SMALL-DATA SCATTERING FOR NONLINEAR WAVES OF CRITICAL DECAY IN TWO SPACE DIMENSIONS

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Abstract. Consider the nonlinear wave equation with zero mass in two space dimensions. When it comes to the associated Cauchy problem with small initial data, the known existence results are already sharp; those require the data to decay at a rate $k \geq k_c$, where k_c is a critical decay rate that depends on the order of the nonlinearity. However, the known scattering results treat only the supercritical case $k > k_c$. In this paper, we prove the existence of the scattering operator for the full optimal range $k \geq k_c$.

1. Introduction

We study the scattering problem for the nonlinear wave equation

$$\partial_t^2 u - \Delta u = F(u) \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}, \tag{1.1}$$

where $F(u)$ behaves like $|u|^p$ for some $p > 1$. When it comes to the associated Cauchy problem, it is known that both the size of p and the decay rate k of the initial data play a crucial role in the existence theory for small initial data. In fact, the condition $k \geq 2/(p-1)$ is one of the sharp conditions needed to ensure the existence of small-amplitude solutions. However, the scattering operator for (1.1) has been constructed only in the supercritical case $k > 2/(p-1)$. In this paper, we construct the scattering operator for the full optimal range $k \geq 2/(p-1)$.

Let us first focus on the associated Cauchy problem and prescribe initial data

$$u(x,0) = \varphi(x), \qquad \partial_t u(x,0) = \psi(x). \tag{1.2}$$

A sharp existence result for (1.1) was obtained by Glassey [1] under the assumption that φ, ψ are compactly supported. An extension of Glassey's result to more general data was obtained by Kubota [5] and independently by Tsutaya [10, 11]. In these results, one assumes that

$$\sum_{|\alpha|\leq 3} |\partial_x^\alpha \varphi(x)| + \sum_{|\beta|\leq 2} |\partial_x^\beta \psi(x)| \leq \varepsilon(1+|x|)^{-k-1} \tag{1.3}$$

for some $k > 0$ and some small $\varepsilon > 0$. To ensure the existence of classical solutions to the associated Cauchy problem, it then suffices to require that

$$p > \frac{3+\sqrt{17}}{2}, \qquad k \geq \frac{2}{p-1}\,. \tag{1.4}$$

Recall that p denotes the order of the nonlinear term. Conditions (1.4) are also known to be necessary for the existence of small-amplitude solutions. That is, there exist arbitrarily small initial data satisfying (1.3) for which the solution to (1.1) blows up in finite time, if either $1 < p \leq (3+\sqrt{17})/2$ or else $0 < k < 2/(p-1)$ for some $p > 1$; see [2, 7, 11].

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When it comes to the associated scattering problem, Tsutaya [12] established the existence of the scattering operator for small initial data under the assumptions

$$p > \frac{3+\sqrt{17}}{2}, \qquad k > \frac{2}{p-1}. \tag{1.5}$$

A similar scattering result was obtained by Kubota and Mochizuki [6], however the additional assumption $k > 1/2$ was imposed there. In view of the existence results stated above, the only possible improvement of Tsutaya's result [12] amounts to replacing his conditions (1.5) by the conditions (1.4) which are necessary for the existence of solutions. Our goal in this paper is to show that such an improvement is feasible for the two-dimensional problem (1.1). Namely, the conditions which are necessary for the existence of solutions are also sufficient for the existence of the scattering operator. It is perhaps worth mentioning that this is no longer the case for the three-dimensional version of (1.1). In three space dimensions, that is, the sharp conditions needed for the existence of the scattering operator are

$$p > 1+\sqrt{2}, \qquad k \geq \frac{2}{p-1}, \qquad kp > \frac{5}{2} \tag{1.6}$$

and only the first two of those conditions are necessary for the existence of solutions; see our previous work [4] for more details. Based on the results of [4], one would expect

$$p > \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}, \quad k \geq \frac{2}{p-1}, \quad kp > \frac{n}{2}+1$$

to be the analogous sharp conditions in n space dimensions. Here, the rightmost condition is redundant only when $n=2$, as the middle condition already implies that $kp \geq k+2 > 2$. For a list of the known results in higher dimensions, see [3] and the references cited therein.

Let us now focus on the two-dimensional scattering problem for (1.1). In what follows, we denote by u_0^- the solution to the homogeneous wave equation

$$\partial_t^2 u_0 - \Delta u_0 = 0 \qquad \text{in } \mathbb{R}^2 \times \mathbb{R} \tag{1.7}$$

subject to the initial data (1.2). As it is well-known, one can obtain a solution u to (1.1) by solving the associated integral equation

$$u = u_0^- + \mathscr{L}F(u), \tag{1.8}$$

where the Duhamel operator $\mathscr{L}$ is defined by the formula

$$[\mathscr{L}F(u)](x,t) = \frac{1}{2\pi}\int_{-\infty}^t (t-\tau)\int_{|y|<1} \frac{F(u(x+(t-\tau)y,\tau))}{\sqrt{1-|y|^2}}\,dy\,d\tau. \tag{1.9}$$

Regarding the existence of solutions to (1.8), we shall establish the following

Theorem 1 (Existence)**.** *Let u_0^- be the solution to the homogeneous wave equation*

$$\partial_t^2 u_0 - \Delta u_0 = 0, \quad u_0(x,0) = \varphi(x), \quad \partial_t u_0(x,0) = \psi(x).$$

Assume (1.3), (1.4) *and take $F(u) = \pm|u|^p$ or $F(u) = \pm|u|^{p-1}u$. If ε is sufficiently small, then the integral equation* (1.8) *has a unique $\mathcal{C}^2$-solution.*

Remark 2. The solutions we construct lie in a certain Banach space that we introduce in the next section; see (2.4). We can similarly construct solutions for more general nonlinear terms than the ones listed here; our precise assumptions on F appear in (2.8), (2.9).

The proof of Theorem 1 is essentially based on the approach of [6, 12], where the existence of solutions was shown for supercritical decay rates $k > 2/(p-1)$. To extend these results to the critical decay rate, however, we need to establish a new estimate for the kernel associated with the homogeneous wave equation in two space dimensions; see Lemmas 16 and 17. This new estimate plays a key role in treating the logarithmic singularity that arises in the kernel, and it also provides a crucial refinement of the estimates used in [6, 12].

As an immediate consequence of Theorem 1, we shall also establish the following

Theorem 3 (Scattering)**.** *Let the assumptions of Theorem 1 hold and define the energy norm*

$$||w||_e = \left(\int_{\mathbb{R}^2} |\partial_t w(x,t)|^2 \, dx + \int_{\mathbb{R}^2} |\nabla w(x,t)|^2 \, dx \right)^{1/2} . \tag{1.10}$$

Then the unique solution u provided by Theorem 1 satisfies

$$||u - u_0^-||_e \to 0 \qquad as \;\; t \to -\infty, \tag{1.11}$$

and there exists a unique solution u_0^+ to the homogeneous equation (1.7) *which satisfies*

$$||u - u_0^+||_e \to 0 \qquad as \;\; t \to +\infty. \tag{1.12}$$

In particular, one can define the scattering operator $S\colon u_0^- \to u_0^+$.

The remaining of this paper is organized as follows. In section 2, we give the well-known weighted L^∞-estimates for the homogeneous wave equation and then we establish some useful estimates involving our weight function (2.6). In section 3, we obtain a new estimate for the associated kernel and we also establish the basic estimate for the existence proof. Finally, the proofs of our main results, Theorem 1 and Theorem 3, are given in section 4.

2. A priori estimates

In this section, we gather some estimates that will be needed in the proof of our existence result, Theorem 1. Let us first focus on the homogeneous wave equation

$$\partial_t^2 u_0 - \Delta u_0 = 0 \qquad \text{in } \; \mathbb{R}^2 \times \mathbb{R} \tag{2.1}$$

and impose the conditions

$$u_0(x,0) = \varphi(x), \qquad \partial_t u_0(x,0) = \psi(x). \tag{2.2}$$

When it comes to the initial data, we shall assume that

$$\sum_{|\alpha| \le 3} |\partial_x^\alpha \varphi(x)| + \sum_{|\alpha| \le 2} |\partial_x^\alpha \psi(x)| \le \varepsilon \, \langle x \rangle^{-k-1} \,, \tag{2.3}$$

where $\langle x \rangle = 1 + |x|$ and the constants ε, k are both positive. To study the homogeneous wave equation with such data, it is convenient to introduce the Banach space

$$X = \Big\{ u(x,t) \; : \; \partial_x^\alpha u(x,t) \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}) \quad \text{for } \; |\alpha| \le 2, \quad ||u|| < \infty \Big\} \, . \tag{2.4}$$

Here, the norm $|| \cdot ||$ is defined by

$$||u|| = \sum_{|\alpha| \le 2} \sup_{\substack{x \in \mathbb{R}^2 \\ t \in \mathbb{R}}} |\partial_x^\alpha u(x,t)| \cdot w_k(|x|, |t|), \tag{2.5}$$

where the weight function w_k is of the form

$$w_k(|x|,|t|) = \langle |x|+|t|\rangle^{\beta} \langle |x|-|t|\rangle^{\gamma} \left(1+\ln\frac{\langle |x|+|t|\rangle}{\langle |x|-|t|\rangle}\right)^{-\delta_{k,1/2}} \tag{2.6}$$

with $\beta = \min(k, 1/2)$, $\gamma = \max(k-1/2, 0)$ and $\delta_{k,1/2}$ the usual Kronecker delta.

