RECONSTRUCTION OF INCLUSIONS FOR THE INVERSE BOUNDARY VALUE PROBLEM OF HEAT EQUATION USING PROBE METHOD

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Table of Contents

Chapter 1

Introduction

This dissertation studies about the probe method for stationary/non-stationary heat equation.

The probe method was originally introduced by M. Ikehata [16] for reconstructing an unknown inclusion inside an isotropic stationary heat conductive medium by many boundary measurements (*i.e.* that is the so called Dirichlet-to-Neumann map).

In Chapter 2, we developed the theory of the probe method for stationary, anisotropic heat equation with mixed type boundary condition and source term. Here the unknown inclusion D and the back ground $\Omega \setminus \overline{D}$ with an ambient region Ω have anisotropic conductivity $\gamma_0 + \gamma_1$ and γ_0, respectively. We assume γ_1 is either positive or negative definite almost everywhere in the closure $\overline{D}$ of D. In case of γ_0 and γ_1 are conformal to each other, the probe method had been already studied in [10].

We also have to point out an analogous method called singular sources method done by R. Potthast ([12]) and his collaborators ([36]). Recently, K. Erhard and R. Potthast ([12]) gave a numerical realization of the probe method and it was carefully examined by J. Cheng, J. J. Liu and G. Nakamura ([32]). They did it for the inverse boundary value problem for the Helmholtz equation identifying an obstacle.

When we search back the origin of the probe method and singular sources method, they both stem from the uniqueness result by V. Isakov [21] for identifying an unknown inclusion inside a conductive medium. The conductivities for the inclusion and known back ground were assumed to be isotropic.

Next we point out two new ingredients of our results.

(i) We give the reconstruction for identifying D under the minimum regularity assumption on

γ_0 and γ_1. That is we assume $\gamma_1 \in L^\infty(\Omega), \gamma_0 \in C^{0,1}(\overline{\Omega})$. It seems it is very hard to weaken these assumptions, because the unique continuation property was used in most of our arguments for the background medium and its necessary and sufficient regularity assumption on γ_0 is $\gamma_0 \in C^{0,1}(\overline{\Omega})$ for $n = 3$ and $\gamma_0 \in L^\infty(\Omega)$ for $n = 2$.

(ii) Due to the mixed type boundary condition and the existence of the source term, we had to prove the L^2 boundedness of the Green function of the boundary value problem associated to our inverse problem in order to analyze the behavior of indicator function which is a mathematical testing machine for the identification. This was already given by M. Grüter and K-O. Widman for $n \geq 3$, but it was missing for $n = 2$.

In Chapter 3, we consider about the probe method for identifying unknown inclusion for non-stationary heat equation. As for the known results, H. Bellout prove the local uniqueness and stability in [2] when the inclusion is independent of time. Also, A. Elayyan and V. Isakov proved the uniqueness for the localized Dirichlet-to-Neumann map ([11]).

There was not any result for reconstructing the inclusion. We developed a theory of probe method for 1 space dimension, non-stationary heat equation to reconstruct the unknown inclusion. As far as we know, this is the first attempt which gave the reconstruction of the unknown inclusion.

Likewise the argument for the stationary heat equation, we have to define the indicator function. But we cannot estimate indicator function directly, because the heat operator doesn't have the coercivity. But we can obtain representation formula of indicator function using reflected solution which is the Green function minus the fundamental solution of heat equation. This enables to analyze the behavior of indicator function by that of the reflected solution. Therefore we need to analyze reflected solution more carefully.

The behavior of the reflected solution $w(x,t)$ can be obtained in the following way.

(i) When D is independent of time, we can obtain the reflected solution by using the Laplace transformation in time and solving some transmission boundary value problem for ordinary different equation. Also, we can extract the dominant part in its behavior of the reflected solution.

(ii) When D depends on time, we freeze the coefficient of non-stationary heat equation at time τ and denote the associated reflected solution by $w^\tau(x,t)$. Then, we can prove that $w(x,t) - w^\tau(x,t)$ is in the Sobolev space $H^{1,0}$ (see Section 3.1). So, the dominant part of $w(x,t)$ can be obtained from that of $w^t(x,t)$.

In the last chapter, we deal with the numerical realization of probe method for the time-independent case. The key for this is the numerical realization of Runge's approximation theorem based on the single layer potential. We proposed a scheme for the probe method based on the optimization technique for Runge's approximation. The numerical scheme of this is also given.

Chapter 2

Stationary Heat Equation Case

2.1 Statement of the Problem and Result

Let Ω be an bounded domain in $\mathbb{R}^n$ $(n = 2$ or $3)$ with C^2 boundary Γ. Ω is considered as a conductive heat medium with heat conductivity

$$\gamma = \gamma_0 + \chi_D \gamma_1 \tag{2.1.1}$$

with matrices $\gamma_0(x) = \big(\gamma_{0ij}(x)\big), \gamma_1(x) = \big(\gamma_{1ij}(x)\big)$. The regularity assumption for γ_0 is $\gamma_0 \in C^{0,1}(\overline{\Omega})$. As for γ_1, we only assume $\gamma_1 \in L^\infty(\Omega)$. Here D is a bounded domain with Lipschitz boundary ∂D such that $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ is connected, χ_D is the characteristic function of D and $C^{0,1}(\overline{\Omega})$ is the space of functions which are Lipschitz continuous on $\overline{\Omega}$.

We assume that $\gamma = \big(\gamma_{ij}(x)\big)$ and $\gamma_0 = \big(\gamma_{0ij}(x)\big)$ are symmetric matrices satisfying

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} \gamma_{0ij}(x)\xi_i\xi_j \geq C_1|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \quad \text{a.e. } x \in \overline{\Omega}\big) \\ \displaystyle\sum_{i,j=1}^{n} \gamma_{ij}(x)\xi_i\xi_j \geq C_1|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \ \text{a.e. } x \in \overline{\Omega}\big) \end{cases} \tag{2.1.2}$$

for some constant $C_1 > 0$. Moreover, we assume that for any $a \in \partial D$, there exists a $\delta > 0$ such that either

$$\sum_{i,j=1}^{n} \gamma_{1ij}\xi_i\xi_j \geq C_2|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \ \text{a.e. } x \in B_\delta(a) \cap D\big) \tag{2.1.3}$$

or

$$\sum_{i,j=1}^{n} \gamma_{1ij}\xi_i\xi_j \leq -C_2|\xi|^2 \quad \big(\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \ \text{a.e. } x \in B_\delta(a) \cap D\big) \tag{2.1.4}$$

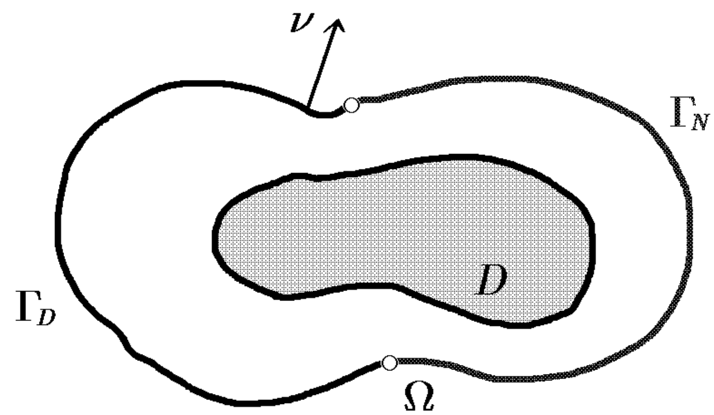
holds for some constant $C_2 > 0$, where $B_\delta(a) := \{x \in \mathbb{R}^n; |x - a| < \delta\}$.

Let Γ consist of two parts. That is

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \tag{2.1.5}$$

where Γ_D, Γ_N are open subsets of Γ such that $\Gamma_D \cap \Gamma_N = \phi$, $\Gamma_D \neq \phi$, $\Gamma_N \neq \phi$ and for $n = 3$, the boundaries $\partial\Gamma_D$ of Γ_D and $\partial\Gamma_N$ of Γ_N are C^2. We have assumed $\Gamma_D \neq \phi$ here for simplicity. For the case $\Gamma_D = \phi$, consult [18].

The two dimensional figure of Ω, D, Γ_D and Γ_N is given below.



Consider the mixed type boundary value problem of stationary heat equation:

$$\begin{cases} \big(L_D u\big)(x) := \operatorname{div}(\gamma(x)\nabla u(x)\big) = F(x) \text{ in } \Omega \\ u = f \text{ on } \Gamma_D, \quad \partial_{L_D} u = g \text{ on } \Gamma_N \end{cases} \tag{2.1.6}$$

for given $f \in \overline{H}^{\frac{1}{2}}(\Gamma_D), g \in \overline{H}^{-\frac{1}{2}}(\Gamma_N), F \in L^2(\Omega)$ where

$$\big(\partial_{L_D} u\big)(x) := \gamma(x)\partial_\nu u(x) \tag{2.1.7}$$

with the unit outer normal vector $\nu = (\nu_1, \cdots, \nu_n)$ of Γ. We also define

$$(L_\phi u)(x) := \operatorname{div}(\gamma_0(x)\nabla u(x)) \tag{2.1.8}$$

and

$$(\partial_{L_\phi} u)(x) := \gamma_0(x)\partial_\nu u(x). \tag{2.1.9}$$

Here we have used the notations given in [14] to denote Sobolev spaces.

The physical meaning of f, g and u are the temperature, heat flux and heat, respectively. The mixed type boundary condition appears in many practical applications. For example, if Γ_D is grounded or iced to have voltage or temperature equal zero on Γ_D, we have $f = 0$. Then, the measurement Π_D is to measure temperature induced from inputting heat flux infinitely many times. When $\Gamma_D = \phi$, the many measurements Π_D correspond to the so called continuous model ([4]).

There is an another model called the complete model which is more practical than the continuous model. The complete model ([4]) has some mixed type boundary condition, but it is not exactly the same as ours. Recently, Hyvönen ([15]) showed that the complete model can be approximated by the continuous model.

From the Lax-Milgram theorem (see Section 2.5), (2.1.6) has a unique solution $u = u(f, g, F) \in \overline{H}^1(\Omega)$ with the estimate:

$$\|u\|_{\overline{H}^1(\Omega)} \le C\Big(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{L^2(\Omega)}\Big), \tag{2.1.10}$$

where the constant $C > 0$ does not depend on f, g, F.

Moreover, even for $F \in W^*$ with $W := \{w \in \overline{H}^1(\Omega);\ \ w = 0 \text{ on } \Gamma_D\}$ and $\operatorname{supp} F \subset \Omega$, we have a similar result and in this case $\|F\|_{L^2(\Omega)}$ in (2.1.10) has to be replaced by $\|F\|_{W^*}$. Hereafter, the norm $\|\cdot\|_W$ and inner product $(\ ,\ )$ of W are those of $\overline{H}^1(\Omega)$, and the norm of the dual space W^* of W is denoted by $\|\cdot\|_{W^*}$.

Next, we define the Dirichlet to Neumann map Λ_D and the Neumann to Dirichlet map Π_D as follows.

Definition 2.1.1. Let $u(f, g, F)$ be the solution to (2.1.6).

(i) Fixing g and F, define $\Lambda_D : \overline{H}^{\frac{1}{2}}(\Gamma_D) \to \overline{H}^{-\frac{1}{2}}(\Gamma_D)$ by

$$\Lambda_D(f) := \partial_{L_D} u(f, g, F) \text{ on } \Gamma_D. \tag{2.1.11}$$

(ii) Fixing f and F, define $\Pi_D : \overline{H}^{-\frac{1}{2}}(\Gamma_N) \to \overline{H}^{\frac{1}{2}}(\Gamma_N)$ by

$$\Pi_D(g) := u(f, g, F) \text{ on } \Gamma_N. \tag{2.1.12}$$

And we also define $\Lambda_\phi := \Lambda_D, \Pi_\phi := \Pi_D$ if $D = \phi$.

