ON RADIAL SOLUTIONS OF SEMI-RELATIVISTIC HARTREE EQUATIONS

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Abstract. We consider the semi-relativistic Hartree type equation with nonlocal nonlinearity $F(u) = \lambda(|x|^{-\gamma} * |u|^2)u, 0 < \gamma < n, n \ge 1$. In [2], the global well-posedness (GWP) was shown for the value of $\gamma \in (0, \frac{2n}{n+1}), n \ge 2$ with large data and $\gamma \in (2, n), n \ge 3$ with small data. In this paper, we extend the previous GWP result to the case for $\gamma \in (1, \frac{2n-1}{n}), n \ge 2$ with radially symmetric large data. Solutions in a weighted Sobolev space are also studied.

1. Introduction

We consider the following Cauchy problem describing boson stars

$$\begin{cases} i\partial_t u = \sqrt{1-\Delta}u + F(u) \ \text{ in } \mathbb{R}^n \times \mathbb{R},\ n \ge 1 \\ u(0) = \varphi \in H^s(\mathbb{R}^n). \end{cases} \tag{1}$$

Here $F(u) = (V_\gamma * |u|^2)u$ is Hartree type nonlinearity, where $*$ denotes the convolution in $\mathbb{R}^n$ and $V_\gamma(x) = \lambda|x|^{-\gamma}$ with $\lambda \in \mathbb{R}$, $0 < \gamma < n$. $H^s = (1-\Delta)^{-\frac{s}{2}}L^2$ is the Sobolev space of order $s \in \mathbb{R}$. We consider (1) in the form of the integral equation

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \tag{2}$$

where

$$(U(t)\varphi)(x) = (e^{-it\sqrt{1-\Delta}}\varphi)(x) \equiv \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t\sqrt{1+|\xi|^2})}\widehat{\varphi}(\xi)\,d\xi$$

and $\widehat{\varphi}$ denotes the Fourier transform defined as $\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$.

If the solution u of (1) or (2) has sufficient decay at infinity and smoothness, it satisfies two conservation laws:

$$\begin{aligned} \|u(t)\|_{L^2} &= \|\varphi\|_{L^2}, \\ E(u) \equiv K(u) + V(u) &= E(\varphi), \end{aligned} \tag{3}$$

where $K(u) = \langle\sqrt{1-\Delta}\,u, u\rangle$, $V(u) = \frac{1}{4}\langle F(u), u\rangle$ and $\langle , \rangle$ is the complex inner product in L^2.

Recently, the equation (1) has been extensively studied. Elgart and Schlein [5] and Fröhlich and Lenzmann [6] considered the mean field limit problem of boson stars with Coulomb potential. The finite time blowup solution with negative energy was studied by Fröhlich and Lenzmann [6]. Fröhlich, Jonsson and Lenzmann [7] proved the existence of traveling solitary waves in $\mathbb{R}^3$.

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For the well-posedness results, we refer the readers to the papers [10, 2]. Lenzmann in [10] established the global well-posedness in $H^{\frac12}(\mathbb{R}^3)$ for $\gamma = 1$. In [2], we considered the general case $0 < \gamma < n, n \ge 1$ and showed the local and global existence by utiliizing the Strichartz estimates. In particular, we showed the global existence for $0 < \gamma < \frac{2n}{n+1}$ in $H^{\frac12}$ with large data and for $2 < \gamma < n$ in $H^s, s > \frac{\gamma}{2} - \frac{n-2}{2n}$ with small data.

The first result of this paper is on the global existence of radially symmetric solutions of (2) for $\frac{2n}{n+1} \le \gamma < \frac{2n-1}{n}, n \ge 2$.

Theorem 1.1. *Let γ satisfy $1 < \gamma < \frac{2n-1}{n}, n \ge 2, s \ge \frac12$. If $\lambda > 0$, then for any radially symmetric function $\varphi \in H^s$, (2) has a unique radially symmetric solution $u \in C(\mathbb{R}; H^s) \cap L^q_{loc}\widetilde{H}^{\frac12, \frac12 - \sigma}_{\frac{2n}{n-1}}$ for $q = \frac{2n}{n-1} + \varepsilon$ and $\sigma = \frac12 + \varepsilon'$ with sufficiently small $\varepsilon, \varepsilon' > 0$. For all time the energy and L^2 norm of $u(t)$ are conserved. If $\lambda < 0$, then there exists $\rho > 0$ such that the same conclusion holds for φ with $\|\varphi\|_{L^2} \le \rho$. Moreover,*

$$\|u(t)\|_{H^s} \lesssim \|\varphi\|_{H^s} \exp\left(C|t|(1 + \|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac{q}{q-2}}\right). \tag{4}$$

Here $H^s_r = (1-\Delta)^{-s/2}L^r$ and

$$\widetilde{H}^{s,s'}_r = \{v : \|v\|_{\widetilde{H}^{s,s'}_r} \equiv \|(-\Delta)^{\frac{s}{2}}P_{\le 1}v\|_{L^r} + \|(1-\Delta)^{\frac{s'}{2}}P_{1>1}v\|_{L^r} < \infty\}$$

are the usual Sobolev space and a hibrid Sobolev space, where $P_{\le 1}$ and $P_{>1}$ are frequency projection over frequency less than 1 and greater than 1. We mean H^s by H^s_2 and $\dot{H}^s$ by $\dot{H}^s_2$. Hereafter, we denote the space $L^q_T(B)$ by $L^q(-T,T;B)$ and its norm by $\|\cdot\|_{L^q_T B}$ for some Banach space B, and also $L^q(B)$ with norm $\|\cdot\|_{L^q B}$ by $L^q(\mathbb{R}; B)$, $1 \le q \le \infty$.