For the proof of the following lemma, we refer the reader to [5, 10].

Lemma 4. *Let $\varphi \in \mathcal{C}^3(\mathbb{R}^2)$ and $\psi \in \mathcal{C}^2(\mathbb{R}^2)$ be subject to (2.3) for some $\varepsilon > 0$ and $0 < k < 1$. Then the Cauchy problem (2.1)-(2.2) admits a unique solution $u_0^- \in X$, where X is defined by (2.4). Moreover, one has $||u_0^-|| \le C_0\varepsilon$ for some constant C_0 that depends solely on k.*

Remark 5. Although Lemma 4 applies for the case $k \ge 1$ as well, the definition (2.6) of the weight function needs to be slightly modified for that case. When it comes to the nonlinear problem we wish to address, however, we need only treat the case $0 < k < 1$ because the decay rate k may be decreased without loss of generality; see (2.10).

Next, we turn to the nonlinear wave equation

$$\partial_t^2 u - \Delta u = F(u) \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}. \tag{2.7}$$

When it comes to the nonlinear term $F(u)$, we assume that

$$F \in \mathcal{C}^2(\mathbb{R}); \qquad F(0) = F'(0) = F''(0) = 0 \tag{2.8}$$

and that the estimate

$$|F''(u) - F''(v)| \le A(|u|+|v|)^{p-3} \cdot |u-v| \tag{2.9}$$

holds for some $A > 0$ and some $p > \frac{3+\sqrt{17}}{2}$ whenever $|u|, |v| \le 1$.

Recall that we seek a solution to the integral equation (1.8), where u_0^- is the solution of Lemma 4. One of our assumptions ensures that $k \ge 2/(p-1)$, where k is the decay rate of the initial data. There is no loss of generality in decreasing the decay rate k, as long as the lower bound is not contradicted. Since we actually have

$$\frac{2}{p-1} < \frac{1}{2} + \frac{1}{p} < \frac{p}{2} - 1$$

whenever $p > \frac{3+\sqrt{17}}{2}$, this means there is no loss of generality in assuming

$$\frac{2}{p-1} \le k < \frac{1}{2} + \frac{1}{p} < \frac{p}{2} - 1, \qquad p > \frac{3+\sqrt{17}}{2}\,. \tag{2.10}$$

Now, in the definition (2.6) of our weight function, we also introduced the parameters

$$\beta = \min(k, 1/2), \qquad \gamma = \max(k-1/2, 0). \tag{2.11}$$

Under our assumption (2.10), those are easily seen to satisfy the conditions

$$0 \le \gamma p < 1, \qquad \beta + \gamma = k < 1, \qquad \beta p \ge \min(k+2, p/2) > 3/2. \tag{2.12}$$

In what follows, we shall frequently need to use the following four elementary facts.

Lemma 6. *Let $r,t>0$ be arbitrary. Assuming that $0\le a\le 1/2$ and $0<k<1$, one has*

$$r^{-1/2}\int_{|t-r|}^{t+r}\frac{\langle y\rangle^{a-1/2-k}}{(r-t+y)^a}\,dy\le C(a,k)\cdot w_k(r,t)^{-1},$$

where the weight function w_k is given by (2.6).

Lemma 7. *Let $b\ge 0$, $y\in\mathbb{R}$ and $z\ge |y|$ be arbitrary. Assuming that $c<-1$, one has*

$$\int_z^\infty \langle x\rangle^c\left(1+\ln\frac{\langle x\rangle}{\langle y\rangle}\right)^b dx\le C(b,c)\cdot\langle z\rangle^{c+1}\left(1+\ln\frac{\langle z\rangle}{\langle y\rangle}\right)^b.$$

Lemma 8. *Let $w\le 0$ be arbitrary. Assuming that $0\le a<1$ and $a+c>1$, one has*

$$\int_{-\infty}^w\frac{\langle y\rangle^{-c}}{(w-y)^a}\,dy\le C(a,c)\cdot\langle w\rangle^{1-a-c}.$$

Lemma 9. *Let $w\ge 0$ be arbitrary. Assuming that $0\le a<1$, $b\ge 0$ and $c<1$, one has*

$$A_\pm\equiv\int_0^w\frac{\langle y\rangle^{-c}}{(w\pm y)^a}\left(1+\ln\frac{\langle w\rangle}{\langle y\rangle}\right)^b dy\le C(a,b,c)\cdot w^{1-a}\langle w\rangle^{-c}.$$

Essentially, the proof of the first fact can be found in [10], where the case $a=0$ is treated in Lemma 3.5 and the case $a=1/2$ is treated in Lemma 3.6. The proof of the second fact can be found in [4], while the proof of the third fact appears in [3]. Finally, the fourth fact follows easily from Lemma 3 in [4]; we only include its derivation here for the sake of completeness.

Proof of Lemma 9. Note that $w\pm y$ is equivalent to w whenever $0\le y\le w/2$, while $\langle y\rangle$ is equivalent to $\langle w\rangle$ whenever $w/2\le y\le w$. This gives

$$A_\pm\le Cw^{-a}\int_0^{w/2}\langle y\rangle^{-c}\left(1+\ln\frac{\langle w\rangle}{\langle y\rangle}\right)^b dy+C\langle w\rangle^{-c}\int_{w/2}^w(w\pm y)^{-a}\,dy,$$

while the estimate

$$\int_0^w\langle y\rangle^{-c}\left(1+\ln\frac{\langle w\rangle}{\langle y\rangle}\right)^b dy\le Cw\langle w\rangle^{-c}$$

is provided by Lemma 3 in [4]. Since $a<1$ by assumption, the result follows easily. ■

Lemma 10. *Let $r>0$ and $t\in\mathbb{R}$. Assume* (2.10) *through* (2.12) *and fix some*

$$0<\delta<\min(\beta p-3/2,\ 1/2). \tag{2.13}$$

Then we have

$$\mathcal{I}_\delta\equiv r^{\delta-1/2}\int_{-\infty}^t\int_{|\lambda_-|}^{\lambda_+}\frac{\lambda^{\delta+1/2}\,w_k(\lambda,|\tau|)^{-p}}{\lambda_+^\delta(\lambda-\lambda_-)^\delta}\,d\lambda\,d\tau\le Cw_k(r,|t|)^{-1},$$

where $\lambda_\pm=t-\tau\pm r$, w_k is given by (2.6) *and the constant C is independent of r,t.*

Proof. Since $\lambda_+=t-\tau+r\ge r$ within the region of integration, it is clear that

$$\mathcal{I}_\delta\le r^{-1/2}\int_{-\infty}^t\int_{|\lambda_-|}^{\lambda_+}\frac{\lambda^{\delta+1/2}\,w_k(\lambda,|\tau|)^{-p}}{(\lambda-\lambda_-)^\delta}\,d\lambda\,d\tau.$$

First, we treat the part in which $\tau \geq 0$. For this part, we have to show that

$$\mathcal{I}'_\delta \equiv r^{-1/2} \int_0^t \int_{|\lambda_-|}^{\lambda_+} \frac{\lambda^{\delta+1/2}\, w_k(\lambda,\tau)^{-p}}{(\lambda-\lambda_-)^\delta}\; d\lambda\, d\tau \leq C w_k(r,t)^{-1} \tag{2.14}$$

whenever $t > 0$. Let us recall the definition (2.6) of our weight function w_k and write

$$\mathcal{I}'_\delta = r^{-1/2} \int_0^t \int_{|t-\tau-r|}^{t-\tau+r} \frac{\lambda^{\delta+1/2}\, \langle \lambda+\tau\rangle^{-\beta p}}{(\lambda+\tau+r-t)^\delta}\; \langle \lambda-\tau\rangle^{-\gamma p} \left(1 + \ln \frac{\langle\lambda+\tau\rangle}{\langle\lambda-\tau\rangle}\right)^{p\delta_{k,1/2}} d\lambda\, d\tau.$$

Changing variables by $x = \lambda - \tau$ and $y = \lambda + \tau$, we then get

$$\mathcal{I}'_\delta \leq C r^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{-\beta p}}{(r-t+y)^\delta} \int_{-y}^{y} (x+y)^{\delta+1/2} \cdot \langle x\rangle^{-\gamma p} \left(1 + \ln \frac{\langle y\rangle}{\langle x\rangle}\right)^{p\delta_{k,1/2}} dx\, dy.$$

Since $x + y \leq 2y$ within the region of integration, this trivially leads to

$$\mathcal{I}'_\delta \leq C r^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{\delta+1/2-\beta p}}{(r-t+y)^\delta} \int_0^y \langle x\rangle^{-\gamma p} \left(1 + \ln \frac{\langle y\rangle}{\langle x\rangle}\right)^{p\delta_{k,1/2}} dx\, dy.$$

Since $\gamma p < 1$ by (2.12), we may now apply Lemma 9 with $a = 0$ to find

$$\mathcal{I}'_\delta \leq C r^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{\delta+3/2-\beta p-\gamma p}}{(r-t+y)^\delta}\; dy.$$

Noting that $\beta p + \gamma p = kp \geq k + 2$ by assumption, this also implies

$$\mathcal{I}'_\delta \leq C r^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{\delta-1/2-k}}{(r-t+y)^\delta}\; dy \leq C w_k(r,t)^{-1}$$

by means of Lemma 6. In particular, the proof of (2.14) is complete.