Remark 2.1.1. The trace of $\partial_{L_D} u(f, g, F) \in \overline{H}^{\frac{1}{2}}(\Omega)$ exists, because $F \in L^2(\Omega)$ or $F \in W^*$ with $\operatorname{supp} F \subset \Omega$.

Now, we consider the two kinds of inverse problems *(IP1)* and *(IP2)*:

(IP1) Suppose γ_0 is known and γ_1, D are unknown. Reconstruct D from Λ_D.

(IP2) Suppose γ_0 is known and γ_1, D are unknown. Reconstruct D from Π_D.

Theorem 2.1.1. *There are reconstruction procedures for the both inverse problems (IP1) and (IP2).*

Remark 2.1.2. ([3])
(i) Calderón started the inverse problem for identifying the conductivity γ from Λ_D when $\Gamma_N = \phi$ and $F = 0$.
(ii) Let Ω_1 be subdomain of Ω such that $D \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$, $\Omega \setminus \overline{\Omega_1}$ and $\Omega_1 \setminus \overline{D}$ are connected and its boundary Γ_1 is Lipschitz smooth. Define the Dirichlet to Neumann map $\Lambda_{1D}: \overline{H}^{\frac{1}{2}}(\Gamma_1) \to \overline{H}^{-\frac{1}{2}}(\partial\Omega_1)$ by $\Lambda_{1D}(\varphi) := \partial_{L_D} v(\varphi)$ on Γ_1 for any $\varphi \in \overline{H}^{\frac{1}{2}}(\Gamma_1)$ where $v = v(\varphi) \in \overline{H}^1(\Omega_1)$ is the solution to $L_D v = 0$ in Ω_1, $v = \varphi$ on Γ_1. Knowing Λ_{1D}, D can be reconstructed from Λ_{1D} by an argument analogous to that given in [17]. However, to relate Λ_{1D} to Λ_D or Π_D, the usual way is to solve the Cauchy problem iteratively which is very ill-posed. Therefore, we focuss on obtaining a reconstruction procedure which directly uses Λ_D or Π_D.

2.2 Indicator Functions and Reconstruction Procedure

Definition 2.2.1 (Needle). We call a nonselfintersecting piecewise C^1 curve $\mathcal{C} := \{c(\theta); 0 \le \theta \le 1\}$ joining $c(0), c(1) \in \Gamma$ needle if it satisfies $\mathcal{C} \setminus \{c(0), c(1)\} \subset \Omega$.

Definition 2.2.2 (Singular Solution).
(i) Fix $x^0 \in \Omega$ and $G(x - x^0) \in \mathcal{D}'(\mathbb{R}^n)$ be a fundamental solution of

$$\mathrm{div}\big(\gamma_0(x^0)\nabla G(x - x^0)\big) + \delta(x - x^0) = 0 \ \text{ in } \mathbb{R}^n. \tag{2.2.1}$$

(ii) Let $H_j(x, x^0) \in \mathcal{D}'(\mathbb{R}^n_x)$ $(j = 1, 2)$ be solutions of

$$L_\phi H_j(x, x^0) + \delta(x - x^0) = 0 \ \text{ in } \Omega \tag{2.2.2}$$

such that

$$H_j(x, x^0) - G(x - x^0) \in \overline{H}^1(\Omega) \tag{2.2.3}$$

and

$$\begin{cases} \partial_{L_\phi} H_1(x, x^0) = 0 \ \text{ on } \Gamma_N \\ H_2(x, x^0) = 0 \ \text{ on } \Gamma_D. \end{cases} \tag{2.2.4}$$

We call each $H_j(x, x^0)$ singular solution.

Remark 2.2.1. The construction of singular solution can be done similarly as Lemma 3 in [19]

Let $\mathcal{C} := \{c(\theta); 0 \leq \theta \leq 1\}$ be a needle. By the Runge's approximation theorem in Section 2.6, there exist sequences of approximate functions $\{v_{1k}\}, \{v_{2k}\} \subset \overline{H}^1(\Omega)$ such that

$$v_{jk} \to V_j(\,\cdot\,, c(\theta)) := v'_j + H_j\big(\,\cdot\,, c(\theta)\big) \quad (k \to \infty) \quad \text{ in } \overline{H}^1_{\text{loc}}(\Omega \setminus \mathcal{C}_\theta) \quad (j = 1, 2) \tag{2.2.5}$$

where

$$\begin{cases} L_\phi v_{1k} = F \text{ in } \Omega \\ \partial_{L_\phi} v_{1k} = g \text{ on } \Gamma_N \end{cases} \tag{2.2.6}$$

and

$$\begin{cases} L_\phi v_{2k} = F \text{ in } \Omega \\ v_{2k} = f \text{ on } \Gamma_D \end{cases} \tag{2.2.7}$$

where $\mathcal{C}_\theta := \{c(\vartheta) : 0 \leq \vartheta \leq \theta\}$ and $v'_j \in \overline{H}^1(\Omega)$ $(j = 1, 2)$ are the solutions to

$$\begin{cases} L_\phi v'_1 = F \text{ in } \Omega \\ v'_1 = 0 \text{ on } \Gamma_D, \quad \partial_{L_\phi} v'_1 = g \text{ on } \Gamma_N \end{cases} \tag{2.2.8}$$

and

$$\begin{cases} L_\phi v'_2 = F \text{ in } \Omega \\ v'_2 = f \text{ on } \Gamma_D, \quad \partial_{L_\phi} v'_2 = 0 \text{ on } \Gamma_N. \end{cases} \tag{2.2.9}$$

we call $\{v_{1k}\}, \{v_{2k}\} \subset \overline{H}^1(\Omega)$ Runge's approximation functions.

Definition 2.2.3 (Indicator Functions for Stationary Heat Equation Case). Let $\mathcal{C} = \{c(\theta); 0 \leq \theta \leq 1\}$ be a needle, θ $(0 < \theta < 1)$ satisfy $\mathcal{C}_\theta \cap \overline{D} = \phi$ and $\{v_{jk}\} \subset \overline{H}^1(\Omega)$ $(j = 1, 2)$ be the Runge's approximation functions. Then, we define two indicator functions for stationary heat equation case $I_1(\theta, \mathcal{C})$ and $I_2(\theta, \mathcal{C})$ associated with *(IP1)* and *(IP2)*:

$$I_1(\theta, \mathcal{C}) := \lim_{k \to \infty} \big\langle (\Lambda_D - \Lambda_\phi)\big(v_{1k}\big|_{\Gamma_D}\big), v_{1k}\big|_{\Gamma_D} \big\rangle_1 \tag{2.2.10}$$

and

$$I_2(\theta, \mathcal{C}) := \lim_{k \to \infty} \big\langle \big(\partial_{L_D} v_{2k}\big)\big|_{\Gamma_N}, (\Pi_D - \Pi_\phi)\big((\partial_{L_D} v_{2k})\big|_{\Gamma_N}\big) \big\rangle_2 \tag{2.2.11}$$

where $\langle\, , \,\rangle_1$ and $\langle\, , \,\rangle_2$ are the pairings for the pair $\{\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D}), \overline{H}^{\frac{1}{2}}(\Gamma_D)\}$ and $\{\overline{H}^{-\frac{1}{2}}(\Gamma_N), \dot{H}^{\frac{1}{2}}(\overline{\Gamma_N})\}$, respectively.

Remark 2.2.2. From (2.3.7), we can see that the definitions of the indicator functions do not depend on the choice of $\{v_{jk}\}$.

Definition 2.2.4 (First Hitting Point). Let $\mathcal{C} = \{c(\theta); 0 \leq \theta \leq 1\}$ be a needle such that $\mathcal{C} \cap \overline{D} \neq \phi$. We define $\Theta(\mathcal{C}, D)$ by

$$\Theta(\mathcal{C}, D) := \sup\{\,\theta\,; 0 < \theta < 1,\;\; c(\vartheta) \notin \overline{D}\;\; (0 \leq \vartheta < \theta)\}. \tag{2.2.12}$$

We call $c(\Theta(\mathcal{C}, D))$ the first hitting point of $\mathcal{C}$ to D.

Definition 2.2.5 (Detecting Point). Let $\mathcal{C}$ be as in Definition 2.2.4. For the indicator functions $I_j(\theta, \mathcal{C})$ $(j = 1, 2)$, we define their detecting point $c(\theta_j(\mathcal{C}, D))$ $(j = 1, 2)$ by

$$c(\theta_j(\mathcal{C}, D)) := \sup\Big\{\, c(\theta)\,; 0 < \theta < 1,\;\; \sup_{0<\vartheta<\theta} |I_j(\vartheta, \mathcal{C})| < \infty \Big\}. \tag{2.2.13}$$

Then, we have our main theorem.

Theorem 2.2.1. *For each j $(j = 1, 2)$, we obtain detecting point is first hitting point. i.e*

$$\Theta(\mathcal{C}, D) = \theta_j(\mathcal{C}, D) \quad \textit{if } \mathcal{C} \cap \overline{D} \neq \phi. \tag{2.2.14}$$

Since we can reconstruct D by knowing $\theta_j(\mathcal{C}, D)$ for all possible $\mathcal{C}$, Theorem 2.2.1 implies Theorem 2.1.1.

Before ending this section, we summarize all the steps necessary for our reconstruction procedure. For simplicity, we only give them for the inverse problem *(IP1)*.

Step 1. Consider a needle $\mathcal{C} = \{c(\theta); 0 \leq \theta \leq 1\}$ and the domain $\Omega \setminus \mathcal{C}_\theta$.

$(\mathcal{C}_\theta := \{c(\vartheta); 0 \leq \vartheta \leq \theta\})$

Step 2. Take $v_{1k} \in \overline{H}^1(\Omega)$ $(k \in \mathbb{N})$ which approximates $V_1(\,\cdot\,, c(\theta)) = v_1' + H_1(\,\cdot\,, c(\theta))$.

(See Section 2.6 for the details).

Step 3. Compute the indicator function $I_1(\theta, \mathcal{C}) = \lim_{k\to\infty} \big\langle (\Lambda_D - \Lambda_\phi)\big(v_{1k}\,|_{\Gamma_D}\big), v_{1k}\,|_{\Gamma_D} \big\rangle_1$ for small θ.

Step 4. Increase θ and search for the detecting point $c(\theta_1(\mathcal{C}, D))$ at which $|I_1(\theta, \mathcal{C})|$ blows up.

By Theorem 2.2.1, this gives the first hitting point $c(\Theta(\mathcal{C}, D))$.

Step 5. Take many $\mathcal{C}$'s and repeat all the previous steps. Plot all the $c(\Theta(\mathcal{C}, D))$ for these γ's.

Then these points generate the boundary ∂D of D.

2.3 Estimates of Indicator Functions

In this Section we give some estimates for the indicator functions for stationary heat equation case $I_j(\theta, \mathcal{C})$ $(j = 1, 2)$.

Let $u_{jk} \in \overline{H}^1(\Omega)$ $(j = 1, 2;\, k \in \mathbb{N})$ be

$$\begin{cases} u_{1k} := u\big(v_{1k}\big|_{\Gamma_D}, g, F\big) \\ u_{2k} := u\big(f, (\partial_{L_\phi} v_{2k})\big|_{\Gamma_N}, F\big), \end{cases} \tag{2.3.1}$$

where $u = u(f, g, F)$ is the solution to (2.1.6).