In order to prove Theorem 1.1, we pursue the contraction mapping argument. For this purpose, we use the energy and L^2 conservation laws and the Strichartz estimate for radial functions. By the Strichartz estimate we mean (see [11, 12]):

$$\begin{aligned} &\|U(t)\varphi\|_{L^{q_0}_T H^{s-\sigma_0}_{r_0}} \lesssim \|\varphi\|_{H^{s_0}}, \\ &\left\|\int_0^t U(t-t')F(t')\,dt'\right\|_{L^{q_1}_T H^{s_1-\sigma_1}_{r_1}} \lesssim \|F\|_{L^1_T H^{s_1}}, \end{aligned} \tag{5}$$

where $(q_i, r_i), i = 0, 1$, satisfy that for any $\theta \in [0,1]$

$$\begin{aligned} &\frac{2}{q_i} = (n-1+\theta)\left(\frac12 - \frac{1}{r_i}\right), \quad 2\sigma_i = (n+1+\theta)\left(\frac12 - \frac{1}{r_i}\right), \\ &2 \le q_i, r_i \le \infty, \quad (q_i, r_i) \ne (2, \infty). \end{aligned} \tag{6}$$

If φ and F are radially symmetric, then by the well-known decay property of the Fourier transform of measure on unit sphere the estimate (5) can be extended as:

$$\begin{aligned} &\|U(t)\varphi\|_{L^p_T \widetilde{H}^{\frac12, s-\sigma}_p} \lesssim \|\varphi\|_{H^s}, \\ &\left\|\int_0^t U(t-t')F(t')\,dt'\right\|_{L^p_T \widetilde{H}^{\frac12, s-\sigma}_p} \lesssim \|F\|_{L^1_T H^s}, \end{aligned} \tag{7}$$

where $s \in \mathbb{R}$ and $\frac{2n}{n-1} < p < \infty, \sigma = \frac{n}{2} - \frac{n+1}{p}$. The second estimate does not follow simply from the first one. We prove this via low-diagonal operator estimate. These will be shown in Section 2.

Interpolating (5) and (7)[1], we get wider range of pairs (q, r). For Theorem 1.1, we need only the pairs $(q, \frac{2n}{n-1})$ with q slightly larger than $\frac{2n}{n-1}$. To put is another way, given $\varepsilon > 0$ we can find q and σ such that $\frac{2n}{n-1} < q < \frac{2n}{n-1} + \varepsilon$, $\frac{1}{2n} < \sigma < \frac{1}{2n} + \varepsilon$ and

$$\|U(t)\varphi\|_{L_T^q \widetilde{H}^{\frac12, \frac12 - \sigma}_{\frac{2n}{n-1}}} \lesssim \|\varphi\|_{H^{\frac12}},$$

$$(8) \qquad \left\| \int_0^t U(t-t')F(t')\, dt' \right\|_{L_T^q \widetilde{H}^{\frac12, \frac12 - \sigma}_{\frac{2n}{n-1}}} \lesssim \|F\|_{L_T^1 H^{\frac12}},$$

With these pairs we can make the value of σ close to $\frac{1}{2n}$ and the value γ to $\frac{2n-1}{n}$.

Next we consider a radial solution in weighted Sobolev space $H^{s,r} = \{v \in H^s : \|v\|_{H^{s,r}} \equiv \|(1+|x|^2)^{\frac{r}{2}}(1-\Delta)^{\frac{s}{2}} v\|_{L^2} < \infty\}$.

Theorem 1.2. *Let $n \ge 2$ and $1 < \gamma < \frac{2n-1}{n}$. Let φ and u be as in Theorem 1.1. If in addition $\varphi \in H^{1,1}$, then $u \in C(\mathbb{R}; H^{1,1}) \cap L^q_{loc} \widetilde{H}^{\frac12, \frac12 - \sigma}_{\frac{2n}{n-1}}$, where q and σ are the numbers as stated in Theorem 1.1. Moreover, if $n \ge 3$, then*

$$(9) \qquad \|u(t)\|_{H^{1,1}} \lesssim \|\varphi\|_{H^{1,1}} \exp\left(C|t|(1 + \|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac{q}{q-2}} \right).$$

The essential parts of the proof for the global existence in $H^{1,1}$ are the estimate (4) and the following estimates

$$(10) \qquad \|x\sqrt{1-\Delta}F(u)(t)\|_{L^2} \lesssim (\|u(t)\|_{H^1}^2 + \|u(t)\|^2_{\widetilde{H}^{\frac12, \frac12 - \sigma}_4}) \|u(t)\|_{H^{1,1}} \quad \text{for } n = 2,$$

$$(11) \qquad \|x\sqrt{1-\Delta}F(u)(t)\|_{L^2} \lesssim (\|u(t)\|^2_{H^{\frac12}} + \|u(t)\|^2_{\widetilde{H}^{\frac12, \frac12 - \sigma}_{\frac{2n}{n-1}}}) \|u(t)\|_{H^{1,1}} \quad \text{for } n \ge 3,$$

$$(12) \qquad \|x\sqrt{1-\Delta}u(t)\|_{L^2} \lesssim \|\varphi\|_{H^{1,1}} + \int_0^t (\|\nabla u\|_{L^2} + \|x\sqrt{1-\Delta}F(u)\|_{L^2})\, dt'.$$

For the inequality (12), we do not need the radial symmetry. To obtain (10) and (11), we need to estimate $\||\nabla V_\gamma(u)||x|u\|_{L^2}$ for which we establish the pointwise estimate of fractional integral of radial function for $n \ge 3$. See Lemma 3.1 below.