Next, we treat the part in which $\tau \leq 0$. For this part, we have to show that

$$\mathcal{I}''_\delta \equiv r^{-1/2} \int_{-\infty}^{\min(0,t)} \int_{|\lambda_-|}^{\lambda_+} \frac{\lambda^{\delta+1/2}\, w_k(\lambda,-\tau)^{-p}}{(\lambda-\lambda_-)^\delta}\; d\lambda\, d\tau \leq C w_k(r,|t|)^{-1} \tag{2.15}$$

for any $t \in \mathbb{R}$ whatsoever. Proceeding as above, let us first write

$$\mathcal{I}''_\delta = r^{-1/2} \int_{-\infty}^{\min(0,t)} \int_{|\lambda_-|}^{\lambda_+} \frac{\lambda^{\delta+1/2}\, \langle \lambda-\tau\rangle^{-\beta p}}{(\lambda+\tau+r-t)^\delta}\; \langle \lambda+\tau\rangle^{-\gamma p} \left(1 + \ln \frac{\langle\lambda-\tau\rangle}{\langle\lambda+\tau\rangle}\right)^{p\delta_{k,1/2}} d\lambda\, d\tau.$$

Since $\tau \leq t$ within the region of integration, we have $\lambda \geq |\lambda_-| \geq t - \tau - r$. Since $\tau \leq 0$, we also have $\lambda \geq |\lambda_-| \geq |t-r| + \tau$. Changing variables by $x = \lambda - \tau$ and $y = \lambda + \tau$, we then get

$$\mathcal{I}''_\delta \leq C r^{-1/2} \int_{t-r}^{t+r} \frac{\langle y\rangle^{-\gamma p}}{(r-t+y)^\delta} \int_z^\infty (x+y)^{\delta+1/2}\; \langle x\rangle^{-\beta p} \left(1 + \ln \frac{\langle x\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dx\, dy,$$

where we have set $z = \max(|y|, |t-r|)$ for convenience. Since $x + y \leq 2x$ here, we find

$$\mathcal{I}''_\delta \leq C r^{-1/2} \int_{t-r}^{t+r} \frac{\langle y\rangle^{-\gamma p}}{(r-t+y)^\delta} \int_z^\infty \langle x\rangle^{\delta+1/2-\beta p} \left(1 + \ln \frac{\langle x\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dx\, dy.$$

Moreover, $\delta + 3/2 - \beta p < 0$ by our assumption (2.13), so Lemma 7 applies to give

$$\mathcal{I}''_\delta \le Cr^{-1/2} \int_{t-r}^{t+r} \frac{\langle y\rangle^{-\gamma p}}{(r-t+y)^\delta} \cdot \langle z\rangle^{\delta+3/2-\beta p} \left(1 + \ln \frac{\langle z\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dy \tag{2.16}$$

with $z = \max(|y|, |t-r|)$ as above.
Case 1: When $t > 0$, we have $t - r \le |t-r| \le t + r$, so equation (2.16) reads

$$\begin{aligned}\mathcal{I}''_\delta &\le Cr^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{\delta+3/2-kp}}{(r-t+y)^\delta}\, dy \\ &\quad + Cr^{-1/2} \langle t-r\rangle^{\delta+3/2-\beta p} \int_{t-r}^{|t-r|} \frac{\langle y\rangle^{-\gamma p}}{(r-t+y)^\delta} \cdot \left(1 + \ln \frac{\langle t-r\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dy\end{aligned}$$

because $z = |y|$ within the former integral and $z = |t-r|$ within the latter. When it comes to the former integral, we have $kp \ge k+2$, hence also

$$r^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{\delta+3/2-kp}}{(r-t+y)^\delta}\, dy \le r^{-1/2} \int_{|t-r|}^{t+r} \frac{\langle y\rangle^{\delta-1/2-k}}{(r-t+y)^\delta}\, dy \le Cw_k(r,t)^{-1}$$

by Lemma 6. In particular, it suffices to treat the latter integral

$$\mathcal{I}'''_\delta \equiv r^{-1/2} \langle r-t\rangle^{\delta+3/2-\beta p} \int_0^{r-t} \frac{\langle y\rangle^{-\gamma p}}{(r-t\pm y)^\delta} \cdot \left(1 + \ln \frac{\langle r-t\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dy$$

whenever $r \ge t$. Since $\gamma p < 1$ by (2.12), an application of Lemma 9 gives

$$\begin{aligned}\mathcal{I}'''_\delta &\le Cr^{-1/2}(r-t)^{1-\delta} \langle r-t\rangle^{\delta+3/2-kp} \\ &\le Cr^{-1/2}(r-t)^{1-\delta} \langle r-t\rangle^{\delta-1/2-k}.\end{aligned}$$

Since $\beta + \gamma = k$, we may thus deduce the desired (2.15) once we know that

$$r^{-1/2}(r-t)^{1-\delta} \langle r-t\rangle^{\delta-1/2-\beta} \le C \langle r+t\rangle^{-\beta} \qquad \text{when } r \ge t > 0. \tag{2.17}$$

If $r \ge t$ and $r \le 1$, then each of $r \pm t$ is bounded and we easily get the desired

$$r^{-1/2}(r-t)^{1-\delta} \langle r-t\rangle^{\delta-1/2-\beta} \le Cr^{1/2-\delta} \le C$$

because $\delta < 1/2$. If $r \ge \max(t,1)$, then r is equivalent to $\langle r+t\rangle$ and we similarly get

$$r^{-1/2}(r-t)^{1-\delta} \langle r-t\rangle^{\delta-1/2-\beta} \le r^{-1/2} \langle r-t\rangle^{1/2-\beta} \le C \langle r+t\rangle^{-\beta}$$

because $\beta \le 1/2$ by (2.11).
Case 2: When $t \le 0$, we have $|r+t| \le r + |t| = r - t$, so equation (2.16) reads

$$\mathcal{I}''_\delta \le Cr^{-1/2} \langle r-t\rangle^{\delta+3/2-\beta p} \int_{t-r}^{t+r} \frac{\langle y\rangle^{-\gamma p}}{(r-t+y)^\delta} \cdot \left(1 + \ln \frac{\langle r-t\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dy. \tag{2.18}$$

Subcase 2a: If it happens that $|t| \le 3r$, we proceed as in the previous case to obtain

$$\begin{aligned}\mathcal{I}''_\delta &\le Cr^{-1/2} \langle r-t\rangle^{\delta+3/2-\beta p} \int_{t-r}^{r-t} \frac{\langle y\rangle^{-\gamma p}}{(r-t+y)^\delta} \cdot \left(1 + \ln \frac{\langle r-t\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dy \\ &\le Cr^{-1/2}(r-t)^{1-\delta} \langle r-t\rangle^{\delta-1/2-k}\end{aligned}$$

using Lemma 9. Since $r-t=r+|t|\leq 4r$ for this subcase, we then get

$$\mathcal{I}''_\delta \leq Cr^{1/2-\delta}\left\langle r+|t|\right\rangle^{\delta-1/2-k} \leq C\left\langle r+|t|\right\rangle^{-k}$$

because $\delta<1/2$. This estimate is actually stronger than the desired (2.15).
Subcase 2b: If it happens that $-t=|t|\geq 3r$, then $\langle r+t\rangle$ is equivalent to $\langle r-t\rangle$ because

$$|r+t|\leq r+|t|=r-t\leq -2(r+t)$$

for this subcase. Since $\delta<1/2$, equation (2.18) then trivially leads to

$$\begin{aligned}\mathcal{I}''_\delta &\leq Cr^{-1/2}\left\langle r+|t|\right\rangle^{\delta-1/2-k}\int_{t-r}^{t+r}(r-t+y)^{-\delta}\,dy\\ &\leq Cr^{1/2-\delta}\left\langle r+|t|\right\rangle^{\delta-1/2-k}\leq C\left\langle r+|t|\right\rangle^{-k}.\end{aligned}$$

This estimate already implies the desired (2.15), so the proof is finally complete. ■

Lemma 11. *Under the assumptions of Lemma 10, one also has*

$$\mathcal{J}_\theta \equiv \int_{\min(t-r,0)}^{t-r}\int_0^{\lambda_-}\frac{\lambda\, w_k(\lambda,|\tau|)^{-p}\,d\lambda\,d\tau}{\lambda_+^\theta\,\lambda_-^{1/2-\theta}(\lambda_--\lambda)^\theta}\leq Cr^{-\nu}\left\langle t-r\right\rangle^{1/2-\theta+\nu-k},$$

where $0<\theta\leq 1/2$ *is arbitrary and* $\nu=0,\theta$.