To estimate indicator functions, we need the following Lemma:

Lemma 2.3.1 (Weak Solution). *Let*

$$w_{jk} := u_{jk} - v_{jk} \quad (j = 1, 2;\, k \in \mathbb{N}), \tag{2.3.2}$$

then for j=1,2, w_{jk} has a limit $w_j \in \overline{H}^1(\Omega)$ satisfying

$$\begin{cases} L_D w_j = \operatorname{div}\Big((\gamma_0 - \gamma)\nabla V_j\big(\,\cdot\,, c(\theta)\big)\Big) \quad \text{in } \Omega \\ w_j = 0 \quad \text{on } \Gamma_D, \quad \partial_{L_D} w_j = 0 \quad \text{on } \Gamma_N. \end{cases} \tag{2.3.3}$$

Proof w_{jk} satisfy

$$\begin{cases} L_D w_{jk} = \operatorname{div}((\gamma_0 - \gamma)\nabla v_{jk}) \text{ in } \Omega \\ w_{jk} = 0 \text{ on } \Gamma_D, \quad \partial_{L_D} w_{jk} = 0 \text{ on } \Gamma_N. \end{cases} \tag{2.3.4}$$

More precisely, $w_{jk} \in W$ is the solution of the variational equation:

$$\int_\Omega \gamma \nabla w_{jk} \cdot \nabla\varphi\, dx = \int_\Omega (\gamma_0 - \gamma)\nabla v_{jk} \cdot \nabla\varphi\, dx \quad (\varphi \in W). \tag{2.3.5}$$

Since

$$\sup_{\|\varphi\|_W \le 1} \left| \int_\Omega (\gamma_0 - \gamma)\nabla(v_{jk} - v_{jl}) \cdot \nabla\varphi\, dx \right| \le \|\gamma_1\|_{L^\infty(D)} \|v_{jk} - v_{jl}\|_{\overline{H}^1(D)} \to 0 \tag{2.3.6}$$

as $k, l \to \infty$ by $\mathcal{C}_\theta \cap \overline{D} = \phi$ and from (2.2.5), $v_{jk} \to V_j(\,\cdot\,, c(\theta))$ $(k \to \infty)$ in $\overline{H}^1_{\mathrm{loc}}(\Omega \setminus \mathcal{C}_\theta)$, we have from (2.1.10)

$$\|w_{jk} - w_{jl}\|_{\overline{H}^1(\Omega)} \to 0 \quad (k, l \to \infty). \tag{2.3.7}$$

Hence, there exist limits $w_j := \lim_{k\to\infty} w_{jk} \in \overline{H}^1(\Omega)$ $(j = 1, 2)$ and they satisfy (2.3.3). □

Also, we use the following blow-up properties to estimate indicator functions.

Theorem 2.3.2 (Blow-up Property). *Let $u, v \in \overline{H}^1(\Omega)$ be the solution to*

$$\begin{cases} L_D u = F \ \text{in } \Omega \\ u = f \ \text{on } \Gamma_D, \quad \partial_{L_D} u = g \ \text{on } \Gamma_N, \end{cases} \quad \begin{cases} L_\phi v = F \ \text{in } \Omega \\ u = f \ \text{on } \Gamma_D, \quad \partial_{L_\phi} u = g \ \text{on } \Gamma_N, \end{cases} \tag{2.3.8}$$

respectively, then u, v satisfy the following estimation:
(i)

$$\langle (\Lambda_D - \Lambda_\phi) f, f \rangle_1 \leq \int_D \gamma_1 \nabla v \cdot \nabla v \, dx - \int_\Omega F(u-v)\, dx + \langle g, u-v \rangle_2 \tag{2.3.9}$$

and

$$\langle (\Lambda_D - \Lambda_\phi) f, f \rangle_1 \geq \int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla v) \cdot (\gamma_0 \nabla v) \, dx - \int_\Omega F(u-v)\, dx + \langle g, u-v \rangle_2. \tag{2.3.10}$$

(ii)

$$\langle g, (\Pi_D - \Pi_\phi) g \rangle_2 \leq \int_D \gamma_1 \nabla v \cdot \nabla v \, dx - \int_\Omega F(u-v)\, dx - \langle \partial_{L_\phi}(u-v), f \rangle_1 \tag{2.3.11}$$

and

$$\langle g, (\Pi_D - \Pi_\phi) g \rangle_2 \geq \int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla v) \cdot (\gamma_0 \nabla v) \, dx - \int_\Omega F(u-v)\, dx - \langle \partial_{L_\phi}(u-v), f \rangle_1. \tag{2.3.12}$$

The proof is given in Section 2.8.

Therefore,

$$\begin{aligned} &\int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla V_1(\,\cdot\,, c(\theta))) \cdot (\gamma_0 \nabla V_1(\,\cdot\,, c(\theta)))\, dx - \int_\Omega F w_1 \, dx + \langle g, w_1 \rangle_2 \\ &\leq I_1(\theta, \mathcal{C}) \leq \int_D \gamma_1 \nabla V_1(\,\cdot\,, c(\theta)) \cdot \nabla V_1(\,\cdot\,, c(\theta))\, dx - \int_\Omega F w_1 \, dx + \langle g, w_1 \rangle_2 \end{aligned} \tag{2.3.13}$$

and

$$\begin{aligned} &\int_D {\gamma_0}^{-1} \gamma_1 \gamma^{-1} (\gamma_0 \nabla V_2(\,\cdot\,, c(\theta))) \cdot (\gamma_0 \nabla V_2(\,\cdot\,, c(\theta)))\, dx - \int_\Omega F w_2 \, dx + \langle \partial_{L_\phi} w_2, f \rangle_1 \\ &\leq I_2(\theta, \mathcal{C}) \leq \int_D \gamma_1 \nabla V_2(\,\cdot\,, c(\theta)) \cdot \nabla V_2(\,\cdot\,, c(\theta))\, dx - \int_\Omega F w_2 \, dx - \langle \partial_{L_\phi} w_2, f \rangle_1. \end{aligned} \tag{2.3.14}$$

2.4 Behavior of the Indicator Functions

In this section we analyze the behavior of the indicator functions $I_j(\theta, \mathcal{C})$ as $\theta \uparrow \Theta(\mathcal{C}, D)$ when $\mathcal{C} \cap \overline{D} \neq \phi$. Hereafter, constants C, C' which will appear in the estimates are general constants.

Let $\mathcal{C} \cap \overline{D} \neq \phi$ and $0 < \theta < 1$ satisfy $\mathcal{C}_\theta \cap \overline{D} = \phi$.

Lemma 2.4.1. *There exists a constant $M > 0$ independent of θ such that*

$$\|w_j\|_{L^2(\Omega)} \leq M \quad (j = 1, 2) \ \text{as } \theta \uparrow \Theta(\mathcal{C}, D). \tag{2.4.1}$$

Proof For simplicity, put $\theta_0 := \Theta(\mathcal{C}, D)$ and $c_{\theta_0} := c(\theta_0)$. For $0 < \theta < \theta_0$, $K = K\big(x, c(\theta)\big)$ be the Green function given in Section 2.7 for $A = L_D$.

K satisfies

$$\begin{cases} L_D K + \delta\big(\,\cdot\, - c(\theta)\big) = 0 \text{ in } \Omega \\ K = 0 \text{ on } \Gamma. \end{cases} \tag{2.4.2}$$

Define W_j $(j = 1, 2)$ by

$$W_j = w_j + V_j \quad (j = 1, 2). \tag{2.4.3}$$

Then, from (2.2.2), (2.2.4), (2.2.8), (2.2.9) and (2.3.3), we have

$$\begin{cases} L_D W_1 + \delta\big(\,\cdot\, - c(\theta)\big) = F \text{ in } \Omega \\ W_1 = H_1\big(\,\cdot\,, c(\theta)\big) \text{ on } \Gamma_D, \quad \partial_{L_D} W_1 = g \text{ on } \Gamma_N \end{cases} \tag{2.4.4}$$

and

$$\begin{cases} L_D W_2 + \delta\big(\,\cdot\, - c(\theta)\big) = F \text{ in } \Omega \\ W_2 = f \text{ on } \Gamma_D, \quad \partial_{L_D} W_2 = \partial_{L_D} H_2\big(\,\cdot\,, c(\theta)\big) \text{ on } \Gamma_N. \end{cases} \tag{2.4.5}$$

Hence, defining Z_j $(j = 1, 2)$ by

$$Z_j = W_j - K, \tag{2.4.6}$$

we have

$$\begin{cases} L_D Z_1 = F \text{ in } \Omega \\ Z_1 = H_1\big(\,\cdot\,, c(\theta)\big) \text{ on } \Gamma_D, \quad \partial_{L_D} Z_1 = g - \partial_{L_D} w \text{ on } \Gamma_N \end{cases} \tag{2.4.7}$$

and

$$\begin{cases} L_D Z_2 = F \text{ in } \Omega \\ Z_2 = f \text{ on } \Gamma_D, \quad \partial_{L_D} Z_2 = \partial_{L_D} H_2\big(\,\cdot\,, c(\theta)\big) - \partial_{L_D} w \text{ on } \Gamma_N. \end{cases} \tag{2.4.8}$$

Next we prove that $\partial_{L_D} K$ is uniformly bounded in $\overline{H}^{-\frac{1}{2}}(\Gamma)$ as $\theta \uparrow \theta_0$. In order to do that let $\eta \in C_0^\infty(\Omega)$, $\eta = 1$ in an open neighborhood of $\overline{D}$ and $\zeta := 1 - \eta$. Then, we have

$$L_D(\zeta K) = \big(L_\phi \zeta\big) K + 2\gamma_0(x) \nabla\zeta \cdot \nabla K. \tag{2.4.9}$$

Here, we can assume $c(\theta) \notin \operatorname{supp} \zeta$. Hence, from Theorem 2.7.1, the right hand side of (2.4.9) is uniformly bounded in $L^2(\Omega)$ as $\theta \uparrow \theta_0$. Then, $(\zeta K)\big|_\Gamma = 0$ and the well-posedness of the Dirichlet boundary value problem imply $\partial_{L_D} w = \partial_{L_D}(\zeta K)$ is uniformly bounded in $\overline{H}^{-\frac{1}{2}}(\Gamma)$ as $\theta \uparrow \theta_0$.

Now, by (2.1.6) and what we have just proven, we have that for each j $(j = 1, 2)$, Z_j is uniformly bounded in $\overline{H}^1(\Omega)$ as $\theta \to \theta_0$. Hence, by (2.2.3), (2.4.3), (2.4.6) and (2.7.12), $w_j = Z_j + K - V_j$ is uniformly bounded in $L^2(\Omega)$ as $\theta \uparrow \theta_0$.

Let $\alpha \in C_0^\infty(\Omega)$ satisfy $\alpha = 1$ in an open neighborhood of $\overline{D}$ and $\widetilde{w}_j := w_j - \alpha w_j$ $(j = 1, 2)$. From (2.3.3) and $\mathrm{supp}(\gamma - \gamma_0) \subset \overline{D}$, we have

$$(1-\alpha)L_D w_j = 0 \text{ in } \Omega \tag{2.4.10}$$

and

$$L_D \alpha = L_\phi \alpha. \tag{2.4.11}$$

Then $\widetilde{w}_j$ satisfies

$$\begin{cases} L_\phi \widetilde{w}_j = F_j \text{ in } \Omega \\ \widetilde{w}_j = 0 \text{ on } \Gamma_D, \quad \partial_{L_D} \widetilde{w}_j = 0 \text{ on } \Gamma_N, \end{cases} \tag{2.4.12}$$

where $F_j := -(L_D\alpha)w_j - 2\gamma\nabla\alpha \cdot \nabla w_j$ satisfies $\mathrm{supp}\, F_j \subset \Omega$ and $\|F_j\|_{W^*}$ is uniformly bounded for any θ $(0 < \theta - \theta_0 \leq \eta)$. Therefore, by the continuity of the trace and $\partial_{L_\phi} w_2 = \partial_{L_D} \widetilde{w}_2$,

$$\|w_1\|_{\overline{H}^{\frac{1}{2}}(\Gamma_N)} + \|\partial_{L_\phi} w_2\|_{\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D})} \leq M \text{ as } \theta \uparrow \Theta(\mathcal{C}, D) \tag{2.4.13}$$

for another constant $M > 0$ independent of θ.