We can estimate $\||\nabla V_\gamma(u)||x|u\|_{L^2}$ for $n \ge 2$ without using Lemma 3.1. For example, $\||\nabla V_\gamma(u)||x|u\|_{L^2} \le \|\nabla V_\gamma(u)\|_{L^n} \|xu\|_{L^{\frac{2n}{n-2}}} \lesssim \|\nabla u\|_{L^2} \|u\|^2_{L^{\frac{2n}{n-2(\gamma-1)}}} \|u\|_{H^{1,1}}$. Hence combining this with (4), we have at least $\|u(t)\|_{H^{1,1}} \lesssim \exp(C|t|\exp(C|t|))$.

If $0 < \gamma \le 1$, then in view of GWP in $H^s, s \ge \frac12$ of [10, 2], from the estimate $\||\nabla V_\gamma(u)||x|u\|_{L^2} \lesssim \|\nabla u\|_{L^2} \|u\|_{H^{\frac12}} \|u\|_{H^{1,1}}$ we deduce the GWP in $H^{1,1}$ without radial symmetry condition. One can also prove the global existence of radial solutions $H^{k,l}$ with integers $k, l > 1$ by the same method as in Section 3.

If not specified, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \le CB$ and $A \ge C^{-1}B$, respectively. Different positive constants possibly depending on n, λ and γ might be denoted by the same letter C. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For simplicity, we only consider the positive time because the proof for negative time can be treated in the same way.

[1] We can proceed the complex interpolation after changing $U(t)$ an operator mapping from 1-dimensional function space to n-dimensional space in a similar way to the proof of (7) below.

Let us first define a complete metric space $X_{T,\rho}$ with metric $d(u,v) = \|u-v\|_{X_T}$, where $X_T = C([0,T]; H^s) \cap L_T^q \widetilde{H}^{\frac12, \frac12-\sigma}_{\frac{2n}{n-1}}$ by

$$X_{T,\rho} \equiv \{v \in X_T : \ v \ \text{ is radially symmetric and } \ \|v\|_{X_T} \le \rho\}.$$

As stated in the introduction, given $\varepsilon > 0$ we can find q and σ satisfying (8).

Now we define a mapping $N : u \mapsto N(u)$ on $X_{T,\rho}$ by

$$N(u)(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')\,dt'. \tag{13}$$

For any $u \in X_{T,\rho}$, $N(u)$ is radially symmetric. By Strichartz estimate (8), we have

$$\|N(u)\|_{X_T} \lesssim \|\varphi\|_{H^s} + \|F(u)\|_{L_T^1 H^s}. \tag{14}$$

To estimate of the second term on the RHS of (14), let us introduce a generalized Leibniz rule (see Lemma A1 $\sim$ Lemma A4 in Appendix of [8]).

Lemma 2.1. *For any $s \ge 0$ we have*

$$\|(-\Delta)^{\frac{s}{2}}(uv)\|_{L^r} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^{r_1}}\|v\|_{L^{q_1}} + \|u\|_{L^{q_2}}\|(-\Delta)^{\frac{s}{2}}v\|_{L^{r_2}},$$
$$\frac1r = \frac1{r_i} + \frac1{q_i}, \quad r_i \in (1,\infty),\ q_i \in (1,\infty], \quad i = 1,2.$$

Since $\gamma \le 2$, we use Lemma 2.1 with $(r_1,q_1) = (\infty, 2)$, $(r_2,q_2) = (\frac{2n}{\gamma}, \frac{2n}{n-\gamma})$ and $(r_1,q_1) = (2, \frac{2n}{n-\gamma}) = (q_2,r_2)$, and Hardy-Littlewood-Sobolev inequality to obtain

$$\begin{aligned}
\|F(u)\|_{L_T^1H^s} &\lesssim \|V_\gamma(u)\|_{L_T^1L^\infty}\|u\|_{L_T^1H^s} + \|V_\gamma(u)\|_{L_T^{q'}H^s_{\frac{2n}{\gamma}}}\|u\|_{L_T^qL^{\frac{2n}{n-\gamma}}}\\
&\lesssim T^{1-\frac2q}\|u\|_{L_T^qL^{\frac{2n}{n-\gamma-\varepsilon_0}}}\|u\|_{L_T^qL^{\frac{2n}{n-\gamma+\varepsilon_0}}}\|u\|_{L_T^\infty H^s}\\
&\qquad + T^{1-\frac2q}\||u|^2\|_{L_T^qH^s_{\frac{2n}{2n-\gamma-\varepsilon_0}}}\|u\|_{L_T^qL^{\frac{2n}{n-\gamma+\varepsilon_0}}}\\
&\lesssim T^{1-\frac2q}\left(\|u\|_{L_T^qL^{\frac{2n}{n-\gamma-\varepsilon_0}}}\|u\|_{L_T^qL^{\frac{2n}{n-\gamma+\varepsilon_0}}}\right)\|u\|_{L_T^\infty H^s},
\end{aligned} \tag{15}$$

where $0 < \varepsilon_0 < n-\gamma$ and we have used the inequality that for any $x \in \mathbb{R}^n$

$$|V_\gamma(u)(x)| \lesssim \|u\|_{L^{\frac{2n}{n-\gamma-\varepsilon_0}}}\|u\|_{L^{\frac{2n}{n-\gamma+\varepsilon_0}}}. \tag{16}$$

