Integrable four-vortex motion on sphere with zero moment of vorticity

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Abstract

We consider the motion of four vortex points on sphere, which defines a Hamiltonian dynamical system. When the moment of vorticity vector, which is a conserved quantity, is zero at the initial moment, the motion of the four vortex points is integrable. The present paper gives a description of the integrable system by reducing it to a three-vortex problem. At the same time, we discuss if the vortex points collide self-similarly in finite time.

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1 Introduction

Let (θ_m, ϕ_m) denote the position of the mth vortex point in the spherical coordinates. The motion of the N-vortex points with the strength Γ_m on a sphere with unit radius is governed by

$$\dot{\theta}_m = -\frac{1}{4\pi}\sum_{j\neq m}^{N}\frac{\Gamma_j \sin\theta_j \sin(\phi_m-\phi_j)}{1-\cos\gamma_{mj}}, \tag{1}$$

$$\dot{\phi}_m = -\frac{1}{4\pi\sin\theta_m}\sum_{j\neq m}^{N}\frac{\Gamma_j[\cos\theta_m\sin\theta_j\cos(\phi_m-\phi_j)-\sin\theta_m\cos\theta_j]}{1-\cos\gamma_{mj}}, \tag{2}$$

in which γ_{mj} represents the central angle between the mth and the jth vortex points, and $\cos\gamma_{mj} = \cos\Theta_m \cos\Theta_j + \sin\Theta_m \sin\Theta_j \cos(\Psi_m - \Psi_j)$. With the Hamiltonian

$$H = -\frac{1}{4\pi} \sum_{m<j}^{N} \Gamma_m \Gamma_j \log(1 - \cos\gamma_{mj}), \tag{3}$$

we rewrite the equations (1) and (2) in the following form[6, 8]:

$$\begin{aligned} \frac{d\cos\theta_m}{dt} &= \{\cos\theta_m, H\}, \\ \frac{d\phi_m}{dt} &= \{\phi_m, H\}, \end{aligned}$$

in which the Poisson bracket between two functions f and g is defined by

$$\{f, g\} = \sum_{m=1}^{N} \frac{1}{\Gamma_m} \left(\frac{\partial f}{\partial \phi_m} \frac{\partial g}{\partial \cos\theta_m} - \frac{\partial g}{\partial \phi_m} \frac{\partial f}{\partial \cos\theta_m} \right). \tag{4}$$

Let us introduce the total vorticity Γ and the moment of vorticity vector $\mathbf{M} = (Q, P, S)$ by

$$\begin{aligned} \Gamma &= \sum_{m=1}^{N} \Gamma_m, \\ Q &= \sum_{m=1}^{N} \Gamma_m \sin\theta_m \cos\phi_m, \\ P &= \sum_{m=1}^{N} \Gamma_m \sin\theta_m \sin\phi_m, \\ S &= \sum_{m=1}^{N} \Gamma_m \cos\theta_m. \end{aligned}$$

Each component of $\mathbf{M}$ is an invariant quantity due to $\{H, Q\} = \{H, P\} = \{H, S\} = 0$. Since the invariants $P^2 + Q^2$ and S satisfy $\{H, P^2 + Q^2\} = \{P^2 + Q^2, S\} = 0$, the motion of three vortex points is integrable and it has been investigated[4, 5, 11]. Moreover, we have $\{Q, P\} = S$, $\{P, S\} = Q$ and $\{S, Q\} = P$, and thus they are in involution with each other when $Q = P = S = 0$ holds at the initial moment. This indicates that the four-vortex problem on the sphere is also integrable, if the moment of vorticity is zero.

The four-vortex motion in an unbounded plane is integrable when the total vorticity and the total impulse are zero[3, 8]. Aref and Stremeler[2] have given the complete description of the integrable system with the reduction method developed by Rott[10]. The purpose of the article is to give a description of the integrable four-vortex motion on the sphere with the reduction method. We must note that it is unnecessary to assume that the total vorticity is zero to obtain the integrability for the spherical case, which is a big difference from the planar case.

The article consists of six sections. In §2, we reduce the four vortex problem to a three vortex problem with the Rott's reduction method. After discussing the existence of the self-similar collapse in §3, we give a complete description of the four-vortex problem for $\Gamma(\Gamma - 2\Gamma_4) \neq 0$ in §4, and for $\Gamma(\Gamma - 2\Gamma_4) = 0$ in §5. We summarize the results in the last section.

2 Reduction to a three vortex problem

The reduction method is based on the relative configuration of the four vortex points. Suppose that we define the distance l_{ij} between two vortex points (θ_i, ϕ_i) and (θ_j, ϕ_j) by $l_{ij}^2 = 2(1 - \cos\gamma_{ij})$. Then the equations (1) and (2) conserve the Hamiltonian H

$$\begin{aligned} H &= \Gamma_1\Gamma_2 \log l_{12}^2 + \Gamma_2\Gamma_3 \log l_{23}^2 + \Gamma_3\Gamma_1 \log l_{31}^2 \\ &\quad + \Gamma_1\Gamma_4 \log l_{14}^2 + \Gamma_2\Gamma_4 \log l_{24}^2 + \Gamma_3\Gamma_4 \log l_{34}^2, \end{aligned} \tag{5}$$

and the invariant C

$$\begin{aligned} C &= \Gamma^2 - (P^2 + Q^2 + R^2) \\ &= \Gamma_1\Gamma_2 l_{12}^2 + \Gamma_2\Gamma_3 l_{23}^2 + \Gamma_3\Gamma_1 l_{31}^2 + \Gamma_1\Gamma_4 l_{14}^2 + \Gamma_2\Gamma_4 l_{24}^2 + \Gamma_3\Gamma_4 l_{34}^2. \end{aligned} \tag{6}$$

Note that $C = \Gamma^2$, since the moment of vorticity vector is always zero.

Now, we introduce the other invariants as follows. Owing to $Q = P = S = 0$, separating the terms for the vortex 1 and for the vortex triple 234, we have

$$\Gamma_1^2 \sin^2\theta_1 \cos^2\phi_1 = (\Gamma_2 \sin\theta_2 \cos\phi_2 + \Gamma_3 \sin\theta_3 \cos\phi_3 + \Gamma_4 \sin\theta_4 \cos\phi_4)^2, \tag{7}$$

$$\Gamma_1^2 \sin^2\theta_1 \sin^2\phi_1 = (\Gamma_2 \sin\theta_2 \sin\phi_2 + \Gamma_3 \sin\theta_3 \sin\phi_3 + \Gamma_4 \sin\theta_4 \sin\phi_4)^2, \tag{8}$$

$$\Gamma_1^2 \cos^2\theta_1 = (\Gamma_2 \cos\theta_2 + \Gamma_3 \cos\theta_3 + \Gamma_4 \cos\theta_4)^2. \tag{9}$$

Adding both sides of (7), (8) and (9), we define the invariant L_1 by

$$L_1 \equiv \Gamma_2\Gamma_3 l_{23}^2 + \Gamma_2\Gamma_4 l_{24}^2 + \Gamma_3\Gamma_4 l_{34}^2 = (\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1^2. \tag{10}$$

Applying the similar algebraic steps to the vortex triples 134, 124 and 123 yields the other invariants L_2, L_3 and L_4:

$$L_2 \equiv \Gamma_3\Gamma_1 l_{31}^2 + \Gamma_1\Gamma_4 l_{14}^2 + \Gamma_3\Gamma_4 l_{34}^2 = (\Gamma_1 + \Gamma_3 + \Gamma_4)^2 - \Gamma_2^2, \tag{11}$$

$$L_3 \equiv \Gamma_1\Gamma_2 l_{12}^2 + \Gamma_1\Gamma_4 l_{14}^2 + \Gamma_2\Gamma_4 l_{24}^2 = (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - \Gamma_3^2, \tag{12}$$

$$L_4 \equiv \Gamma_1\Gamma_2 l_{12}^2 + \Gamma_2\Gamma_3 l_{23}^2 + \Gamma_3\Gamma_1 l_{31}^2 = (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4^2. \tag{13}$$

Then, we obtain

$$
\begin{aligned}
L_1 - L_2 - L_3 + L_4 &= 2\Gamma_2\Gamma_3 l_{23}^2 - 2\Gamma_1\Gamma_4 l_{14}^2 = 2\Gamma(-\Gamma_1 + \Gamma_2 + \Gamma_3 - \Gamma_4), \quad (14)\\
L_1 - L_2 + L_3 - L_4 &= 2\Gamma_2\Gamma_4 l_{24}^2 - 2\Gamma_3\Gamma_1 l_{31}^2 = 2\Gamma(-\Gamma_1 + \Gamma_2 - \Gamma_3 + \Gamma_4), \quad (15)\\
L_1 + L_2 - L_3 - L_4 &= 2\Gamma_3\Gamma_4 l_{34}^2 - 2\Gamma_1\Gamma_2 l_{12}^2 = 2\Gamma(-\Gamma_1 - \Gamma_2 + \Gamma_3 + \Gamma_4). \quad (16)
\end{aligned}
$$

Hence, the distances l_{14}^2, l_{24}^2 and l_{34}^2 are derived from the relative distances of the vortex triple 123 by

$$
\begin{aligned}
\Gamma_1\Gamma_4 l_{14}^2 &= \Gamma_2\Gamma_3 l_{23}^2 + \Gamma(\Gamma_1 - \Gamma_2 - \Gamma_3 + \Gamma_4), &(17)\\
\Gamma_2\Gamma_4 l_{24}^2 &= \Gamma_3\Gamma_1 l_{31}^2 + \Gamma(-\Gamma_1 + \Gamma_2 - \Gamma_3 + \Gamma_4), &(18)\\
\Gamma_3\Gamma_4 l_{34}^2 &= \Gamma_1\Gamma_2 l_{12}^2 + \Gamma(-\Gamma_1 - \Gamma_2 + \Gamma_3 + \Gamma_4). &(19)
\end{aligned}
$$

3 Self-similar collapsing solution

3.1 Non existence of the four vortex self-similar collapse

In the planar vortex problem, it is known that the four vortex points collapse self-similarly in finite time[7]. As for the three-vortex problem on the sphere, the self-similar collapsing solution has been found[5]. Here, we consider whether or not the self-similar collapse of the four vortex points exists when the moment of vorticity is zero.

Suppose that the four vortex points converge on a point in finite time, i.e. $l_{12}^2 = l_{23}^2 = l_{31}^2 = l_{14}^2 = l_{24}^2 = l_{34}^2 = 0$. Then we have $C = \Gamma^2 = 0$ at the critical time, which yields $\Gamma = 0$. Moreover, we assume the self-similarity for the vortex triple 123,

$$
l_{12}^2 = \lambda_1 l_{31}^2, \qquad l_{23}^2 = \lambda_2 l_{31}^2, \qquad \lambda_1, \lambda_2 \in \mathbb{R}. \tag{20}
$$

Then, it follows from (17), (18) and (19) with $\Gamma = 0$ that we have

$$
l_{14}^2 = \frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4}\lambda_2 l_{31}^2, \qquad l_{24}^2 = \frac{\Gamma_3\Gamma_1}{\Gamma_2\Gamma_4} l_{31}^2, \qquad l_{34}^2 = \frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4}\lambda_1 l_{31}^2. \tag{21}
$$

Substitution of (20) and (21) into (5) leads to

$$
H = \log\left(l_{31}^2\right)^{\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_3\Gamma_1+\Gamma_1\Gamma_4+\Gamma_2\Gamma_4+\Gamma_3\Gamma_4}.
$$

Since the Hamiltonian must remain finite as l_{31}^2 tends to zero, the necessary conditions for the existence of the self-similar collapse are given by

$$
\Gamma = 0, \qquad \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_4 + \Gamma_3\Gamma_4 = 0. \tag{22}
$$

As a matter of fact, the conditions never hold simultaneously. This is because

$$
\begin{aligned}
0 &= \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_4 + \Gamma_3\Gamma_4\\
&= \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 - (\Gamma_1 + \Gamma_2 + \Gamma_3)^2\\
&= -\frac{1}{2}(\Gamma_1 + \Gamma_2)^2 - \frac{1}{2}(\Gamma_2 + \Gamma_3)^2 - \frac{1}{2}(\Gamma_3 + \Gamma_1)^2.
\end{aligned}
$$

This leads to $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$, which contradicts $\Gamma_i \neq 0$. Accordingly, there exists no self-similar collapse of the four vortex points.

3.2 On the partial self-similar collapse

We discuss the necessary conditions for the existence of the partial collapse. There are two possibilities where not all but part of the vortex points collapse self-similarly in finite time: The first case is that the three vortex points collide at a point, while the other stays at another position. The second case is that the pairs of two vortex points collide at different positions.