Now it is easy to see that the dominant parts of $\int_D {\gamma_0}^{-1}\gamma_1\gamma^{-1}(\gamma_0\nabla V_j(\,\cdot\,, c(\theta))) \cdot (\gamma_0\nabla V_j(\,\cdot\,, c(\theta)))\, dx$ and $\int_D \gamma_1 \nabla V_j(\,\cdot\,, c(\theta)) \cdot \nabla V_j(\,\cdot\,, c(\theta))\, dx$ are $\int_{D\cap B_\delta(c_{\theta_0})} {\gamma_0}^{-1}\gamma_1\gamma^{-1}\Big(\gamma_0\nabla G_j\big(\,\cdot\, - c(\theta)\big)\Big) \cdot \Big(\gamma_0\nabla G_j\big(\,\cdot\, - c(\theta)\big)\Big)\, dx$ and $\int_{D\cap B_\delta(c_{\theta_0})} \gamma_1 \nabla G_j\big(\,\cdot\, - c(\theta)\big) \cdot \nabla G_j\big(\,\cdot\, - c(\theta)\big)\, dx$ which are positive or negative according to (2.1.3) or (2.1.4) and blow up as $\theta \uparrow \theta_0$. Here we have used the identity:

$${\gamma_0}^{-1}\gamma_1(\gamma_0+\gamma_1)^{-1} = (\gamma_0+\gamma_1)^{-1}\gamma_1(\gamma_0+\gamma_1)^{-1} + (\gamma_0+\gamma_1)^{-1}\gamma_1{\gamma_0}^{-1}\gamma_1(\gamma_0+\gamma_1)^{-1} \tag{2.4.14}$$

Therefore, by (2.2.5), (2.3.13), (2.3.14), (2.4.1), (2.4.13), and the definition of the singular solution and its property given in Defintion 2.2.2,

$$|I_j(\theta, \mathcal{C})| \to \infty \quad (\theta \uparrow \theta_0). \tag{2.4.15}$$

Finally, (2.2.14) can be proven by the standard argument given in [17], So we omit its proof.

2.5 Boundary Problem for Forward Problem

In this section we discuss about the mixed type boundary problem for forward problem. Now we assume $\gamma \in L^\infty(\Omega)$ satisfies $\gamma \geq \delta$ in Ω

Theorem 2.5.1. *If f, g, F as follows there exists unique solution of (2.1.6) . Moreover u satisfies (2.1.10)*

Proof For $f \in \overline{H}^{\frac{1}{2}}(\Gamma_D)$, there exists $\widetilde{f} \in \overline{H}^{\frac{1}{2}}(\Gamma)$ which is extension of f.

Let $\widetilde{u} \in \overline{H}^1(\Omega)$ be the solution to

$$L_D \widetilde{u} = 0 \text{ in } \Omega, \qquad \widetilde{u}\big|_{\Gamma} = \widetilde{f}. \tag{2.5.1}$$

Then,

$$\|\widetilde{u}\|_{\overline{H}^1(\Omega)} \leq C\|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)} \leq C'\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} \tag{2.5.2}$$

Also, by the continuity of trace,

$$\|\partial_{L_D}\widetilde{u}\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} \leq C\|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)} \leq C'\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} \tag{2.5.3}$$

Let $v := u - \widetilde{u}$, v satisfies

$$\begin{cases} L_D v = G \text{ in } \Omega \\ v = 0 \text{ on } \Gamma_D, \quad \partial_{L_D} v = \text{ on } \Gamma_N \end{cases} \tag{2.5.4}$$

where $G = F - L_D\widetilde{u}, \widetilde{g} := g - \partial_{L_D}\widetilde{u}$. Now define

$$\langle G, w\rangle := \langle F, w\rangle + \int_\Omega \gamma \nabla\widetilde{u} \cdot \nabla w \, dx + \int_{\Gamma_N} \widetilde{g} w\big|_{\Gamma_N} \, d\Gamma \tag{2.5.5}$$

and

$$B[v, w] := \int_\Omega \gamma \nabla v \cdot \nabla w \, dx \tag{2.5.6}$$

for any $v, w \in W$. By the Schwarz inequality,

$$|B[v, w]| \leq \int_\Omega |\gamma||\nabla v||\nabla w| \, dx \leq M\|v\|_{\overline{H}^1(\Omega)}\|w\|_{\overline{H}^1(\Omega)}. \tag{2.5.7}$$

By the Poincaré inequality

$$B[v, v] \geq \int_\Omega \gamma|\nabla v|^2 \, dx \geq \delta\|\nabla v\|_{L^2(\Omega)} \geq \delta'\|v\|_{\overline{H}^1(\Omega)}. \tag{2.5.8}$$

for some constant $\delta' > 0$ independent of v, w.

Now we remind the Lax-Milgram theorem.

Theorem 2.5.2 (Lax-Milgram Theorem). *Let X be real Hilbert space and $B : X \times X \to \mathbb{R}$ be bilinear map satisfying*

$$|B[x, y]| \leq \gamma \|x\| \|y\| \tag{2.5.9}$$

$$B[x, x] \geq \delta \|x\|^2, \tag{2.5.10}$$

then there exists a unique bounded linear bijective operator $S : X \to X$ such that

$$(x, y) = B[Sx, y] \tag{2.5.11}$$

and

$$\|S\| \leq \delta^{-1}, \ \ \|S^{-1}\| \leq \gamma \tag{2.5.12}$$

By applying Theorem 2.5.2, there exists a unique bounded linear bijective operator $S : W \to W$ such that

$$(S^{-1}v, w) = B[v, w] \quad (v, w \in W), \quad \|S\| \leq (\delta')^{-1}, \ \ \|S^{-1}\| \leq M \tag{2.5.13}$$

where (,) is the inner product in $W \times W$.

As immediate estimates, we have

$$|\langle F, w\rangle| \leq \|F\|_{W^*} \|w\|_W, \tag{2.5.14}$$

$$\left| \int_\Omega \gamma \nabla \widetilde{u} \cdot \nabla w \, dx \right| \leq M \|\widetilde{u}\|_{\overline{H}^1(\Omega)} \|w\|_{\overline{H}^1(\Omega)}. \tag{2.5.15}$$

Hence, by (2.5.3), (2.5.14) and (2.5.15), the continuity of the trace $\widetilde{u}\big|_\Gamma = \widetilde{f}$ and extension $\widetilde{f}$ of f,

$$|\langle G, w\rangle| \leq C\big(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{W^*}\big)\|w\|_W. \tag{2.5.16}$$

By the Riesz representation theorem, the exists a unique $\widetilde{v} \in W$ such that

$$\langle G, w\rangle = -(\widetilde{v}, w), \quad \|\widetilde{v}\|_W = \|G\|_{W^*} \tag{2.5.17}$$

Let $v_0 \in W$ be $v_0 = S^{-1}\widetilde{v}$. Then, by (2.5.13),

$$B[v_0, w] + \langle G, w\rangle = 0 \quad (w \in W) \tag{2.5.18}$$

Therefore, v_0 is the solution to (2.5.1).

$$\begin{aligned} \|u\|_{\overline{H}^1(\Omega)} &\leq \|v_0\|_{\overline{H}^1(\Omega)} + \|\widetilde{u}\|_{\overline{H}^1(\Omega)} \leq C\big(\|\widetilde{v}\|_{\overline{H}^1(\Omega)} + \|\widetilde{f}\|_{\overline{H}^{\frac{1}{2}}(\Gamma)}\big) \\ &\leq C'\big(\|G\|_{W^*} + \|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)}\big) \leq C''\big(\|f\|_{\overline{H}^{\frac{1}{2}}(\Gamma_D)} + \|g\|_{\overline{H}^{-\frac{1}{2}}(\Gamma_N)} + \|F\|_{W^*}\big) \end{aligned} \tag{2.5.19}$$

□

2.6 Runge's Approximation Theorem

In this section two Runge's approximation theorems are given and they are applied to construct construct the two sequences of approximate functions $\{v_{1k}\}$ and $\{v_{2k}\}$ given in Section 2.2.

Theorem 2.6.1 (Runge's Approximation Theorem to L_ϕ 1). *Let U be an open subset of Ω such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected.*

$$\begin{cases} X := \left\{ u\big|_U ; u \in \overline{H}^1(V), L_\phi u = 0 \text{ in an open neighborhood } V \text{ of } U \right\} \\ Y := \left\{ v\big|_U ; v \in \overline{H}^1(\Omega), L_\phi v = 0 \text{ in } \Omega, \partial_{L_\phi} v\big|_{\Gamma_N} = 0,\ \operatorname{supp}\big(v\big|_{\Gamma_D}\big) \subset \Gamma_0 \right\} \end{cases} \tag{2.6.1}$$

where V is an open subset of Ω depending on u such that

$$\overline{U} \subset V \subset \overline{V} \subset \Omega \tag{2.6.2}$$

and Γ_0 is a fixed open subset of Γ_D. Then, Y is dense in X with respect to $\overline{H}^1(U)$ topology.

Proof By Hahn-Banach theorem, it is enough to prove.

$$f \in \overline{H}^1(V)^*, f(v\big|_U) = 0 \ (v \in Y) \Longrightarrow f(u\big|_U) = 0 \quad (u \in X) \tag{2.6.3}$$

Suppose $f \in \overline{H}^1(V)^*$, $f(v\big|_U) = 0 \quad (v \in Y)$. Let $y \in \Gamma_0$ and take a small open ball B centered at y and $\Omega_0 := \Omega \cup B$. We extend $\gamma_0 \in C^{0,1}(\overline{\Omega})$ to a neighborhood of $\overline{\Omega_0}$ preserving its regularity . Also, let

$$T: \{\Psi \in \overline{H}^1(V);\ \Psi\big|_{\Gamma_D} = 0\} \to \mathbb{R}, \quad T(\Psi) = f(\Psi|_U). \tag{2.6.4}$$

T has a bounded linear extension $\widetilde{T} \in \overline{H}^1(\Omega)^*$. Hence, by the unique solvability of a unique solution to variational problem, there exists

$$w \in \overline{H}^1(\Omega),\ w\big|_{\Gamma_D} = 0;\ -\int_\Omega \gamma_0 \nabla w \cdot \nabla \Psi \, dx = \widetilde{T}(\Psi) \quad \big(\Psi \in \overline{H}^1(\Omega),\ \Psi\big|_{\Gamma_D} = 0\big) \tag{2.6.5}$$

Therefore

$$-\int_\Omega \gamma_0 \nabla w \cdot \nabla \Psi \ dx = f(\Psi\big|_U) \qquad \big(\Psi \in \overline{H}^1(\Omega),\ \Psi\big|_{\Gamma_D} = 0\big) \tag{2.6.6}$$

□

Define $\widetilde{w}$ by

$$\widetilde{w} = \begin{cases} w & \text{in } \Omega \\ 0 & \text{in } \Omega_0 \setminus \Omega \end{cases} \tag{2.6.7}$$

Since $w\big|_{\Gamma_D} = 0$, we obtain

$$\widetilde{w} \in \overline{H}^1(\Omega_0). \tag{2.6.8}$$

Claim

$$\int_{\Omega_0} \gamma_0 \nabla\widetilde{w} \cdot \nabla\varphi = f(\varphi\big|_U) \qquad (\varphi \in \dot{H}^1(\overline{\Omega})) \tag{2.6.9}$$

The proof of this claim is given later.