Now if we choose ε and ε_0 so small that $\gamma < 2-2\sigma$ and

$$\frac{2n}{n-1} \le \frac{2n}{n-(\gamma-\varepsilon_0)} < \frac{2n}{n-(\gamma+\varepsilon_0)} \le \frac{2n}{n-1-2(\frac12-\sigma)},$$

then we have from (14), (15) and embeddings $H^{\frac12} \cap \widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}} \hookrightarrow L^r$ for any $\frac{2n}{n-1} \le r \le \frac{2n}{n-1-2(\frac12-\sigma)} < \frac{2n}{n-2}$ that $\|u\|_{L^{\frac{2n}{n-(\gamma\pm\varepsilon_0)}}} \lesssim \|u\|_{H^{\frac12}} + \|u\|_{\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}}$ and

$$\|N(u)\|_{X_T} \le C(\|\varphi\|_{H^s} + (T+T^{1-\frac2q})\|u\|_{X_T}^3) \le C(\|\varphi\|_{H^s} + (T+T^{1-\frac2q})\rho^3)$$

for some constant C. Thus if we choose ρ and T so that $C\|\varphi\|_{H^s} \le \frac{\rho}{2}$ and $C(T + T^{1-\frac2q})\rho^3 \le \frac{\rho}{2}$, then we conclude that N maps from $X_{T,\rho}$ to itself.

For any $u, v \in X_{T,\rho}$, we have

$$\begin{aligned}
d(N(u),N(v)) &\lesssim \|F(u)-F(v)\|_{L_T^1H^s}\\
&\lesssim \|(|\cdot|^{-\gamma} * (|u|^2-|v|^2))u\|_{L_T^1H^s} + \|(|\cdot|^{-\gamma} * |v|^2)(u-v)\|_{L_T^1H^s}.
\end{aligned} \tag{17}$$

By (16) and Hölder's inequality, we have for sufficiently small $\varepsilon_0 > 0$

$$\begin{aligned}
&\|(|\cdot|^{-\gamma}*(|u|^2-|v|^2))u\|_{L^1_T H^s}\\
&\lesssim T^{\frac12}\||\cdot|^{-\gamma}*(|u|^2-|v|^2)\|_{L^2_T L^\infty}\|u\|_{L^\infty_T H^s}\\
&\qquad + \||\cdot|^{-\gamma}*(|u|^2-|v|^2)\|_{L^{q'}_T H^s_{\frac{2n}{\gamma+\varepsilon_0}}}\|u\|_{L^q_T L^{\frac{2n}{n-(\gamma+\varepsilon_0)}}}\\
&\lesssim T^{\frac12}\rho\||u|^2-|v|^2\|^{\frac12}_{L^1_T L^{\frac{n}{n-(\gamma+\varepsilon_0)}}}\||u|^2-|v|^2\|^{\frac12}_{L^1_T L^{\frac{n}{n-(\gamma-\varepsilon_0)}}} \qquad (18)\\
&\quad + T^{1-\frac2q}\rho\|u-v\|_{L^\infty_T H^s}(\|u\|_{L^q_T L^{\frac{2n}{n-(\gamma-\varepsilon_0)}}}+\|v\|_{L^q_T L^{\frac{2n}{n-(\gamma-\varepsilon_0)}}})\\
&\quad + T^{1-\frac2q}\rho\|u-v\|_{L^q_T L^{\frac{2n}{n-(\gamma-\varepsilon_0)}}}(\|u\|_{L^\infty_T H^s}+\|v\|_{L^\infty_T H^s}).
\end{aligned}$$

Now by another Hölder's inequality in time, we have

$$\|(|\cdot|^{-\gamma}*(|u|^2-|v|^2))u\|_{L^1_T H^s} \lesssim (T+T^{1-\frac2q})\rho^2 d(u,v).$$

Similarly,

$$\begin{aligned}
&\|(|\cdot|^{-\gamma}*|v|^2)(u-v)\|_{L^1_T H^s}\\
&\lesssim \|(|\cdot|^{-\gamma}*|v|^2)\|_{L^1_T L^\infty}\|u-v\|_{L^\infty_T H^s}\\
&\quad + T^{1-\frac2q}\|(|\cdot|^{-\gamma}*|v|^2)\|_{L^q_T L^{\frac{2n}{\gamma+\varepsilon_0}}}\|u-v\|_{L^q_T L^{\frac{2n}{n-(\gamma+\varepsilon_0)}}} \qquad (19)\\
&\lesssim T^{1-\frac2q}\|v\|_{L^q_T L^{\frac{2n}{n-(\gamma-\varepsilon_0)}}}\|v\|_{L^q_T L^{\frac{2n}{n-(\gamma+\varepsilon_0)}}} d(u,v)\\
&\quad + T^{1-\frac2q}\|v\|_{L^\infty_T H^s}\|v\|_{L^q_T L^{\frac{2n}{n-(\gamma-\varepsilon_0)}}}\|u-v\|_{L^q_T L^{\frac{2n}{n-(\gamma+\varepsilon_0)}}}.
\end{aligned}$$

Hence by Sobolev embedding we get

$$\|(|\cdot|^{-\gamma}*|v|^2)(u-v)\|_{L^1_T H^s} \lesssim (T+T^{1-\frac2q})\rho^2 d(u,v).$$

Substituting these two estimates into (17) and then using the fact $C(T+T^{1-\frac2q})\rho^2 \le \frac12$ for small T, we conclude that N is a contraction mapping on $X_{T,\rho}$. The energy and L^2 conservations follow by the method in [13].