First suppose that the vortex triple 123 converges on a point at a certain time, i.e. $l_{12}^2 = l_{23}^2 = l_{31}^2 = 0$, whereas the fourth vortex point is still located at another position, i.e. $l_{14}^2 \neq 0$, $l_{24}^2 \neq 0$ and $l_{34}^2 \neq 0$. Then we obtain $\Gamma \neq 0$, otherwise it follows from (17) with $l_{23}^2 = 0$ that $\Gamma_1\Gamma_4 l_{14}^2 = 0$ at the collapsing time, which gives a contradiction to $l_{14}^2 \neq 0$. On the other hand, substituting (17), (18) and (19) into (6) leads to

$$C = 2\Gamma_1\Gamma_2 l_{12}^2 + 2\Gamma_2\Gamma_3 l_{23}^2 + 2\Gamma_3\Gamma_1 l_{31}^2 + \Gamma(4\Gamma_4 - \Gamma). \tag{23}$$

Hence, we have $C = \Gamma(4\Gamma_4 - \Gamma) = \Gamma^2$ at the singular time. Due to $\Gamma \neq 0$, we obtain a necessary condition for the existence of the partial self-similar collapse, which is $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Gamma_4$. Then the relations (17), (18) and (19) become

$$l_{14}^2 = \frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4} l_{23}^2 + 4, \qquad l_{24}^2 = \frac{\Gamma_3\Gamma_1}{\Gamma_2\Gamma_4} l_{31}^2 + 4, \qquad l_{34}^2 = \frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4} l_{12}^2 + 4. \tag{24}$$

They indicate that if the vortex triple 123 collapses, the fourth vortex point is located at the opposite pole-position of the collapsing point. On the other hand, since we assume the self-similar collapse, i.e. $l_{12}^2 = \lambda_1 l_{31}^2$ and $l_{23}^2 = \lambda_2 l_{31}^2$ with the constant ratio $\lambda_1, \lambda_2 \in \mathbb{R}$, the Hamiltonian is equivalent to

$$\log \left(l_{31}^2\right)^{\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_3\Gamma_1} \left(\frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4}\lambda_2 l_{31}^2 + 4\right)\left(\frac{\Gamma_3\Gamma_1}{\Gamma_2\Gamma_4} l_{31}^2 + 4\right)\left(\frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4}\lambda_1 l_{12}^2 + 4\right). \tag{25}$$

The Hamiltonian must remain finite as $l_{31}^2 \to 0$, so we have another necessary condition, $\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 = 0$. In the similar manner, considering the partial collapse of the vortex triple 124, 134 and 234, we have the necessary conditions as follows:

$$\Gamma_1 + \Gamma_2 + \Gamma_3 = \Gamma_4, \qquad \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 = 0, \tag{26}$$
$$\Gamma_1 + \Gamma_2 + \Gamma_4 = \Gamma_3, \qquad \Gamma_1\Gamma_2 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_4 = 0, \tag{27}$$
$$\Gamma_1 + \Gamma_3 + \Gamma_4 = \Gamma_2, \qquad \Gamma_1\Gamma_3 + \Gamma_1\Gamma_4 + \Gamma_3\Gamma_4 = 0, \tag{28}$$
$$\Gamma_2 + \Gamma_3 + \Gamma_4 = \Gamma_1, \qquad \Gamma_2\Gamma_3 + \Gamma_2\Gamma_4 + \Gamma_3\Gamma_4 = 0. \tag{29}$$

Next we deal with the case when pairs of two vortex points collide at some different positions self-similarly in finite time; suppose that the pair of vortices

1 and 2 converges on a point, and the vortex 3 collides with the vortex 4 at a different point at the same time. Then since $l_{12}^2 = l_{34}^2 = 0$ at the critical time, the relation (19) becomes $\Gamma(-\Gamma_1 - \Gamma_2 + \Gamma_3 + \Gamma_4) = 0$. Hence we have either $\Gamma = 0$ or $\Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4$. As a matter of fact, we can show that Γ is not zero. If $\Gamma = 0$, the relations (17), (18) and (6) at the collapsing time are represented by

$$\Gamma_1\Gamma_4 l_{14}^2 = \Gamma_2\Gamma_3 l_{23}^2, \tag{30}$$

$$\Gamma_2\Gamma_4 l_{24}^2 = \Gamma_3\Gamma_1 l_{31}^2, \tag{31}$$

$$\Gamma_2\Gamma_3 l_{23}^2 + \Gamma_3\Gamma_1 l_{31}^2 + \Gamma_2\Gamma_4 l_{24}^2 + \Gamma_1\Gamma_4 l_{14}^2 = 0. \tag{32}$$

We note that all the vortex strengths cannot be of the same sign due to $\Gamma = 0$. If one of the vortex strength, say Γ_4 for instance, is negative and the others have the positive sign, the both sides of (30) have the opposite signs and thus (30) is invalid. On the other hand, it follows from (30), (31) and (32) that we have $\Gamma_3 l_{23}^2 + \Gamma_4 l_{24}^2 = 0$, which never holds when both Γ_3 and Γ_4 are negative. Consequently, we have $\Gamma \neq 0$, and thus a necessary condition for the pairing collapse is given by $\Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4$. Furthermore, we can see from the self-similarity assumption, $l_{12}^2 = \lambda l_{34}^2$, $(\lambda \in \mathbb{R})$, that the Hamiltonian remains finite even at the singular time, that is to say

$$\log\left(l_{12}^2\right)^{\Gamma_1\Gamma_4 + \Gamma_2\Gamma_3} < \infty,$$

as $l_{12}^2 \to 0$. Hence, we have another necessary condition $\Gamma_1\Gamma_4 + \Gamma_2\Gamma_3 = 0$ for the partial collapse. With the similar argument, we have the other necessary conditions for the existence of the pairing self-similar collapse:

$$\Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4 \qquad \Gamma_1\Gamma_2 + \Gamma_3\Gamma_4 = 0, \tag{33}$$

$$\Gamma_1 + \Gamma_3 = \Gamma_2 + \Gamma_4 \qquad \Gamma_1\Gamma_3 + \Gamma_2\Gamma_4 = 0, \tag{34}$$

$$\Gamma_1 + \Gamma_4 = \Gamma_2 + \Gamma_3 \qquad \Gamma_1\Gamma_4 + \Gamma_2\Gamma_3 = 0. \tag{35}$$

For the time being, we are unable to rule out the possibility of the partial self-similar collapse. It is generally difficult to show the existence of the collapsing solution mathematically. Hence we will verify it numerically in the following sections.

4 Four-vortex motion for non-degenerate case

4.1 Trilinear representation and physical region

We deal with the dynamics of the four vortex points when $L_4 = (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4^2 = \Gamma(\Gamma - 2\Gamma_4) \neq 0$. Let us introduce the variables

$$b_1 = \frac{3\Gamma_2\Gamma_3 l_{23}^2}{\Gamma(\Gamma - 2\Gamma_4)}, \qquad b_2 = \frac{3\Gamma_3\Gamma_1 l_{31}^2}{\Gamma(\Gamma - 2\Gamma_4)}, \qquad b_3 = \frac{3\Gamma_1\Gamma_2 l_{12}^2}{\Gamma(\Gamma - 2\Gamma_4)}. \tag{36}$$

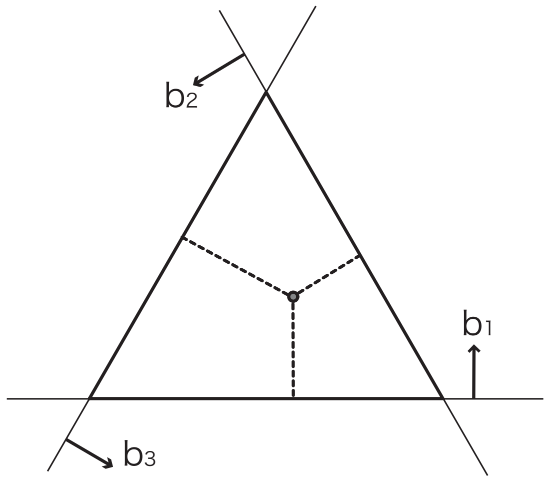

Figure 1: Trilinear coordinates representation for the non-degenerate case.

Then the definition of L_4 with these variables yields $b_1 + b_2 + b_3 = 3$, which indicates that the relative configuration of the vortex triple 123 corresponds to a point in the phase plane as is shown in Figure 1. Each component of the trilinear coordinates (b_1, b_2, b_3) represents the distance from one of the sides of the equilateral triangle with height 3, which is called the *trilinear triangle*. The trilinear representation is a standard tool to describe the integrable three-vortex motion[1, 2, 4, 11].

In the four-vortex problem, we need to consider the distances between the vortex triple 123 and the fourth vortex point. Thus we define the other variables B_1, B_2 and B_3 from (17), (18) and (19) as follows:

$$B_1 = \frac{3\Gamma_1\Gamma_4 l_{14}^2}{\Gamma(\Gamma - 2\Gamma_4)} = b_1 + \frac{3}{\Gamma - 2\Gamma_4}(\Gamma - 2\Gamma_2 - 2\Gamma_3), \tag{37}$$

$$B_2 = \frac{3\Gamma_2\Gamma_4 l_{24}^2}{\Gamma(\Gamma - 2\Gamma_4)} = b_2 + \frac{3}{\Gamma - 2\Gamma_4}(\Gamma - 2\Gamma_3 - 2\Gamma_1), \tag{38}$$

$$B_3 = \frac{3\Gamma_3\Gamma_4 l_{34}^2}{\Gamma(\Gamma - 2\Gamma_4)} = b_3 + \frac{3}{\Gamma - 2\Gamma_4}(\Gamma - 2\Gamma_1 - 2\Gamma_2). \tag{39}$$

Then we note that

$$B_1 + B_2 + B_3 = b_1 + b_2 + b_3 + \frac{3}{\Gamma - 2\Gamma_4}(3\Gamma - 4\Gamma_1 - 4\Gamma_2 - 4\Gamma_3) = \frac{6\Gamma_4}{\Gamma - 2\Gamma_4},$$

which indicates that the triple (B_1, B_2, B_3) also defines another trilinear coordinates; each component B_i represents the distance from one of the sides of the equilateral triangle with height $\frac{6\Gamma_4}{\Gamma - 2\Gamma_4}$. As in the integrable planar four-vortex problem[2], the triangle is called the *physical triangle*.

Since the vortex triple 123 forms a triangle on the sphere, the values of (b_1, b_2, b_3) are restricted by the triangle inequalities, which is equivalent to the following inequality[4],

$$2l_{12}^2 l_{23}^2 + 2l_{23}^2 l_{31}^2 + 2l_{31}^2 l_{12}^2 - l_{12}^4 - l_{23}^4 - l_{31}^4 \geq l_{12}^2 l_{23}^2 l_{31}^2. \tag{40}$$

In terms of the trilinear representation, the condition is equivalent to

$$3V_p - \Gamma(\Gamma - 2\Gamma_4)b_1 b_2 b_3 \geq 0, \tag{41}$$

where

$$V_p = 2\Gamma_2\Gamma_3 b_2 b_3 + 2\Gamma_3\Gamma_1 b_3 b_1 + 2\Gamma_1\Gamma_2 b_1 b_2 - (\Gamma_1 b_1)^2 - (\Gamma_2 b_2)^2 - (\Gamma_3 b_3)^2.$$

The region in which the condition (41) is satisfied is called the *physical region*, where the motion of the reduced three vortex points is restricted. Any point at the boundary of the physical region corresponds to the collinear configuration, for which the three vortex points line on a great circle of the sphere.

Now, we consider the singular points at the boundary of the physical region. They are obtained by solving the following equations,

$$b_1 + b_2 + b_3 = 3, \qquad 3V_p - \Gamma(\Gamma - 2\Gamma_4)b_1 b_2 b_3 = 0, \tag{42}$$

with $b_i = 0$. Then we have

$$(b_1, b_2, b_3) = \left(0, \frac{3\Gamma_3}{\Gamma_2 + \Gamma_3}, \frac{3\Gamma_2}{\Gamma_2 + \Gamma_3}\right), \tag{43}$$

$$(b_1, b_2, b_3) = \left(\frac{3\Gamma_3}{\Gamma_1 + \Gamma_3}, 0, \frac{3\Gamma_1}{\Gamma_1 + \Gamma_3}\right), \tag{44}$$

$$(b_1, b_2, b_3) = \left(\frac{3\Gamma_2}{\Gamma_1 + \Gamma_2}, \frac{3\Gamma_1}{\Gamma_1 + \Gamma_2}, 0\right). \tag{45}$$

They correspond to the singular configurations coinciding the vortices 2 and 3, 3 and 1, and 1 and 2, respectively. Since the boundary of the physical region is a conic section, the points are tangent to the sides of the trilinear triangle. Thus we refer to the points as the *points of tangency*.

