

Zero-dimensional Gorenstein algebras with the action of the symmetric group S_k

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Contents

1	Introduction	2
2	Preliminaries	5
2.1	A list of notational conventions	5
2.2	The equi-degree monomial complete intersection as a tensor space	5
2.3	Young tableaux and Specht polynomials	6
2.4	Nilpotent matrices and Jordan bases	7

2.5	The strong Lefschetz property	8
3	$A(n, 2)$ or the two fold tensor of K^n	10
4	$A(2, k)$ or the Boolean algebra	12
5	Application to the theory of Gorenstein rings	16
6	The Hilbert function of the ring of invariants of $A(n, k)$	18
7	Appendix	20

1 Introduction

We start studying zero dimensional Gorenstein algebras over a field K on which the symmetric group S_k acts by the permutation of the variables. For this there are several reasons.

1. Following is an example of a non trivial Gorenstein algebra due to F. S. Macaulay. Observe that it is defined by an ideal fixed by the group S_3 . This can be found in ([1] p.29) and ([14] p.81).

$$K[x, y, z]/(xy, yz, zx, x^2 - y^2, x^2 - z^2). \tag{1}$$

If $K = \mathbb{C}$, this is the only isomorphism type of a Gorenstein algebra with the Hilbert function (1 3 1).

For $K = \mathbb{R}$, there is another isomorphism type:

$$K[x, y, z]/(x^2, y^2, z^2, x(y - z), y(z - x)) \tag{2}$$

Early 70's Sakuma and Okuyama discovered some more such examples:

$$K[x, y, z]/(x^3 - y^3, y^3 - z^3, xy, yz, zx)$$

$$K[x, y, z]/(xy - xz, yx - yz, x^3, y^3, z^3)$$

Surprisingly these are also defined by ideals that are fixed by the symmetric group. These examples suggest at least that Gorenstein algebras with the S_k -action are natural.

2. Gorenstein algebras with the S_k -action have been studied in the representation theory for a long time. For example, Terasoma-Yamada [17] constructed a basis of an irreducible decomposition for the coinvariant algebra of S_k ,

$$K[x_1, \dots, x_k]/(e_1, \dots, e_k), \quad (3)$$

where e_i is the elementary symmetric polynomial of degree i . For related results see [6], [13], [15], [16], [18]. Coinvariant algebras are also treated in [2] and [9].

3. Consider the equi-degree monomial complete intersection:

$$A = K[x_1, \dots, x_k]/(x_1^n, \dots, x_k^n) \quad (4)$$

Obviously this is a zero dimensional Gorenstein algebra with the action of S_k . As a vector space this is isomorphic to the tensor space

$$(K^n)^{\otimes k}.$$

The Schur-Weyl duality deals with the decomposition of the space $(K^n)^{\otimes k}$ as $(GL(n) \times S_k)$ -modules. Thus all results the Schur-Weyl duality implies can be applied to the algebra A . Conversely with the identification $A \cong (K^n)^{\otimes n}$, it is possible to look at the Schur-Weyl duality from the viewpoint of the theory of commutative algebra.

4. Suppose that $f \in K[x_1, \dots, x_k]$ is a (semi-)invariant of S_k . Then obviously

$$A = K[x_1, \dots, x_k]/(x_1^n, \dots, x_k^n): f$$

is a zero dimensional Gorenstein algebra with the S_k -action. It is not difficult to show that a zero dimensional Gorenstein algebra with the S_k -action arises in this way. (cf. [19] Lemma 4.)

Following are some such examples. Here Δ is the difference product of the variables, and p_i is the power sum and h_i the complete symmetric polynomial of degree i .

$$(a) (x^2, y^2, z^2): x + y + z = (x^2, y^2, z^2, xy - xz, xy - yz)$$

$$(b) (x^3, y^3, z^3): x^3y^3 + y^3z^3 + z^3x^3 = (xy, yz, zx, x^3 - y^3, y^3 - x^3)$$

$$(c) (x^{r+2}, y^{r+2}, z^{r+2}): \Delta = (h_r, h_{r+1}, h_{r+2}), \quad r = 1, 2, \dots$$

$$(d) (x^{3r}, y^{3r}, z^{3r}): \Delta(h_{2(r-1)})^3 = (p_r, p_{r+1}, p_{r+2}), \quad r = 1, 2, \dots$$

One might be interested in decomposing such an algebra into irreducible S_k -modules. It seems to be an interesting question to ask what is the Hilbert function of the graded vector space

$$Y^\lambda(A)$$

where Y^λ is a Young symmetrizer corresponding to a partition λ .

The purpose of this paper is to show that the strong Lefschetz property of a Gorenstein algebra can be used very efficiently to deal with these problems mentioned above.

Suppose that A is a Gorenstein ring with a strong Lefschetz element $l \in A$. Put $L = \times l \in \text{End}(A)$. Then it is possible to construct a degree -1 map $D \in \text{End}(A)$ such that $\{L, D, H\}$ is an $\mathfrak{sl}(2)$ -triple, where $H = [LD]$. This means that the eigenspaces of H are precisely the homogeneous parts of A . Suppose moreover that A has the action of S_k and that the invariant linear form $l = x_1 + x_2 + \cdots + x_n$ is a strong Lefschetz element. Then, as is obvious, the vector space $\text{Ker}L$ is fixed, hence the spaces $\text{Ker}L \cap A_i$ are fixed by the action of S_k . Thus in such a case an irreducible decomposition of A can be constructed by first decomposing the kernel of the multiplication map

$$\times(x_1 + \cdots + x_k): A \rightarrow A \quad (5)$$

into irreducible spaces and then applying the map D repeatedly to the constituents of the decomposition of $\text{Ker}L$. (Or equivalently, first decompose $\text{Ker}D$ and then apply L .)

A linear map $D \in \text{End}(A)$ for a given nilpotent endomorphism $L \in \text{End}(A)$ such that $\{L, D, [LD]\}$ is an $\mathfrak{sl}(2)$ -triple can be constructed from a Jordan basis of the vector space A for L . (See [20].) In this sense Proposition 7 is the key proposition of this paper.

An obvious example of an Artinian Gorenstein ring in which $x_1 + \cdots + x_n$ is a strong Lefschetz element is the equi-degree monomial complete intersection

$$A(n, k): = K[x_1, \cdots, x_k]/(x_1^n, \cdots, x_k^n)$$

mentioned above. In this paper we treat the two extremal cases $A(n, 2)$ and $A(2, k)$. If $k = 2$, we have that $\text{Ker}L$ decomposes into one dimensional homogeneous parts. Hence the irreducibility of $(\text{Ker}L)_i$ is trivial. In the case $n = 2$, the irreducibility of $(\text{Ker}L)_i$ is immediate if we take for granted the basic facts of the representation theory of the symmetric group. These results are stated in Theorem 10 and in Theorem 19 respectively.