Now we show that the local solutions can be extended globally in time. For this purpose we prove an a priori estimate in X_T for any $T > 0$. Fixing T, since $\gamma < 2$, from the energy conservation we see that at any $t \le T$, the solution u satisfies that for $\lambda > 0$,

$$\frac12\|u(t)\|^2_{H^{\frac12}} \le \frac12\|u(t)\|^2_{L^2} + E(u) = \frac12\|\varphi\|^2_{L^2} + E(\varphi)$$

and for $\lambda < 0$

$$\begin{aligned}
\frac12\|u(t)\|^2_{H^{\frac12}} &\le \frac12\|u(t)\|^2_{L^2} + |E(u)| + |V(u)|\\
&\le \frac12\|\varphi\|^2_{L^2} + |E(\varphi)| + C\|u\|^2_{L^{\frac{2n}{n-\gamma+1}}}\|u\|^2_{H^{\frac12}}\\
&\le \frac12\|\varphi\|^2_{L^2} + |E(\varphi)| + C\|u\|^{2-\gamma}_{L^2}\|u\|^{1+\gamma}_{H^{\frac12}}\\
&= \frac12\|\varphi\|^2_{L^2} + |E(\varphi)| + C\|\varphi\|^{2-\gamma}_{L^2}\|u\|^{1+\gamma}_{H^{\frac12}}
\end{aligned}$$

and hence by Young's inequality and the smallness of $\|\varphi\|_{L^2}$

$$\|u(t)\|_{H^{\frac12}}^2 \le C(\|\varphi\|_{L^2}^2 + |E(\varphi)|). \tag{20}$$

From the estimates (20) and (16), we have for $\delta > 0$

$$\|u\|_{L^q_\delta \widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}} \lesssim (\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac12} + \delta(\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac32} + \delta^{1-\frac{2}{q}}(\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac12}\|u\|^2_{L^q_\delta \widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}}.$$

Thus for sufficiently small δ but equivalent to the value $(1+\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{-\frac{q}{q-2}}$,

$$\|u\|_{L^q(T_{j-1},T_j;\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}})} \le C(\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac12},$$

where $T_j - T_{j-1} = \delta$ for $j \le k-1$, $T_k = T$ and $T_k - T_{k-1} \sim \delta$. This implies that

$$\begin{aligned} \|u\|^q_{L^q(0,T;\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}})} &\le \sum_{1\le j\le k} \|u\|^q_{L^q(T_{j-1},T_j;\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}})} \\ &\lesssim k\delta(1+\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac{q^2}{2q-4}} \lesssim T(1+\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac{q^2}{2q-4}}. \end{aligned} \tag{21}$$

Finally, we have from (15)

$$\|u(t)\|_{H^s} \le \|\varphi\|_{H^s} + \|F(u)\|_{L^1_t H^s} \lesssim \|\varphi\|_{H^s} + \int_0^t (\|u\|^2_{H^{\frac12}} + \|u\|^2_{\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}})\|u\|_{H^s}\,dt'.$$

Hence by Gronwall's inequality and (21),

$$\begin{aligned} \|u(t)\|_{H^s} &\lesssim \|\varphi\|_{H^s} \exp\left(Ct(\|\varphi\|_{L^2}^2 + |E(\varphi)|) + Ct^{1-\frac{2}{q}}\|u\|^2_{L^q_t \widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}}\right) \\ &\lesssim \|\varphi\|_{H^s} \exp\left(Ct(1+\|\varphi\|_{L^2}^2 + |E(\varphi)|)^{\frac{q}{q-2}}\right). \end{aligned} \tag{22}$$

This completes the proof of Theorem 1.1.

Proof of Strichartz estimate (7) *of radial functions.* For the first inequality, we follow the proof of Proposition 6.3 in [14]. By the spherical coordinate,

$$(U(t)\varphi)(x) = c_n \int_0^\infty e^{-it\sqrt{1+\rho^2}}\widehat{d\sigma}(r\rho)\widehat{\varphi}(\rho)\rho^{n-1}\,d\rho,$$

where $r = |x|$, $\rho = |\xi|$ and

$$\widehat{d\sigma}(r\rho) = \int_{S^{n-1}} e^{-ix\cdot\xi}\,d\sigma = \int_{S^{n-1}} e^{ix\cdot\xi}\,d\sigma.$$

Let us define a one-dimensional function f by $f(\rho) = w(\rho)\widehat{\varphi}(\rho)\rho^{\frac{n-1}{2}}$ for some positive function to be chosen later and an operator $W(t)$ by $(W(t)f)(r) = r^{\frac{n-1}{p}}(U(t)\varphi)(r)$. Then since the space $\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}$ is equivalent to the space $(-\Delta)^{-\frac14}(1-\Delta)^{\frac14}H^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}$, we have only to show that for $w = \rho^{-\frac12}(1+\rho^2)^{\frac14}(1+\rho^2)^{\frac{n}{4}-\frac{n+1}{2p}}$,

$$\|W(\cdot)f\|_{L^p((0,T)\times(0,\infty))} \lesssim \|f\|_{L^2}. \tag{23}$$

By the change of variable $\sqrt{1+\rho^2} \mapsto \rho$, $W(t)f$ is written as

$$(W(t)f)(r) = c_n r^{\frac{n-1}{p}} \int_{\mathbb{R}} e^{-it\rho}\chi_{(1,\infty)}(\rho)\widehat{d\sigma}(r\sqrt{\rho^2-1})\frac{f(\sqrt{\rho^2-1})}{w(\sqrt{\rho^2-1})^2}\frac{\rho(\rho^2-1)^{\frac{n-1}{4}}}{\sqrt{\rho^2-1}}\,d\rho.$$