Finally, we obtain the points tangent to the sides of the physical triangle. Solving the equations (42) with $B_1 = 0$, i.e.

$$b_1 = \frac{3}{\Gamma - 2\Gamma_4}(2\Gamma_2 + 2\Gamma_3 - \Gamma),$$

we have

$$(b_1, b_2, b_3) = \left(\frac{3(2\Gamma_2 + 2\Gamma_3 - \Gamma)}{\Gamma - 2\Gamma_4}, \frac{3\Gamma_1(2\Gamma_2 - \Gamma)}{(\Gamma - 2\Gamma_4)(\Gamma_2 + \Gamma_3 - \Gamma)}, \frac{3\Gamma_1(2\Gamma_3 - \Gamma)}{(\Gamma - 2\Gamma_4)(\Gamma_2 + \Gamma_3 - \Gamma)}\right). \tag{46}$$

Regarding $B_2 = 0$ and $B_3 = 0$, we have the points of tangency in the similar manner:

$$(b_1, b_2, b_3) = \left(\frac{3\Gamma_2(2\Gamma_1 - \Gamma)}{(\Gamma - 2\Gamma_4)(\Gamma_3 + \Gamma_1 - \Gamma)}, \frac{3(2\Gamma_3 + 2\Gamma_1 - \Gamma)}{\Gamma - 2\Gamma_4}, \frac{3\Gamma_2(2\Gamma_3 - \Gamma)}{(\Gamma - 2\Gamma_4)(\Gamma_3 + \Gamma_1 - \Gamma)} \right), \tag{47}$$

$$(b_1, b_2, b_3) = \left(\frac{3\Gamma_3(2\Gamma_1 - \Gamma)}{(\Gamma - 2\Gamma_4)(\Gamma_1 + \Gamma_2 - \Gamma)}, \frac{3\Gamma_3(2\Gamma_2 - \Gamma)}{(\Gamma - 2\Gamma_4)(\Gamma_1 + \Gamma_2 - \Gamma)}, \frac{3(2\Gamma_1 + 2\Gamma_2 - \Gamma)}{\Gamma - 2\Gamma_4} \right). \tag{48}$$

These three points (46) – (48) represent the singular configurations where the vortices 1 and 4, 2 and 4, and 3 and 4 coincide together.

As the four vortex points evolve, the corresponding point in the trilinear phase space follows a contour curve of the Hamiltonian,

$$\begin{aligned} H &= \Gamma_2\Gamma_3 \log|b_1| + \Gamma_3\Gamma_1 \log|b_2| + \Gamma_1\Gamma_2 \log|b_3| \\ &\quad + \Gamma_1\Gamma_4 \log|B_1| + \Gamma_2\Gamma_4 \log|B_2| + \Gamma_3\Gamma_4 \log|B_3|. \end{aligned} \tag{49}$$

Therefore, in order to observe the four-vortex motion, we plot the contour curves of the Hamiltonian inside the physical region (41) with the points of tangency at the boundary.

4.2 Restriction on the vortex strengths

In the planar four-vortex problem[2], the zero total vorticity condition is required to guarantee the integrability. On the contrary, we have no such condition on the vortex strengths for the spherical case. Nevertheless, as a matter of fact, we need to restrict the vortex strengths so that the invariants L_1, L_2, L_3 and L_4 are bounded, since the distances between any two vortex points on the sphere are bounded by

$$0 \le l_{12}^2, l_{23}^2, l_{31}^2, l_{14}^2, l_{24}^2, l_{34}^2 \le 4. \tag{50}$$

Without loss of generality, we may assume that $\Gamma_1 \ge \Gamma_2 \ge \Gamma_3 \ge \Gamma_4$. Besides, noting that the equations (1) and (2) are symmetric with respect to the discrete transformation $\Gamma_i \to -\Gamma_i$ and $t \to -t$, it is sufficient to deal with the following three cases:

$$\Gamma_1 \ge \Gamma_2 \ge \Gamma_3 \ge \Gamma_4 > 0, \quad \Gamma_1 \ge \Gamma_2 \ge \Gamma_3 > 0 > \Gamma_4, \quad \Gamma_1 \ge \Gamma_2 > 0 > \Gamma_3 \ge \Gamma_4.$$

For each case, we discuss the existence region of Γ_3 and Γ_4 when Γ_1 and Γ_2 are specified, which is called the *possible region*.

In the meantime, we consider the singular cases when the points of tangency (43) – (48) take the same position. They play an important role in the description of the four-vortex motion, since the change of the distribution of the points results in the change of the topological structure of the contour lines of the Hamiltonian. For $B_1 = b_1 = 0$, which corresponds to the point when the vortices 1 and 4,

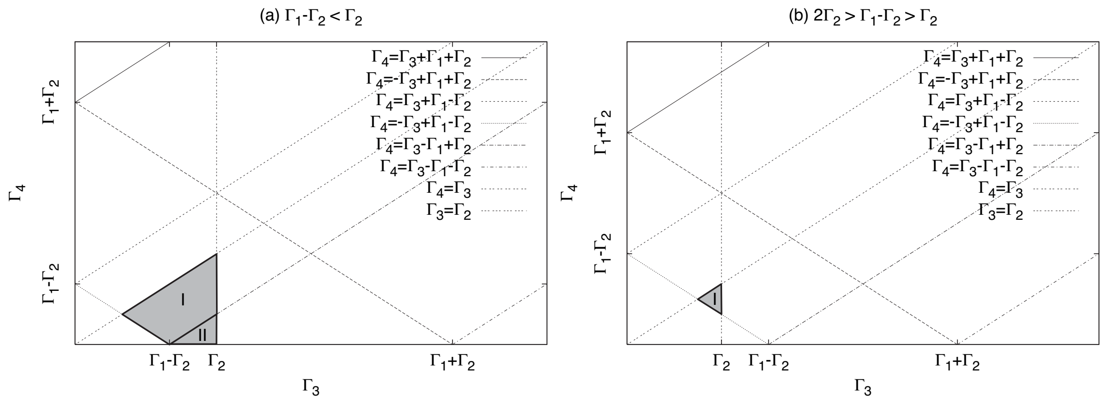

Figure 2: The possible regions for $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq \Gamma_4 > 0$.

and 2 and 3 collide simultaneously, we have $\Gamma_1 + \Gamma_4 = \Gamma_2 + \Gamma_3$. It follows from $B_1 = b_2 = 0$ and $B_1 = b_3 = 0$, for which the vortex triples 134 and 124 collide, that we have $\Gamma_2 = \Gamma_1 + \Gamma_3 + \Gamma_4$ and $\Gamma_3 = \Gamma_1 + \Gamma_2 + \Gamma_4$. In the same way, solving $B_2 = b_i = 0$ and $B_3 = b_i = 0$, we obtain the following six singular cases:

$$\Gamma_3 = \Gamma_1 + \Gamma_2 + \Gamma_4, \qquad \Gamma_2 = \Gamma_1 + \Gamma_3 + \Gamma_4, \qquad \Gamma_1 = \Gamma_2 + \Gamma_3 + \Gamma_4, \tag{51}$$
$$\Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4, \qquad \Gamma_1 + \Gamma_3 = \Gamma_2 + \Gamma_4, \qquad \Gamma_1 + \Gamma_4 = \Gamma_2 + \Gamma_3. \tag{52}$$

We must note that they are part of the necessary conditions (27) – (29) and (33) – (35) for the existence of the partial self-similar collapses given in §3.

4.3 Case I : $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq \Gamma_4 > 0$

We derive the conditions of the vortex strengths so that the invariants L_i are well-defined. Since $\min L_i = 0$ due to $\Gamma_i > 0$, the necessary conditions are given by

$$L_1 = (\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1 > 0, \qquad L_2 = (\Gamma_1 + \Gamma_3 + \Gamma_4)^2 - \Gamma_2 > 0, \tag{53}$$
$$L_3 = (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - \Gamma_3 > 0, \qquad L_4 = (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4 > 0. \tag{54}$$

Owing to $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq \Gamma_4 > 0$, they are reduced to

$$\Gamma_1 < \Gamma_2 + \Gamma_3 + \Gamma_4, \qquad 0 < \Gamma_4 \leq \Gamma_3 \leq \Gamma_2. \tag{55}$$

Strictly speaking, it is necessary to show that the condition (55) assures $L_i < \max L_i$, namely

$$L_1 < 4\Gamma_2\Gamma_3 + 4\Gamma_2\Gamma_4 + 4\Gamma_3\Gamma_4, \qquad L_2 < 4\Gamma_1\Gamma_3 + 4\Gamma_1\Gamma_4 + 4\Gamma_3\Gamma_4, \tag{56}$$
$$L_3 < 4\Gamma_1\Gamma_2 + 4\Gamma_1\Gamma_4 + 4\Gamma_2\Gamma_4, \qquad L_4 < 4\Gamma_1\Gamma_2 + 4\Gamma_2\Gamma_3 + 4\Gamma_3\Gamma_1. \tag{57}$$

The proof of the upper bound is important but lengthy, so we give it in the appendix. Figure 2 shows the possible regions for $\Gamma_1 - \Gamma_2 \leq \Gamma_2$ and for $\Gamma_2 <$

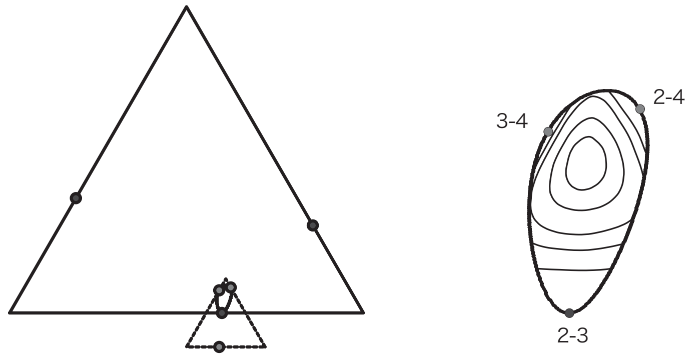

Figure 3: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = 2$ and $\Gamma_4 = 1$.

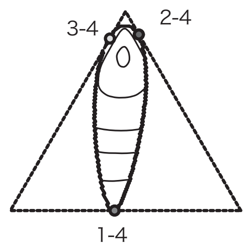

Figure 4: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = \frac{5}{2}$ and $\Gamma_1 = \frac{1}{4}$.

$\Gamma_1 - \Gamma_2 \leq 2\Gamma_2$. No possible region exists for $\Gamma_1 - \Gamma_2 > 2\Gamma_2$. The singular conditions (51) and (52) divide the possible region into two regions I and II for $\Gamma_1 - \Gamma_2 \leq \Gamma_2$. The possible region consists of just one region for $\Gamma_1 - \Gamma_2 > \Gamma_2$, which is still denoted by the region I, since the distribution of the points of tangency is the same as that for $\Gamma_1 - \Gamma_2 \leq \Gamma_2$. We give examples of the contour lines of the Hamiltonian, when we choose the vortex strengths Γ_3 and Γ_4 from the regions I and II for $\Gamma_1 = 5$ and $\Gamma_2 = 3$.

Figure 3 shows the contour lines for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = 2$ and $\Gamma_4 = 1$ in the region I. The trilinear triangle and the physical triangle are plotted by the solid line and the dotted line respectively. Since the physical region, which is contained in the physical triangle, is too small to discern, we magnify it to the right side of the figure. The points of tangency at the boundary of the physical region are designated by $2-3$, $2-4$ and $3-4$, which correspond to $b_1 = 0$, $B_2 = 0$ and $B_3 = 0$ respectively. The topological structure of the contour curves around the singular points is elliptic. There is an elliptic center inside the physical region, which corresponds to a relative fixed configuration. There exist points at the boundary where two level curves meet, which indicates the existence of the hyperbolic collinear fixed points.