One other purpose of this paper is to apply Theorem 19 to determine the minimal number of generators of the ideal $(x_1^2, \cdots, x_k^2): (x_1 + \cdots + x_k)$ in the polynomial ring. By analyzing Specht polynomials involving only square-free monomials, it is possible to determine a minimal generating set of the ideal. We do this in Section 5. Theorem 21 may be regarded as a generalization of Macaulay's example mentioned earlier.

It is well known that the group algebra $K[S_k]$ is semi-simple (assuming $\text{char } K = 0$) and the primitive idempotents are precisely the Young symmetrizers. Thus given an S_k -module, say A , one obtains an irreducible decomposition of A by applying various Young symmetrizers to it. With this principle it is possible to construct an irreducible decomposition of the tensor space $A = (K^n)^{\otimes k}$. We exhibit the decomposition in Appendix. We would like to note that the bases treated in this paper are different from those obtained in this way.

In section 6 we exhibit the Hilbert function of the module $Y^\lambda(A)$ for arbitrary n, k and $\lambda \vdash n$. Note that if λ is trivial, then the Young symmetrizer Y^λ is the map

$$Y^\lambda: A \rightarrow A$$

defined by

$$Y^\lambda a = \frac{1}{k!} \sum_{\sigma \in S_k} a^\sigma.$$

Hence $Y^\lambda(A) = A^{S_k}$ is the ring of invariants.

As one will see this shows that to consider the ring structure in the vector space $(K^n)^{\otimes k}$ has much advantage than to treat the tensor space as it is.

2 Preliminaries

2.1 A list of notational conventions

Here is a list of notation which we are going to fix throughout the paper. Details are following the list.

- $\lambda = (k_1, k_2, \dots, k_r) \vdash k$ indicates that λ is a partition of a positive integer k . The same notation is used to indicate a Young diagram of size k . It is assumed that $k_1 \geq k_2 \geq \dots \geq k_r > 0$. The length r of λ is denoted by $l(\lambda)$.
- If $J \in \text{End}(V)$ is nilpotent, the partition $\lambda(J) = (k_1, \dots, k_r)$ is called the conjugacy class of J , indicating the sizes of Jordan blocks in the Jordan decomposition of J .
- A Young tableau T is a Young diagram λ with a numbering of boxes with integers $1, 2, \dots, k$. In this case $\lambda = |T|$ is the shape of T .
- Δ_T denotes the Specht polynomial defined by a Young tableau T .
- $\mu(I)$ denotes the minimal number of generators of ideal.

2.2 The equi-degree monomial complete intersection as a tensor space

Throughout this paper we denote by K an algebraically closed field of characteristic 0. Let $R = K[x_1, \dots, x_k]$ be the polynomial ring. The partial degree of a homogeneous polynomial $f \in R$ is the maximum degree of f with respect to a single variable. By $A(n, k)$ we denote the vector subspace of R consisting of polynomials of partial degree at most $n - 1$. Let $I = (x_1^n, \dots, x_k^n)$ be the ideal of R . Then as vector spaces we have the decomposition

$$R = A(n, k) \oplus I.$$

The vector space $A(n, k)$ may be regarded as the tensor space $(K^n)^{\otimes k}$. Since $A(n, k) \cong R/I$ the vector space $A(n, k)$ has the structure of a commutative ring. Put $A = A(n, k)$. A basis

of A can be the set of monomials of partial degree at most $(n - 1)$:

$$\{x_1^{i_1} \cdots x_k^{i_k} \mid 0 \leq i_1, \dots, i_k \leq n - 1\}$$

An element of A is expressed uniquely as

$$\sum F(i_1, i_2, \dots, i_k) x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}.$$

With the identification $A \cong (K^n)^{\otimes k}$ the general linear group $GL(n)$ acts on the vector space A as the tensor representation. Let

$$\Phi: GL(n) \rightarrow GL(A) \tag{6}$$

be the representation. Explicitly, if $g = (g_{\alpha\beta}) \in GL(n)$ then

$$\Phi(g)(x_1^{i_1} \cdots x_k^{i_k}) = \left(\sum_{\beta=0}^{n-1} g_{i_1\beta} x_1^\beta \right) \cdots \left(\sum_{\beta=0}^{n-1} g_{i_k\beta} x_k^\beta \right). \tag{7}$$

Here the indices α, β of the matrix entries for $g = (g_{\alpha\beta}) \in GL(n)$ range over $0, 1, \dots, n - 1$. At the same time the symmetric group S_k acts on A as the permutation of the variables.

We are interested in the decomposition of A into irreducible S_k modules. According to the Schur-Weyl duality it will give us an irreducible decomposition of A as $(GL(n) \times S_k)$ -modules.

2.3 Young tableaux and Specht polynomials

A partition of a positive integer k is a way to express k as a sum of positive integers. If we say that $\lambda = (k_1, \dots, k_r)$ is a partition of k , it means that $k = k_1 + \dots + k_r$ and $k_1 \geq \dots \geq k_r > 0$. A partition of k is identified with a Young diagram of size k in a well known manner. Thus the same notation $\lambda = (k_1, \dots, k_r)$ denotes a Young diagram of size $\sum k_i$ with rows of k_i boxes, $i = 1, \dots, r$, aligned left. A Young tableau T is a Young diagram λ whose boxes are numbered with integers $1, \dots, k$ in any order. In this case we say that λ is the shape of T and write $\lambda = |T|$. A Young tableau is standard if every row and column is numbered increasingly.

A Young tableau T defines a Specht polynomial, denoted Δ_T , as follows:

Put $I = \{1, 2, \dots, k\}$ and

$$I_j = \{\alpha \in I \mid \alpha \text{ is in the } j\text{th column of } |T|\}.$$

Define

$$\Delta_j = \prod_{\substack{\alpha < \beta \\ \alpha, \beta \in I_j}} (x_\alpha - x_\beta),$$

and finally,

$$\Delta_T = \prod_j \Delta_j, \tag{8}$$

where j runs over all columns. This is the Specht polynomial defined by the Young tableau T . (We disregard the signs of Specht polynomials.)

2.4 Nilpotent matrices and Jordan bases

Let $\mathbf{M}(k)$ denote the set of $k \times k$ matrices with entries in K .

Let $J \in \mathbf{M}(k)$ be a nilpotent matrix. Let $\nu_i = \text{rank}J^i - \text{rank}J^{i+1}$ for $i = 0, 1, \dots, p$, where p is the least integer such that $J^{p+1} = O$. Then $\widehat{\lambda} := (\nu_0, \nu_1, \dots, \nu_p)$ is a partition of k . We denote the dual partition of $\widehat{\lambda}$ by $\lambda(J)$. (cf. Definition 1 below.)

Now suppose that T is a Young tableau of size k . Using the numbering of T define the matrix $J = (a_{ij}) \in \mathbf{M}(k)$ by

$$a_{ij} = \begin{cases} 1 & \text{if } j \text{ is next to the right of } i, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

It is easy to see that the matrix J is nilpotent and $\lambda(J) = |T|$. We call any matrix defined as above for a Young tableau T a **Jordan canonical form** (of a nilpotent matrix).