Using Sobolev embedding $\dot{H}^{\frac{1}{2}-\frac{1}{p}}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ and Plancherel theorem, it follows from change of variable $\sqrt{\rho^2-1} \mapsto \rho$ that

$$\begin{aligned}
&\|(W(\cdot)f)(r)\|_{L^p(0,T)}^2 \\
&\lesssim r^{\frac{2(n-1)}{p}} \int_1^\infty \rho^{2(\frac{3}{2}-\frac{1}{p})}(\rho^2-1)^{\frac{n-3}{2}}\left|\widehat{d\sigma}(r\sqrt{\rho^2-1})\right|^2 \frac{|f(\sqrt{\rho^2-1})|^2}{w(\sqrt{\rho^2-1})^2}\,d\rho \qquad (24)\\
&\lesssim r^{\frac{2(n-1)}{p}} \int_0^\infty (1+\rho^2)^{1-\frac{1}{p}}\rho^{n-2}\left|\widehat{d\sigma}(r\rho)\right|^2 \frac{|f(\rho)|^2}{w(\rho)^2}\,d\rho.
\end{aligned}$$

Hence by taking $L^{\frac{p}{2}}$ norm in r-variable to (24), we have

$$\|W(\cdot)f\|_{L^p((0,T)\times(0,\infty))}^2 \lesssim \int_0^\infty (1+\rho^2)^{1-\frac{1}{p}}\rho^{n-2}\frac{|f(\rho)|^2}{w(\rho)^2}A(\rho)^2\,d\rho, \qquad (25)$$

where

$$A(\rho) = \left(\int_0^\infty r^{n-1}\left|\widehat{d\sigma}(r\rho)\right|^p\,dr\right)^{\frac{1}{p}}.$$

From the well-known decay of Fourier transform of measure on the unit sphere (see [16]), we have for $p > \frac{2n}{n-1}$

$$A(\rho)^p \lesssim \int_0^\infty r^{n-1}(1+r\rho)^{-\frac{p(n-1)}{2}}\,dr \lesssim \rho^{-n}.$$

Substituting this into (25), we finally have

$$\begin{aligned}
\|W(\cdot)f\|_{L^p((0,T)\times(0,\infty))}^2 &\lesssim \int_0^\infty (1+\rho^2)^{1-\frac{1}{p}}\rho^{n-2-\frac{2n}{p}}\frac{|f(\rho)|^2}{w(\rho)^2}\,d\rho \\
&\lesssim \int_0^\infty (1+\rho^2)^{\frac{n}{2}-\frac{n+1}{p}}\rho^{-1}\sqrt{1+\rho^2}\frac{|f(\rho)|^2}{w(\rho)^2}\,d\rho,
\end{aligned}$$

since $p > \frac{2n}{n-1}$. Therefore, if we choose $w(\rho) = \rho^{-\frac{1}{2}}(1+\rho^2)^{\frac{1}{4}}(1+\rho^2)^{\frac{n}{4}-\frac{n+1}{2p}}$, then we prove the claim.

Now we prove the second inequality. For this purpose, it suffices to show that

$$\left\|\int_0^t W(t-t')G(t')\,dt'\right\|_{L^p((0,T)\times(0,\infty))} \lesssim \|G\|_{L_T^1L^2}, \qquad (26)$$

where $G(\rho,t') = w(\rho)\widehat{F}(\rho,t)\rho^{\frac{n-1}{2}}$ and

$$\begin{aligned}
W(t-t')G(t') &= c_n r^{\frac{n-1}{p}}\int_0^\infty e^{-i(t-t')\sqrt{1+\rho^2}}\widehat{d\sigma}(r\rho)\frac{G(\rho,t')}{w(\rho)}\,d\rho \\
&= r^{\frac{n-1}{p}}U(t-t')F(r,t').
\end{aligned}$$

To show (26), let us introduce the low-diagonal operator estimate. For instance, see [3, 15, 1].

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach spaces. Let K be an operator such that $\|KG\|_{L_T^q(\mathcal{A})} \le C\|G\|_{L_T^p(\mathcal{B})}$ for $1 \le p \le q \le \infty$ with kernel k defined by $KG(t) = \int_0^T k(t-t')G(t')\,dt'$, where C does not depend on T. If $p < q$, then the low-diagonal operator $\widetilde{K}$ defined by $\widetilde{K}G = \int_0^t k(t-t')G(t')\,dt'$ satisfies that $\|\widetilde{K}G\|_{L_T^q(\mathcal{A})} \le \widetilde{C}\|G\|_{L_T^q(\mathcal{B})}$, where $\widetilde{C}$ does not depend on T.*

By Lemma 2.2 with kernel $k(t) = W(t)$, $\mathcal{A} = L^2$ and $\mathcal{B} = L^p$, it suffices to show that

$$\|\int_0^T W(t-t')G(t')\,dt'\|_{L^p((0,T)\times(0,\infty))} \lesssim \|G\|_{L_T^1 L^2}. \tag{27}$$

