for any distinct points $\tau_1, ..., \tau_n \in M(A)$ and complex numbers $w_1, ..., w_n$, then we say that A/I satisfies the Pick property.
(1) $[(1 - w_i\overline{w_j})k_{ji}]_{i,j=1}^n \geq 0$, where $k_{ij} = \langle k_i, k_j \rangle_\mu$, and $\tau_j(f) = \langle f, k_j \rangle_\mu$, $(f \in A)$.
(2) There exists $f \in A$ such that $\tau_j(f) = w_j$, $(1 \leq j \leq n)$ and $\|f + I\| \leq 1$.
The following proposition is essentially known.

Proposition 4.6. *Let A/I be an n-dimensional semisimple commutative Banach algebra. Then $S^\mu : A/I \to B(H^2(\mu) \cap I^\perp)$ is isometric if and only if A/I satisfies the Pick property.*

Proof. Suppose S^μ is isometric. For any $w_1, ..., w_n \in \mathbf{C}$, there exists an $f \in A$ such that $\tau_j(f) = w_j$, $(1 \leq j \leq n)$. Suppose $[(1 - w_i\overline{w_j})k_{ji}]_{i,j=1}^n \geq 0$. For any complex numbers $\alpha_1, ..., \alpha_n$, let $k = \sum_{j=1}^n \alpha_j k_j$. Then $\|k\|_\mu^2 = \sum_{i,j=1}^n \overline{\alpha_i}\alpha_j k_{ji}$. Since $(S_f^\mu)^* k_j = \overline{\tau_j(f)} k_j$, $(S_f^\mu)^* k = \sum_{j=1}^n \alpha_j \overline{\tau_j(f)} k_j$. By (1),

$$\|k\|_\mu^2 - \|(S_f^\mu)^* k\|_\mu^2 = \sum_{i,j=1}^n \overline{\alpha_i}\alpha_j (1 - w_i\overline{w_j}) k_{ji} \geq 0.$$

Since $H^2(\mu) \cap I^\perp$ is spanned by $k_1, ..., k_n$, this implies that $\|(S_f^\mu)^*\| \leq 1$. Since S^μ is isometric, $\|f + I\| = \|S_f^\mu\| \leq 1$. Therefore A/I satisfies the Pick property. Conversely, suppose A/I satisfies the Pick property and $\|S_f^\mu\| = 1$. Since $(S_f^\mu)^* k_j = \overline{\tau_j(f)} k_j$ and $\|(S_f^\mu)^*\| = 1$, it follows that

$$\sum_{i,j=1}^n \overline{\alpha_i}\alpha_j (1 - \tau_i(f)\overline{\tau_j(f)}) k_{ji} = \|k\|_\mu^2 - \|(S_f^\mu)^* k\|_\mu^2 \geq 0,$$

and hence $[(1 - \tau_i(f)\overline{\tau_j(f)})k_{ji}]_{i,j=1}^n \geq 0$. By the Pick property, there exists $g \in A$ such that $\|g + I\| \leq 1$ and $\tau_j(g) = \tau_j(f), (1 \leq j \leq n)$. Therefore $\|f + I\| = \|g + I\| \leq 1 = \|S_f^\mu\|$. Since the reverse inequality $\|S_f^\mu\| \leq \|f + I\|$ is always holds, $\|S_f^\mu\| = \|f + I\|$. This completes the proof.

5. Q-Algebras of a Disc Algebra

In this section, we assume that A is the disc algebra and $\dim A/I = 3$. For $f \in A$, let $\|f+I\| = \|f+I\|_{A/I}$. Since $M(A) = \bar{\mathbf{D}} = \{|z| \leq 1\}$, for each $1 \leq j \leq 3$, τ_j is just an evaluation functional at a point of $\bar{\mathbf{D}}$ and so we write that $\tau_1 = a$, $\tau_2 = b$ and $\tau_3 = c$, where a, b and c are in $\bar{\mathbf{D}}$. By Theorem 3.2, we may assume that a, b and c are in $\mathbf{D} = \{|z| < 1\}$. Theorem 5.2 shows that the set of all 3-dimensional semisimple Q-algebras of the disc algebra is a proper subset in the set of all 3-dimensional semisimple commutative operator algebras with unit on a Hilbert space of dimension 3. However Theorem 5.2 has not solved Problem 2 yet. We use Lemma 5.1 to prove Theorem 5.2. Let a, b, c be the distinct points in the open unit disc $\mathbf{D}$. Let $T(a,b,c)$ denote the subset of $\mathbf{C}^3$ which consists of all $(x,y,z) \in \mathbf{C}^3$ satisfying