Let V be a vector space of dimension k . If a basis of V is fixed, we may identify $\mathbf{M}(k)$ and $\text{End}(V)$. Suppose that $J \in \text{End}(V)$ is nilpotent. A **Jordan basis** for J is a basis of V on which J is put in a Jordan canonical form. (According to the definition of a Jordan basis just defined above, any permutation of basis elements of a Jordan basis is a Jordan basis.) It is an elementary fact that there exists a Jordan basis. Note that if two nilpotent elements J and J' are conjugate, then $\lambda(J) = \lambda(J')$. We make a definition of the ‘‘conjugacy class’’ of a nilpotent endomorphism, with a slight abuse of language, as follows.

Definition 1. Let V be a k -dimensional vector space over K . Suppose that $J \in \text{End}(V)$ is nilpotent. We say that a partition (k_1, \dots, k_r) of k is the **conjugacy class** of J if the Jordan canonical form of J consists of the Jordan blocks of sizes k_1, \dots, k_r . We denote by $\lambda(J)$ the conjugacy class of J .

Remark 2. Note that the notation $\lambda(J)$ coincides with the previously defined $\lambda(J)$ for a nilpotent matrix. In fact, if we put $\nu_i = \dim \text{Im}J^i / \text{Im}J^{i+1}$, then the sequence $\widehat{\lambda} = (\nu_0, \nu_1, \dots, \nu_p)$ is a partition of k . One sees easily that the dual partition λ to $\widehat{\lambda}$ is the conjugacy class of J .

Let $J \in \text{End}(V)$ be nilpotent with the conjugacy class $\lambda = \lambda(J)$. Suppose that $B \subset V$ is a Jordan basis for J . Then it is possible to place the elements of B into the boxes of the Young diagram $\lambda(J)$ bijectively in such a way that it satisfies the following conditions:

$$\begin{cases} e, e' \in B \text{ and } e' = Je \Leftrightarrow e' \text{ is placed next to the right of } e. \\ e \in \text{Ker}J \Leftrightarrow e \text{ is placed at the end of a row.} \end{cases} \quad (10)$$

(cf. Equation (9).)

Remark 3. As explained above, by choosing a bijection between the elements of a Jordan basis for J and the boxes of the Young diagram $\lambda(J)$, it is possible to identify a Jordan basis B for J and the Young diagram $\lambda(J)$. With this identification the rightmost boxes of $\lambda(J)$ form a basis for $\text{Ker}J$. Also the boxes of the first column of $\lambda(J)$ coincide with $\{b \in B | b \notin \text{Im}J\}$. Once $\lambda(J)$ is known, $\text{Ker}J \cap B$ determines B . Similarly the diagram $\lambda(J)$ and the subset $B \setminus \text{Im}J \subset B$ determine B .

Definition 4. Suppose that $\lambda = (k_1, \dots, k_r)$ is a Young diagram. Let f_1, \dots, f_s be the finest subsequence of (k_1, \dots, k_r) such that $f_1 > \dots > f_s > 0$. Then it is possible to write

$$(k_1, \dots, k_r) = (\underbrace{f_1, \dots, f_1}_{m_1}, \underbrace{f_2, \dots, f_2}_{m_2}, \dots, \underbrace{f_s, \dots, f_s}_{m_s}).$$

We call m_1, \dots, m_s the multiplicity sequence of the Young diagram λ .

2.5 The strong Lefschetz property

Let $V = \bigoplus_{i=0}^c V_i$ be a finite dimensional graded vector space. The Hilbert function of V is the map $i \mapsto \dim V_i$, which we usually write as the polynomial $\sum (\dim V_i) q^i$. (For convention we let $\dim V_i = 0$ for $i < 0$ or $i > c$.) Let $J \in \text{End}(V)$ be a degree one map so J consists of the graded pieces $J|_{A_i} : A_i \rightarrow A_{i+1}$. Then J is nilpotent. We say that $J \in \text{End}(V)$ is a **strong Lefschetz element** if the restricted map

$$J^{c-2i}|_{V_i} : V_i \rightarrow V_{c-i}$$

is bijective for all $i = 0, 1, \dots, [c/2]$. (Such an endomorphism exists only if the Hilbert function of V is symmetric and unimodal.)

Definition 5. Let $A = \bigoplus_{i=0}^c A_i$ be an Artinian graded K -algebra. Denote by $\times : A \rightarrow \text{End}(A)$ the regular representation of A . (I.e., $\times a(b) = ab$ for $a, b \in A$.) We say that A has the strong Lefschetz property, if there exists a linear form $l \in A$ such that $\times l \in \text{End}(A)$ is a strong Lefschetz element. We call such a linear form $l \in A$ a strong Lefschetz element of A as well as $\times l \in \text{End}(A)$.

Proposition 6. Suppose that $A = \bigoplus_{i=0}^c A_i$ is a graded Artinian K -algebra with a symmetric Hilbert function $\sum h_i q^i$. Let l be a linear form of A . Then $\times l$ is the strong Lefschetz element if and only if $\lambda(\times l)$ is the dual partition to $(h'_0, h'_1, \dots, h'_c)$, which is a permutation of (h_0, h_1, \dots, h_c) put in the decreasing order.

Proof. See [8] Proposition 18.

Proposition 7. With the same notation as above, suppose that $J \in \text{End}(V)$ is a strong Lefschetz element. Then any homogeneous basis of $\text{Ker}J$ can be extended uniquely to a Jordan basis for J .

Proof. Let $\sum h_i q^i$ be the Hilbert function of A . Since J is a strong Lefschetz element, we have

$$\dim(\text{Ker}J \cap A_i) = \begin{cases} 0 & \text{if } h_i \leq h_{i+1}, \\ h_i - h_{i+1} & \text{if } h_i > h_{i+1}. \end{cases} \quad (11)$$

Let s be the greatest integer such that $h_{s-1} < h_s$ and let

$$m_i = h_i - h_{i-1}, \text{ for } i = 0, 1, 2, \dots, s.$$

Since $h_i = h_{c-i}$ for $0 \leq i \leq c$, we may rewrite the equation (11) as

$$\dim(\text{Ker}J \cap A_{c-i}) = \begin{cases} m_i & \text{for } i = 0, 1, 2, \dots, s, \\ 0 & \text{otherwise.} \end{cases}$$

Now let B be a homogeneous basis of $\text{Ker}J$ given arbitrarily. We have to find a basis of A containing B on which J is put in the Jordan canonical form. Put $B_{c-i} = B \cap A_{c-i}$ for $i = 0, 1, \dots, s$. Then B_{c-i} is a basis of $\text{Ker}J \cap A_{c-i}$. Since the restricted map $J^{c-2i}: A_i \rightarrow A_{c-i}$ is bijective, there is a finite set $B_i \subset A_i$ such that $\#B_i = m_i$ and such that $J^{c-2i}(B_i) = B_{c-i}$. It is easy to show that the set

$$\tilde{B} := \bigsqcup_{i=0}^s \{J^j(B_i) \mid j = 0, 1, 2, \dots, c-2i\}$$

is linearly independent and hence is a basis of A . It is easy to see that \tilde{B} is a desired basis and that it is unique. \square

Remark 8. In the same notation and assumption of Proposition 7, the conjugacy class of J is given by

$$\lambda(J) = (\underbrace{c+1}_1, \underbrace{c-1, \dots, c-1}_{m_1}, \underbrace{c-3, \dots, c-3}_{m_2}, \dots, \underbrace{c-2s+1, \dots, c-2s+1}_{m_s}).$$

The multiplicity sequence of $\lambda(J)$ is

$$m_0 = 1, m_1, m_2, \dots, m_s$$

where $m_i = h_i - h_{i-1}$ for $i = 0, 1, \dots, s$. \square

Following is proved in [20] and plays an important role in this paper.