Figure 4 shows the contour lines of the Hamiltonian for $\Gamma_1 = 5$, $\Gamma_2 = 3$,

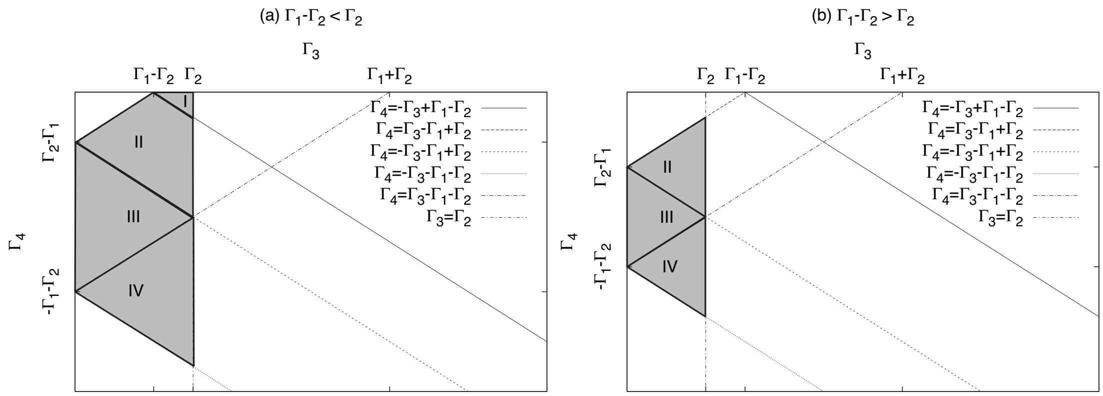

Figure 5: The possible regions for $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 > 0 > \Gamma_4$.

$\Gamma_3 = \frac{5}{2}$ and $\Gamma_4 = \frac{1}{4}$ in the region II. We just show the magnified physical region with the physical triangle, since the trilinear triangle is too large to show in the figure. When we compare the level curves with those in Figure 3, the point of tangency $2-3$ at the boundary is replaced by $1-4$, but the topological structure of the contour curves is the same.

We discuss the existence of the partial collapse. The possible region in Figure 2 indicates that the partial collapse possibly occurs when the vortex strengths Γ_3 and Γ_4 are on the singular boundary, $\Gamma_1 + \Gamma_4 = \Gamma_2 + \Gamma_3$. This is part of the necessary conditions of the pairing collapse (35). However, the other necessary condition $\Gamma_1\Gamma_4 + \Gamma_2\Gamma_3 = 0$ never holds, since all the vortex strengths are positive. Hence, no partial collapse is possible.

4.4 Case II : $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 > 0 > \Gamma_4$

Since $\Gamma_i\Gamma_4$ for $i = 1, 2, 3$ are negative due to $\Gamma_4 < 0$, we have $\min L_1 = 4\Gamma_3\Gamma_4 + \Gamma_2\Gamma_4$, $\min L_2 = 4\Gamma_1\Gamma_4 + \Gamma_3\Gamma_4$, $\min L_3 = 4\Gamma_1\Gamma_4 + \Gamma_2\Gamma_4$ and $\min L_4 = 0$. Hence the vortex strengths should satisfy

$$\begin{aligned} &4\Gamma_3\Gamma_4 + 4\Gamma_2\Gamma_4 < (\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1^2, \qquad &&4\Gamma_1\Gamma_4 + 4\Gamma_3\Gamma_4 < (\Gamma_1 + \Gamma_3 + \Gamma_4)^2 - \Gamma_2^2, \\ &4\Gamma_1\Gamma_4 + 4\Gamma_2\Gamma_4 < (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - \Gamma_3^2, \qquad &&0 < (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4^2. \end{aligned} \tag{58}$$

Owing to $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 > 0 > \Gamma_4$, they are reduced to

$$\Gamma_1 < \Gamma_2 + \Gamma_3 - \Gamma_4, \quad -\Gamma_4 < \Gamma_1 + \Gamma_2 + \Gamma_3, \quad \Gamma_4 < 0 < \Gamma_3 \leq \Gamma_2. \tag{59}$$

The possible region is shown in Figure 5. The proof of the sufficiency for (59) is provided in the appendix. The possible region is divided into four regions by the singular lines (51) and (52) for $\Gamma_1 - \Gamma_2 \leq \Gamma_2$, and three regions for $\Gamma_1 - \Gamma_2 > \Gamma_2$. The divided regions are referred to as the regions I to IV.

Figure 6 shows the physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = \frac{9}{4}$ and $\Gamma_4 = -\frac{1}{4}$ in the region I. There are three points of

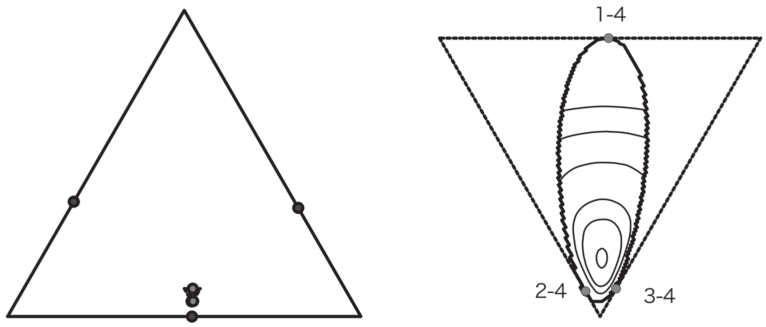

Figure 6: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = \frac{9}{4}$ and $\Gamma_4 = -\frac{1}{4}$.

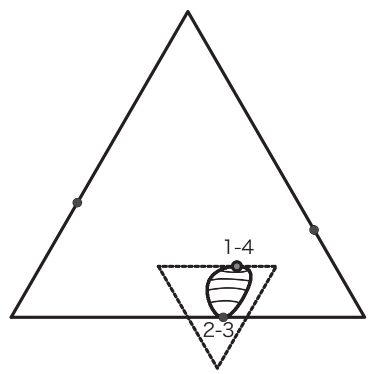

Figure 7: The physical region and the contour lines for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = 2$ and $\Gamma_4 = -2$.

tangency $1-4$, $2-4$ and $3-4$ at the boundary of the physical region, which are elliptic singular points. There is an elliptic center inside the physical region. The topological structure of the contour curves is similar to that we have observed in §4.3.

Figure 7 shows the contour lines of the Hamiltonian for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = 2$ and $\Gamma_4 = -2$ in the region II. As the vortex strengths Γ_3 and Γ_4 pass across the singular boundary $\Gamma_2 + \Gamma_3 + \Gamma_4 = \Gamma_1$ between the region I and II, the points of tangency $2-4$ and $3-4$ are united, and then the new point of tangency $2-3$ emerges. The topological structure of the contour is simple; there is no fixed configuration.

For $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = 2$ and $\Gamma_4 = -5$ in the region III, the contour lines are given in Figure 8. On crossing the singular boundary $\Gamma_1 + \Gamma_3 + \Gamma_4 = \Gamma_2$ between the region II and III, the point of tangency $1-4$ observed in the region II is split into two elliptic points of tangency $1-3$ and $3-4$. There is a hyperbolic fixed point inside the physical region, which is connected by homoclinic orbits. The topological structure of the contour curves suggest that there exists another elliptic fixed point at the boundary near the side $B_1 = 0$ of the physical triangle.

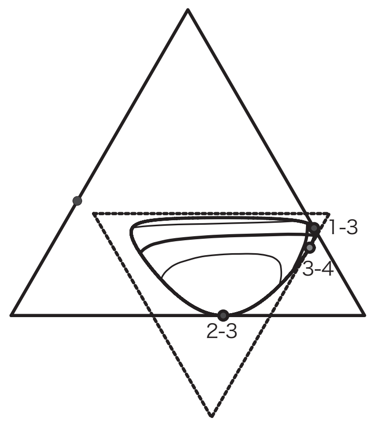

Figure 8: The physical region and the contour lines for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = 2$ and $\Gamma_4 = -5$.

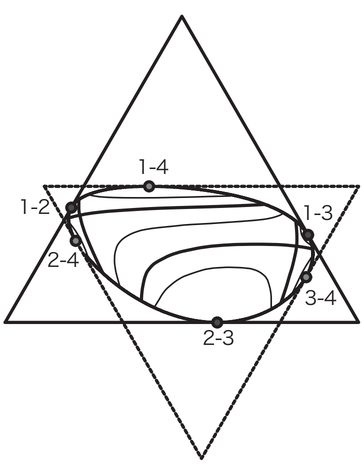

Figure 9: The physical region and the contour lines for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = 2$ and $\Gamma_4 = -8$.

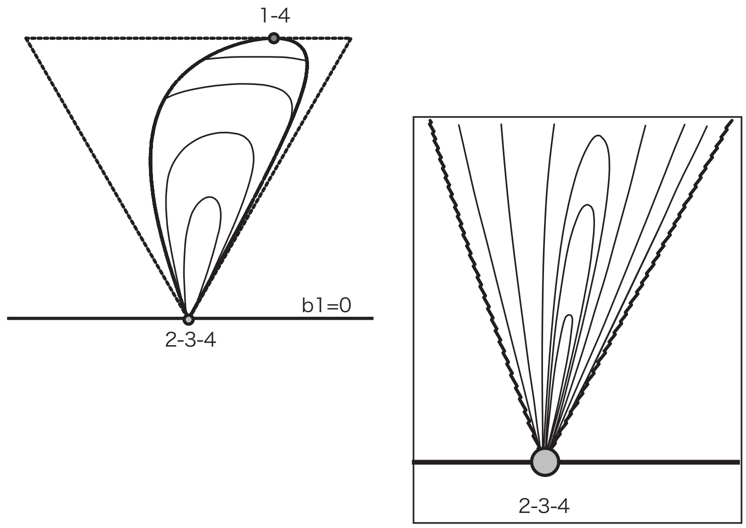

Figure 10: The physical region and the contour lines for $\Gamma_1 = 6, \Gamma_2 = 5, \Gamma_3 = \frac{1}{2}(1+\sqrt{21})$ and $\Gamma_4 = \frac{1}{2}(1-\sqrt{21})$.

The contour lines for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = 2$ and $\Gamma_4 = -8$ in the region IV is shown in Figure 9. The boundary of the physical region touches the side of the trilinear triangle $b_3 = 0$, and the three points of tangency $1-2$, $2-4$ and $1-4$ emerge. There are two hyperbolic fixed points inside the physical region with homoclinic connections.

The self-similar partial collapse is possible when the vortex strengths are at the singular boundaries within the possible region:

$$\Gamma_2 + \Gamma_3 + \Gamma_4 = \Gamma_1, \quad \Gamma_1 + \Gamma_2 + \Gamma_4 = \Gamma_3, \quad \Gamma_1 + \Gamma_3 + \Gamma_4 = \Gamma_2.$$

The corresponding necessary conditions for the existence of the self-similar collapsing solutions are given in (27) – (29); we examine each of them separately. First, we note that the singular boundary $\Gamma_2 + \Gamma_3 + \Gamma_4 = \Gamma_1$ is included in the possible region only when $\Gamma_1 - \Gamma_2 \leq \Gamma_2$. Then substituting $\Gamma_1 = \Gamma_2 + \Gamma_3 + \Gamma_4$ into $\Gamma_2\Gamma_3 + \Gamma_2\Gamma_4 + \Gamma_3\Gamma_4 = 0$ in (29) leads to the equation of Γ_3,

$$\Gamma_3^2 + \Gamma_3(\Gamma_2 - \Gamma_1) + \Gamma_2(\Gamma_2 - \Gamma_1) = 0.$$

For instance, $\Gamma_1 = 6$, $\Gamma_2 = 5$, $\Gamma_3 = \frac{1}{2}(1+\sqrt{21})$ and $\Gamma_4 = \frac{1}{2}(1-\sqrt{21})$ satisfy (29). We plot the physical triangle and the contour curves of the Hamiltonian for this case in Figure 10. One of the corner of the physical triangle touches the line $b_1 = 0$, at which there is the point of tangency $2-3-4$. If there exists a contour line connecting the point of tangency, it implies the existence of the self-similar collapsing solution. However, the contour curves in the neighborhood of the point $2-3-4$, which is shown in the left side of the figure, shrink to the point $2-3-4$ as the value of the Hamiltonian tends to that at the point of tangency. Therefore,

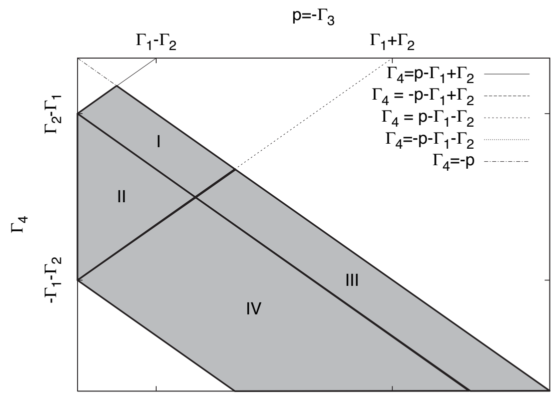

Figure 11: The possible region for $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3 \geq \Gamma_4$.

we hardly make sure the existence of the self-similar collapse. Second, it follows from $\Gamma_1 + \Gamma_2 + \Gamma_4 = \Gamma_3$ and $\Gamma_1\Gamma_2 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_4 = 0$ that we obtain

$$\Gamma_3 = \frac{\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2}{\Gamma_1 + \Gamma_2}.$$

However, since $\Gamma_2 - \Gamma_3 = -\frac{\Gamma_1^2}{\Gamma_1+\Gamma_2} < 0$, it contradicts the assumption $\Gamma_2 \geq \Gamma_3$. Hence, the necessary conditions (27) never hold together. Last, solving the equations $\Gamma_2 = \Gamma_1 + \Gamma_3 + \Gamma_4$ and $\Gamma_1\Gamma_3 + \Gamma_1\Gamma_4 + \Gamma_3\Gamma_4 = 0$, we have the equation of the vortex strength Γ_3,

$$\Gamma_3^2 + \Gamma_3(\Gamma_1 - \Gamma_2) + \Gamma_1(\Gamma_1 - \Gamma_2) = 0.$$

Since the determinant of the quadratic equation is $(3\Gamma_1 + \Gamma_2)(\Gamma_2 - \Gamma_1) < 0$, there is no real solution of the equation and thus no self-similar partial collapse.