In fact, since for any $G \in L_T^1(L^2(0,\infty))$ we can find a unique radial function $F \in L_T^1(H^{\frac{n}{2}-\frac{n+1}{p}}(\mathbb{R}^n))$ such that $G(\rho,t) = w(\rho)\widehat{F}(\rho,t)\rho^{\frac{n-1}{2}}$ and hence

$$\|\int_0^T W(t-t')G(t')\,dt'\|_{L^p((0,T)\times(0,\infty))} = \|L(\Delta)\int_0^T U(t-t')F(t')\,dt'\|_{L^p((0,T)\times\mathbb{R}^n)},$$

where $L(\Delta) = (-\Delta)^{\frac14}(1-\Delta)^{-\frac14}$. Then by the Strichartz estimate (7), we have

$$\begin{aligned}&\|L(\Delta)\int_0^T U(t-t')F(t')\,dt'\|_{L^p((0,T)\times\mathbb{R}^n)}\\ &= \|L(\Delta)U(t)\int_0^T U(-t')F(t')\,dt'\|_{L^p((0,T)\times\mathbb{R}^n)} \lesssim \|F\|_{L_T^1 H^{\frac{n}{2}-\frac{n+1}{p}}} = \|G\|_{L_T^1 L^2}.\end{aligned}$$

This proves (27) and thus the claim (26). □

3. Proof of Theorem 1.2

We proceed a similar line to the proof of Theorem 1.1(contraction scheme for local existence, energy and L^2 conservation for global time extension) except for $H^{1,1}$ estimate. For this purpose, we will prove only a priori estimates (10), (11) and (12).

Let us begin with proof of (11). Using the commutator relation

$$[\,x,\,\sqrt{1-\Delta}\,] = \nabla(1-\Delta)^{-\frac12}, \tag{28}$$

we have

$$\begin{aligned}\|x\sqrt{1-\Delta}F(u)\|_{L^2} &\le \|\nabla(1-\Delta)^{-\frac12}F\|_{L^2} + \|\sqrt{1-\Delta}(xF(u))\|_{L^2}\\ &\lesssim \|F\|_{L^2} + \|xF(u)\|_{L^2} + \|\nabla(V_\gamma(u)xu)\|_{L^2}.\end{aligned}$$

By using the estimate (15), we have

$$\|F\|_{L^2} + \|xF(u)\|_{L^2} \lesssim (\|u\|_{H^{\frac12}}^2 + \|u\|_{\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}}^2)\|u\|_{H^{1,1}}$$

and

$$\begin{aligned}\|\nabla(V_\gamma(u)xu)\|_{L^2} &\le \|V_\gamma(u)\|_{L^\infty}\|\nabla(xu)\|_{L^2} + \||\nabla V_\gamma(u)||x|u\|_{L^2}\\ &\lesssim (\|u\|_{H^{\frac12}}^2 + \|u\|_{\widetilde{H}^{\frac12,\frac12-\sigma}_{\frac{2n}{n-1}}}^2)\|u\|_{H^{1,1}} + \||\nabla V_\gamma(u)||x|u\|_{L^2}.\end{aligned}$$

It remains to estimate the last term. To do this, let us introduce a fractional integration of radial function in $\mathbb{R}^n, n \ge 3$ which will be proven later.

Lemma 3.1. *Let γ satisfy $0 < \gamma < n-1$ for $n \geq 3$. If $f \in H^\alpha$ and $g \in H^\beta$ are radially symmetric functions with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq \gamma$, then for any $x \neq 0$*

$$\int_{\mathbb{R}^n} \frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy \lesssim |x|^{\alpha+\beta-\gamma}\|f\|_{H^\alpha}\|g\|_{H^\beta}.$$

Now letting $f = |\nabla u|$ and $g = |u|$, by Lemma 3.1 with $\alpha = 0, \beta = 0$ for $1 < \gamma \leq \frac{3}{2}$ and $\alpha = 0, \beta = \frac{1}{2}$ for $\frac{3}{2} < \gamma < \frac{2n-1}{n}$, we get $|\nabla V_\gamma(u)| \lesssim |x|^{-\gamma}\|u\|_{H^1}\|u\|_{L^2}$ for $1 < \gamma \leq \frac{3}{2}$ and $|\nabla V_\gamma(u)| \lesssim |x|^{-\gamma+\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}$ for $\frac{3}{2} < \gamma < \frac{2n-1}{n}$ and hence

$$\||\nabla V_\gamma(u)||x|u\|_{L^2} \lesssim \|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}\left\|\frac{|u|}{|x|^\theta}\right\|_{L^2}$$

for some $\theta \in (0, \frac{1}{2})$. Therefore the Hardy inequality yields $\||\nabla V_\gamma(u)||x|u\|_{L^2} \lesssim \|u\|_{H^1}\|u\|^2_{H^{\frac{1}{2}}}$. This proves the estimate (11). Combining the argument in the introduction and the above one, we have (10).

Now we prove (12). From the identity (28) we have

$$\begin{aligned} x\sqrt{1-\Delta}u(t) = U(t)x\sqrt{1-\Delta}\varphi - i\int_0^t x\sqrt{1-\Delta}F(u(t'))\,dt' \\ - itU(t)\nabla\varphi - \int_0^t U(t-t')(t-t')\nabla F(u(t'))\,dt', \end{aligned}$$

where the last two terms are rewritten as

$$\begin{aligned} &- i\int_0^t U(t)\nabla\varphi\,dt' - \int_0^t\int_0^{t'} U(t-t'')\nabla F(u(t''))\,dt'dt'' \\ &= -i\int_0^t U(t-t')\nabla(U(t')\varphi - i\int_0^{t'} U(t-t')F(u(t''))\,dt'')dt' \\ &= -i\int_0^t U(t-t')\nabla u(t')\,dt'. \end{aligned}$$

Hence

$$x\sqrt{1-\Delta}u(t) = U(t)x\sqrt{1-\Delta}\varphi - i\int_0^t U(t-t')(-\nabla u + x\sqrt{1-\Delta}F(u))\,dt'.$$

This implies the estimate (12).