From this claim,

$$L_\phi \widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{U}. \tag{2.6.10}$$

Note that

$$\begin{cases} \widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{\Omega} \supset \Omega_0 \setminus \overline{U}, \\ \Omega_0 \setminus \overline{U} \text{ is connected.} \end{cases} \tag{2.6.11}$$

Hence, and we have the weak unique continuation theorem for L_ϕ due to $\gamma_0 \in C^{0,1}(\overline{\Omega_0})$.

$$\widetilde{w} = 0 \text{ in } \Omega_0 \setminus \overline{U} \tag{2.6.12}$$

Therefore

$$w = 0 \text{ in } \Omega \setminus \overline{U} \tag{2.6.13}$$

Now let $v \in X$. Then, for some V which is an open neighborhood of $\overline{U}$, there exists $u \in \overline{H}^1(V)$ such that

$$L_\phi u = 0 \text{ in } V,\ u\big|_U = v. \tag{2.6.14}$$

By taking a cut off function, for some $\widetilde{V} \subset V$ which an open neighborhood of V, there exists $\widetilde{u} \in \dot{H}^1(\overline{\Omega})$ such that

$$\widetilde{u}\big|_{\widetilde{V}} = u\big|_{\widetilde{V}} \tag{2.6.15}$$

Hence, by reminding (2.6.6) and (2.6.13), $w \in \dot{H}^1(\overline{\widetilde{V}})$ and $L_\phi u = 0$ in $\widetilde{V}$

$$\begin{aligned} f(v) = f\big(u\big|_U\big) = f\big(\widetilde{u}\big|_U\big) &= \int_\Omega \gamma_0 \nabla w \cdot \nabla\widetilde{u}\, dx \\ &= \int_{\widetilde{V}} \gamma_0 \nabla w \cdot \nabla\widetilde{u}\, dx = \int_{\widetilde{V}} \gamma_0 \nabla w \cdot \nabla u\, dx = 0 \end{aligned} \tag{2.6.16}$$

Finally, we prove the claim. For any $\varphi \in \dot{H}^1(\overline{\Omega}_0)$,

$$\int_{\Omega_0} \gamma_0 \nabla \widetilde{w} \cdot \nabla u \, dx = \int_{\Omega_0 \setminus \overline{\Omega}} \gamma_0 \nabla \widetilde{w} \cdot \nabla u \, dx + \int_{\Omega} \gamma_0 \nabla \widetilde{w} \cdot \nabla u \, dx = \int_{\Omega} \gamma_0 \nabla \widetilde{w} \cdot \nabla u \, dx \tag{2.6.17}$$

Let $v \in \overline{H}^1(\Omega)$ be the solution of

$$L_\phi v = 0, \text{ in } \Omega, \;\; \partial_{L_\phi} v\big|_{\Gamma_N} = 0, \;\; v\big|_{\Gamma_D} = \varphi\big|_{\Gamma_D} \tag{2.6.18}$$

Clearly,

$$v - \varphi \in \overline{H}^1(\Omega), \;\; (v - \varphi)\big|_{\Gamma_D} = 0 \tag{2.6.19}$$

By (2.6.6),

$$-\int_{\Omega} \gamma_0 \nabla w \cdot \nabla (v - \varphi) \, dx = f\big(v\big|_U - \varphi\big|_U\big) \tag{2.6.20}$$

Here note that $v\big|_U \in Y$ by $\operatorname{supp}\big(v\big|_{\Gamma_D}\big) \subset \Gamma_0$,

$$f\big(v\big|_U\big) = 0 \tag{2.6.21}$$

On the other hand, remind that

$$w \in \overline{H}^1(\Omega), w\big|_{\Gamma_D} = 0; L_\phi v = 0 \text{ in } \Omega, \;\; \partial_{L_\phi} v\big|_{\Gamma_N} = 0, \;\; v\big|_{\Gamma_D} = \varphi \tag{2.6.22}$$

By the definition of weak solution,

$$\int_{\Omega} \gamma_0 \nabla w \cdot \nabla v \, dx = 0 \tag{2.6.23}$$

By (2.6.7), (2.6.20), (2.6.21) and (2.6.23)

$$-\int_{\Omega} \gamma_0 \nabla \widetilde{w} \cdot \nabla \varphi \, dx = f\big(\varphi\big|_U\big) \tag{2.6.24}$$

□

Likewise the proof given in [17] we have the second Runge's approximation theorem.

Theorem 2.6.2 (Runge's Approximation Theorem to L_ϕ 2). *Let U be an open subset of Ω such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected. Define the two spaces X, Y of functions by*

$$\begin{cases} X := \{u\big|_U; \;\; u \in \overline{H}^1(V), L_\phi \widetilde{u} = 0 \textit{ in } V\}, \\ Y := \{v\big|_U; \;\; v \in \overline{H}^1(\Omega), L_\phi v = 0 \textit{ in } \Omega, \;\; \operatorname{supp}\big(v\big|_\Gamma\big) \subset \Gamma_0\}, \end{cases} \tag{2.6.25}$$

where V is an open subset of Ω depending on u such that $\overline{U} \subset V \subset \overline{V} \subset \Omega$ and Γ_0 is a fixed open subset of Γ_N. Then, Y is dense in X with respect to $\overline{H}^1(U)$ norm.

Next we construct $\{v_{jk}\}$ $(j=1,2)$. By Theorem 2.6.1 and Theorem 2.6.2, there exist $\{v''_{1k}\}, \{v''_{2k}\} \subset \overline{H}^1(\Omega)$ such that $v''_{jk} \to H\big(\,\cdot\,, c(\theta)\big)$ in $\overline{H}^1_{\mathrm{loc}}(\Omega \setminus \mathcal{C}_\theta)$ for each $j=1,2$,

$$\begin{cases} L_\phi v''_{1k} = 0 \text{ in } \Omega \\ \partial_{L_\phi} v''_{2k} = 0 \text{ on } \Gamma_N, \quad \operatorname{supp}\big(v''_{1k}\big|_\Gamma\big) \subset \Gamma_{10} \end{cases} \tag{2.6.26}$$

and

$$\begin{cases} L_\phi v''_{2k} = 0 \text{ in } \Omega \\ \operatorname{supp}\big(v''_{2k}\big|_\Gamma\big) \subset \Gamma_{20}, \end{cases} \tag{2.6.27}$$

where $\Gamma_{10} \subset \Gamma_D,\ \Gamma_{20} \subset \Gamma_N$ are open subsets.

Then, we only have to define each $\{v_{jk}\}$ $(j=1,2)$ by

$$v_{jk} := v'_j + v''_{jk}, \tag{2.6.28}$$

2.7 The Green Function of Elliptic Operator

In this section we give the proof of the existence of the Green function which we used in Lemma 2.4.1. In [13], the existence is only proven for $n \geq 3$. So we have given here the proof of the existence including the case $n=2$.

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$.

Definition 2.7.1. For a measurable set $\mathcal{A} \subset \Omega$ and $u \in L^1(\mathcal{A})$, we define

$$⨍_{\mathcal{A}} u(x)\,dx := \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} u(x)\,dx \tag{2.7.1}$$

where μ is Lebesgue measure in $\mathbb{R}^n$.

Definition 2.7.2. For $p>0$, we define $L^p_*(\Omega)$ and $\|f\|_{L^p_*(\Omega)}$ by

$$L^p_*(\Omega) := \big\{ f : \text{measurable function on } \Omega;\, \|f\|_{L^p_*(\Omega)} < \infty \big\}, \tag{2.7.2}$$

$$\|f\|_{L^p_*(\Omega)} = \sup_{\sigma>0} \Big\{ \sigma \mu(\{x \in \Omega;\, |f(x)| > \sigma\})^{\frac{1}{p}} \Big\}. \tag{2.7.3}$$

Let $a_{ij} \in L^\infty(\Omega)$ $(1 \leq i,j \leq n)$ satisfy

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad (x \in \Omega,\ \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n) \tag{2.7.4}$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\eta_j \leq \Lambda|\xi||\eta| \quad (x \in \Omega,\ \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n,\ \eta = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n) \tag{2.7.5}$$

for some constants $0 < \lambda \leq \Lambda < \infty$.

Theorem 2.7.1. *There exists a nonnegative function $K : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ such that for each $y \in \Omega$ and any $r > 0$*

$$K(\,\cdot\,, y) \in \overline{H}^1\big(\Omega \setminus \overline{B_r(y)}\big) \cap \dot{W}^{1,1}(\overline{\Omega}) \tag{2.7.6}$$

and for all $\varphi \in C_0^\infty(\Omega)$

$$a[K(\,\cdot\,, y), \varphi] = \varphi(y), \tag{2.7.7}$$

where $a[u,v] := \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\partial_i u \partial_j v\, dx$.

This is called Green function for $A\cdot := \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x)\partial_j\,\cdot\big)$ and it satisfies the following properties: For each fix $y \in \Omega$, denote the function $K(x) := K(x,y)$ by K. Let $\varepsilon > 4$ and define χ_i $(i = 1,2,3)$ by

$$\chi_1 = \begin{cases} \dfrac{\varepsilon}{2} & \text{for } n = 2 \\ \dfrac{n}{n-2} & \text{for } n \geq 3, \end{cases} \quad \chi_2 = \begin{cases} \dfrac{2\varepsilon}{4+\varepsilon} & \text{for } n = 2 \\ \dfrac{n}{n-1} & \text{for } n \geq 3 \end{cases} \quad \text{and } \chi_3 = \begin{cases} \dfrac{\varepsilon}{4} & \text{for } n = 2 \\ n-2 & \text{for } n \geq 3. \end{cases} \tag{2.7.8}$$

Then, we have

$$K \in L_*^{\chi_1}(\Omega) \text{ with } \|K\|_{L_*^{\chi_1}(\Omega)} \leq C(n)\lambda^{-1} \tag{2.7.9}$$

for some constant $C(n) > 0$ depending only on n,

$$\nabla K \in L_*^{\chi_2}(\Omega) \text{ with } \|\nabla K\|_{L_*^{\chi_2}(\Omega)} \leq C(n, \lambda, \Lambda) \tag{2.7.10}$$

for some constant $C(n, \lambda, \Lambda) > 0$ depending only on n, λ, Λ,

$$K \in \dot{W}^{1,p}(\overline{\Omega}) \text{ for each } 1 \leq p \leq \chi_2, \tag{2.7.11}$$

$$K(x,y) \leq C(n, \Lambda/\lambda)\lambda^{-1}|x-y|^{-\chi_3}. \tag{2.7.12}$$

Here, $C(n), C(n, \lambda, \Lambda)$ and $C(n, \Lambda/\lambda)$ are positive constants which depend only on $n, \{n, \lambda, \Lambda\}$ and $\{n, \Lambda/\lambda\}$, respectively. Moreover, $\dot{W}^{1,p}(\overline{\Omega})$ is the Sobolev space with "$\cdot$" having the same meaning as "$\cdot$" of $\dot{H}^{-\frac{1}{2}}(\overline{\Gamma_D})$.

Remark 2.7.1. For $n \geq 3$, the uniqueness of K is given in ([13]).

Proof of Theorem 2.7.1

Fix $y \in \Omega$ and $\rho > 0$. Write $B_\rho := B_\rho(y)$.