Proof. If $t\leq r$, then there is nothing to prove. Assume that $t\geq r$ and write

$$\mathcal{J}_\theta=\int_0^{t-r}\int_0^{\lambda_-}\frac{\lambda\, w_k(\lambda,\tau)^{-p}\;d\lambda\,d\tau}{\lambda_+^\theta\,\lambda_-^{1/2-\theta}(\lambda_--\lambda)^\theta}. \tag{2.19}$$

In order to estimate the integrand, we use the fact that

$$\frac{\lambda}{\lambda_-}=\frac{\lambda}{t-\tau-r}\leq C\left(\frac{\lambda+\tau}{t-r}\right). \tag{2.20}$$

This holds if $0\leq\tau\leq(t-r)/2$, in which case $t-r-\tau$ is equivalent to $t-r$, but it also holds if $(t-r)/2\leq\tau$ and $0\leq\lambda\leq t-r-\tau$, in which case $\lambda+\tau$ is equivalent to $t-r$. Using (2.20) and our assumption $0<\theta\leq 1/2$, one now easily finds that

$$\frac{\lambda}{\lambda_+^\theta\,\lambda_-^{1/2-\theta}}\leq\frac{\lambda}{\lambda_-^{1/2}}\leq C\lambda^{1/2}\left(\frac{\lambda+\tau}{t-r}\right)^{1/2}.$$

Using (2.20) and the fact that $\lambda_+=t-\tau+r\geq 2r$, one similarly finds

$$\frac{\lambda}{\lambda_+^\theta\,\lambda_-^{1/2-\theta}}\leq\frac{C\lambda^{1/2+\theta}}{r^\theta}\cdot\left(\frac{\lambda+\tau}{t-r}\right)^{1/2-\theta}.$$

This proves the estimate

$$\frac{\lambda}{\lambda_+^\theta\,\lambda_-^{1/2-\theta}}\leq\frac{C\lambda^{1/2+\nu}}{r^\nu}\cdot\left(\frac{\lambda+\tau}{t-r}\right)^{1/2-\nu},\qquad \nu=0,\theta. \tag{2.21}$$

Inserting this estimate in (2.19) and changing variables by $x = \lambda - \tau$, $y = \lambda + \tau$, we now get

$$\mathcal{J}_\theta \leq \frac{Cr^{-\nu}}{(t-r)^{1/2-\nu}} \int_0^{t-r} \frac{y^{1/2-\nu} \langle y \rangle^{-\beta p}}{(t-r-y)^\theta} \times$$
$$\int_{-y}^{y} (x+y)^{1/2+\nu} \cdot \langle x \rangle^{-\gamma p} \left(1 + \ln \frac{\langle y \rangle}{\langle x \rangle}\right)^{p\delta_{k,1/2}} dx\, dy.$$

Since $\gamma p < 1$ by (2.12), we may apply Lemma 9 with $a = 0$ to obtain the estimate

$$\int_{-y}^{y} (x+y)^{1/2+\nu} \cdot \langle x \rangle^{-\gamma p} \left(1 + \ln \frac{\langle y \rangle}{\langle x \rangle}\right)^{p\delta_{k,1/2}} dx \leq Cy^{1/2+\nu} \langle y \rangle^{1-\gamma p}$$

for the inner integral. Since $\beta p + \gamma p = kp \geq k + 2$ by assumption, this implies

$$\mathcal{J}_\theta \leq \frac{Cr^{-\nu}}{(t-r)^{1/2-\nu}} \int_0^{t-r} \frac{y \langle y \rangle^{-k-1}}{(t-r-y)^\theta}\, dy.$$

Noting that $y/\langle y \rangle$ is an increasing function for all y, we thus arrive at

$$\mathcal{J}_\theta \leq \frac{Cr^{-\nu}(t-r)^{1/2+\nu}}{\langle t-r \rangle} \int_0^{t-r} \frac{\langle y \rangle^{-k}}{(t-r-y)^\theta}\, dy$$
$$\leq Cr^{-\nu} \langle t-r \rangle^{-1/2+\nu} \int_0^{t-r} \frac{\langle y \rangle^{-k}}{(t-r-y)^\theta}\, dy.$$

Since $k < 1$ by (2.12), we may then apply Lemma 9 with $b = 0$ to get

$$\mathcal{J}_\theta \leq Cr^{-\nu} \langle t-r \rangle^{-1/2+\nu+(1-\theta)-k}.$$

This is precisely the desired estimate for $\mathcal{J}_\theta$, so the proof is finally complete. ■

Corollary 12. *Under the assumptions of Lemma 10, one also has*

$$\mathcal{J}'_\theta \equiv \int_{-|t-r|}^{t-r} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^\theta \lambda_-^{1/2-\theta} (\lambda_- - \lambda)^\theta} \leq Cr^{-\nu} \langle t-r \rangle^{1/2-\theta+\nu-k},$$

where $0 < \theta \leq 1/2$ *is arbitrary and* $\nu = 0, \theta$.

Proof. If $t \leq r$, then there is nothing to prove. Assume that $t \geq r$ and write

$$\mathcal{J}'_\theta = \mathcal{J}_\theta + \int_{r-t}^{0} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, -\tau)^{-p}\, d\lambda\, d\tau}{\lambda_+^\theta \lambda_-^{1/2-\theta} (\lambda_- - \lambda)^\theta}, \tag{2.22}$$

where $\mathcal{J}_\theta$ is given by the previous lemma. Since $\mathcal{J}_\theta$ is known to satisfy the desired estimate, it suffices to treat the remaining part $\mathcal{J}'_\theta - \mathcal{J}_\theta$. Since $\tau \leq 0$ for this part, one clearly has

$$\frac{\lambda}{\lambda_-} = \frac{\lambda}{t-\tau-r} \leq \frac{\lambda}{t-r} \leq \frac{\lambda-\tau}{t-r}$$

within the region of integration. Using this analogue of (2.20), one obtains the estimate

$$\frac{\lambda}{\lambda_+^\theta \lambda_-^{1/2-\theta}} \leq \frac{\lambda^{1/2+\nu}}{r^\nu} \cdot \left(\frac{\lambda-\tau}{t-r}\right)^{1/2-\nu}, \qquad \nu = 0, \theta \tag{2.23}$$

in the same way that we obtained (2.21). Once we now insert this estimate in (2.22), the change of variables $x = \lambda - \tau$, $y = \lambda + \tau$ leads us to

$$\mathcal{J}'_\theta - \mathcal{J}_\theta \leq \frac{Cr^{-\nu}}{(t-r)^{1/2-\nu}} \int_{r-t}^{t-r} \frac{\langle y \rangle^{-\gamma p}}{(t-r-y)^\theta} \times \int_{|y|}^{3(t-r)} (x+y)^{1/2+\nu} \cdot x^{1/2-\nu} \langle x \rangle^{-\beta p} \left(1 + \ln \frac{\langle x \rangle}{\langle y \rangle} \right)^{p\delta_{k,1/2}} dx\, dy. \tag{2.24}$$

Let us denote the inner integral by $\mathcal{J}_{\text{in}}$. Since $x + y \leq 2x$ here, we certainly have

$$\mathcal{J}_{\text{in}} \leq C \left(1 + \ln \frac{\langle t-r \rangle}{\langle y \rangle} \right)^p \int_{|y|}^{3(t-r)} \langle x \rangle^{1-\beta p}\, dx,$$

and this trivially leads to the estimate

$$\mathcal{J}_{\text{in}} \leq C \left(1 + \ln \frac{\langle t-r \rangle}{\langle y \rangle} \right)^{p+1} \left(\langle t-r \rangle^{2-\beta p} + \langle y \rangle^{2-\beta p} \right).$$

Next, we insert this fact in (2.24). Since $\beta p + \gamma p = kp \geq k + 2$, we arrive at

$$\mathcal{J}'_\theta - \mathcal{J}_\theta \leq \frac{Cr^{-\nu} \langle t-r \rangle^{2-\beta p}}{(t-r)^{1/2-\nu}} \int_0^{t-r} \frac{\langle y \rangle^{-\gamma p}}{(t-r \pm y)^\theta} \left(1 + \ln \frac{\langle t-r \rangle}{\langle y \rangle} \right)^{p+1} dy + \frac{Cr^{-\nu}}{(t-r)^{1/2-\nu}} \int_0^{t-r} \frac{\langle y \rangle^{-k}}{(t-r \pm y)^\theta} \left(1 + \ln \frac{\langle t-r \rangle}{\langle y \rangle} \right)^{p+1} dy.$$

Recalling that $\gamma p, k < 1$ by (2.12), we may then apply Lemma 9 to get

$$\mathcal{J}'_\theta - \mathcal{J}_\theta \leq Cr^{-\nu}(t-r)^{1/2-\theta+\nu} \cdot \langle t-r \rangle^{-k}.$$

Since $1/2 - \theta + \nu \geq 0$, this does imply the desired estimate. ■

Lemma 13. *Under the assumptions of Lemma 10, one can always find some* $0 < \theta \leq 1/2$ *such that the estimate*

$$\mathcal{K}_\theta \equiv r^{\theta-1/2} \int_{-\infty}^{t-r} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^\theta \lambda_-^{1/2-\theta} (\lambda_- - \lambda)^\theta} \leq C w_k(r, |t|)^{-1}$$

holds. In fact, one can simply take $\theta = 1/2$, *except when* $0 \leq t \leq 2r$ *and* $r \geq 1$ *and* $k > 1/2$, *in which case one can simply take* $\theta = \delta$ *with* δ *as in* (2.13).