Theorem 9. *Let $A = A(n, k)$ be the same as defined in Section 1. Then A has the strong Lefschetz property and $x_1 + x_2 + \dots + x_k$ is a strong Lefschetz element of A .*

3 $A(n, 2)$ or the two fold tensor of K^n

If $k = 2$, then the Gorenstein algebra $A = K[x_1, \dots, x_k]/(x_1^n, \dots, x_k^n)$ takes the form

$$A \cong K[x_1, x_2]/(x_1^n, x_2^n).$$

Write x, y for x_1, x_2 . Recall that we identify $A = A(n, 2)$ as a vector subspace of $R = K[x, y]$ and $R = A \oplus (x^n, y^n)$. Denote by $\times: A \rightarrow \text{End}(A)$ the regular representation of the Artinian algebra A . Put $J = \times(x+y)$. Since J is a strong Lefschetz element, we may use Proposition 7 to construct a Jordan basis for J . First we would like to construct a homogeneous basis for $\text{Ker}J$.

The Hilbert function of A is the following sequence.

degree	0	1	2	\dots	n-2	n-1	n	\dots	2n-3	2n-2
dim	1	2	3	\dots	n-1	n	n-1	\dots	2	1

Since J is a strong Lefschetz element, $J: A_i \rightarrow A_{i+1}$ is either injective or surjective. Hence we have

$$\dim(\text{Ker}J \cap A_i) = \begin{cases} 0, & \text{if } i < n - 1, \\ 1, & \text{if } n - 1 \leq i \leq 2n - 2. \end{cases}$$

Since $\dim(\text{Ker}J \cap A_i)$ is at most one, a homogeneous basis of $\text{Ker}J$ is uniquely determined up to constant multiple.

For $d = n - 1, n, n + 1, \dots, 2n - 2$, put

$$b_d = \sum_{j=0}^d (-1)^j x^{d-j} y^j, \text{ and } \bar{b}_d = b_d \text{ mod } (x^n, y^n).$$

Then since $b_d(x+y) = x^{d+1} \pm y^{d+1}$, we have $\bar{b}_d \in \text{Ker}J$ for $d \geq n - 1$. Thus we have

$$\text{Ker}J \cap A_d = \langle \bar{b}_d \rangle \text{ for } d = n - 1, n, \dots, 2n - 2. \quad (12)$$

Because J is a strong Lefschetz element, there is an element $a_i \in A_i$ for each $i \leq n - 1$ such that $J^{2n-2-2i}(a_i) = \bar{b}_{2n-2-i}$. Note that $J^j(a_i)$ are all symmetric if i is even and are alternative if i is odd. Now we have proved the following.

Theorem 10. *The set*

$$\bigsqcup_{i=0}^{n-1} \{J^j(a_i) | j = 0, 1, 2, \dots, 2n - 2 - 2i\}$$

is a homogeneous Jordan basis for $J \in \text{End}(A)$. The basis element $J^j a_i$ is symmetric if i is even and alternative if i is odd. The conjugacy class of J is given by

$$\lambda(J) = (2n - 1, 2n - 3, \dots, 3, 1).$$

Proof. The first part was treated more generally in Proposition 7. The second part follows immediately from the definition of b_{2n-2-i} and a_i . The third statement follows from Proposition 6. \square

Now we consider the representation

$$\Phi: GL(n) \rightarrow GL(A)$$

as mentioned in Section 2.2. (For the definition of Φ see (7) and (18).) Recall that the special linear group $SL(2)$ has a unique irreducible module of dimension i for each $i > 0$. We denote it by $V(i - 1)$. Fix $n > 0$ and let

$$\Psi: SL(2) \rightarrow GL(n) \tag{13}$$

be the irreducible representation corresponding to the module $V(n - 1)$. We may consider $A = A(n, 2)$ as an $SL(2)$ -module via the composition

$$SL(2) \xrightarrow{\Psi} GL(n) \xrightarrow{\Phi} GL(A).$$

Proposition 11. *With the same notation above A decomposes into $SL(2)$ -modules as*

$$A \cong V(2n - 2) \oplus V(2n - 4) \oplus \dots \oplus V(0).$$

Proof. Abbreviate $\rho = \Phi \circ \Psi$, so $\rho: SL(2) \rightarrow GL(A)$. The group homomorphism ρ induces a Lie algebra homomorphism

$$d\rho: \mathfrak{sl}(2) \rightarrow \mathfrak{gl}(A).$$

It is well known that the irreducible decomposition of ρ is determined by that of $d\rho$. Now recall that the Lie algebra $\mathfrak{sl}(2)$ is the vector space spanned by three elements e, f, h with the bracket relations

$$[ef] = h, [he] = 2e, [hf] = -2f.$$

It is easy to see that to decompose A into irreducible $\mathfrak{sl}(2)$ -modules is to decompose $d\rho(e)$ into Jordan blocks. Notice that $d\Psi(e)$ is nilpotent and may be considered as a single Jordan block by conjugation since Ψ is irreducible. Consequently $d\rho(e) \in \mathfrak{gl}(A)$ may be considered as the multiplication map $\times(x + y)$ by definition of ρ and $d\rho$. Hence the assertion follows from Theorem 10. \square

With the same notation as above let $W = V(n - 1) \otimes V(n - 1)$. Let $W = W_s \oplus W_a$ be the decomposition of W into the symmetric and alternate tensors respectively. (It is well known that the spaces W_s and W_a are irreducible $GL(n)$ -modules, which we take for granted.) Via the representation (13) the spaces W_s and W_a are also $SL(2)$ -modules. The following proposition shows how W_s and W_a decompose into irreducible $SL(2)$ -modules.

Proposition 12. *If n is even, then*

$$W_s \cong \underbrace{V(2n-2) \oplus V(2n-6) \oplus \cdots \oplus V(2)}_{n/2},$$

and

$$W_a \cong \underbrace{V(2n-4) \oplus V(2n-8) \oplus \cdots \oplus V(0)}_{n/2}.$$

If n is odd, then

$$W_s \cong \underbrace{V(2n-2) \oplus V(2n-6) \oplus \cdots \oplus V(0)}_{(n+1)/2},$$

and

$$W_a \cong \underbrace{V(2n-4) \oplus V(2n-8) \oplus \cdots \oplus V(2)}_{(n-1)/2}.$$

Proof. Identify $W = A(n, 2)$. Then W_s is the space spanned by the symmetric polynomials in A and W_a the alternate polynomials. Hence the assertion follows immediately from Theorem 10 and Proposition 11. \square

4 $A(2, k)$ or the Boolean algebra

Throughout this section we fix $A = A(2, k)$. So A is the subspace of the polynomial ring $K[x_1, \dots, x_k]$ spanned by square-free monomials. At the same time A is endowed with the algebra structure

$$A = K[x_1 \cdots, x_k] / (x_1^2, \dots, x_k^2).$$

Usually the set $2^{\{x_1, \dots, x_k\}}$ is called the Boolean algebra with two the operations \wedge, \vee . Here we call A above the Boolean algebra as a commutative algebra over the field K .