4.5 Case III: $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3 \geq \Gamma_4$

Since $\min L_1 = 4\Gamma_2\Gamma_3 + 4\Gamma_2\Gamma_4$, $\min L_2 = 4\Gamma_1\Gamma_3 + 4\Gamma_1\Gamma_4$, $\min L_3 = 4\Gamma_1\Gamma_4 + 4\Gamma_2\Gamma_4$ and $\min L_4 = 4\Gamma_1\Gamma_3 + 4\Gamma_2\Gamma_3$, the vortex strengths should satisfy the following conditions,

$$\begin{aligned} &\min L_1 < (\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1^2, &&\min L_2 < (\Gamma_1 + \Gamma_3 + \Gamma_4)^2 - \Gamma_2^2, \\ &\min L_3 < (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - \Gamma_3^2, &&\min L_4 < (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4^2. \end{aligned} \quad (60)$$

Hence, the possible region is described by

$$\Gamma_3 + \Gamma_4 - \Gamma_2 + \Gamma_1 < 0, \quad \Gamma_1 + \Gamma_2 - \Gamma_3 + \Gamma_4 > 0, \quad \Gamma_4 \leq \Gamma_3 < 0. \quad (61)$$

Defining $p = -\Gamma_3 > 0$ for the sake of convenience, we draw the possible region of (p, Γ_4) in Figure 11. It consists of the four subregions I, II, III and IV, separated

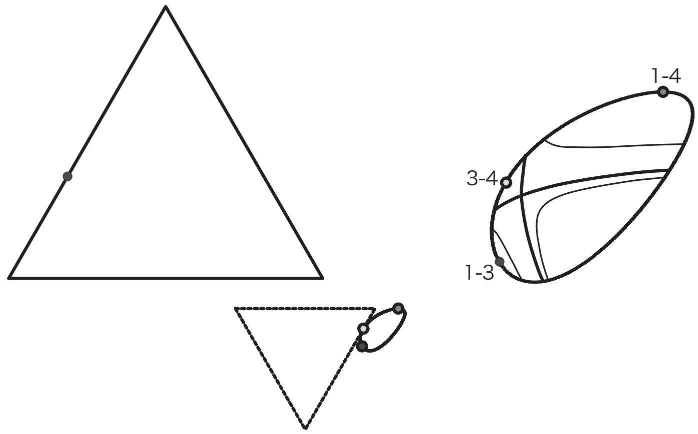

Figure 12: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = -1$ and $\Gamma_4 = -2$.

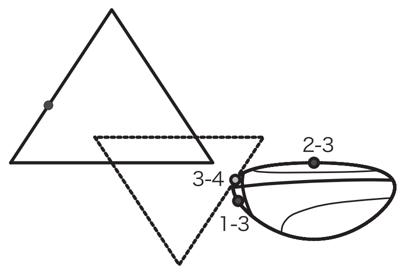

Figure 13: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = -1$ and $\Gamma_4 = -5$.

by the singular boundaries $\Gamma_1 + \Gamma_4 = \Gamma_2 + \Gamma_3$ and $\Gamma_4 = -\Gamma_3 - \Gamma_1 - \Gamma_2$. We now give the contour curves of the Hamiltonian for each region in what follows.

When $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = -1$ and $\Gamma_4 = -2$ in the region I, the physical region is located outside of the trilinear triangle and the physical triangle as we observe in Figure 12. There are three elliptic points of tangency $1-3$, $1-4$ and $3-4$. We can observe an elliptic fixed point at the boundary and a hyperbolic fixed point with homoclinic connections inside the physical region.

Figure 13 shows the contour lines of the Hamiltonian for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = -1$ and $\Gamma_4 = -5$ in the region II. The points of tangency $1-3$, $3-4$ and $2-3$ are on the boundary of the physical region. The topological structure is the same as that in the region I.

When we choose the vortex strengths in the possible region III and IV, the physical region is located on the left side of the trilinear triangle. Figure 14 and 15 show the contour lines of the Hamiltonian for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = -4$ and $\Gamma_4 = -5$, and for $\Gamma_1 = 5$, $\Gamma_2 = 3$, $\Gamma_3 = -2$ and $\Gamma_4 = -8$ respectively. The topological structures of the contour curves are the same except for the the points of tangency $2-3$ and $1-4$ at the boundary of the physical region.

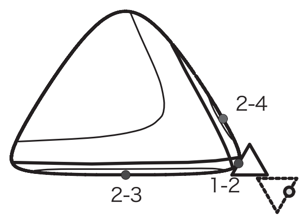

Figure 14: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = -4$ and $\Gamma_4 = -5$.

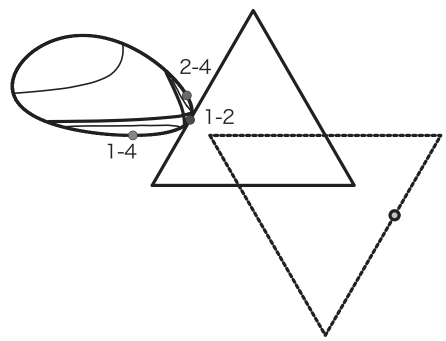

Figure 15: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 5, \Gamma_2 = 3, \Gamma_3 = -2$ and $\Gamma_4 = -8$.

The singular boundaries in the possible region are represented by

$$\Gamma_4 = p - \Gamma_1 - \Gamma_2, \qquad \Gamma_4 = -p - \Gamma_1 + \Gamma_2.$$

It is unnecessary to deal with the first case, since $\Gamma = 0$ is excluded from the non-degenerate case. As for $\Gamma_4 = -p - \Gamma_1 + \Gamma_2$, the other necessary condition of the partial collapse (35) is $\Gamma_1\Gamma_4 + \Gamma_2\Gamma_3 = 0$. However, since both terms are negative due to $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3 \geq \Gamma_4$, the conditions (35) are never satisfied at the same time. Consequently, the self-similar partial collapse never occurs for this case.

5 Four-vortex motion for degenerate case

5.1 Trilinear coordinates and the physical region

We deal with the four-vortex motion for the degenerate case $L_4 = 0$, i.e. $(\Gamma_1+\Gamma_2+\Gamma_3)^2 - \Gamma_4^2 = \Gamma(\Gamma - 2\Gamma_4) = 0$. The trilinear coordinates b_1, b_2 and b_3 are introduced by $b_1 = \Gamma_2\Gamma_3 l_{23}^2$, $b_2 = \Gamma_3\Gamma_1 l_{31}^2$ and $b_3 = \Gamma_1\Gamma_2 l_{12}^2$. They satisfy $b_1 + b_2 + b_3 = 0$ owing to $L_4 = 0$. The trilinear representation (b_1, b_2, b_3) defines a point in the planar phase space as shown in Figure 16. Combining these variables with (40)

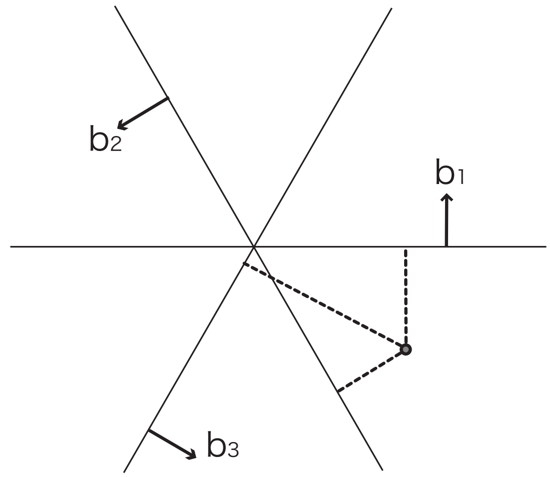

Figure 16: Trilinear coordinates representation for the degenerate case.

gives the physical region

$$2\Gamma_1\Gamma_2 b_1 b_2 + 2\Gamma_2\Gamma_3 b_2 b_3 + 2\Gamma_3\Gamma_1 b_3 b_1 - (\Gamma_1 b_1)^2 - (\Gamma_2 b_2)^2 - (\Gamma_3 b_3)^2 \geq b_1 b_2 b_3. \quad (62)$$

The variables B_1, B_2 and B_3 are also defined by

$$B_1 \equiv \Gamma_1\Gamma_4 l_{14}^2 = b_1 + \Gamma(\Gamma_1 - \Gamma_2 - \Gamma_3 + \Gamma_4), \quad (63)$$
$$B_2 \equiv \Gamma_2\Gamma_4 l_{24}^2 = b_2 + \Gamma(-\Gamma_1 + \Gamma_2 - \Gamma_3 + \Gamma_4), \quad (64)$$
$$B_3 \equiv \Gamma_3\Gamma_4 l_{34}^2 = b_3 + \Gamma(-\Gamma_1 - \Gamma_2 + \Gamma_3 + \Gamma_4). \quad (65)$$

We show the contour lines of the Hamiltonian in the physical region for $\Gamma = 0$ and $\Gamma = 2\Gamma_4$ separately in the following subsections.

5.2 Case I: $\Gamma = 0$

For $\Gamma = 0$, we have $b_i = B_i$ due to (63), (64) and (65). The point of tangency at the boundary is only $b_i = B_i = 0$, which corresponds to the origin in degenerate the trilinear phase space. Since l_{14}^2, l_{24}^2 and l_{34}^2 defined by (63), (64) and (65) are bounded, the condition of the vortex strengths is derived from

$$0 < \frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4} l_{23}^2, \quad \frac{\Gamma_3\Gamma_1}{\Gamma_2\Gamma_4} l_{24}^2, \quad \frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4} l_{12}^2 \leq 4. \quad (66)$$

Since the strength Γ_4 is automatically given by $\Gamma_4 = -\Gamma_1 - \Gamma_2 - \Gamma_3$, we consider the condition of the three vortex strengths for the following three cases: (1) $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 > 0$, (2) $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3$ and (3) $\Gamma_1 > 0 > \Gamma_2 \geq \Gamma_3$. For the first case, since $\frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4} l_{23}^2 < 0$ due to $\Gamma_4 < 0$ and $l_{23}^2 \geq 0$, the condition (66) is never satisfied. For the second case, Γ_4 must be negative, otherwise $\frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4} l_{23}^2 < 0$. For the similar reason, Γ_4 is positive for the third case. Remembering that the

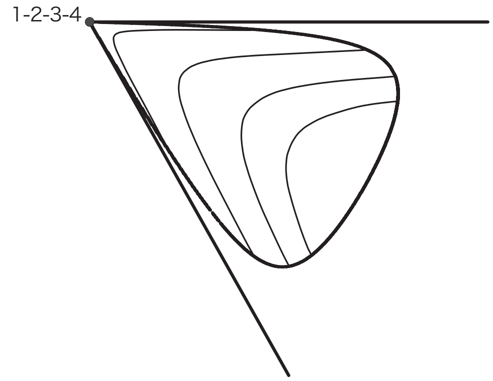

Figure 17: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 4, \Gamma_2 = 2, \Gamma_3 = -3$ and $\Gamma_4 = -3$.

equations (1) and (2) have the symmetry with respect to the the sign changing of the vortex strengths and the time reversal, we have only to treat just one case, $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3$ and $\Gamma_4 < 0$, i.e. $\Gamma_3 > -\Gamma_1 - \Gamma_2$.