$$1 + |x|^2 = \left|\frac{1-\bar{b}a}{a-b}\right|^2, \quad 1 + |z|^2 = \left|\frac{1-\bar{c}b}{b-c}\right|^2,$$

$$1 + |y|^2 \left|\frac{a-b}{1-\bar{b}a}\right|^2 = \left|\frac{1-\bar{a}c}{c-a}\right|^2.$$

This implies that $x \neq 0$, $y \neq 0$, and $z \neq 0$. $T(a,b,c)$ is characterized by saying that the absolute values of x, y, z are fixed and that their argument are arbitrary. In the following, we consider some inequalities of x, y, and z. For $j = 1, 2, 3$, there exists $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Hence, $f_1(a) = f_2(b) = f_3(c) = 1$, and $f_1(b) = f_1(c) = f_2(a) = f_2(c) = f_3(a) = f_3(b) = 0$.

Lemma 5.1. *Let a, b, c be the distinct points in $\mathbf{D}$. Let $f \in A$. Let $I = \{g \in A\ ;\ g(a) = g(b) = g(c) = 0\}$. Let $d\mu = \frac{d\theta}{2\pi}$.*
(1) $S_f^\mu = f(a)S_{f_1}^\mu + f(b)S_{f_2}^\mu + f(c)S_{f_3}^\mu$, and

$$(S_{f_1}^\mu)^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^\mu)^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^\mu)^* = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x,y,z) \in T(a,b,c)$.
(2) $\|f+I\| = \|S_f^\mu\|$, $(f \in A)$. That is, A/I is isometrically isomorphic to the 3-dimensional semisimple commutative operator algebra on $H^2(\mu) \cap I^\perp$ which is spanned by

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x,y,z) \in T(a,b,c)$.

Proof. $H^2(\mu) \cap I^\perp$ is a 3-dimensional Hilbert space which is spanned by

$$k_1(z) = \frac{1}{1-\bar{a}z}, \quad k_2(z) = \frac{1}{1-\bar{b}z}, \quad k_3(z) = \frac{1}{1-\bar{c}z}.$$

For orthonormal basis ψ_1, ψ_2, ψ_3 defined in Proposition 2.3,

$$\psi_1(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad \psi_2(z) = \gamma_2 \frac{z-a}{1-\bar{a}z}\frac{\sqrt{1-|b|^2}}{1-\bar{b}z}, \quad \psi_3(z) = \gamma_3 \frac{z-a}{1-\bar{a}z}\frac{z-b}{1-\bar{b}z}\frac{\sqrt{1-|c|^2}}{1-\bar{c}z},$$

where

$$\gamma_2 = -\left(\frac{a-b}{1-\bar{a}b}\right)^{-1}\left|\frac{a-b}{1-\bar{a}b}\right|, \quad \gamma_3 = \left(\frac{a-c}{1-\bar{a}c}\right)^{-1}\left|\frac{a-c}{1-\bar{a}c}\right|\left(\frac{b-c}{1-\bar{b}c}\right)^{-1}\left|\frac{b-c}{1-\bar{b}c}\right|.$$

Since

$$k_2 - (k_2, \psi_1)\psi_1 = \frac{(\bar{b}-\bar{a})(z-a)}{(1-\bar{b}a)(1-\bar{a}z)(1-\bar{b}z)},$$

it follows that

$$\|k_2 - (k_2, \psi_1)\psi_1\| = \left|\frac{\bar{b}-\bar{a}}{1-\bar{b}a}\right|\frac{1}{\sqrt{1-|b|^2}}.$$