We put $L = \times(x_1 + x_2 + \cdots + x_k)$ and

$$D = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_k}.$$

We think of L and D as operating on the polynomial ring $R = K[x_1, \dots, x_k]$. Recall that $R = A \oplus I$, where $I = (x_1^2, \dots, x_k^2)$. We denote by $D|_A$ the restricted map D on A . Similarly by $L|_A$ we denote the map $R/I \rightarrow R/I$ induced by L . Thus $L|_A, D|_A \in \text{End}(A)$. Let H be the commutator $H = [L|_A, D|_A]$. So $H \in \text{End}(A)$.

Proposition 13. (a) $\langle L|_A, D|_A, H \rangle$ is an $\mathfrak{sl}(2)$ -triple.

(b) There exists a Jordan basis B for $L|_A$ such that $B \setminus \text{Im}(L|_A)$ is a basis of $\text{Ker}(D|_A)$.

Proof. (a) Let $J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then the three elements $\langle J_+, J_-, H \rangle$, where $H := [J_+, J_-] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, is an $\mathfrak{sl}(2)$ -triple. This proves the case $k = 1$. Let E_2 be the 2×2 identity matrix. Then one sees that, using square free monomials as a basis of A , the map $L|_A$ is represented by the matrix

$$\sum_{i=0}^k \underbrace{E_2 \otimes \cdots \otimes E_2}_{i-1} \otimes J_+ \otimes \underbrace{E_2 \otimes \cdots \otimes E_2}_{k-i}$$

and similarly $D|_A$ by

$$\sum_{i=0}^k \underbrace{E_2 \otimes \cdots \otimes E_2}_{i-1} \otimes J_- \otimes \underbrace{E_2 \otimes \cdots \otimes E_2}_{k-i}.$$

We induct on k to show that the three matrices $L|_A, D|_A$, and $H := [L|_A, D|_A]$ satisfy the required relations $[H, L|_A] = 2L|_A$, and $[H, D|_A] = 2D|_A$.

(b) See Humphrey ([10] pp.31-34). □

The following theorem enables us to construct a Jordan basis for $L|_A$.

Theorem 14. *For $i = 0, 1, 2, \dots, [k/2]$, the vector space $(\text{Ker}D) \cap A_i$ is spanned by the Specht polynomial of degree i . The Specht polynomials arising from the standard Young tableaux form a basis of $(\text{Ker}D) \cap A$.*

Proof of Theorem 14 is postponed to the end of Proposition 18.

Lemma 15. $\text{Ker}D = K[\{x_i - x_j | 1 \leq i, j \leq k\}]$

Proof. Recall that $R = K[x_1, \dots, x_k]$ and $\text{Ker}D = \{f \in R | Df = 0\}$. Since $\text{Ker}D$ is a subalgebra of R , we have

$$\text{Ker}D \supset K[\{x_i - x_j | 1 \leq i, j \leq k\}].$$

On the other hand the RHS is isomorphic to the polynomial ring in $(k-1)$ variables. Noticing that D is surjective of degree -1 , it is easily verified that they coincide by comparing the Hilbert functions.

Lemma 16. *Put $V = (\text{Ker}D) \cap A$. Then $\dim V_i = \text{Max}\{\binom{k}{i} - \binom{k}{i-1}, 0\}$, where $V_i = V \cap A_i$.*

Proof. First note that the Hilbert function of A is given by $\sum \binom{k}{i} q^i$. The graded vector space A has the SLP with $L|_A$ a strong Lefschetz element. By Proposition 13, the maps $D|_A$ and $L|_A$ are alike except that $D|_A$ is a degree -1 map. Thus $D|_A: A_i \rightarrow A_{i-1}$ is either injective or surjective. Thus the assertion follows. □

Lemma 17. *Let T be a Young tableau with the shape λ . Suppose that λ has size k . Let Δ_T be the Specht polynomial defined by T .*

- (1) $\Delta_T = 1 \iff \lambda$ has one row.
- (2) $\Delta_T \in A$ and $\Delta_T \neq 1 \iff \lambda$ has two rows.
- (3) Suppose that $\Delta_T \in A$. Then the degree of Δ_T is equal to the second term of λ .
- (4) $\Delta_T \in \text{Ker}[D|_A : A \rightarrow A] \cap A_i$ if $\lambda = (k - i, i)$.

Proof. (1), (2) and (3) are immediate from the definition of Δ_T . (4) follows from Lemma 15. \square

We need one more proposition to prove Theorem 14.

Proposition 18. *Suppose that $\lambda = (k - i, i)$ is a Young diagram.*

- (a) *The number of standard Young tableaux of shape λ is $\binom{k}{i} - \binom{k}{i-1}$.*
- (b) *The set of Specht polynomials defined by the standard Young tableaux of a fixed shape λ is linearly independent.*

Proof. (a) Suppose that T is a standard Young tableau. If the box k is removed from T it is a Standard Young tableau of size $k - 1$. Thus the induction works. (Details are left to the reader.)

(b) Suppose that T is standard. Then one notices easily that the head term of Δ_T in the reverse lexicographic monomial order is the product of monomials in the second row. It means that two Specht polynomials arising from different standard Young tableaux are different. Thus the proof is complete.

Proof of Theorem 14. Immediate by Lemmas 15, 16, 17 and Proposition 18. \square

Now our main theorem is stated as follows.

Theorem 19. *Let $J = L|_A$ and $V_d = (\text{Ker}D|_A) \cap A_d$. Put $h = [k/2]$, and $m_i = \binom{k}{i} - \binom{k}{i-1}$ for $i = 0, 1, \dots, h$.*

- (1) *The conjugacy class $\lambda(J)$ of J is given by*

$$\lambda(J) = \begin{cases} \left(\underbrace{(k+1)}_{m_0}, \underbrace{k-1, \dots, k-1}_{m_1}, \underbrace{k-3, \dots, k-3}_{m_{-2}}, \dots, \underbrace{1, \dots, 1}_{m_h} \right), & \text{if } k \text{ is even,} \\ \left(\underbrace{(k+1)}_{m_0}, \underbrace{k-1, \dots, k-1}_{m_1}, \underbrace{k-3, \dots, k-3}_{m_{-2}}, \dots, \underbrace{2, \dots, 2}_{m_h} \right), & \text{if } k \text{ is odd.} \end{cases}$$

(2) For $0 \leq d \leq h$, and $0 \leq i \leq k - 2d$, the vector space

$$J^i(V_d)$$

is an irreducible S_k -module of isomorphism type $\lambda = (k - d, d)$.