Figure 17 shows the contour lines of the Hamiltonian for $\Gamma_1 = 4$, $\Gamma_2 = 2$, $\Gamma_3 = -3$ and $\Gamma_4 = -3$. The origin of the degenerate trilinear phase space, which corresponds to the four-vortex collision, is the point of tangency at the boundary of the physical region. The topological structure of the contour is simple; there is an elliptic fixed point at the boundary.

5.3 Case II: $\Gamma = 2\Gamma_4$

Due to (63), (64) and (65), l_{14}^2, l_{24}^2 and l_{34}^2 are equal to

$$l_{14}^2 = \frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4} l_{23}^2 + 4, \quad l_{24}^2 = \frac{\Gamma_3\Gamma_1}{\Gamma_2\Gamma_4} l_{31}^2 + 4, \quad l_{34}^2 = \frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4} l_{12}^2 + 4. \tag{67}$$

Hence, the conditions for the existence of the fourth vortex point on the sphere are derived from

$$-4 \leq \frac{\Gamma_2\Gamma_3}{\Gamma_1\Gamma_4} l_{23}^2, \quad \frac{\Gamma_3\Gamma_1}{\Gamma_2\Gamma_4} l_{31}^2, \quad \frac{\Gamma_1\Gamma_2}{\Gamma_3\Gamma_4} l_{12}^2 \leq 0. \tag{68}$$

With the same argument as we have in the previous subsection, we have only to deal with the case when $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3$ and $\Gamma_4 = \Gamma_1 + \Gamma_2 + \Gamma_3 > 0$.

The points of tangency for $B_i = 0$ are given as follows. Taking $B_1 = 0$ in (63), we have $b_1 = -4\Gamma_1\Gamma_4$. By solving the equations,

$$b_1 + b_2 + b_3 = 0,$$
$$2\Gamma_1\Gamma_2 b_1 b_2 + 2\Gamma_2\Gamma_3 b_2 b_3 + 2\Gamma_3\Gamma_1 b_3 b_1 - (\Gamma_1 b_1)^2 - (\Gamma_2 b_2)^2 - (\Gamma_3 b_3)^2 - b_1 b_2 b_3 = 0,$$

with it, we have the point of tangency in the degenerate trilinear coordinates,

$$(b_1, b_2, b_3) = \left(-4\Gamma_1\Gamma_4, \frac{4\Gamma_1\Gamma_4(\Gamma_3 + \Gamma_1)}{2\Gamma_1 + \Gamma_2 + \Gamma_3}, \frac{4\Gamma_1\Gamma_4(\Gamma_2 + \Gamma_1)}{2\Gamma_1 + \Gamma_2 + \Gamma_3} \right). \tag{69}$$

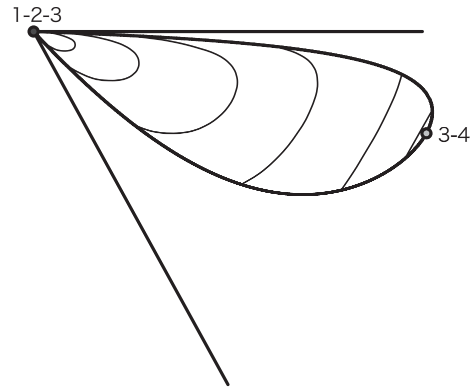

Figure 18: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 4, \Gamma_2 = 2, \Gamma_3 = -1$ and $\Gamma_4 = 5$.

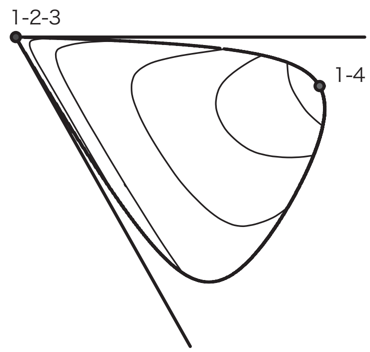

Figure 19: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 4, \Gamma_2 = 2, \Gamma_3 = -3$ and $\Gamma_4 = 3$.

As for $B_2 = 0$ and $B_3 = 0$, the points of tangency are represented by

$$(b_1, b_2, b_3) = \left(\frac{4\Gamma_2\Gamma_4(\Gamma_3 + \Gamma_2)}{\Gamma_1 + 2\Gamma_2 + \Gamma_3}, -4\Gamma_2\Gamma_4, \frac{4\Gamma_2\Gamma_4(\Gamma_1 + \Gamma_2)}{\Gamma_1 + 2\Gamma_2 + \Gamma_3} \right), \quad (70)$$

$$(b_1, b_2, b_3) = \left(\frac{4\Gamma_3\Gamma_4(\Gamma_2 + \Gamma_3)}{\Gamma_1 + \Gamma_2 + 2\Gamma_3}, \frac{4\Gamma_3\Gamma_4(\Gamma_1 + \Gamma_3)}{\Gamma_1 + \Gamma_2 + 2\Gamma_3}, -4\Gamma_3\Gamma_4 \right). \quad (71)$$

The points of tangency $B_1 = 0$ and $B_3 = 0$ coincide with the trilinear axis $b_2 = 0$ when $\Gamma_3 + \Gamma_1 = 0$, and the points $B_1 = 0$ and $B_2 = 0$ agree with $b_3 = 0$ when $\Gamma_3 + \Gamma_2 = 0$. Therefore, the distribution of the points of tangency changes at $\Gamma_3 = -\Gamma_1$ and $\Gamma_3 = -\Gamma_2$. Thus we show the contour plots of the Hamiltonian for the following three cases: $0 > \Gamma_3 > -\Gamma_2$, $-\Gamma_2 > \Gamma_3 > -\Gamma_1$ and $-\Gamma_2 > \Gamma_3 > -\Gamma_1 - \Gamma_2$.

Figure 18 shows the contour lines for $\Gamma_1 = 4$, $\Gamma_2 = 2$, $\Gamma_3 = -1$ and $\Gamma_4 = 5$. The point of tangency $3-4$ is elliptic and the structure of the contours is simple. Second, we show the contour plot of the Hamiltonian in Figure 19 for $\Gamma_1 = 4$, $\Gamma_2 = 2$, $\Gamma_3 = -3$ and $\Gamma_4 = 3$, which gives the similar topological structure as the

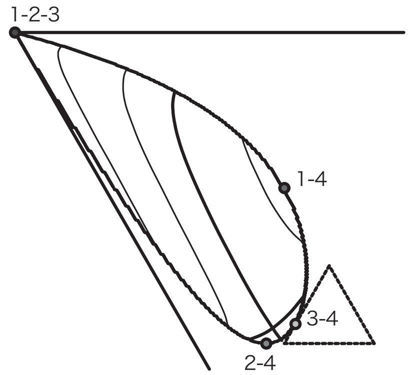

Figure 20: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 4, \Gamma_2 = 2, \Gamma_3 = -5$ and $\Gamma_4 = 1$.

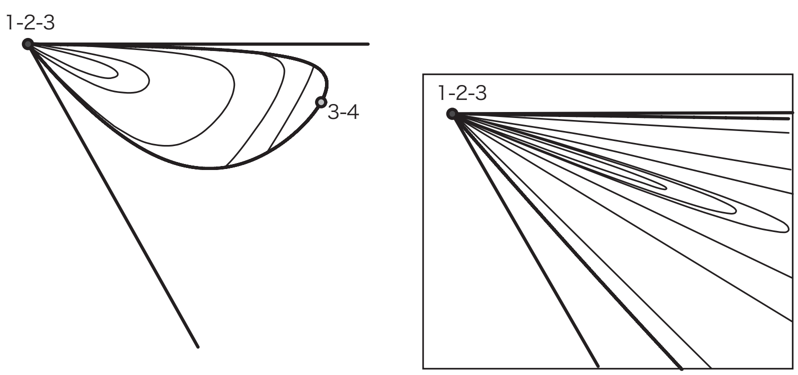

Figure 21: The physical region and the contour lines of the Hamiltonian for $\Gamma_1 = 4, \Gamma_2 = 2, \Gamma_3 = -\frac{4}{3}$ and $\Gamma_4 = \frac{14}{3}$.

first case except for the points of tangency $1-4$. Third, the contour lines of the Hamiltonian for $\Gamma_1 = 4$, $\Gamma_2 = 2$, $\Gamma_3 = -5$ and $\Gamma_4 = 1$ in Figure 20 indicate that there are elliptic points of tangency $1-4$, $2-4$ and $3-4$ and one hyperbolic fixed point with homoclinic connections in the physical region.

In the very last part of the subsection, we consider the special case that allows the partial self-similar collapse of the vortex triple 123 considered in §3. The necessary conditions for the existence of the self-similar collision (26) give $\Gamma_3 = -\frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}$. As an example, Figure 21 shows the contour lines of the Hamiltonian for $\Gamma_1 = 4$, $\Gamma_2 = 2$, $\Gamma_3 = -\frac{4}{3}$ and $\Gamma_4 = \frac{14}{3}$. There is the elliptic point of tangency $3-4$. The contour lines shrink to the origin. In order to see the contour lines in the neighborhood of the origin, the close-up picture is shown in the left of the figure. The contour curves shrink to the origin as the value of the Hamiltonian tends to that at the origin, which suggests no existence of the self-similar collapse of the three vortex points.

6 Summary

We have investigated the integrable motion of the four vortex points on the sphere, when the moment of vorticity is zero. The article completes the study of the integrable vortex-problem on the sphere together with the integrable three-vortex problem[4, 5, 11]. As in the four-vortex problem in the unbounded plane[2], we have successfully applied the reduction method to a three-vortex problem proposed by Rott[10]. The evolution is observed by plotting the level curves of the Hamiltonian in the reduced trilinear coordinates.

In the meantime, the zero total vorticity condition is unnecessary to guarantee the integrability. However, the vortex strengths are strongly restricted by the fact that the distance between two vortex points is bounded.

We have considered whether or not the self-similar collapse is possible. No existence of the four-vortex collapse has been proved mathematically. On the other hand, we found no numerical evidence showing the partial collapse of three vortex points and of the pairs of two vortex points occur.

Acknowledgment

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A Deduction of the upper bound for L_i.

The possible regions for the vortex strengths are derived from the lower bounds $\min L_i < L_i$. So we need to show that they are the sufficient conditions that ensure the upper bounds $L_i < \max L_i$. In this appendix, we give a detailed proof of this fact.

A.1 Case I: $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq \Gamma_4 > 0$

The possible region for $\Gamma_1 - \Gamma_2 \leq \Gamma_2$ in Figure 2 is expressed in the following two ways:

$$0 < \Gamma_4 \leq \frac{1}{2}(\Gamma_1 - \Gamma_2), \qquad \Gamma_1 - \Gamma_2 - \Gamma_4 < \Gamma_3 \leq \Gamma_2, \tag{72}$$

$$\frac{1}{2}(\Gamma_1 - \Gamma_2) < \Gamma_4 \leq \Gamma_2, \qquad \Gamma_4 \leq \Gamma_3 \leq \Gamma_2, \tag{73}$$

or

$$\frac{1}{2}(\Gamma_1 - \Gamma_2) < \Gamma_3 \leq \Gamma_1 - \Gamma_2, \qquad \Gamma_1 - \Gamma_2 - \Gamma_3 < \Gamma_4 \leq \Gamma_3, \tag{74}$$

$$\Gamma_1 - \Gamma_2 < \Gamma_3 \leq \Gamma_2, \qquad 0 < \Gamma_4 \leq \Gamma_3. \tag{75}$$

Then we need to show that if the vortex strengths are in the possible region, the invariants L_i satisfy the following upper bounds:

$$L_1 = (\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1^2 < \max L_1 = 4\Gamma_2\Gamma_3 + 4\Gamma_2\Gamma_4 + 4\Gamma_3\Gamma_4, \tag{76}$$

$$L_2 = (\Gamma_1 + \Gamma_3 + \Gamma_4)^2 - \Gamma_2^2 < \max L_2 = 4\Gamma_1\Gamma_3 + 4\Gamma_1\Gamma_4 + 4\Gamma_3\Gamma_4, \tag{77}$$

$$L_3 = (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - \Gamma_3^2 < \max L_3 = 4\Gamma_1\Gamma_2 + 4\Gamma_1\Gamma_4 + 4\Gamma_2\Gamma_4, \tag{78}$$

$$L_4 = (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4^2 < \max L_4 = 4\Gamma_1\Gamma_2 + 4\Gamma_2\Gamma_3 + 4\Gamma_3\Gamma_1. \tag{79}$$