Proof. We divide our analysis into three cases.
Case 1: Suppose $t \leq 0$ or $t \geq 2r$ or $r \leq 1$. Then we need only show that

$$\mathcal{K}_{1/2} = \int_{-\infty}^{t-r} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^{1/2} (\lambda_- - \lambda)^{1/2}} \leq C \langle r-t \rangle^{-k}. \tag{2.25}$$

First, we employ Corollary 12 with $\theta = 1/2$ and $\nu = 0$ to get the estimate

$$\mathcal{J}'_{1/2} = \int_{-|t-r|}^{t-r} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^{1/2} (\lambda_- - \lambda)^{1/2}} \leq C \langle r-t \rangle^{-k}.$$

Next, we treat the remaining part

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} = \int_{-\infty}^{-|t-r|} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^{1/2}(t-r-\lambda-\tau)^{1/2}}\,.$$

We note that $\lambda \le \lambda_+$ and $\lambda \le \lambda - \tau$ within the region of integration. Once we now switch to characteristic coordinates $x = \lambda - \tau$ and $y = \lambda + \tau$, we find

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} \le C \int_{-\infty}^{t-r} \frac{\langle y\rangle^{-\gamma p}}{(t-r-y)^{1/2}} \int_z^\infty \langle x\rangle^{1/2-\beta p} \left(1 + \ln\frac{\langle x\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dx\, dy$$

with $z = \max(|y|, |t-r|)$. Since $\beta p > 3/2$ by (2.12), an application of Lemma 7 gives

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} \le C \int_{-\infty}^{t-r} \frac{\langle y\rangle^{-\gamma p}}{(t-r-y)^{1/2}} \cdot \langle z\rangle^{3/2-\beta p} \left(1 + \ln\frac{\langle z\rangle}{\langle y\rangle}\right)^{p\delta_{k,1/2}} dy.$$

Recalling that $\beta p + \gamma p = kp \ge k+2$, we then get

$$\begin{aligned}\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} &\le C \int_{-\infty}^{-|t-r|} \frac{\langle y\rangle^{-k-1/2}}{(t-r-y)^{1/2}}\, dy \\ &\quad + C\,\langle t-r\rangle^{3/2-\beta p} \int_{-|t-r|}^{t-r} \frac{\langle y\rangle^{-\gamma p}}{(t-r-y)^{1/2}} \left(1 + \ln\frac{\langle t-r\rangle}{\langle y\rangle}\right)^p dy\end{aligned}$$

because $z = |y|$ within the former integral and $z = |t-r|$ within the latter. Using Lemma 8 for the former integral and Lemma 9 for the latter, we deduce the desired (2.25).
Case 2: Suppose $0 \le t \le 2r$ and $r \ge 1$ and $0 < k \le 1/2$. Then we need only show that

$$\mathcal{K}_{1/2} \le C w_k(r,t)^{-1} + C r^{-1/2} \langle t-r\rangle^{1/2-k}\,. \tag{2.26}$$

Namely, r is equivalent to $\langle t+r\rangle$ for this case, so one also has

$$r^{-1/2}\langle t-r\rangle^{1/2-k} \le C\langle t+r\rangle^{-\beta}\langle t-r\rangle^{-\gamma} \le C w_k(r,t)^{-1} \tag{2.27}$$

because $\beta + \gamma = k$ by (2.12) and $\beta \le 1/2$ by (2.11).

To establish (2.26), we first use Corollary 12 with $\theta = \nu = 1/2$ to get the estimate

$$\mathcal{J}'_{1/2} = \int_{-|t-r|}^{t-r} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^{1/2}(\lambda_- - \lambda)^{1/2}} \le C r^{-1/2}\langle t-r\rangle^{1/2-k}\,.$$

Next, we focus on the remaining part

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} = \int_{-\infty}^{-|t-r|} \int_0^{\lambda_-} \frac{\lambda\, w_k(\lambda, |\tau|)^{-p}\, d\lambda\, d\tau}{\lambda_+^{1/2}(t-r-\lambda-\tau)^{1/2}}\,.$$

Note that $2\lambda_+ \ge \lambda + \lambda_+ = t + r + \lambda - \tau$ within the region of integration. Once we now switch to characteristic coordinates $x = \lambda - \tau$ and $y = \lambda + \tau$, we find

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} \le C \int_{-\infty}^{t-r} \frac{\langle y\rangle^{-\gamma p}}{(t-r-y)^{1/2}} \int_z^\infty \frac{\langle x\rangle^{1-\beta p}}{(t+r+x)^{1/2}} \left(1 + \ln\frac{\langle x\rangle}{\langle y\rangle}\right)^p dx\, dy$$

with $z = \max(|y|, |t-r|)$. Since we are assuming that $0 < k \le 1/2$ for this case, we have

$$\beta p = \min(k, 1/2)\cdot p = kp \ge k+2 > 2$$

and then an application of Lemma 7 leads us to

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} \leq C \int_{-\infty}^{t-r} \frac{\langle y \rangle^{-\gamma p} \langle z \rangle^{2-\beta p}}{(t-r-y)^{1/2} (t+r+|y|)^{1/2}} \left(1 + \ln \frac{\langle z \rangle}{\langle y \rangle} \right)^p dy.$$

Since we also have $t+r \geq r \geq 1$ for this case, we trivially get

$$\begin{aligned} \mathcal{K}_{1/2} - \mathcal{J}'_{1/2} \leq C \int_{-\infty}^{-t-r} \frac{\langle y \rangle^{-k-1/2}}{(t-r-y)^{1/2}} \, dy + Cr^{-1/2} \int_{-t-r}^{-|t-r|} \frac{\langle y \rangle^{-k}}{(t-r-y)^{1/2}} \, dy \\ + Cr^{-1/2} \langle t-r \rangle^{2-\beta p} \int_{-|t-r|}^{t-r} \frac{\langle y \rangle^{-\gamma p}}{(t-r-y)^{1/2}} \left(1 + \ln \frac{\langle t-r \rangle}{\langle y \rangle} \right)^p dy, \end{aligned}$$

as $z = |y|$ within the first two integrals and $z = |t-r|$ within the third one. Using Lemma 8 for the first integral, Lemma 6 for the second and Lemma 9 for the third, we then find

$$\mathcal{K}_{1/2} - \mathcal{J}'_{1/2} \leq C \langle t+r \rangle^{-k} + Cw_k(r,t)^{-1} + Cr^{-1/2} \langle t-r \rangle^{1/2-k}.$$

Moreover, $\beta + \gamma = k$ by (2.12) and $\gamma \geq 0$ by (2.11), so we also have

$$\langle t+r \rangle^{-k} \leq \langle t+r \rangle^{-\beta} \langle t-r \rangle^{-\gamma} \leq w_k(r,t)^{-1}.$$

Combining the last two equations, we may thus deduce the desired estimate (2.26).

Case 3: Suppose $0 \leq t \leq 2r$ and $r \geq 1$ and $k > 1/2$. Since (2.27) remains valid for this case as well, it suffices to establish the estimate

$$\mathcal{K}_\delta \leq Cr^{-1/2} \langle t-r \rangle^{1/2-k}, \tag{2.28}$$

where $0 < \delta < 1/2$ is given by (2.13). Once again, we divide $\mathcal{K}_\delta$ into two parts to be treated separately. To treat the first part

$$\mathcal{K}'_\delta \equiv r^{\delta-1/2} \int_{-|t-r|}^{t-r} \int_0^{\lambda_-} \frac{\lambda \, w_k(\lambda, |\tau|)^{-p} \, d\lambda \, d\tau}{\lambda_+^\delta \, \lambda_-^{1/2-\delta} (\lambda_- - \lambda)^\delta},$$

we need only apply Corollary 12 with $\theta = \nu = \delta$ to get the desired

$$\mathcal{K}'_\delta = r^{\delta-1/2} \cdot \mathcal{J}'_\delta \leq Cr^{-1/2} \langle t-r \rangle^{1/2-k}.$$

Let us now worry about the remaining part

$$\mathcal{K}_\delta - \mathcal{K}'_\delta = r^{\delta-1/2} \int_{-\infty}^{-|t-r|} \int_0^{\lambda_-} \frac{\lambda \, w_k(\lambda, |\tau|)^{-p} \, d\lambda \, d\tau}{\lambda_+^\delta \, \lambda_-^{1/2-\delta} (\lambda_- - \lambda)^\delta}.$$