(3) For any Specht polynomial $\Delta_T \in V_d$, the vector space

$$\langle \Delta_T, J(\Delta_T), J^2(\Delta_T) \cdots, J^{k-2d}(\Delta_T) \rangle$$

is an irreducible $GL(2)$ -module of isomorphism type $\lambda = (k - d, d)$.

(4) An irreducible decomposition of A as S_k -modules is given by

$$A = \bigoplus_{d=0}^h \left(\bigoplus_{i=0}^{k-2d} J^i(V_d) \right).$$

In particular the irreducible module of type $\lambda = (k - d, d)$ occurs $(k + 1 - 2d)$ times, and the irreducible $GL(2)$ -module of type $(k - d, d)$ occurs m_d times.

Proof. (1) See Proposition 6. (2) In Lemma 17 and Proposition 18 we showed that the space V_d is spanned by the Specht polynomials defined by the standard Young tableau of shape $(k - d, d)$. It is well known that this is irreducible. Also $J^i(V_d)$ is isomorphic to V_d unless it is trivial because $x_1 + \cdots + x_k$ is S_k -invariant. (3) Let ρ be the composition

$$SL(2) \xrightarrow{\Psi} GL(2) \xrightarrow{\Phi} GL(A)$$

where Ψ is the natural injection and Φ is the tensor representation. To decompose A into irreducible $GL(2)$ -modules is the same as to decompose it as $SL(2)$ -modules. This is obtained by decomposing the Lie algebra representation:

$$d\rho : \mathfrak{sl}(2) \rightarrow \mathfrak{gl}(A).$$

Now the assertion follows from Lemma 13. (4) Clear from (1), (2) and (3). \square

Example 20. Let $k = 4$. Let $l = \times(x_1 + x_2 + x_3 + x_4) \in \text{End}(A(2, 4))$. We exhibit the Jordan basis of $\times l$. The Hilbert function of A is $(1, 4, 6, 4, 1)$. The derived sequence is $(1, 3, 2)$.

1. The Specht polynomial of degree 0 is 1.
2. The Specht polynomials of degree 1 (corresponding to the standard Young tableau with shape $\lambda = \lambda(3, 1)$) are $a := x_1 - x_2$, $b := x_1 - x_3$ and $c := x_1 - x_4$.
3. The Specht polynomials of degree 2 (corresponding to the standard Young tableau with shape $\lambda = \lambda(2, 2)$) are $f := (x_1 - x_2)(x_3 - x_4)$ and $g := (x_1 - x_3)(x_2 - x_4)$.

The bases for the irreducible decomposition of $A = A(2, 4)$ as $GL(2)$ -modules are:

1. $\langle 1, l, l^2, l^3, l^4 \rangle$
2. $\langle a, la, l^2a \rangle$ and $\langle b, lb, l^2b \rangle$ and $\langle c, lc, l^2c \rangle$
3. $\langle f \rangle$ and $\langle g \rangle$

We have $2^4 = 5 \times 1 + 3 \times 3 + 1 \times 2$. When l^i is expanded, all terms which contain a square of a variable should be regarded zero. With this convention l^i is equal to the i th elementary symmetric polynomial multiplied by $i!$.

5 Application to the theory of Gorenstein rings

Put $A = R/(x_1^2, \dots, x_k^2)$ and $l = x_1 + x_2 + \dots + x_k \in A$. Using the notation of Proposition 13, we have $\times l = L|_A$. Since we have obtained a Jordan basis for $D|_A: A \rightarrow A$, it is a Jordan basis for $\times l$ as well. (cf. Proposition 13.) Thus it in particular gives us a basis for $0: l \subset A$ as a vector space. However, it does not necessarily determine a minimal ideal basis for $0: l$. In this section we would like to exhibit a minimal generating set of $0: l$. Denote by $(\)^*: A \rightarrow A$ the ‘‘Hodge dual’’ of A . Namely, define

$$M^* = (x_1 \cdots x_k)/M$$

for $M \in A$. By linearity this is extended to define the dual map $A \rightarrow A$.

If Δ is a Specht polynomial, we call Δ^* the dual Specht polynomial. (We assume that this is defined only for Specht polynomials of partial degree at most one.)

In the next theorem we use the notation fixed in Section 2. Namely, $\lambda = \lambda(r, s)$ is a Young diagram with two rows of length r, s , and if T is a Young tableau then $|T|$ is the shape of T and Δ_T is the Specht polynomial defined by T . μ is the minimal number of generators.

Theorem 21. *In the polynomial ring R , put*

$$I = (x_1^2, \dots, x_k^2) : (x_1 + \dots + x_k).$$

Then we have

$$\mu(I) = k + \binom{k}{h} - \binom{k}{h+1}.$$

Here h is such that $h = k/2$ or $h = (k+1)/2$ according as k is even or odd.

If k is even, then

$$I = (x_1^2, \dots, x_k^2) + \{\Delta_T^* \mid |T| = \lambda(h, h)\}R$$

and if k is odd,

$$I = (x_1^2, \dots, x_k^2) + \{\Delta_T^* \mid |T| = \lambda(h, h-1)\}R.$$

(If k is even, it is the same if \star is dropped.)

We need a lemma before proving this theorem.

Lemma 22. *Let T be a Young tableau of the shape $\lambda = \lambda(k - s, s)$. Let $\Delta = \Delta_T$ be the Specht polynomial defined by T . Suppose two integers (i, j) appear in a same column of λ . Then $(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})\Delta$ is a Specht polynomial. (A constant multiple is disregarded.) Furthermore we have*

$$\left(\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Delta\right)^* = (x_i - x_j)\Delta^*$$

Proof. By hypothesis it immediately follows that $(x_i - x_j)$ is a factor of Δ , and it is the only factor of Δ which involve these two variables. Hence we have $(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j})\Delta = \pm 2\Delta/(x_i - x_j)$. The second assertion is as easy.

Proof of Theorem 21. It is enough to prove the second part.

We have already proved that $\text{Ker}[D: A \rightarrow A]$ is spanned by the Specht polynomials. Thus we have that $\text{Ker}[(\times l): A \rightarrow A]$ is spanned by the dual Specht polynomials. Let \mathfrak{a} be the ideal of A generated by the dual Specht polynomials of the least degree. Then \mathfrak{a} contains all dual Specht polynomials by Lemma 22. \square

Remark 23. Theorem 21 may be regarded as a broad generalization of F. S. Macaulay's example (2) in Introduction.

Corollary 24. *Let $A = K[x_1, \dots, x_k]/(x_1^2, \dots, x_k^2)$, and let l be the linear element $l = x_1 + \dots + x_n$ of A . Then*

$$\mu(0: l) = \binom{k}{h} - \binom{k}{h+1}.$$

Here h is as in Theorem 21 .

Proof. Immediate by Theorem 21.