Hence

$$\psi_2 = \frac{k_2 - (k_2, \psi_1)\psi_1}{\|k_2 - (k_2, \psi_1)\psi_1\|} = \gamma_2 \frac{z-a}{1-\bar{a}z}\frac{\sqrt{1-|b|^2}}{1-\bar{b}z}.$$

Since

$$k_3 - \langle k_3, \psi_1\rangle\psi_1 - \langle k_3, \psi_2\rangle\psi_2 = \frac{(\bar{a}-\bar{c})(\bar{b}-\bar{c})(z-a)(z-b)}{(1-\bar{c}a)(1-\bar{c}b)(1-\bar{a}z)(1-\bar{b}z)(1-\bar{c}z)},$$

it follows that

$$\psi_3 = \frac{k_3 - \langle k_3, \psi_1\rangle\psi_1 - \langle k_3, \psi_2\rangle\psi_2}{\|k_3 - \langle k_3, \psi_1\rangle\psi_1 - \langle k_3, \psi_2\rangle\psi_2\|} = \gamma_3 \frac{z-a}{1-\bar{a}z}\frac{z-b}{1-\bar{b}z}\frac{\sqrt{1-|c|^2}}{1-\bar{c}z}.$$

If we calculate x, y, z using the formulas in Proposition 2.3, then it follows that $(x, y, z) \in T(a, b, c)$. Then

$$x = \frac{-\langle k_2, k_1\rangle}{\sqrt{\|k_1\|^2\|k_2\|^2 - |\langle k_1, k_2\rangle|^2}} = \frac{\frac{-1}{1-\bar{b}a}}{\sqrt{\frac{1}{(1-|a|^2)(1-|b|^2)} - \frac{1}{|1-\bar{a}b|^2}}} = \gamma_4 \frac{\sqrt{1-|a|^2}\sqrt{1-|b|^2}}{|a-b|},$$

where

$$\gamma_4 = -\frac{1-\bar{a}b}{|1-\bar{a}b|}.$$

Hence

$$1 + |x|^2 = \left|\frac{1-\bar{b}a}{a-b}\right|^2.$$

Since

$$-\langle k_3, \psi_1\rangle - \langle k_3, \psi_2\rangle x = \frac{\sqrt{1-|a|^2}}{1-\bar{c}a}\frac{1-\bar{a}b}{\bar{b}-\bar{a}}\frac{\bar{c}-\bar{b}}{1-b\bar{c}},$$

it follows that

$$y = \frac{-\langle k_3, \psi_1\rangle - \langle k_3, \psi_2\rangle x}{\|k_3 - \langle k_3, \psi_1\rangle\psi_1 - \langle k_3, \psi_2\rangle\psi_2\|} = \gamma_5 \frac{1-\bar{a}b}{\bar{a}-\bar{b}}\frac{\sqrt{1-|a|^2}\sqrt{1-|c|^2}}{|a-c|},$$

where

$$\gamma_5 = \left(\frac{a-b}{1-a\bar{b}}\right)\left|\frac{a-b}{1-a\bar{b}}\right|^{-1}\left(\frac{b-c}{1-\bar{b}c}\right)^{-1}\left|\frac{b-c}{1-\bar{b}c}\right|\frac{|1-\bar{c}a|}{1-\bar{c}a}.$$

Since

$$\langle k_3, \psi_2\rangle = \overline{\gamma_2}\frac{\bar{c}-\bar{a}}{1-a\bar{c}}\frac{\sqrt{1-|b|^2}}{1-b\bar{c}},$$

it follows that

$$z = \frac{-\langle k_3, \psi_2\rangle}{\|k_3 - \langle k_3, \psi_1\rangle\psi_1 - \langle k_3, \psi_2\rangle\psi_2\|} = \gamma_6\frac{\sqrt{1-|b|^2}\sqrt{1-|c|^2}}{|b-c|},$$

where

$$\gamma_6 = \left(\frac{a-b}{1-\bar{a}b}\right)\left|\frac{a-b}{1-\bar{a}b}\right|^{-1}\left(\frac{c-a}{1-\bar{a}c}\right)^{-1}\left|\frac{c-a}{1-\bar{a}c}\right|\frac{|1-b\bar{c}|}{1-b\bar{c}}.$$