Proof of Lemma 3.1. Fixing x, we divide the integration into three parts as follows.

$$\int_{\mathbb{R}^n}\frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy = \int_{|y|>2|x|} + \int_{\frac{|x|}{2}\leq|y|\leq 2|x|} + \int_{|y|<\frac{|x|}{2}} \equiv I + II + III.$$

For I, since $|x-y| \geq \frac{|y|}{2}$ for $|y| > 2|x|$ and $\alpha+\beta\leq\gamma$, we have

$$I \lesssim \int_{|y|>2|x|}|y|^{\alpha+\beta-\gamma}\frac{|f(y)|}{|y|^\alpha}\frac{|g(y)|}{|y|^\beta}\,dy \lesssim |x|^{\alpha+\beta-\gamma}\|f\|_{H^\alpha}\|g\|_{H^\beta}.$$

Since u is radially symmetric, we may assume that $x = re_1 = r(0,0,\cdots,0,1)$. Using the spherical coordinates $(\rho,\theta_1,\theta_2,\cdots,\theta_{n-1}) \in (0,\infty)\times[0,\pi]\times[0,\pi]\times\cdots\times[0,2\pi]$ for y variable, the integrals II and III are converted into

$$II + III = \left(\int_{\frac{r}{2}}^{2r} + \int_0^{\frac{r}{2}}\right)\rho^{\alpha+\beta+n-1}\frac{|f(\rho)|}{\rho^\alpha}\frac{|g(\rho)|}{\rho^\beta}\Omega(r,\rho)d\rho,$$

where

$$\Omega(r,\rho) = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(\rho^2\sin^2\theta_1 + (\rho\cos\theta_{n-1} - r)^2)^{-\frac{\gamma}{2}}$$
$$\times \sin^{n-2}\theta_1 \sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\,d\theta_1\cdots d\theta_{n-2}d\theta_{n-1}.$$

If $\frac{r}{2} \le \rho \le 2r$, then by the fact[2] $n-2-\gamma > -1$

$$\Omega(r,\rho) \lesssim \rho^{-\gamma}\int_0^{\pi}\sin^{n-2-\gamma}\theta_1\,d\theta_1 \lesssim \rho^{-\gamma}.$$

If $\rho < \frac{r}{2}$, then $\Omega(r,\rho) \lesssim r^{-\gamma}$, since $|\rho\cos\theta_{n-1} - r| \ge \frac{r}{2}$. Therefore by Hölder and Hardy inequalities we have

$$II + III \lesssim r^{\alpha+\beta-\gamma}\int_0^{\infty}\rho^{n-1}\frac{|f(\rho)|}{\rho^{\alpha}}\frac{|g(\rho)|}{\rho^{\beta}}d\rho \lesssim r^{\alpha+\beta-\gamma}\|f\|_{H^\alpha}\|g\|_{H^\beta}.$$

This completes the proof of the lemma. □

References

[1] C. Ahn and Y. Cho, *Lorentz space extension of Strichartz estimate*, Proc. Amer. Math. Soc. 133 (2005) pp. 3497–3503.

[2] Y. Cho and T. Ozawa, *On the semi-relativistic Hartree type equation*, Hokkaido University Preprint Series in Math ♯773.

[3] M. Christ and A. Kiselev, *Maximal functions associated to filtrations*, J. Func. Anal., 179 (2001), pp. 409–425.

[4] F. M. Christ and M. I. Weinstein, *Dispersion of small amplitude solution of the generalized Korteweg-de Vries equation*, J. Func. Anal., 100 (1991), pp. 87–109.

[5] A. Elgart and B. Schlein, *Mean field dynamics of boson stars*, to appeer in Comm. Pure Appl. Math., arXiv:math-ph/0504051.

[6] J. Fröhlich and E. Lenzmann, *Mean-field limit of quantum bose gases and nonlinear Hartree equation*, Sémin. Equ. Dériv., Partielles XIX (2003-2004), pp. 1–26, arXiv:math-ph/0409019.

[7] J. Fröhlich, B. L. G. Jonsson and E. Lenzmann, *Boson stars as solitary waves*, to appear in Comm. Pure Appl. Math., arXiv:math-ph/0512040.

[8] T. Kato, *On nonlinear Schrödinger equations II. H^s-solutions and unconditional well-posedness*, J. Anal. Math., 67 (1995), pp. 281–306.

[9] E. H. Lieb and H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys., 112 (1987), pp. 147–174.

[10] E. Lenzmann, *Well-posedness for semi-relativistic Hartree equations of critical type*, arXiv:math.AP/0505456.

[11] S. Machihara, K. Nakanishi and T. Ozawa, *Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations*, Math. Ann., 322 (2002), pp. 603–621.

[12] S. Machihara, K. Nakanishi and T. Ozawa, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation*, Rev. Mat. Iberoamericana, 19 (2003), pp. 179–194.

[13] T. Ozawa, *Remarks on proofs of conservation laws for nonlinear Schrödinger equations*, Cal. Var. PDE. 25 (2006), pp. 403–408.

[14] C. D. Sogge, *Lectures on Nonlinear Wave Equations*, International Press, 1995.

[15] H. Smith and C. D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations 25 (2000), pp. 2171–2183.

[16] E. M. Stein, *Harmonic Analysis*, Princeton University Press, New Jersey, 1993.

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[2]This is the reason for the condition $n \ge 3$ in Theorem 1.2.