Theorem 25. *As before let $A = R/(x_1^2, \dots, x_k^2)$, and let $y \in A$ be a general element of A . Then the Macaulay type of $A/(y)$ is the h -th Catalan number $\frac{1}{h+1}\binom{2h}{h}$. Here h is as in Theorem 21. Equivalently if we put $I = (x_1^2, \dots, x_k^2, Y)$, where Y is a general element of the polynomial ring and if we write a minimal free resolution of R/I as*

$$0 \rightarrow F_k \rightarrow F_{k-1} \rightarrow \dots \rightarrow F_1 \rightarrow R \rightarrow R/I \rightarrow 0$$

then, we have $\text{rank } F_k = \frac{1}{h+1}\binom{2h}{h}$.

Proof. The Macaulay type of $A/(y)$ is equal to the minimal number of generators of $0: y$. It is well known that this is also equal to the last rank of the minimal free resolution of $0: y$. Now it suffices to notice that

$$\binom{k}{h} - \binom{k}{h+1} = \frac{1}{h+1}\binom{2h}{h},$$

where $h = k/2$ or $(k+1)/2$ as in Theorem 21.

6 The Hilbert function of the ring of invariants of $A(n, k)$

In this section we let $A = K[x_1, \dots, x_k]/(x_1^n, \dots, x_k^n)$ where n and k are arbitrary positive integers. Let $G := S_k$ act on A by the permutation of the variables. In the next theorem we would like to exhibit the ring of invariants A^G and the Hilbert function of A^G .

Theorem 26.

$$A^G = K[e_1, \dots, e_n]/(p_n, p_{n+1}, \dots, p_{n+k-1}).$$

Here e_d is the elementary symmetric polynomial of degree d and p_d is the power sum $p_d = x_1^d + \dots + x_k^d$. Hence the Hilbert function of A^G is:

$$h_{A^G}(q) = \frac{(1 - q^n)(1 - q^{n+1}) \dots (1 - q^{n+k-1})}{(1 - q^1)(1 - q^2) \dots (1 - q^k)}$$

Proof. Consider the exact sequence

$$0 \rightarrow (x_1^n, \dots, x_k^n) \rightarrow R \rightarrow A \rightarrow 0.$$

Since $\text{ch}K = 0$, we have the exact sequence

$$0 \rightarrow (x_1^n, \dots, x_k^n)^G \rightarrow R^G \rightarrow A^G \rightarrow 0.$$

Note that $R^G = K[e_1, \dots, e_k]$. The socle degree of A is $nk - k$, and $A_{nk-k} = \langle (e_k)^{n-1} \rangle$. Since e_k^{n-1} is fixed under the action of G , this shows that the maximum degree of elements of A^G is $nk - k$.

Put $A' = R^G/(p_n, p_{n+1}, \dots, p_{n+k-1})$. Obviously we have a natural surjection:

$$A' \rightarrow A^G \rightarrow 0 \tag{14}$$

which we would like to prove to be an isomorphism. First note that the rational function in the statement of this Theorem is the Hilbert function of A' . (This can be obtained using the fact $R^G = K[e_1, e_2, \dots, e_k]$.) This shows that A' and A^G have the same socle degree, which is equal to $nk - k$. Since A' is an Artinian Gorenstein ring, the one dimensional vector space of the maximum degree is the unique minimal ideal of the ring A' . This shows that the map (14) cannot have a non-trivial kernel. This completes the proof.

Remark 27. 1. Since

$$\lim_{q \rightarrow 1} \frac{(1 - q^n)(1 - q^{n+1}) \dots (1 - q^{n+k-1})}{(1 - q^1)(1 - q^2) \dots (1 - q^k)} = \binom{n+k-1}{k},$$

we have $\dim A^G = h_{A^G}(1) = \binom{n+k-1}{k}$. This is expected for A^G is the irreducible $GL(n)$ -module corresponding to the trivial λ , which is the symmetric tensor space.

2. One may conceive that the Hilbert function of $Y^\lambda(A)$ where Y^λ is a Young symmetrizer corresponding to $\lambda = (k_1, \dots, k_r) \vdash k$ should be obtained as a q -analog of the dimension formula of the irreducible $GL(n)$ -module in the decomposition of $(K^n)^{\otimes k}$. This can be proved using the q -dimension formula ([11] Proposition 10.10). We indicate an outline of proof in the next proposition. Note that it is a q -analog of the dimension formula for W^λ as shown at the end of Appendix.

Recall that we have the isomorphism $A(n, k) \cong (K^n)^{\otimes k}$ as vector spaces. The symmetric group S_k acts on $A(n, k)$ as permutations of variables. Also the general linear group $GL(n)$ acts on $A(n, k)$ as the tensor representation. (See Appendix (18)). Put $W^\lambda = Y^\lambda(A(n, k))$, where the map $Y^\lambda : A(n, k) \rightarrow A(n, k)$ is a Young symmetrizer corresponding to $\lambda = (k_1, \dots, k_r) \vdash k$ ($r \leq n$). It is well known that W^λ is an irreducible $GL(n)$ -module and every irreducible $GL(n)$ -module is obtained in this way. (For more details see, e.g., [7] pp. 336-339.)

Proposition 28. *For $\lambda = (k_1, \dots, k_r) \vdash k$ with $r \leq n$, the Hilbert function $h_{W^\lambda}(q)$ of the Weyl module W^λ regarded as a submodule of the graded algebra A is*

$$h_{W^\lambda}(q) = q^{k_2+2k_3+\dots+(r-1)k_r} \prod_{1 \leq i < j \leq n} \frac{[k_i - k_j + j - i]}{[j - i]}.$$

Here $[a]$ denotes $\frac{1-q^a}{1-q}$ for any positive integer a , and $k_j = 0$ for $j > r$.

We first review the q -dimension formula [11, §10.9, §10.10] for $GL(n)$. Let W be a finite dimensional irreducible $GL(n)$ -module with highest weight λ , and $W = \bigoplus_{\mu} W_{\mu}$ its weight space decomposition. The q -dimension of W is defined by

$$\dim_q W = \sum_{\mu: \text{weight of } W} (\dim W_{\mu}) q^{\langle \lambda - \mu, \delta \rangle}, \quad (15)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^n , and δ is ‘the half sum of the positive roots’ defined by $(n-1, n-3, \dots, -n+1)/2 \in \mathbb{R}^n$. Note that the exponent $\langle \lambda - \mu, \delta \rangle$ is the number of the simple roots occurring in $\mu = \lambda - (\text{sum of simple roots})$, since $\langle \alpha, \delta \rangle = 1$ for any simple root α and since the inner product is bilinear.

The q -dimension formula [11, Proposition 10.10] is

$$\dim_q W = \prod_{1 \leq i < j \leq n} \frac{[k_i - k_j + j - i]}{[j - i]}, \quad (16)$$

where W is a finite dimensional irreducible $GL(n)$ -module with highest weight $\lambda = (k_1, \dots, k_n)$.