Since $\lambda_+ = t - \tau + r \geq 2r$ and $\lambda \leq \lambda_-$ within the region of integration, we easily find

$$\mathcal{K}_\delta - \mathcal{K}'_\delta \leq Cr^{-1/2} \int_{-\infty}^{-|t-r|} \int_0^{\lambda_-} \frac{\lambda^{1/2+\delta} \, w_k(\lambda, |\tau|)^{-p} \, d\lambda \, d\tau}{(t-r-\lambda-\tau)^\delta}.$$

Switching to characteristic coordinates $x = \lambda - \tau$ and $y = \lambda + \tau$, we thus find

$$\mathcal{K}_\delta - \mathcal{K}'_\delta \leq Cr^{-1/2} \int_{-\infty}^{t-r} \frac{\langle y \rangle^{-\gamma p}}{(t-r-y)^\delta} \int_z^\infty \langle x \rangle^{1/2+\delta-\beta p} \left(1 + \ln \frac{\langle x \rangle}{\langle y \rangle} \right)^p dx \, dy$$

with $z = \max(|y|, |t-r|)$. Since $\delta < \beta p - 3/2$ by (2.13), we may apply Lemma 7 to get

$$\begin{aligned}\mathcal{K}_\delta - \mathcal{K}'_\delta &\le Cr^{-1/2} \int_{-\infty}^{-|t-r|} \frac{\langle y\rangle^{3/2+\delta-kp}\,dy}{(t-r-y)^\delta} \\ &\quad + Cr^{-1/2}\langle t-r\rangle^{3/2+\delta-\beta p} \int_{-|t-r|}^{t-r} \frac{\langle y\rangle^{-\gamma p}}{(t-r-y)^\delta} \left(1 + \ln\frac{\langle t-r\rangle}{\langle y\rangle}\right)^p dy.\end{aligned}$$

As $kp \ge k+2$ by assumption and $\gamma p < 1$ by (2.12), this actually implies

$$\mathcal{K}_\delta - \mathcal{K}'_\delta \le Cr^{-1/2} \int_{-\infty}^{-|t-r|} \frac{\langle y\rangle^{-1/2+\delta-k}\,dy}{(t-r-y)^\delta} + Cr^{-1/2}\langle t-r\rangle^{1/2-k}$$

in view of Lemma 9. Since we also have $k > 1/2$ for this case, we may then apply Lemma 8 to deduce the desired estimate (2.28). This finally completes the proof. ∎

3. Basic Estimate for the Existence Proof

In this section, we turn our attention to the Duhamel operator

$$[\mathscr{L}F(u)](x,t) = \frac{1}{2\pi}\int_{-\infty}^{t} (t-\tau) \int_{|y|<1} \frac{F(u(x+(t-\tau)y,\tau))}{\sqrt{1-|y|^2}}\; dy\, d\tau \tag{3.1}$$

and prove the following basic estimate for the existence proof.

Lemma 14. *Suppose that F satisfies* (2.8), (2.9) *and assume that* (2.10), (2.12) *hold. Given an element $u \in X$ of the Banach space* (2.4) *such that $||u|| \le 1$, one then has the estimate*

$$||\mathscr{L}F(u)|| \le C_1 ||u||^p$$

for some constant C_1 which is independent of u.

To prove this lemma, we first use a direct differentiation to write

$$\partial_x^\alpha [\mathscr{L}F(u)](x,t) = \frac{1}{2\pi}\int_{-\infty}^{t} \frac{1}{t-\tau} \int_{|z-x|<t-\tau} \partial_z^\alpha F(u(z,\tau))\; dz\, d\tau$$

for each multi-index α. When it comes to the integrand, we have the estimate

$$|\partial_z^\alpha F(u(z,\tau))| \le C||u||^p \cdot w_k(|z|,|\tau|)^{-p}, \qquad |\alpha| \le 2.$$

One can easily obtain this estimate using our assumptions (2.8), (2.9) on F and the definition of our norm (2.5), so we omit its derivation. Combining the last two equations, we now get

$$|\partial_x^\alpha \mathscr{L}F(u)| \le C||u||^p \int_{-\infty}^{t} (t-\tau) \int_{|y|<1} \frac{w_k(|x+(t-\tau)y|,|\tau|)^{-p}}{\sqrt{1-|y|^2}}\; dy\, d\tau.$$

Switching to polar coordinates $y = \rho\xi$ with $|\xi| = 1$, we thus get

$$\begin{aligned}|\partial_x^\alpha \mathscr{L}F(u)| &\le C||u||^p \int_{-\infty}^{t} (t-\tau) \int_0^1 \int_{|\xi|=1} \frac{w_k(|x+(t-\tau)\rho\xi|,|\tau|)^{-p}}{\sqrt{1-\rho^2}}\; \rho\, dS_\xi\, d\rho\, d\tau \\ &= C||u||^p \int_{-\infty}^{t}\int_0^{t-\tau} \frac{\sigma}{\sqrt{(t-\tau)^2-\sigma^2}} \int_{|\xi|=1} w_k(|x+\sigma\xi|,|\tau|)^{-p}\; dS_\xi\, d\sigma\, d\tau.\end{aligned}$$

In order to proceed, we need to invoke the following elementary lemma from [5].

Lemma 15. *Let* $\sigma > 0$ *and* $x \in \mathbb{R}^2$. *Given a continuous function* $g\colon \mathbb{R} \to \mathbb{R}$, *one has*

$$\int_{|\xi|=1} g(|x+\sigma\xi|)\, dS_\xi = \int_{|\sigma-r|}^{\sigma+r} \frac{4\lambda\, g(\lambda)}{H(\lambda,r,\sigma)}\, d\lambda,$$

where $r = |x|$ *and we have also set*

$$H(\lambda,r,\sigma) = \sqrt{\sigma^2-(\lambda-r)^2}\,\sqrt{(\lambda+r)^2-\sigma^2}. \tag{3.2}$$

Applying this lemma, we now arrive at

$$|\partial_x^\alpha \mathscr{L}F(u)| \le C||u||^p \int_{-\infty}^t \int_0^{t-\tau} \int_{|\sigma-r|}^{\sigma+r} \frac{\lambda w_k(\lambda,|\tau|)^{-p}}{\sqrt{(t-\tau)^2-\sigma^2}} \cdot \frac{\sigma}{H(\lambda,r,\sigma)}\, d\lambda\, d\sigma\, d\tau.$$

Switching the order of integration in the two innermost integrals, we then get

$$\begin{aligned} |\partial_x^\alpha \mathscr{L}F(u)| &\le C||u||^p \int_{-\infty}^t \int_{|\lambda_-|}^{\lambda_+} \int_{|\lambda-r|}^{t-\tau} \frac{\lambda w_k(\lambda,|\tau|)^{-p}}{\sqrt{(t-\tau)^2-\sigma^2}} \cdot \frac{\sigma}{H(\lambda,r,\sigma)}\, d\sigma\, d\lambda\, d\tau \\ &\quad + C||u||^p \int_{-\infty}^t \int_0^{\max(\lambda_-,0)} \int_{|\lambda-r|}^{\lambda+r} \frac{\lambda w_k(\lambda,|\tau|)^{-p}}{\sqrt{(t-\tau)^2-\sigma^2}} \cdot \frac{\sigma}{H(\lambda,r,\sigma)}\, d\sigma\, d\lambda\, d\tau, \end{aligned}$$

where we have set $\lambda_\pm = t-\tau \pm r$ for convenience. Write this equation as

$$\begin{aligned} |\partial_x^\alpha \mathscr{L}F(u)| &\le C||u||^p \int_{-\infty}^t \int_{|\lambda_-|}^{\lambda_+} \lambda w_k(\lambda,|\tau|)^{-p} \cdot K(\lambda,r,t-\tau)\, d\lambda\, d\tau \\ &\quad + C||u||^p \int_{-\infty}^{t-r} \int_0^{\lambda_-} \lambda w_k(\lambda,|\tau|)^{-p} \cdot K(\lambda,r,t-\tau)\, d\lambda\, d\tau, \end{aligned} \tag{3.3}$$

where the kernel $K(\lambda,r,t)$ is defined by

$$K(\lambda,r,t) = \int_{|\lambda-r|}^{\min(\lambda+r,t)} \frac{\sigma}{\sqrt{t^2-\sigma^2}} \cdot H(\lambda,r,\sigma)^{-1}\, d\sigma. \tag{3.4}$$

To estimate this kernel, we shall use the following elementary fact.

Lemma 16. *Let* $r,t > 0$ *and suppose that* $\max(0,r-t) \le \lambda \le r+t$. *Using the notation above, one can then write the kernel* (3.4) *in the form*

$$K(\lambda,r,t) = (8r\lambda)^{-1/2} \cdot J(\mu(\lambda,r,t)), \tag{3.5}$$

where $\mu(\lambda,r,t)$ *denotes the rational function*

$$\mu(\lambda,r,t) = \frac{\lambda^2+r^2-t^2}{2r\lambda} \tag{3.6}$$

and we have also set

$$J(\mu) = \int_{\max(-1,\mu)}^1 (s-\mu)^{-1/2}\,(1-s^2)^{-1/2}\, ds$$

for convenience. Here, the function $J(\mu)$ *is well-defined for each* $\mu \le 1$ *and satisfies*

$$J(\mu) \le C\ln\Big(1+|\mu+1|^{-1/2}\Big) \qquad \text{near } \mu=-1 \tag{3.7}$$

as well as

$$J(\mu) = O(|\mu|^{-1/2}) \qquad \text{as } \mu \to -\infty. \tag{3.8}$$

Proof. We shall merely establish the identity (3.5), as the proof of our remaining assertions can be found in section 8.2 of [8].