Since $|\gamma_2| = |\gamma_3| = |\gamma_4| = |\gamma_5| = |\gamma_6| = 1$, it follows that

$$1+|y|^2\left|\frac{a-b}{1-\bar{b}a}\right|^2 = \left|\frac{1-\bar{a}c}{c-a}\right|^2, \quad 1+|z|^2 = \left|\frac{1-\bar{c}b}{b-c}\right|^2.$$

Hence, (1) follows. It is sufficient to prove (2). By the theorem of D. Sarason (cf. [2], p.125, [10], Vol.1, p.231, [11]), $\|f+I\| = \|S^\mu_f\|$. Then $(S^\mu_{f_1})^*k_1 = k_1$, $(S^\mu_{f_1})^*k_2 = (S^\mu_{f_1})^*k_3 = 0$, $(S^\mu_{f_2})^*k_2 = k_2$, $(S^\mu_{f_2})^*k_3 = (S^\mu_{f_2})^*k_1 = 0$, $(S^\mu_{f_3})^*k_3 = k_3$, and $(S^\mu_{f_3})^*k_1 = (S^\mu_{f_3})^*k_2 = 0$. By Proposition 2.3,

$$(S^\mu_{f_1})^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S^\mu_{f_2})^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S^\mu_{f_3})^* = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $f - f(a)f_1 - f(b)f_2 - f(c)f_3 \in I$ and $I(H^2(\mu)\cap I^\perp) \subset IH^2(\mu) \subset H^2(\mu)\cap I^\perp$, it follows that

$$(S^\mu_f - S^\mu_{f(a)f_1+f(b)f_2+f(c)f_3})\psi = S^\mu_{f-f(a)f_1-f(b)f_2-f(c)f_3}\psi = 0, \quad (\psi \in I^\perp_\mu).$$

Hence

$$S^\mu_f = S^\mu_{f(a)f_1+f(b)f_2+f(c)f_3} = f(a)S^\mu_{f_1} + f(b)S^\mu_{f_2} + f(c)S^\mu_{f_3}.$$

This completes the proof.

For example, if $(a,b,c) = (0, \frac{1}{2}, \frac{1}{3})$ and $(x,y,z) = (-\sqrt{3}, 4\sqrt{2}, -2\sqrt{6})$, then the algebra $\text{span}\{P_1, P_2, P_3\}$ is isometrically isomorphic to A/I which is a Q-algebra of a disc algebra.

Theorem 5.2. *Let a, b, c be the distinct points in $\mathbf{D}$. Let $f \in A$. Let $d\mu = \frac{d\theta}{2\pi}$. Let $I = \{g \in A\ ;\ g(a) = g(b) = g(c) = 0\}$. If a 3-dimensional semisimple commutative operator algebra $\mathcal{B}$ on $H^2(\mu)\cap I^\perp$ is isometrically isomorphic to A/I, then $\mathcal{B}$ is unitarily equivalent to the 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space H spanned by P_1, P_2, P_3 such that*

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y, z satisfy (1) $\sim$ (3).

$$xyz \neq 0, \tag{1}$$

$$\frac{1}{\sqrt{1+|y|^2}} < \frac{1}{\sqrt{1+|x|^2}} + \frac{1}{\sqrt{1+|z|^2}}, \tag{2}$$

$$|y| > \frac{|xz|}{\sqrt{1+|z|^2}+1}, \tag{3}$$

Proof. By the theorem of B. Cole and J. Wermer (cf. [3]) and (2) of Theorem 2.4, we may assume that H is spanned by the orthonormal basis ψ_1, ψ_2, ψ_3 which are calculated in the proof of Lemma 5.1. By Lemma 5.1, there are complex numbers x, y, z satisfying $(x, y, z) \in T(a, b, c)$. Since

$$1+|x|^2 = \left|\frac{1-\bar{b}a}{a-b}\right|^2 > 1, \quad 1+|z|^2 = \left|\frac{1-\bar{c}b}{b-c}\right|^2 > 1,$$

$$1+|y|^2\left|\frac{a-b}{1-\bar{b}a}\right|^2 = \left|\frac{1-\bar{a}c}{c-a}\right|^2 > 1,$$