Proof of Proposition 28. Let $GL(n)$ act on n -dimensional K -vector space $K[x]/(x^n)$ through the vector representation with the basis $1, x, x^2, \dots, x^{n-1}$. Then x^{j-1} is a weight vector with weight $\varepsilon_j = (0, \dots, 1, \dots, 0)$ (only j -th entry is 1). Therefore the weight of a weight vector decreases by a single simple root (like $\varepsilon_j - \varepsilon_{j+1}$), as its degree increases by one. This principle

holds also for $A(n, k)$, since $A(n, k) = K[x_1, \dots, x_k]/(x_1^n, \dots, x_k^n)$ is isomorphic to the tensor product $K[x_1]/(x_1^n) \otimes_K \dots \otimes_K K[x_k]/(x_k^n)$ as $GL(n)$ -modules. In particular, the homogeneous polynomial with the lowest degree in $W^\lambda \subset A(n, k)$ is the highest weight vector.

It follows from the principle above and the note after (15) that the Hilbert function $h_{W^\lambda}(q)$ of W^λ coincides with the q -dimension $\dim_q W^\lambda$ except for a multiple of a power of q . More precisely, $\dim_q W^\lambda$ starts with q^0 followed by higher degree terms, while $h_{W^\lambda}(q)$ starts with q^d , where d is the degree of the monomial of the highest weight vector. Therefore it suffices to show that the degree of monomial with weight $\lambda = (k_1, \dots, k_r) \vdash k$ ($r \leq n$) is equal to $k_2 + 2k_3 + \dots + (r-1)k_r$.

Since the weight of 1 is $\varepsilon_1 = (1, 0, \dots, 0)$ in the $GL(n)$ -module $K[x]/(x^n)$, the weight of 1 in $A \simeq (K[x]/(x^n))^{\otimes k}$ is $k\varepsilon_1 = (k, 0, \dots, 0)$. The degree of monomial with weight λ is

$$\langle k\varepsilon_1 - \lambda, \delta \rangle = \langle (k - k_1, -k_2, -k_3, \dots, -k_r), \delta \rangle. \quad (17)$$

It follows from $k = \sum_j k_j$ that (17) does not change when δ is added by $(a, a, \dots, a) \in \mathbb{R}^k$. Hence (17) equals

$$\begin{aligned} & \langle k\varepsilon_1 - \lambda, \delta - (n-1, n-1, \dots, n-1)/2 \rangle \\ &= \langle k\varepsilon_1 - \lambda, (0, -1, \dots, -n+1) \rangle \\ &= \langle \lambda, (0, 1, \dots, n-1) \rangle \\ &= k_2 + 2k_3 + \dots + (r-1)k_r. \end{aligned}$$

We thus have proved the assertion. □

7 Appendix

We exhibit a basis of the tensor space

$$(K^n)^{\otimes k}$$

so that it decomposes into irreducible $(GL(n) \times S_k)$ -modules.

In addition to the notation fixed at the beginning of Section 2, we use the following notation.

- A tableau T (of shape λ) is said to be **standard** if the boxes of T are filled with the integers $\{1, 2, \dots, k\}$, where k is the size of λ , and they increase strictly in every row and column. We denote by $\text{STab}(\lambda)$ the set of standard Young tableaux of shape λ . For example, if $\lambda = (2, 1)$, then

$$\text{STab}(\lambda) = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \right\}.$$

- A tableau T of shape λ is said to be **semi-standard** (with letters at most n) if the boxes of T are filled with integers at most n , and these integers strictly increasing in each column, and not strictly decreasing in each row. We denote by $\text{SSTab}_n(\lambda)$ the set of semi-standard tableaux of shape λ with letters at most n . For example, if $\lambda = (2, 1)$ and $n = 3$, then we have

$$\text{SSTab}_n(\lambda) = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array} \right\}.$$

Put $A = V^{\otimes k}$ where V is an n dimensional vector space over K .

The general linear group $GL(n)$ acts on the vector space A as the tensor representation

$$\Phi: GL(n) \rightarrow GL(A) \quad (18)$$

by

$$\Phi(g)(a_1 \otimes a_2 \otimes \cdots \otimes a_k) = ga_1 \otimes ga_2 \otimes \cdots \otimes ga_k,$$

for $g \in GL(n)$ and $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A$.

The symmetric group S_k acts on A as the permutation of the components, i.e., for each $\sigma \in S_k$ and $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A$, we let

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_k)^\sigma = a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(k)}.$$

According to the Schur-Weyl duality, the vector space A decomposes as $(GL(n) \times S_k)$ -modules as

$$A \cong_{GL(n) \times S_k} \bigoplus_{\substack{\lambda \vdash k \\ l(\lambda) \leq n}} W^\lambda \otimes V^\lambda,$$

where W^λ and V^λ are the λ -th irreducible representation modules of $GL(n)$ and S_k respectively. The dimensions and the multiplicities are given as follows:

	dimension	multiplicity
W^λ	$\#\text{SSTab}_n(\lambda)$	$\#\text{STab}(\lambda)$
V^λ	$\#\text{STab}(\lambda)$	$\#\text{SSTab}_n(\lambda)$

See [4], [7], [21] for details of these modules.

Let $T \in \text{SSTab}_n(\lambda)$ and $S \in \text{STab}(\lambda)$. Let Y^S be the Young symmetrizer corresponding to S . Furthermore let $\{e_1, e_2, \dots, e_n\}$ be a basis of V so the basis of A is given by

$$\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \mid 1 \leq i_1, i_2, \dots, i_k \leq n\}.$$

Then we define $e_T^S \in A$ to be

$$e_T^S = e_{T(S^{-1}(1))} \otimes e_{T(S^{-1}(2))} \otimes \cdots \otimes e_{T(S^{-1}(k))},$$

where the tableaux S and T are regarded as maps from the set of boxes in the Young diagram λ to the set of integers $\{1, 2, \dots, k\}$.

For example, if $k = n = 3$ and $S = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ and $T = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$, then $e_T^S = e_2 \otimes e_3 \otimes e_2$.

Now the irreducible constituents V^T and W^S of A are given by

$$V^T = \bigoplus_{S \in \text{STab}(\lambda)} KY^S e_T^S,$$

and

$$W^S = \bigoplus_{T \in \text{SSTab}_n(\lambda)} KY^S e_T^S.$$

Furthermore

$$\bigoplus_{S \in \text{STab}(\lambda)} W^S \cong W^\lambda \otimes V^\lambda \cong \bigoplus_{T \in \text{SSTab}_n(\lambda)} V^T,$$

and finally,

$$A = \bigoplus_{\substack{\lambda \vdash k \\ l(\lambda) \leq n}} \bigoplus_{S \in \text{STab}(\lambda)} W^S = \bigoplus_{\substack{\lambda \vdash k \\ l(\lambda) \leq n}} \bigoplus_{T \in \text{SSTab}_n(\lambda)} V^T.$$

It is well known that

$$\dim W^\lambda = \prod_{1 \leq i < j \leq n} \frac{k_i - k_j + j - i}{j - i}. \quad (19)$$

Here $\lambda = (k_1, \dots, k_r) \vdash k$. (See, for example, [7] p.303.)

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