First we show that (72) implies (76). Comparing $4\Gamma_2\Gamma_4 + \Gamma_1^2$ with the square of $2\Gamma_2 + 2\Gamma_4 - \Gamma_1$, we have

$$(4\Gamma_2\Gamma_4 + \Gamma_1^2) - (2\Gamma_2 + 2\Gamma_4 - \Gamma_1)^2 = 4\Gamma_4(\Gamma_1 - \Gamma_2 - \Gamma_4) + 4\Gamma_2(\Gamma_1 - \Gamma_2) \geq 0,$$

since $\Gamma_1 \geq \Gamma_2$ and $\Gamma_1 - \Gamma_2 - \Gamma_4 \geq \frac{1}{2}(\Gamma_1 - \Gamma_2) \geq 0$ due to (72). Hence, it follows from $\Gamma_1^2 + 4\Gamma_2\Gamma_4 > 0$ and $2\Gamma_2 + 2\Gamma_4 - \Gamma_1 \geq 2\Gamma_2 - \Gamma_1 > 0$ that

$$\sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_4} \geq 2\Gamma_2 + 2\Gamma_4 - \Gamma_1 > 0.$$

Thus we have

$$\Gamma_2 + \Gamma_4 - \sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_4} \leq \Gamma_1 - \Gamma_2 - \Gamma_4 < \Gamma_3,$$

due to (72), which results in

$$\sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_4} > \Gamma_2 - \Gamma_3 + \Gamma_4 > 0. \tag{80}$$

On the other hand, when the vortex strengths satisfy (73), since

$$4\Gamma_2\Gamma_4 + \Gamma_1^2 - \Gamma_2^2 > 2\Gamma_2(\Gamma_1 - \Gamma_2) + \Gamma_1^2 - \Gamma_2^2 = (\Gamma_1 + 3\Gamma_2)(\Gamma_1 - \Gamma_2) \geq 0,$$

we have

$$\Gamma_2 + \Gamma_4 - \sqrt{4\Gamma_2\Gamma_4 + \Gamma_1^2} < \Gamma_4 \leq \Gamma_3.$$

Hence, we have (80) again. When both sides of (80) are squared, we obtain

$$(\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1^2 < 4\Gamma_2\Gamma_3 + 4\Gamma_2\Gamma_4 + 4\Gamma_3\Gamma_4.$$

Second, we have the following the comparisons

$$4\Gamma_1\Gamma_4 + \Gamma_2^2 - (2\Gamma_4 + \Gamma_2)^2 = 4\Gamma_4(\Gamma_1 - \Gamma_2 - \Gamma_4) \geq 0,$$

for (72) and

$$4\Gamma_1\Gamma_4 + \Gamma_2^2 - \Gamma_1^2 > 2\Gamma_1(\Gamma_1 - \Gamma_2) + \Gamma_2^2 - \Gamma_1^2 = (\Gamma_1 - \Gamma_2)^2 \geq 0,$$

for (73). Each of the comparisons leads to

$$\Gamma_4 + \Gamma_1 - \sqrt{4\Gamma_1\Gamma_4 + \Gamma_2^2} \leq \Gamma_1 - \Gamma_2 - \Gamma_4 < \Gamma_3,$$

and

$$\Gamma_1 + \Gamma_4 - \sqrt{4\Gamma_1\Gamma_4 + \Gamma_2^2} < \Gamma_4 \leq \Gamma_3.$$

Hence, we have

$$\sqrt{4\Gamma_1\Gamma_4 + \Gamma_2^2} > \Gamma_1 - \Gamma_3 + \Gamma_4 > 0.$$

Taking square of both sides, we have (77).

Third, we use the expressions (74) and (75) to prove (78). Since (74) implies

$$4\Gamma_1\Gamma_2 + \Gamma_3^2 - (2\Gamma_2 + \Gamma_3)^2 = 4\Gamma_2(\Gamma_1 - \Gamma_2 - \Gamma_3) \geq 0,$$

and (75) shows

$$4\Gamma_1\Gamma_2 + \Gamma_3^2 - (\Gamma_1 + \Gamma_2)^2 = (\Gamma_3 - \Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_1 - \Gamma_2) > 0,$$

we have

$$\Gamma_1 + \Gamma_2 - \sqrt{4\Gamma_1\Gamma_2 + \Gamma_3^2} \leq \Gamma_1 - \Gamma_2 - \Gamma_3 < \Gamma_4,$$

for (74) and

$$\Gamma_1 + \Gamma_2 - \sqrt{\Gamma_3^2 + 4\Gamma_1\Gamma_2} < 0 < \Gamma_4,$$

for (75). Hence, we acquire $\sqrt{4\Gamma_1\Gamma_2+\Gamma_3^2} > \Gamma_1+\Gamma_2-\Gamma_4 > 0$, from which the upper bound of L_3 is obtained.

Finally, regarding (79), we have

$$4\Gamma_1\Gamma_2+\Gamma_4^2-(\Gamma_4+2\Gamma_2)^2 = 4\Gamma_2(\Gamma_1-\Gamma_2-\Gamma_4) \geq 0,$$

due to (72) and

$$\begin{aligned} 4\Gamma_1\Gamma_2+\Gamma_4^2-(\Gamma_1+\Gamma_2-\Gamma_4)^2 &= 2\Gamma_1\Gamma_2+2\Gamma_1\Gamma_4+2\Gamma_2\Gamma_4-\Gamma_1^2-\Gamma_2^2 \\ &> 2\Gamma_1\Gamma_2+(\Gamma_1+\Gamma_2)(\Gamma_1-\Gamma_2)-\Gamma_1^2-\Gamma_2^2 \\ &= 2\Gamma_2(\Gamma_1-\Gamma_2) \geq 0, \end{aligned}$$

due to (73), from which we obtain

$$\Gamma_1+\Gamma_2-\sqrt{\Gamma_4^2+4\Gamma_1\Gamma_2} \leq \Gamma_1-\Gamma_2-\Gamma_4 < \Gamma_3,$$

for (72) and

$$\Gamma_1+\Gamma_2-\sqrt{\Gamma_4^2+4\Gamma_1\Gamma_2} < \Gamma_4 \leq \Gamma_3,$$

for (73). Therefore, we have the inequality $0 < \Gamma_1+\Gamma_2-\Gamma_3 < \sqrt{\Gamma_4^2+4\Gamma_1\Gamma_2}$, which gives (79).

As for the possible region for $\Gamma_2 < \Gamma_1-\Gamma_2 \leq 2\Gamma_2$, it is represented by

$$0 < \Gamma_1-2\Gamma_2 \leq \Gamma_4 \leq \frac{1}{2}(\Gamma_1-\Gamma_2), \qquad \Gamma_1-\Gamma_2-\Gamma_4 < \Gamma_3 \leq \Gamma_2, \tag{81}$$

$$\frac{1}{2}(\Gamma_1-\Gamma_2) < \Gamma_4 \leq \Gamma_2 \qquad \Gamma_4 \leq \Gamma_3 \leq \Gamma_2, \tag{82}$$

or

$$\frac{1}{2}(\Gamma_1-\Gamma_2) < \Gamma_3 \leq \Gamma_2 \leq \Gamma_1-\Gamma_2, \quad \Gamma_1-\Gamma_2-\Gamma_3 < \Gamma_4 \leq \Gamma_3. \tag{83}$$

Since they are equivalent to (72), (73) and (74) respectively, the proof is the same.

A.2 Case II : $\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 > 0 > \Gamma_4$

The possible region for $\Gamma_1-\Gamma_2 \leq \Gamma_2$ in Figure 5 is described by

$$0 < \Gamma_3 \leq \Gamma_1-\Gamma_2, \qquad -\Gamma_3-\Gamma_1-\Gamma_2 < \Gamma_4 < \Gamma_3-\Gamma_1+\Gamma_2, \tag{84}$$

$$\Gamma_1-\Gamma_2 < \Gamma_3 \leq \Gamma_2, \qquad -\Gamma_3-\Gamma_1-\Gamma_2 < \Gamma_4 < 0, \tag{85}$$

or

$$-\Gamma_1+\Gamma_2 \leq \Gamma_4 < 0, \qquad \Gamma_1-\Gamma_2+\Gamma_4 < \Gamma_3 \leq \Gamma_2, \tag{86}$$

$$-\Gamma_1-\Gamma_2 \leq \Gamma_4 < -\Gamma_1+\Gamma_2, \qquad 0 < \Gamma_3 \leq \Gamma_2, \tag{87}$$

$$-\Gamma_1-2\Gamma_2 < \Gamma_4 < -\Gamma_1-\Gamma_2, \qquad -\Gamma_1-\Gamma_2-\Gamma_4 < \Gamma_3 \leq \Gamma_2, \tag{88}$$

from which we show the following upper bounds for L_i:

$$L_1 = (\Gamma_2 + \Gamma_3 + \Gamma_4)^2 - \Gamma_1^2 < 4\Gamma_2\Gamma_3, \tag{89}$$
$$L_2 = (\Gamma_1 + \Gamma_3 + \Gamma_4)^2 - \Gamma_2^2 < 4\Gamma_1\Gamma_3, \tag{90}$$
$$L_3 = (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - \Gamma_3^2 < 4\Gamma_1\Gamma_2, \tag{91}$$
$$L_4 = (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - \Gamma_4^2 < 4\Gamma_1\Gamma_2 + 4\Gamma_2\Gamma_3 + 4\Gamma_3\Gamma_1. \tag{92}$$

Since $-\Gamma_3 - \Gamma_1 - \Gamma_2 < \Gamma_4$ holds for both (84) and (85), we have

$$-\Gamma_2 - \Gamma_3 - \sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_3} < -\Gamma_1 - \Gamma_2 - \Gamma_3 < \Gamma_4,$$

and thus $-\sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_3} < \Gamma_2 + \Gamma_3 + \Gamma_4$. On the other hand, the comparison

$$\Gamma_1^2 + 4\Gamma_2\Gamma_3 - (\Gamma_2 + \Gamma_3)^2 = (\Gamma_1 - \Gamma_2 + \Gamma_3)(\Gamma_1 + \Gamma_2 - \Gamma_3) > 0,$$

and $\Gamma_4 < 0$ reveal that

$$\sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_3} > \Gamma_2 + \Gamma_3 > \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Therefore, we obtain

$$-\sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_3} < \Gamma_2 + \Gamma_3 + \Gamma_4 < \sqrt{\Gamma_1^2 + 4\Gamma_2\Gamma_3},$$

which is followed by (89).

In order to prove (90), we have to show

$$-\sqrt{\Gamma_2^2 + 4\Gamma_1\Gamma_3} < \Gamma_1 + \Gamma_3 + \Gamma_4 < \sqrt{\Gamma_2^2 + 4\Gamma_1\Gamma_3}.$$

The upper bound is derived as follows. Due to (84), the comparison

$$(\Gamma_2^2 + 4\Gamma_1\Gamma_3) - (2\Gamma_3 + \Gamma_2)^2 = 4\Gamma_3(\Gamma_1 - \Gamma_2 - \Gamma_3) \geq 0,$$

implies

$$\Gamma_1 + \Gamma_3 - \sqrt{\Gamma_2^2 + 4\Gamma_1\Gamma_3} \leq \Gamma_1 - \Gamma_2 - \Gamma_3 < -\Gamma_4.$$

As for (85), we see from

$$(\Gamma_2^2 + 4\Gamma_1\Gamma_3) - (\Gamma_1 + \Gamma_3)^2 = (\Gamma_2 - \Gamma_1 + \Gamma_3)(\Gamma_2 + \Gamma_1 - \Gamma_3) > 0.$$

that $\Gamma_1 + \Gamma_3 - \sqrt{\Gamma_2^2 + 4\Gamma_1\Gamma_3} < 0 < -\Gamma_4$. Hence we have the upper bound. The lower bound is simply obtained by

$$-\Gamma_1 - \Gamma_3 - \sqrt{\Gamma_2^2 + 4\Gamma_1\Gamma_3} < -\Gamma_1 - \Gamma_3 - \Gamma_2 < \Gamma_4,$$

for both (84) and (85).