Suppose first that $|t-r| \leq \lambda \leq t+r$, in which case our definition (3.4) reads

$$K(\lambda, r, t) = \int_{|\lambda - r|}^{t} \frac{\sigma}{\sqrt{t^2 - \sigma^2}} \cdot H(\lambda, r, \sigma)^{-1} \, d\sigma. \tag{3.9}$$

We note that $\mu(\lambda, r, |\lambda \pm r|) = \mp 1$ and $\partial_\sigma \mu(\lambda, r, \sigma) = -\sigma/(r\lambda)$. Moreover, we have

$$t^2 - \sigma^2 = 2r\lambda \cdot \Big(\mu(\lambda, r, \sigma) - \mu(\lambda, r, t) \Big),$$

while the equations

$$\mu(\lambda, r, \sigma) \pm 1 = \frac{(\lambda \pm r + \sigma)(\lambda \pm r - \sigma)}{2r\lambda} \tag{3.10}$$

combine to give

$$1 - \mu(\lambda, r, \sigma)^2 = \frac{H(\lambda, r, \sigma)^2}{4r^2\lambda^2}$$

with $H(\lambda, r, \sigma)$ as in (3.2). Using the substitution $s = \mu(\lambda, r, \sigma)$ in (3.9), we now find

$$K(\lambda, r, t) = (8r\lambda)^{-1/2} \int_{\mu(\lambda, r, t)}^{1} \big(s - \mu(\lambda, r, t)\big)^{-1/2} (1 - s^2)^{-1/2} \, ds.$$

This is precisely our assertion (3.5), since $|\mu(\lambda, r, t)| \leq 1$ whenever $|t-r| \leq \lambda \leq t+r$.

Suppose now that $0 \leq \lambda \leq t-r$. Arguing as above, one finds that

$$K(\lambda, r, t) = (8r\lambda)^{-1/2} \int_{-1}^{1} \big(s - \mu(\lambda, r, t)\big)^{-1/2} (1 - s^2)^{-1/2} \, ds.$$

This is precisely our assertion (3.5), since $\mu(\lambda, r, t) \leq -1$ whenever $0 \leq \lambda \leq t-r$. ■

Lemma 17. *Suppose that $0 < \delta$ and $0 < \theta \leq 1/2$. Using the notation above, one then has*

$$J(\mu(\lambda, r, t)) \leq C \left(\frac{r}{t+r} \right)^{\delta} \left(\frac{\lambda}{\lambda + r - t} \right)^{\delta} \tag{3.11}$$

whenever $|t-r| \leq \lambda \leq t+r$; and also

$$J(\mu(\lambda, r, t)) \leq C \left(\frac{r}{t+r} \right)^{\theta} \frac{\lambda^{1/2}(t-r)^{\theta - 1/2}}{(t-r-\lambda)^{\theta}} \tag{3.12}$$

whenever $0 \leq \lambda \leq t-r$. In either case, the constant C is independent of λ, r and t.

Proof. Suppose first that $|t-r| \leq \lambda \leq t+r$, in which case $|\mu(\lambda, r, t)| \leq 1$. Then (3.7) gives

$$J(\mu(\lambda, r, t)) \leq C|1 + \mu(\lambda, r, t)|^{-\delta} = C \left(\frac{r}{t+r+\lambda} \right)^{\delta} \left(\frac{\lambda}{\lambda + r - t} \right)^{\delta}$$

in view of (3.10). Since $t+r+\lambda$ is equivalent to $t+r$ here, our assertion (3.11) follows.

Suppose now that $0 \leq \lambda \leq t-r$. Using the fact that $t \geq r$ here, one can easily check that the rational function (3.6) is increasing in λ with

$$\lim_{\lambda \to 0^+} \mu(\lambda, r, t) = -\infty, \qquad \mu(t-r, r, t) = -1.$$

Thus, we shall need to use the asymptotic expansions of $J(\mu)$ at each of these points. To obtain the desired estimate (3.12), we divide our analysis into several cases.
Case 1: Suppose that $0 \leq \lambda \leq (t-r)/2$. Then we need only establish the estimate

$$J(\mu(\lambda, r, t)) \leq C \left(\frac{r}{t+r} \right)^{\theta} \left(\frac{\lambda}{t-r-\lambda} \right)^{1/2} \tag{3.13}$$

because $t-r$ is equivalent to $t-r-\lambda$. For the values of λ we are considering here,

$$\mu(\lambda, r, t) \leq \mu \left(\frac{t-r}{2}, r, t \right) = -\frac{5r+3t}{4r} \leq -2$$

because $t \geq r$ by above. Thus, the asymptotic expansion (3.8) ensures that

$$J(\mu(\lambda, r, t)) \leq C|1+\mu(\lambda, r, t)|^{-1/2} = C \left(\frac{r}{t+r+\lambda} \right)^{1/2} \left(\frac{\lambda}{t-r-\lambda} \right)^{1/2}$$

in view of (3.10). Since $t+r+\lambda$ is equivalent to $t+r$ here and since $\theta \leq 1/2$ by assumption, this does imply the desired (3.13).
Case 2: When $(t-r)/2 \leq \lambda \leq t-r$ and $r \leq t \leq 2r$, it suffices to show that

$$J(\mu(\lambda, r, t)) \leq C \left(\frac{\lambda}{t-r-\lambda} \right)^{\theta}. \tag{3.14}$$

For the values of λ we are considering here, however, one has

$$\mu(\lambda, r, t) \geq \mu \left(\frac{t-r}{2}, r, t \right) = -\frac{5r+3t}{4r} \geq -\frac{11}{4}$$

because $t \leq 2r$ for this case. Recalling (3.7), one then easily obtains the estimate

$$J(\mu(\lambda, r, t)) \leq C|1+\mu(\lambda, r, t)|^{-\theta} \leq C \left(\frac{r}{t+r} \right)^{\theta} \left(\frac{\lambda}{t-r-\lambda} \right)^{\theta},$$

which trivially implies the desired (3.14).
Case 3: When $(t-r)/2 \leq \lambda \leq t-r$ and $t \geq 2r$ and $\lambda \geq t-3r/2$, it suffices to show that

$$J(\mu(\lambda, r, t)) \leq C \left(\frac{r}{t+r} \right)^{\theta} \left(\frac{\lambda}{t-r-\lambda} \right)^{\theta}. \tag{3.15}$$

Note that λ is equivalent to $t \pm r$ for this case. For the values of λ we are considering here,

$$\mu(\lambda, r, t) \geq \mu \left(t - \frac{3r}{2}, r, t \right) = -\frac{12t-13r}{8t-12r} \geq -\frac{11}{4}$$

because $t \geq 2r$ for this case. Recalling (3.7), one then easily obtains the desired

$$J(\mu(\lambda, r, t)) \leq C|1+\mu(\lambda, r, t)|^{-\theta} \leq C \left(\frac{r}{t+r} \right)^{\theta} \left(\frac{\lambda}{t-r-\lambda} \right)^{\theta}.$$

Case 4: When $(t-r)/2 \leq \lambda \leq t-r$ and $t \geq 2r$ and $\lambda \leq t-3r/2$, it still suffices to show that (3.15) holds. For the values of λ we are considering here, however, one has

$$\mu(\lambda, r, t) \leq \mu \left(t - \frac{3r}{2}, r, t \right) = -\frac{12t-13r}{8t-12r} \leq -\frac{3}{2}.$$

Thus, the asymptotic expansion (3.8) is now applicable and we get

$$J(\mu(\lambda,r,t)) \leq C\left(\frac{r}{t+r}\right)^{1/2}\left(\frac{\lambda}{t-r-\lambda}\right)^{1/2}$$
$$= C\left(\frac{r}{t+r}\right)^{\theta}\left(\frac{r}{t+r}\right)^{1/2-\theta}\left(\frac{\lambda}{t-r-\lambda}\right)^{1/2}.$$

Since λ is equivalent to $t \pm r$ and since $r/2 \leq t-r-\lambda$ for this case, we thus get

$$J(\mu(\lambda,r,t)) \leq C\left(\frac{r}{t+r}\right)^{\theta}\left(\frac{t-r-\lambda}{\lambda}\right)^{1/2-\theta}\left(\frac{\lambda}{t-r-\lambda}\right)^{1/2}$$

because $\theta \leq 1/2$ by assumption. This is precisely the desired estimate (3.15). ■