For the proof of Theorem 2.7.1, we need the following Fact and Lemma 2.7.2

Fact([35]) For $p > 1$,

$$\|f\|_{L^p_*(\Omega)} \le \|f\|_{L^p(\Omega)}. \tag{2.7.13}$$

$$\|f\|_{L^{p-q}(\Omega)} \le \left(\frac{p}{q}\right)^{\frac{1}{p-q}} \mu(\Omega)^{\frac{q}{p(p-q)}} \|f\|_{L^p_*(\Omega)} \text{ for } 0 < q \le p-1. \tag{2.7.14}$$

Lemma 2.7.2. *([13]) Let $u \in \overline{H}^1(\Omega)$ satisfy $u \ge 0$ in Ω and*

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\partial_i u \partial_j \varphi \, dx \le 0 \text{ for any } \varphi \in \dot{H}^1(\overline{\Omega}) \text{ with } \varphi \ge 0 \text{ in } \Omega. \tag{2.7.15}$$

Then, there exists a constant $C(n) > 0$ depending only on n, such that for $\alpha > 1$ and $\overline{B_\rho(x)} \subset \Omega$,

$$\sup_{y \in B_{\frac{\rho}{2}}(x)} u^\alpha(y) \le C(n)\left(\frac{\alpha}{\alpha-1}\right)^2 \left(\frac{\Lambda}{\lambda}\right)^n ⨍_{B_\rho(x)} u^\alpha(y)\, dy. \tag{2.7.16}$$

We define T, which is bounded linear function on $\overline{H}^1(\Omega)$, by

$$T(\varphi) := ⨍_{B_\rho} \varphi \, dx. \tag{2.7.17}$$

For any $u, v \in \dot{H}^1(\Omega)$,

$$|a[u,v]| \le \Lambda \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \le \Lambda \|u\|_{\overline{H}^1(\Omega)} \|v\|_{\overline{H}^1(\Omega)}, \tag{2.7.18}$$

$$a[u,u] \ge \lambda \|\nabla u\|^2_{L^2(\Omega)} \ge \lambda' \|u\|^2_{\overline{H}^1(\Omega)} \tag{2.7.19}$$

for some constant $\lambda' > 0$ independent of u.

By the Lax-Milgram theorem and the Riesz representation theorem, there exists $G_\rho \in \dot{H}^1(\overline{\Omega})$ satisfying

$$a[G_\rho, \varphi] = ⨍_{B_\rho} \varphi \, dx \tag{2.7.20}$$

for all $\varphi \in \dot{H}^1(\overline{\Omega})$. Taking $|G_\rho| \in \dot{H}^1(\overline{\Omega})$ as a test function,

$$a[G_\rho, G_\rho] = \fint_{B_\rho} G_\rho \, dx \leq \fint_{B_\rho} |G_\rho| \, dx = a[G_\rho, |G_\rho|]. \tag{2.7.21}$$

Put $M := \dfrac{a[G_\rho, |G_\rho|]}{a[G_\rho, G_\rho]} \geq 1$, then

$$a[G_\rho, G_\rho] = a\left[\frac{|G_\rho|}{M}, G_\rho\right] = a\left[G_\rho, \frac{|G_\rho|}{M}\right]. \tag{2.7.22}$$

From (2.7.22),

$$\begin{aligned} a\left[\frac{|G_\rho|}{M}, \frac{|G_\rho|}{M}\right] &= \frac{1}{M^2} a[|G_\rho|, |G_\rho|] = \frac{1}{M^2} \sum_{i,j=1}^n \int_\Omega a_{ij}(x) \partial_i |G_\rho| \partial_j |G_\rho| \, dx \\ &= \frac{1}{M^2} \sum_{i,j=1}^n \int_\Omega a_{ij}(x) \partial_i G_\rho \partial_j G_\rho \, dx = \frac{1}{M^2} a[G_\rho, G_\rho] \leq a[G_\rho, G_\rho] = a\left[G_\rho, \frac{|G_\rho|}{M}\right]. \end{aligned} \tag{2.7.23}$$

Note that for $u \in \overline{H}^1(\Omega)$, $\nabla |u| \in L^2(\Omega)$ with $\nabla |u| = \begin{cases} \nabla u \text{ in } \{x \in \Omega;\, u > 0\} \\ 0 \text{ in } \{x \in \Omega;\, u = 0\} \\ -\nabla u \text{ in } \{x \in \Omega;\, u < 0\}. \end{cases}$

Then, we have

$$a\left[\frac{|G_\rho|}{M} - G_\rho, \frac{|G_\rho|}{M} - G_\rho\right] = a\left[\frac{|G_\rho|}{M} - G_\rho, \frac{|G_\rho|}{M}\right] - a\left[\frac{|G_\rho|}{M}, G_\rho\right] \leq 0. \tag{2.7.24}$$

Hence

$$G_\rho = \frac{|G_\rho|}{M} \geq 0. \tag{2.7.25}$$

At first, we prove

$$\|G_\rho\|_{L_*^{\chi_1}(\Omega)} \leq C(n)\lambda^{-1} \tag{2.7.26}$$

for some constant $C(n) > 0$ depending only on n.

Fixing $\sigma_0 > 0$, choose a test function $\varphi(x) = \max\left\{\dfrac{1}{\sigma_0} - \dfrac{1}{G_\rho(x)}, 0\right\}$.

Then we have

$$\frac{1}{\sigma_0} \geq \fint_{B_\rho} \varphi \, dx = \sum_{i,j=1}^n \int_\Omega a_{ij}(x) \partial_i G_\rho \partial_j \varphi \, dx = \sum_{i,j=1}^n \int_{\Omega_{\sigma_0}} a_{ij}(x) \partial_i G_\rho \frac{\partial_j G_\rho}{{G_\rho}^2} \, dx \geq \lambda \int_{\Omega_{\sigma_0}} \frac{|\nabla G_\rho|^2}{{G_\rho}^2} \, dx, \tag{2.7.27}$$

where $\Omega_{\sigma_0} := \{x \in \Omega; G_\rho(x) > \sigma_0\}$. By Sobolev's inequality,

$$\left(\int_{\Omega_{\sigma_0}} \left|\log \frac{G_\rho}{\sigma_0}\right|^{2\chi_1} dx\right)^{\frac{1}{\chi_1}} \leq C(n) \int_{\Omega_{\sigma_0}} \left|\nabla \log \frac{G_\rho}{\sigma_0}\right|^2 dx = C(n) \int_{\Omega_{\sigma_0}} \frac{|\nabla G_\rho|^2}{{G_\rho}^2} \leq \frac{C(n)}{\lambda \sigma_0} \tag{2.7.28}$$

for some constant $C(n) > 0$ depending only on n. Hence,

$$(\log 2)^2 \mu(\Omega_{2\sigma_0})^{\frac{1}{\chi_1}} \leq \left(\int_{\Omega_{2\sigma_0}} \left| \log \frac{G_\rho}{\sigma_0} \right|^{2\chi_1} dx \right)^{\frac{1}{\chi_1}} \leq C(n)\lambda^{-1}\sigma_0^{-1}. \tag{2.7.29}$$

Therefore,

$$2\sigma_0 \mu(\Omega_{2\sigma_0})^{\frac{1}{\chi_1}} \leq \frac{2C(n)}{(\log 2)^2} \lambda^{-1} \tag{2.7.30}$$

and this gives (2.7.26).

Now, we take $G_\rho \in \overline{H}^1(\Omega)$ as a test function. Then we have

$$\begin{aligned} \lambda \int_\Omega |\nabla G_\rho|^2 \, dx &\leq \sum_{i,j=1}^n a_{ij}(x) \partial_i G_\rho \partial_j G_\rho \, dx = \fint_{B_\rho} G_\rho \, dx = \frac{1}{\mu(B_\rho)} \int_{B_\rho} G_\rho \, dx \\ &\leq \frac{1}{\mu(B_\rho)} \|G_\rho\|_{L^{2\chi_1}(B_\rho)} \mu(B_\rho)^{1-\frac{1}{2\chi_1}} \leq C(n) \|\nabla G_\rho\|_{L^2(\Omega)} \mu(B_\rho)^{-\frac{1}{2\chi_1}} = C'(n) \|\nabla G_\rho\|_{L^2(\Omega)} \rho^{-\frac{n}{2\chi_1}}. \end{aligned} \tag{2.7.31}$$

for some constants $C(n), C'(n) > 0$ depending only on n.

Thus

$$\int_\Omega |\nabla G_\rho|^2 \, dx \leq C'(n) \lambda^{-2} \rho^{-\frac{n}{\chi_1}}. \tag{2.7.32}$$

Next we will show

$$G_\rho(x) \leq C(n, \Lambda/\lambda) \lambda^{-1} |x-y|^{-\chi_3} \text{ if } |x-y| \geq 2\rho. \tag{2.7.33}$$

Let $R := |x-y| (\geq 2\rho)$.

First we consider the case: $\overline{B_{\frac{R}{2}}(x)} \subset \Omega$.

Since G_ρ is the solution of $Au = 0$ in $\Omega \setminus B_R$, we have

$$G_\rho(x)^\alpha \leq C(\alpha, n, \Lambda/\lambda) \fint_{B_{\frac{R}{4}}(x)} {G_\rho}^\alpha \, dy \tag{2.7.34}$$

by using Lemma 2.7.2. By (2.7.14) and (2.7.26), we have from $n \geq 3$,

$$\int_{B_{\frac{R}{4}}(x)} {G_\rho}^\alpha \, dx \leq \frac{n}{n - \alpha(n-2)} \mu\Big(B_{\frac{R}{4}}\Big)^{1-\frac{\alpha(n-2)}{n}} \|G_\rho\|_{L_*^{\frac{n}{n-2}}(\Omega)} \leq C(n,\alpha) \lambda^\alpha R^{n-\alpha(n-2)} \tag{2.7.35}$$

for some constant $C(n, \alpha) > 0$ depending only on n, α. Hence, for (2.7.34) and (2.7.35),

$$G_\rho(x) \leq C(n, \Lambda/\lambda) \lambda^{-1} R^{-(n-2)} \tag{2.7.36}$$

for some constant $C(n, \Lambda/\lambda) > 0$ depending only $n, \Lambda/\lambda$.

For $n = 2$, we have from (2.7.14) and (2.7.26),

$$\int_{B_{\frac{R}{4}}(x)} {G_\rho}^\alpha \, dx \leq \frac{\varepsilon}{\varepsilon - 2\alpha} \mu\Big(B_{\frac{R}{4}}\Big)^{1-\frac{2\alpha}{\varepsilon}} \|G_\rho\|_{L_*^{\frac{\varepsilon}{2}}(\Omega)} \leq C(\alpha)\lambda^{-\alpha} R^{2-\frac{4\alpha}{\varepsilon}} \tag{2.7.37}$$

for some constant $C(\alpha) > 0$ depending only on α.

Hence, for (2.7.34) and (2.7.37),

$$G_\rho(x) \leq C(\Lambda/\lambda)\lambda^{-1} R^{-\frac{4}{\varepsilon}} \tag{2.7.38}$$

for some constant $C(\Lambda/\lambda) > 0$ depending only on Λ/λ.

Next we consider the case: $\overline{B_{\frac{R}{2}}(x)} \not\subset \Omega$. Consider a domain $\widetilde{\Omega}$ such that $\overline{B_{\frac{R}{2}}(x)} \subset \widetilde{\Omega}$ and extend operator A to $\widetilde{\Omega}$. Then, likewise G_ρ for A, we have $\widetilde{G}_\rho$ for this extended A. By restricting $\widetilde{G}_\rho$ to Ω, we have

$$A(G_\rho - \widetilde{G}_\rho) = 0 \text{ in } \Omega. \tag{2.7.39}$$

$G_\rho = 0 \leq \widetilde{G}_\rho$ on $\partial\Omega$, therefore the maximum principle implies

$$G_\rho \leq \widetilde{G}_\rho \text{ in } \Omega. \tag{2.7.40}$$

Since $\widetilde{G}_\rho$ satisfies (2.7.33), we have

$$\widetilde{G}_\rho(x) \leq C(n, \Lambda/\lambda)\lambda^{-1} R^{-\chi_3}. \tag{2.7.41}$$

This completes the proof of (2.7.33).