(1) follows. Let

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

Then

$$\rho(a,b) = \frac{1}{\sqrt{1+|x|^2}}, \quad \rho(b,c) = \frac{1}{\sqrt{1+|z|^2}}, \quad \rho(c,a) = \sqrt{\frac{1+|x|^2}{1+|x|^2+|y|^2}} > \frac{1}{\sqrt{1+|y|^2}}.$$

Since $\rho(c,a) \leq \rho(a,b) + \rho(b,c)$, (2) follows. Let

$$d(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

Since $d(c,a) \leq d(a,b) + d(b,c)$,

$$\frac{\sqrt{1+|x|^2+|y|^2}+\sqrt{1+|x|^2}}{\sqrt{1+|x|^2+|y|^2}-\sqrt{1+|x|^2}} \leq \frac{\sqrt{1+|z|^2}+1}{\sqrt{1+|z|^2}-1}\cdot\frac{\sqrt{1+|x|^2}+1}{\sqrt{1+|x|^2}-1}.$$

Hence

$$\frac{\sqrt{1+|x|^2}+1}{|y|} < \frac{\sqrt{1+|x|^2+|y|^2}+\sqrt{1+|x|^2}}{|y|} \leq \frac{\sqrt{1+|z|^2}+1}{|z|}\cdot\frac{\sqrt{1+|x|^2}+1}{|x|}.$$

this implie (3). This completes the proof.

Example 5.3. In Example 2.5, $\mathcal{B}_0$ is isometrically isomorphic to $\mathcal{B}_2$. Since $y_0 = 0$, it follows from Theorem 5.2 that $\mathcal{B}_2$ is not isometrically isomorphic to a 3-dimensional semisimple Q-algebra A/I where A is a disc algebra. Hence $\mathcal{B}_0$ is also not isometrically isomorphic to a Q-algebra A/I. Therefore $\mathcal{B}_0$ and $\mathcal{B}_2$ is the example to show that the set of all 3-dimensional semisimple Q-algebra A/I where A is a disc algebra is smaller than the set of all 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

References

[1] F. Bonsall, J. Duncan, Complete Normed Algebras, Springer, Berlin, 1973.

[2] B. Cole, J. Wermer, Pick interpolation, von Neumann inequalities, and hyperconvex sets, Complex potential theory (Montreal, PQ, 1993), 89-129, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 439, Kluwer Acad.Publ., Dordrecht, 1994.

[3] B. Cole, J. Wermer, Isometries of certain operator algebras, Proc. Amer. Math. Soc. 124 (1996), 3047-3053.

[4] S.W. Drury, Remarks on von Neumann's inequality, Lecture Notes in Mathematics, Vol.995, Springer, Berlin, 1983, pp.14-32.

[5] I. Feldman, N. Krupnik, A. Markus, On the norm of two adjoint projections, Integr. Eq. Oper. Theory 14 (1991), 69-90.

[6] J. Holbrook, Schur norms and the multivariate von Neumann inequality, Operator Theory: Advances and Applications. Vol.127, Birkhäuser 2001, 375-386.

[7] S. McCullough, V. Paulsen, C^*-envelopes and interpolation theory, Indiana Univ. Math. J. 51 (2002), 479-505.

[8] T. Nakazi, Two dimensional Q-algebras, Linear Algebra Appl. 315 (2000), 197-205.

[9] T. Nakazi, K. Takahashi, Two dimensional representations of uniform algebras, Proc. Amer. Math. Soc. 123 (1995), 2777-2784.

[10] N.K. Nikolski, Operators, Functions, and Systems, Vol. 1, 2, Amer. Math. Soc., 2002.

[11] D. Sarason, Generalized interpolation in H^∞, Trans. Amer. Math. Soc. 127 (1967), 179-203.