Let us now return to the proof of Lemma 14. As we already know from (3.3), we have

$$|\partial_x^\alpha \mathscr{L}F(u)| \leq C||u||^p \int_{-\infty}^{t}\int_{|\lambda_-|}^{\lambda_+} \lambda w_k(\lambda,|\tau|)^{-p}\cdot K(\lambda,r,t-\tau)\ d\lambda\, d\tau$$
$$+ C||u||^p \int_{-\infty}^{t-r}\int_{0}^{\lambda_-} \lambda w_k(\lambda,|\tau|)^{-p}\cdot K(\lambda,r,t-\tau)\ d\lambda\, d\tau,$$

where $\lambda_\pm = t-\tau \pm r$ and the kernel $K(\lambda,r,t)$ is given by (3.4). Using (3.5) and the estimates of Lemma 17, we then find

$$|\partial_x^\alpha \mathscr{L}F(u)| \leq C||u||^p\cdot r^{\delta-1/2}\int_{-\infty}^{t}\int_{|\lambda_-|}^{\lambda_+} \frac{\lambda^{\delta+1/2}\, w_k(\lambda,|\tau|)^{-p}}{\lambda_+^\delta(\lambda-\lambda_-)^\delta}\ d\lambda\, d\tau$$
$$+ C||u||^p\cdot r^{\theta-1/2}\int_{-\infty}^{t-r}\int_{0}^{\lambda_-} \frac{\lambda\, w_k(\lambda,|\tau|)^{-p}}{\lambda_+^\theta\,\lambda_-^{1/2-\theta}(\lambda_--\lambda)^\theta}\ d\lambda\, d\tau,$$

where $0<\delta$ and $0<\theta\leq 1/2$ are arbitrary, while C is independent of r,t. Note that the last equation can also be written in the form

$$|\partial_x^\alpha \mathscr{L}F(u)| \leq C||u||^p\cdot(\mathcal{I}_\delta + \mathcal{K}_\theta),$$

where $\mathcal{I}_\delta$ and $\mathcal{K}_\theta$ are the integrals treated in Lemmas 10 and 13, respectively. Once we now fix the parameters δ,θ in accordance with these lemmas, we get

$$|\partial_x^\alpha \mathscr{L}F(u)| \leq C||u||^p\cdot w_k(r,|t|)^{-1}.$$

In view of the definition (2.5) of our norm, this actually implies

$$||\mathscr{L}F(u)|| \leq C||u||^p$$

and also completes the proof of Lemma 14.

4. Existence of the scattering operator

In this section, we turn to the proofs of our main results, Theorem 1 and Theorem 3.

Proof of Theorem 1. Our iteration argument is almost identical with that of [12], so we only give a sketch of the proof. As we have already mentioned earlier, one may decrease the decay rate k of the initial data to ensure that (2.10) and (2.12) hold without loss of generality. We let $u_0 = u_0^-$ be the solution given by Lemma 4 and then recursively define

$$u_{i+1} = u_0^- + \mathscr{L}F(u_i), \qquad i \geq 0. \tag{4.1}$$

According to Lemma 4, we then have $u_0 \in X$ with X as in (2.4), and we also have

$$||u_0|| \leq C_0\varepsilon.$$

In order to proceed, we shall assume that ε is so small that

$$2C_0\varepsilon \leq 1, \qquad 2C_1(2C_0\varepsilon)^{p-1} \leq 1$$

when C_1 is the constant appearing in Lemma 14. Then we have

$$2||u_0|| \leq 1, \qquad 2C_1(2||u_0||)^{p-1} \leq 1.$$

Using Lemma 14 and induction, we now find that $||u_i|| \leq 2||u_0||$ for all i. In particular, the whole sequence $\{u_i\}$ lies in X. Using Lemma 14 and a contraction argument, as in [12], we deduce the existence of a unique solution $u \in X$ to the integral equation (1.8). ∎

Proof of Theorem 3. Our first step is to establish (1.11), which asserts that

$$||u - u_0^-||_e \to 0 \qquad \text{as } t \to -\infty.$$

To prove this fact, as it is well-known, it suffices to obtain an estimate of the form

$$\int_{-\infty}^{t} ||F(u)||_{L^2(\mathbb{R}^2)}\, d\tau \leq C\, \langle t\rangle^{-\varepsilon}, \qquad t \leq 0 \tag{4.2}$$

for some $\varepsilon > 0$; see [9] for more details. In particular, we need only show that

$$G(\tau) \equiv \int_{\mathbb{R}^2} F(u(x,\tau))^2\, dx \leq C\, \langle \tau\rangle^{-2\varepsilon-2}, \qquad \tau \leq 0 \tag{4.3}$$

for some $\varepsilon > 0$. Now, using our assumptions (2.8), (2.9) on F and the definition (2.5) of our norm, one easily finds that

$$F(u(x,\tau))^2 \leq C|u(x,\tau)|^{2p} \leq C||u||^{2p} \cdot w_k(|x|,|\tau|)^{-2p}$$

because $u \in X$ by Theorem 1. Recall that the weight function (2.6) is given by

$$w_k(|x|,|\tau|) = \langle |\tau| + |x|\rangle^{\beta} \langle |\tau| - |x|\rangle^{\gamma} \left(1 + \ln \frac{\langle |\tau| + |x|\rangle}{\langle |\tau| - |x|\rangle}\right)^{-\delta_{k,1/2}},$$

where $\beta = \min(k, 1/2)$, $\gamma = k - \beta$ and $\delta_{k,1/2}$ is the usual Kronecker delta. Inserting the last two equations in our definition (4.3), we now switch to polar coordinates to find that

$$G(\tau) \leq C \int_0^{\infty} \langle |\tau| + r\rangle^{1-2\beta p} \langle |\tau| - r\rangle^{-2\gamma p} \left(1 + \ln \frac{\langle |\tau| + r\rangle}{\langle |\tau| - r\rangle}\right)^{2p\,\delta_{k,1/2}} dr. \tag{4.4}$$

Note that each of $\langle|\tau| \pm r\rangle$ is equivalent to $\langle r\rangle$ whenever $r \geq 2|\tau|$, while each of $\langle|\tau| \pm r\rangle$ is equivalent to $\langle\tau\rangle$ whenever $|\tau| \geq 2r$. Thus, the last equation also implies

$$\begin{aligned} G(\tau) \leq C \int_{2|\tau|}^{\infty} \langle r\rangle^{1-2(\beta+\gamma)p}\, dr + C\,\langle\tau\rangle^{1-2(\beta+\gamma)p} \int_0^{|\tau|/2} dr \\ + C \int_{|\tau|/2}^{2|\tau|} \langle|\tau|+r\rangle^{1-2\beta p}\,\langle|\tau|-r\rangle^{-2\gamma p} \left(1+\ln\frac{\langle|\tau|+r\rangle}{\langle|\tau|-r\rangle}\right)^{2p} dr. \end{aligned}$$

Here, $\beta+\gamma=k$ by definition (2.11), so we actually have

$$2-2(\beta+\gamma)p = 2-2kp \leq -2k-2 < -2$$

because $2kp \geq 2k+4$ by (2.10). Combining the last two equations, we then get

$$G(\tau) \leq C\,\langle\tau\rangle^{2-2kp} + C\,\langle\tau\rangle^{1-2\beta p+\varepsilon} \int_{|\tau|/2}^{2|\tau|} \langle|\tau|-r\rangle^{-2\gamma p}\, dr$$

for any $\varepsilon > 0$ whatsoever. Note that this trivially implies

$$\begin{aligned} G(\tau) &\leq C\,\langle\tau\rangle^{2-2kp} + C\,\langle\tau\rangle^{1-2\beta p+\varepsilon} + C\,\langle\tau\rangle^{2-2(\beta+\gamma)p+2\varepsilon} \\ &\leq C\,\langle\tau\rangle^{2-2kp+2\varepsilon} + C\,\langle\tau\rangle^{1-2\beta p+2\varepsilon} \end{aligned} \tag{4.5}$$

because $\beta+\gamma=k$ by above. In addition, (2.10) and (2.12) ensure that

$$\varepsilon = \frac{1}{2}\cdot\,\min(kp-2,\beta p-3/2)$$

is positive. Invoking (4.5) for this choice of ε, it is now easy to deduce the desired (4.3).

This finally completes the proof of (4.2), which also implies our first assertion (1.11). To prove the remaining assertions of the theorem, we set

$$u_0^+(x,t) = u(x,t) - \frac{1}{2\pi}\int_t^{\infty} (\tau-t)\int_{|y|<1} \frac{F(u(x+(\tau-t)y,\tau))}{\sqrt{1-|y|^2}}\, dy\, d\tau.$$

As one can readily check, u_0^+ is then a $\mathcal{C}^2$-solution to the homogeneous wave equation (1.7). Besides, the expression $u-u_0^+$ bears a close resemblance to the Duhamel operator (1.9), so one may establish the convergence

$$||u-u_0^+||_e \to 0 \qquad \text{as } t\to+\infty$$

in the exact same way that we obtained (1.11). Given some other $\mathcal{C}^2$-solution with the same properties as u_0^+, the difference w of the two must satisfy the homogeneous equation (1.7) and its energy norm $||w||_e$ must tend to zero as $t \to +\infty$. Since this implies that $w \equiv 0$, the uniqueness assertion of the theorem follows as well. ■

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