Next we will show

$$\|\nabla G_\rho\|_{L_*^{\chi_2}(\Omega)} \leq C(\lambda, \Lambda) \tag{2.7.42}$$

for some constant $C(\lambda, \Lambda) > 0$ depending only on λ, Λ.

To show (2.7.42), we will show

$$\int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \leq C(n, \lambda, \Lambda) R^{-\chi_4} \tag{2.7.43}$$

for some constant $C(n, \lambda, \Lambda) > 0$ depending only on n, λ, Λ, where $\chi_4 = \dfrac{8}{\varepsilon}$ for $n = 2$, $\chi_4 = n - 2$ for $n \geq 3$.

Choose a test function $\eta \in C^\infty(\Omega)$ satisfying $\eta = 1$ in $\Omega \setminus B_R$, $\eta = 0$ in $B_{\frac{R}{2}}$ and $|\nabla \eta| \leq \dfrac{C}{R}$ for some constant $C > 0$.

Let $R \geq 4\rho$ and take $G_\rho \eta^2$ as a test function. Then, we have

$$\begin{aligned} 0 = \fint_{B_\rho} G_\rho \eta^2 \, dx &= \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \partial_i G_\rho \partial_j (G_\rho \eta^2) \, dx \\ &\geq \sum_{i,j=1}^{n} \int_{\Omega \setminus B_R} a_{ij}(x) \partial_i G_\rho \partial_j G_\rho \, dx \quad + 2 \sum_{i,j=1}^{n} \int_{\Omega \setminus B_{\frac{R}{2}}} a_{ij}(x) \partial_i G_\rho \partial_j G_\rho G_\rho \eta \, dx. \end{aligned} \tag{2.7.44}$$

This implies

$$\begin{aligned} &\lambda \int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \leq \sum_{i,j=1}^{n} \int_{\Omega \setminus B_R} a_{ij}(x) \partial_i G_\rho \partial_j G_\rho \, dx \leq 2\Lambda \int_{B_R \setminus B_{\frac{R}{2}}} |\nabla G_\rho| \frac{C}{R} G_\rho \eta \, dx \\ &\leq \frac{2\Lambda C}{R} \int_{B_R \setminus B_{\frac{R}{2}}} |\nabla G_\rho| G_\rho \, dx \leq \frac{\lambda}{2} \int_{B_R \setminus B_{\frac{R}{2}}} |\nabla G_\rho|^2 \, dx + \frac{2\Lambda^2 C^2}{\lambda R^2} \int_{B_R \setminus B_{\frac{R}{2}}} {G_\rho}^2 \, dx. \end{aligned} \tag{2.7.45}$$

Hence

$$\int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \leq 4 \left(\frac{\Lambda}{\lambda} \right)^2 \frac{C^2}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} {G_\rho}^2 \, dx. \tag{2.7.46}$$

Combining this with (2.7.33), we have

$$\int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \leq \begin{cases} C(\lambda, \Lambda) R^{-\frac{8}{\varepsilon}} \text{ for } n = 2 \\ C(n, \lambda, \Lambda) R^{-(n-2)} \text{ for } n \geq 3 \end{cases} \tag{2.7.47}$$

for some constants $C(\lambda, \Lambda), C(n, \lambda, \Lambda) > 0$ depending only on $\{\lambda, \Lambda\}, \{n, \lambda, \Lambda\}$, respectively.

Next we consider the case $R < 4\rho$. From (2.7.32), we have

$$\int_{\Omega \setminus B_R} |\nabla G_\rho|^2 \, dx \leq C(n) \lambda^{-2} \rho^{-\frac{n}{\chi_1}} = \begin{cases} C \lambda^{-2} \rho^{-\frac{4}{\varepsilon}} \quad \text{for } n = 2 \\ C(n) \lambda^{-2} \rho^{-(n-2)} \quad \text{for } n \geq 3 \end{cases} \tag{2.7.48}$$

for some constant $C > 0$ and some constant $C(n) > 0$ which depends on n. Observe that, for $n \geq 3$,

$$C(n) \lambda^{-2} \rho^{-(n-2)} \leq C(n, \lambda) R^{-(n-2)} \tag{2.7.49}$$

for some constant $C(n, \lambda)$ depending only on n, λ and for $n = 2$,

$$C \lambda^{-2} \rho^{-\frac{4}{\varepsilon}} \leq C(\lambda) R^{-\frac{4}{\varepsilon}} \leq C(\lambda) R^{-\frac{8}{\varepsilon}} \tag{2.7.50}$$

for some constant $C(\lambda) > 0$ depending only on λ. Therefore we obtain (2.7.43).

Next we return to the proof of (2.7.42).

For $n \geq 3$, we set $\Omega'_\sigma := \{x \in \Omega;\ |\nabla G_\rho(x)| > \sigma\}$ and $R_{\sigma_1} = \sigma_1^{-\frac{1}{n-1}}$ for fixed $\sigma_1 > 0$.

From (2.7.47) and (2.7.49),

$$\sigma_1^2 \mu(\Omega'_{\sigma_1} \cap (\Omega \setminus B_{R_{\sigma_1}})) \leq \int_{\Omega \setminus B_{R_{\sigma_1}}} |\nabla G_\rho|^2 \, dx \leq C(n, \lambda, \Lambda) \sigma_1^{\frac{n-2}{n-1}} \tag{2.7.51}$$

for some constant $C(n,\lambda,\mathit{\Lambda})>0$ depending only on $n,\lambda,\mathit{\Lambda}$. That is

$$\sigma_1\mu(\Omega'_{\sigma_1}\cap(\Omega\setminus B_{R_{\sigma_1}}))^{\frac{n-1}{n}}\leq C(n,\lambda,\mathit{\Lambda}). \tag{2.7.52}$$

for some constant $C(n,\lambda,\mathit{\Lambda})>0$ depending only on n,λ,Λ. Combining this with

$$\mu(\Omega'_{\sigma_1}\cap B_{R_{\sigma_1}})\leq\mu(B_{R_{\sigma_1}})=C(n)R^n_{\sigma_1}=(C'(n)\sigma_1)^{\frac{n}{n-1}} \tag{2.7.53}$$

for some constants $C(n),C'(n)>0$ depending only on n,

$$\sigma_1\mu(\Omega'_{\sigma_1})^{\frac{n-1}{n}}\leq C(n,\lambda,\mathit{\Lambda}). \tag{2.7.54}$$

for some constants $C(n,\lambda,\mathit{\Lambda})>0$ depending only on $n,\lambda,\mathit{\Lambda}$. Hence

$$\|\nabla G_\rho\|_{L_*^{\frac{n-1}{n}}(\Omega)}\leq C(n,\lambda,\Lambda) \tag{2.7.55}$$

for some constants $C(n,\lambda,\mathit{\Lambda})>0$ depending only on n,λ,Λ.

For $n=2$, we set $R_{\sigma_2}=\sigma_2^{-\frac{\varepsilon}{4+\varepsilon}}$ for fixed $\sigma_2>0$.

From (2.7.47) and (2.7.50),

$$\sigma_2^2\mu(\Omega'_{\sigma_2}\cap(\Omega\setminus B_{R_{\sigma_2}}))\leq\int_{\Omega\setminus B_{R_{\sigma_2}}}|\nabla G_\rho|^2\,dx\leq C(\lambda,\mathit{\Lambda})\sigma_2^{\frac{8}{4+\varepsilon}} \tag{2.7.56}$$

for some constant $C(\lambda,\mathit{\Lambda})>0$ depending only on $\lambda,\mathit{\Lambda}$. That is

$$\sigma_2\mu(\Omega'_{\sigma_2}\cap(\Omega\setminus B_{R_{\sigma_2}}))^{\frac{4+\varepsilon}{2\varepsilon}}\leq C'(\lambda,\mathit{\Lambda}) \tag{2.7.57}$$

for some constant $C'(\lambda,\mathit{\Lambda})>0$ depending only on $\lambda,\mathit{\Lambda}$. Combining this with

$$\mu(\Omega'_{\sigma_2}\cap B_{R_{\sigma_2}})\leq\mu(B_{R_{\sigma_2}})=\pi R^2_{\sigma_2}=\pi\sigma_2^{-\frac{2\varepsilon}{4+\varepsilon}}, \tag{2.7.58}$$

$$\sigma_2\mu(\Omega'_{\sigma_2})^{\frac{4+\varepsilon}{2\varepsilon}}\leq C'(\lambda,\mathit{\Lambda}) \tag{2.7.59}$$

for some constant $C'(\lambda,\mathit{\Lambda})>0$ depending only on λ,Λ. Hence

$$\|\nabla G_\rho\|_{L_*^{\frac{2\varepsilon}{4+\varepsilon}}(\Omega)}\leq C'(\lambda,\mathit{\Lambda}) \tag{2.7.60}$$

for some constant $C'(\lambda,\mathit{\Lambda})>0$ depending only on λ,Λ.

Now by (2.7.13),

$$\|G_\rho\|_{L^{\chi_1}(\Omega)} \le C(n)\lambda^{-1} \quad \text{and} \quad \|\nabla G_\rho\|_{L^{\chi_2}(\Omega)} \le C(n,\lambda,\Lambda) \tag{2.7.61}$$

for some constants $C(n), C(n,\lambda,\Lambda) > 0$ depending only on $n, \{n,\lambda,\Lambda\}$, respectively.
Note that $\chi_1 > \chi_2$ and $\chi_2 > 1$, because $\dfrac{2\varepsilon}{4+\varepsilon} < \dfrac{\varepsilon}{2}$, and $\dfrac{2\varepsilon}{4+\varepsilon} > 1$. Hence,

$$G_\rho \in \dot{W}^{1,\chi_2}(\overline{\Omega}). \tag{2.7.62}$$

Reminding Ω is bounded,

$$G_\rho \in \dot{W}^{1,p}(\overline{\Omega}) \text{ for } 1 \le p \le \chi_2. \tag{2.7.63}$$

Hence, fixing $\chi_0 \in [1,\chi_2]$ and applying Rellich's compactness theorem, there exists $K \in \dot{W}^{1,\chi_0}(\overline{\Omega})$ such that

$$G_\rho \to K \text{ weakly in } \dot{W}^{1,p}(\overline{\Omega}) \quad (1 \le p \le \chi_2). \tag{2.7.64}$$

By (2.7.64) and,

$$\fint_{B_\rho} \varphi\, dx \to \varphi(y) \quad \text{as } \rho \to 0 \tag{2.7.65}$$

for any $\varphi \in C_0^\infty(\Omega)$, we have

$$a[K(\,\cdot\,,y),\varphi] = \varphi(y). \tag{2.7.66}$$

Furthermore, from (2.7.26) and (2.7.42), we get (2.7.9), (2.7.10). Also, from (2.7.47) and (2.7.48), we can prove (2.7.6).
Finally (2.7.12) is an easy consequence of (2.7.33), because $K(\,\cdot\,,y)$ is Hölder continuous in $\Omega \setminus \{y\}$. This follows from the famous De Giorgi-Nash-Moser regularity theorem, because $K(\,\cdot\,,y)$ is the solution of $Au = 0$ in $\Omega \setminus B_R(y)$.

2.8 Blow-up Properties

In this section we prove Theorem 2.3.2 which is used used in Section 2.3.