With the similar step, (91) is derived by showing

$$-\sqrt{\Gamma_3^2 + 4\Gamma_1\Gamma_3} < \Gamma_1 + \Gamma_2 + \Gamma_4 < \sqrt{\Gamma_3^2 + 4\Gamma_1\Gamma_2}.$$

The upper bound comes from

$$(\Gamma_3^2 + 4\Gamma_1\Gamma_2) - (\Gamma_3 + 2\Gamma_2)^2 = 4\Gamma_2(\Gamma_1 - \Gamma_2 - \Gamma_3) \geq 0,$$

for (84) and

$$(\Gamma_3^2 + 4\Gamma_1\Gamma_2) - (\Gamma_1 + \Gamma_2)^2 = (\Gamma_3 - \Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_1 - \Gamma_2) > 0,$$

for (85). The lower bound is acquired due to

$$-\Gamma_1 - \Gamma_2 - \sqrt{\Gamma_3^2 + 4\Gamma_1\Gamma_2} < -\Gamma_1 - \Gamma_2 - \Gamma_3 < \Gamma_4.$$

The proof of (92) is accomplished by showing the inequality,

$$0 < \Gamma_1 + \Gamma_2 - \Gamma_3 < \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2},$$

from (86), (87) and (88). When (86) is satisfied, the comparison

$$(\Gamma_4^2 + 4\Gamma_1\Gamma_2) - (2\Gamma_2 - \Gamma_4)^2 = 4\Gamma_2(\Gamma_1 - \Gamma_2 + \Gamma_4) \geq 0,$$

leads to

$$\Gamma_1 + \Gamma_2 - \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2} \leq \Gamma_1 - \Gamma_2 + \Gamma_4 < \Gamma_3.$$

For (87), since

$$(\Gamma_1 + \Gamma_2)^2 - (\Gamma_4^2 + 4\Gamma_1\Gamma_2) = (\Gamma_1 - \Gamma_2 + \Gamma_4)(\Gamma_1 - \Gamma_2 - \Gamma_4) < 0,$$

holds, we also have

$$\Gamma_1 + \Gamma_2 - \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2} < 0 < \Gamma_3.$$

In terms of (88), it follows from

$$\begin{aligned} \Gamma_4^2 + 4\Gamma_1\Gamma_2 - (2\Gamma_1 + 2\Gamma_2 + \Gamma_4)^2 &= -4\Gamma_1\Gamma_2 - 4\Gamma_1^2 - 4\Gamma_2^2 - 4\Gamma_1\Gamma_4 - 4\Gamma_2\Gamma_4 \\ &\geq -4\Gamma_1\Gamma_2 - 4\Gamma_1^2 - 4\Gamma_2^2 + 4(\Gamma_1 + \Gamma_2)^2 = 4\Gamma_1\Gamma_2 > 0, \end{aligned}$$

that we acquire

$$\Gamma_1 + \Gamma_2 - \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2} < -\Gamma_1 - \Gamma_2 - \Gamma_4 < \Gamma_3.$$

Hence, we have proved (92). With the similar algebraic procedures, we also show the same upper bounds for $\Gamma_2 < \Gamma_1 - \Gamma_2 \leq 2\Gamma_2$.

A.3 Case III : $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3 \geq \Gamma_4$

As in §4.5, we use the notation $p = -\Gamma_3 > 0$. The possible region in Figure 11 is expressed by

$$0 < p \leq \frac{1}{2}(\Gamma_1 - \Gamma_2), \qquad -p - \Gamma_1 - \Gamma_2 < \Gamma_4 < p - \Gamma_1 + \Gamma_2, \tag{93}$$

$$\frac{1}{2}(\Gamma_1 - \Gamma_2) < p, \qquad -p - \Gamma_1 - \Gamma_2 < \Gamma_4 \leq -p, \tag{94}$$

or equivalently

$$\frac{1}{2}(-\Gamma_1 + \Gamma_2) > \Gamma_4 \geq -\Gamma_1 + \Gamma_2, \qquad \Gamma_4 + \Gamma_1 - \Gamma_2 < p \leq -\Gamma_4, \tag{95}$$

$$-\Gamma_1 + \Gamma_2 > \Gamma_4 \geq -\Gamma_1 - \Gamma_2, \qquad 0 < p \leq -\Gamma_4, \tag{96}$$

$$-\Gamma_1 - \Gamma_2 > \Gamma_4, \qquad -\Gamma_1 - \Gamma_2 - \Gamma_4 < p \leq -\Gamma_4. \tag{97}$$

The upper bounds for the invariants L_i are given as follows.

$$L_1 = (\Gamma_2 - p + \Gamma_4)^2 - \Gamma_1^2 \quad < \quad -4p\Gamma_4, \tag{98}$$

$$L_2 = (\Gamma_1 - p + \Gamma_4)^2 - \Gamma_2^2 \quad < \quad -4p\Gamma_4, \tag{99}$$

$$L_3 = (\Gamma_1 + \Gamma_2 + \Gamma_4)^2 - p^2 \quad < \quad 4\Gamma_1\Gamma_2, \tag{100}$$

$$L_4 = (\Gamma_1 + \Gamma_2 - p)^2 - \Gamma_4^2 \quad < \quad 4\Gamma_1\Gamma_2. \tag{101}$$

First, it is easy to see from

$$-p - \Gamma_2 + \sqrt{\Gamma_1^2 + 4p\Gamma_2} > -p - \Gamma_2 + \Gamma_1 \geq -p \geq \Gamma_4,$$

and

$$-p - \Gamma_2 - \sqrt{\Gamma_1^2 + 4p\Gamma_2} \leq -p - \Gamma_2 - \Gamma_1 < \Gamma_4,$$

for both (93) and (94) that we have the inequality

$$-\sqrt{\Gamma_1^2 + 4p\Gamma_2} \leq p + \Gamma_2 + \Gamma_4 \leq \sqrt{\Gamma_1^2 + 4p\Gamma_2},$$

Therefore, the upper bound (98) is derived.

Second, the comparison

$$\Gamma_2^2 + 4p\Gamma_1 - (2p + \Gamma_2)^2 = 4p(\Gamma_1 - \Gamma_2 - p) \geq 0,$$

is satisfied due to (93). Hence, we have the inequality

$$-p - \Gamma_1 + \sqrt{\Gamma_2^2 + 4p\Gamma_1} \geq p + \Gamma_2 - \Gamma_1 > \Gamma_4,$$

and thus the upper bound,

$$\Gamma_1 + p + \Gamma_4 < \sqrt{\Gamma_2^2 + 4p\Gamma_1}. \tag{102}$$

It follows from (94) and

$$\Gamma_2^2 + 4p\Gamma_1 - \Gamma_1^2 > \Gamma_2^2 + 2(\Gamma_1 - \Gamma_2)\Gamma_1 - \Gamma_1^2 = (\Gamma_1 - \Gamma_2)^2 \geq 0,$$

that we have

$$-p - \Gamma_1 + \sqrt{\Gamma_2^2 + 4p\Gamma_1} > -p \geq \Gamma_4,$$

which also yields (102). The lower bound $\Gamma_1 + p + \Gamma_4 \geq -\sqrt{\Gamma_2^2 + 4p\Gamma_1}$ is derived from

$$-p - \Gamma_1 - \sqrt{\Gamma_2^2 + 4p\Gamma_1} < -p - \Gamma_1 - \Gamma_2 < \Gamma_4.$$

for both (93) and (94). Hence, we prove $(\Gamma_1 + p + \Gamma_4)^2 < \Gamma_2^2 + 4p\Gamma_1$, and thus (99).

To show the bound (100), we check the inequality

$$-\sqrt{p^2 + 4\Gamma_1\Gamma_2} < \Gamma_1 + \Gamma_2 + \Gamma_4 < \sqrt{p^2 + 4\Gamma_1\Gamma_2}.$$

The lower bound is easy to see from

$$-\Gamma_1 - \Gamma_2 - \sqrt{p^2 + 4\Gamma_1\Gamma_2} < -\Gamma_1 - \Gamma_2 - p < \Gamma_4.$$

On the other hand, the upper bound for (93)

$$-\Gamma_1 - \Gamma_2 + \sqrt{p^2 + 4\Gamma_1\Gamma_2} \geq p + \Gamma_2 - \Gamma_1 > \Gamma_4,$$

comes from the comparison

$$(p^2 + 4\Gamma_1\Gamma_2) - (p + 2\Gamma_2)^2 = 4\Gamma_2(\Gamma_1 - p - \Gamma_2) \geq 2\Gamma_2(\Gamma_1 - \Gamma_2) \geq 0.$$

Regarding the upper bound for (94), we treat the two cases, $\Gamma_1 + \Gamma_2 \geq p > \frac{1}{2}(\Gamma_1 - \Gamma_2)$ and $p > \Gamma_1 + \Gamma_2$, separately. For the former case, since

$$\begin{aligned} p^2 + 4\Gamma_1\Gamma_2 - (\Gamma_1 + \Gamma_2 - p)^2 &= 2p(\Gamma_1 + \Gamma_2) - (\Gamma_1 - \Gamma_2)^2 \\ &> (\Gamma_1 - \Gamma_2)(\Gamma_1 + \Gamma_2) - (\Gamma_1 - \Gamma_2)^2 = 2\Gamma_2(\Gamma_1 - \Gamma_2) \geq 0, \end{aligned}$$

we obtain $-\Gamma_1 - \Gamma_2 + \sqrt{p^2 + 4\Gamma_1\Gamma_2} > -p \geq \Gamma_4$ and thus

$$\Gamma_1 + \Gamma_2 + \Gamma_4 < \sqrt{p^2 + 4\Gamma_1\Gamma_2}. \tag{103}$$

For the latter case $p > \Gamma_1 + \Gamma_2$, since

$$p^2 + 4\Gamma_1\Gamma_2 - (\Gamma_1 + \Gamma_2)^2 = (p - \Gamma_1 + \Gamma_2)(p + \Gamma_1 - \Gamma_2) > 0,$$

we have

$$-\Gamma_1 - \Gamma_2 + \sqrt{p^2 + 4\Gamma_1\Gamma_2} > 0 > \Gamma_4,$$

and thus (103).

Finally, we make use of (95), (96) and (97) to obtain (101). When the vortex strengths satisfy (95), if follows from

$$\Gamma_4^2 + 4\Gamma_1\Gamma_2 - (2\Gamma_2 - \Gamma_4)^2 = 4\Gamma_2(\Gamma_1 - \Gamma_2 + \Gamma_4) \geq 0,$$

that we have

$$\Gamma_1 + \Gamma_2 - \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2} \leq \Gamma_4 + \Gamma_1 - \Gamma_2 < p,$$

and therefore

$$\Gamma_1 + \Gamma_2 - p < \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2}.$$

As for (96) and (97), the comparison

$$\Gamma_4^2 + 4\Gamma_1\Gamma_2 - (\Gamma_1 + \Gamma_2)^2 = (\Gamma_1 - \Gamma_2 - \Gamma_4)(-\Gamma_1 + \Gamma_2 - \Gamma_4) > 0,$$

reveals

$$\Gamma_1 + \Gamma_2 - \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2} \leq 0 < p,$$

Moreover, the inequality

$$\Gamma_1 + \Gamma_2 + \sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2} \geq \Gamma_1 + \Gamma_2 + \sqrt{p^2 + 4\Gamma_1\Gamma_2} > p,$$

holds due to $\Gamma_4 \leq -p < 0$, from which we obtain the lower bound, $\Gamma_1 + \Gamma_2 - p > -\sqrt{\Gamma_4^2 + 4\Gamma_1\Gamma_2}$. Thus the proof of (101) is finished.

References

[1] H. Aref, *Motion of three vortices*, Phys. Fluids, 22 (1979) pp.393–400.

[2] H. Aref and M. S. Stremler, *Four-vortex motion with zero total circulation and impulse*, Phys. Fluids, 11 (1999), pp.3704–3715.

[3] B. Eckhardt, *Integrable four vortex motion*, Phys. Fluids, 31(1988) 2796–2801.

[4] R. Kidambi and P.K. Newton, *Motion of three point vortices on a sphere*, Physica D, 116 (1998), pp.95–134.

[5] R. Kidambi and P.K. Newton, *Collision of three vortices on a sphere*, Nouvo Cimento, 22C (1999), pp. 779–791.

[6] Y. Kimura and H. Okamoto, *Vortex motion on a sphere*, J. Phys. Soc. Japan, 56 (1987), pp. 4203–4206.

[7] Y. Kimura, *Chaos and collapse of point vortices*, Dyn. Res. Research, 3 (1988), pp. 98–104.

[8] P.K. Newton, *The N-vortex problem, Analytical techniques*, Springer-Verlag, New York, 2001.

[9] N. Rott, *Three-vortex motion with zero total circulation*, J. Appl. Math. Phys. (ZAMP), 40 (1989), 473–494.

[10] N. Rott, *Constrained three- and four-vortex problems*, Phys. Fluids A, 2 (1990) 1477–1480.

[11] T. Sakajo, *The motion of three point vortices on a sphere*, Japan J. Indust. Appl. Math., 16 (1999), pp. 321–347.