Proof of Theorem 2.3.2 We use the inequality given in [17]:

$$\gamma_0\nabla(v-u)\cdot\nabla(v-u) + (\gamma-\gamma_0)\nabla u\cdot\nabla u \ge {\gamma_0}^{-1}(\gamma-\gamma_0)\gamma^{-1}(\gamma_0\nabla v)\cdot(\gamma_0\nabla v) \tag{2.8.1}$$

We first prove (i). Observe that

$$\int_\Omega \{\gamma\nabla(u-v)\cdot\nabla(u-v)+(\gamma-\gamma_0)\nabla v\cdot\nabla v\}\,dx$$
$$=\int_\Omega \big(\gamma_0\nabla v\cdot\nabla v-2\gamma\nabla u\cdot\nabla v+\gamma\nabla u\cdot\nabla u\big)\,dx, \quad (2.8.2)$$

$$\int_\Omega \{\gamma_0\nabla(v-u)\cdot\nabla(v-u)+(\gamma-\gamma_0)\nabla u\cdot\nabla u\}\,dx$$
$$=\int_\Omega \big(\gamma_0\nabla v\cdot\nabla v-2\gamma_0\nabla v\cdot\nabla u+\gamma\nabla u\cdot\nabla u\big)\,dx. \quad (2.8.3)$$

By the definition of the Dirichlet-to-Neumann map and Neumann-to-Dirichlet map, we have

$$\int_\Omega \{\gamma\nabla(u-v)\cdot\nabla(u-v)+(\gamma-\gamma_0)\nabla v\cdot\nabla v\}\,dx$$
$$=\begin{cases}-\langle(\Lambda_D-\Lambda_\phi)f,f\rangle_1-\int_\Omega F(u-v)\,dx+\langle g,u-v\rangle_2\\ \langle g,(\Pi_D-\Pi_\phi)g\rangle_2-\int_\Omega F(u-v)\,dx-\langle\partial_{L_\phi}(u-v),f\rangle_1\end{cases} \quad (2.8.4)$$

from (2.8.3) and we have

$$\int_\Omega \{\gamma_0\nabla(v-u)\cdot\nabla(v-u)+(\gamma-\gamma_0)\nabla u\cdot\nabla u\}\,dx$$
$$=\begin{cases}\langle(\Lambda_D-\Lambda_\phi)f,f\rangle_1+\int_\Omega F(u-v)\,dx-\langle g,u-v\rangle_2\\ -\langle g,(\Pi_D-\Pi_\phi)g\rangle_2+\int_\Omega F(u-v)\,dx+\langle\partial_{L_\phi}(u-v),f\rangle_1\end{cases} \quad (2.8.5)$$

from (2.8.2), where $d\sigma$ is line segment for $n=2$ and the surface measure for $n=3$.

Reminding (2.1.2), we have (2.3.9) and (2.3.11) from (2.8.4), respectively. Also, from (2.8.1), we have (2.3.10) and (2.3.12) from (2.8.5).

Chapter 3

Non-stationary Heat Equation Case

3.1 Statement of the Problem and Result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with C^2 boundary Γ if $n \geq 2$. Ω is considered as an isotropic heat conductive medium with heat conductivity

$$\gamma(x,t) = \begin{cases} 1 & \text{in } \Omega \setminus \overline{D(t)} \\ k & \text{in } D(t) \end{cases} \qquad i.e. \quad \gamma(x,t) = 1 + (k-1)\chi_{D(t)} \tag{3.1.1}$$

for each $0 \leq t \leq T$, $0 < T < \infty$. Here $k > 0$ $(k \neq 1)$ is constant, $D(t)$ is a bounded domain such that $\overline{D(t)} \subset \Omega$, $\Omega \setminus \overline{D(t)}$ is connected and $\partial D(t)$ is C^2 if $n \geq 2$, the dependency of $\partial D(t)$ on $t \in [0,T]$ is C^1 and $\chi_{D(t)}$ is the characteristic function of $D(t)$. The two dimensional figure of Ω and $D := \bigcup_{0 \leq t \leq T} D(t) \times \{t\}$ is given below.

We will use the following notations in this paper. For any $E \subset \mathbb{R}^n$ (or $E \subset \mathbb{R}^{n-1}$) and $T_0, T_1 \in \mathbb{R}$ $(T_0 < T_1), T > 0$, we denote $E_{(T_0,T_1)} := E \times (T_0, T_1)$ and $E_T := E \times (0,T)$.

For $p, q \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ or $p = \frac{1}{2}$, $H^p(\Omega), H^p(\Gamma)$ and $H^{p,q}(\Omega_T)$ denote the usual Sobolev spaces where p and q in $H^{p,q}(\Omega_T)$ denote the regularity with respect to x and t, respectively.(cf.[29]) Also, for an open set $U \subset \mathbb{R}^{n+1}$ with Lipschitz boundary and $p, q \in \mathbb{Z}_+$, $H^{p,q}(U)$ is defined likewise $H^{p,q}(\Omega_T)$. That is $g \in H^{p,q}(U)$ if and only if there exists $\boldsymbol{g} \in H^{p,q}(\mathbb{R}^{n+1}) := \{\boldsymbol{g} \in \mathcal{D}'(\mathbb{R}^{n+1}); \|\boldsymbol{g}\|_{H^{p,q}(\mathbb{R}^{n+1})} := \|\{(1+|\xi|^2)^{\frac{p}{2}} + (1+\tau^2)^{\frac{q}{2}}\}\hat{\boldsymbol{g}}(\xi,\tau)\|_{L^2(\mathbb{R}^{n+1})} < \infty\}$ such that $\boldsymbol{g}\big|_U = g$. Here the norm $\|g\|_{H^{p,q}(U)} := \inf_{\boldsymbol{g}\big|_U = g, \boldsymbol{g} \in H^{p,q}(\mathbb{R}^{n+1})} \|\boldsymbol{g}\|_{H^{p,q}(\mathbb{R}^{n+1})}$.

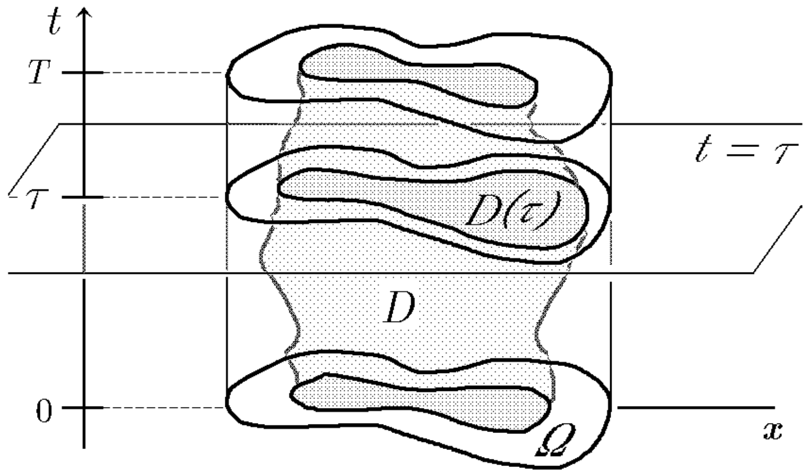


Figure 3.1: 2 dimensional figure of Ω and D

Now, we consider the boundary value problem:

$$\begin{cases} \big(P_D u\big)(x,t) := \partial_t u(x,t) - \operatorname{div}_x\big(\gamma(x,t)\nabla_x u(x,t)\big) = 0 \quad \text{in} \quad \Omega_T \\ \partial_\nu u(x,t) = f(x,t) \quad \text{on} \quad \Gamma_T, \quad u(x,0) = 0. \end{cases} \tag{3.1.2}$$

The physical meaning of u and f are the temperature and heat flux, respectively.

Definition 3.1.1 (Weak Solution to 3.1.2). If $u \in H^{1,0}(\Omega_T)$ satisfies

$$\int_{\Omega_T} (-u\partial_t\varphi + \gamma(x,t)\nabla_x u \cdot \nabla_x\varphi)\, dx\, dt = \int_{\Gamma_T} f\varphi\big|_{\Gamma_T}\, d\sigma\, dt \tag{3.1.3}$$

for all $\varphi \in W(\Omega_T) := \{u \in H^{1,0}(\Omega_T); \partial_t u \in L^2((0,T);(H^1(\Omega))^*)\}$ with $\varphi = 0$ at $t = T$, we call u a weak solution to (3.1.2).

Here we have used the notations given in [29] to denote Sovolev spaces.

Theorem 3.1.1 ([39]) (Unique Solvability of Parabolic Equation 1).
For given $f \in L^2((0,T);(H^{\frac{1}{2}}(\Gamma))^*)$, there exists a unique solution $u = u(f) \in W(\Omega_T)$ to (3.1.2).

Next, we define the Neumann to Dirichlet map Π_D as follow.

Definition 3.1.2 (Neumann-to-Dirichlet Map).
Let $u(f) \in W(\Omega_T)$ be the solution to (3.1.2). We define the Neumann-to-Dirichlet map Π_D : $L^2((0,T);(H^{\frac{1}{2}}(\Gamma))^*) \to L^2((0,T);H^{\frac{1}{2}}(\Gamma))$ by

$$\Pi_D(f) := u(f) \quad \text{on} \quad \Gamma_T. \tag{3.1.4}$$

The measurement Π_D is to measure the temperature induced from inputting current or heat flux infinitely many times.

Now, we consider the inverse problem:

(IP) **Suppose k, D are unknown. Reconstruct D from Π_D.**

Our main theorem is the following.

Theorem 3.1.2. *If $n = 1$, there is a reconstruction procedure for the inverse problem (IP). The details of the reconstruction procedure will be given later.*

The proof will be given later.

This is the first attempt to study the probe method for the inverse boundary value problem for non-stationary heat equations.

We end this section by giving the Runge's approximation theorem which we need in our reconstruction procedure.

Theorem 3.1.3 (Runge's Approximation Theorem for P_ϕ).
For $T_0' < T_0 < T_1 < T_1'$, let U be an open subset of $\Omega_{(T_0',T_1')}$ such that

$$\begin{cases} \partial U \text{ is Lipschitz,} \\ \overline{U} \subset \Omega_{(T_0',T_1')} \text{ and } \Omega_{(T_0',T_1')} \setminus \overline{U} \text{ is connected.} \end{cases} \tag{3.1.5}$$

Then, for any open subset V of $\Omega_{(T_0',T_1')}$ such that $\overline{U} \subset V \subset \overline{V} \subset \Omega_{(T_0',T_1')}$ and any $v \in H^{2,1}(V)$ satisfying

$$P_\phi v := \partial_t v - \Delta_x v = 0 \quad in \quad V, \;\; v\big|_{(T_0',T_0]} = 0, \tag{3.1.6}$$

there exists a sequence $\{v^j\} \subset H^{2,1}(\Omega_{(T_0',T_1')})$ such that

$$P_\phi v^j = 0 \quad in \quad \Omega_{(T_0',T_1')}, \;\; v^j\big|_{(T_0',T_0]} = 0 \tag{3.1.7}$$

and

$$v^j \to v \;\; (j \to \infty) \quad in \quad L^2(U). \tag{3.1.8}$$

Henceforth, for example, $v\big|_{(-T_0',T_0]}$ denote the restriction of the function v to $\mathbb{R}^n \times (-T_0', T_0]$. We also have the same kind of theorem for the dual problem.

Theorem 3.1.4 (Runge's Approximation Theorem for P_ϕ^*).
For $T_0' < T_0 < T_1 < T_1'$, let Ω and U be as above. Then, for any open subset V of $\Omega_{(T_0',T_1')}$ such that $\overline{U} \subset V \subset \overline{V} \subset \Omega_{(T_0',T_1')}$ and any $\varphi \in H^{2,1}(V)$ satisfying

$$P_\phi^* \varphi := -\partial_t \varphi - \Delta_x \varphi = 0 \quad in \quad V, \;\; \varphi\big|_{[T_1,T_1')} = 0, \tag{3.1.9}$$

there exists a sequence $\{\varphi^j\} \subset H^{2,1}(\Omega_{(T_0',T_1')})$ *such that*

$$P_\phi^* \varphi^j = 0 \quad in \quad \Omega_{(T_0',T_1')}, \ \ \varphi^j\big|_{[T_1,T_1')} = 0 \tag{3.1.10}$$

and

$$\varphi^j \to \varphi \ (j \to \infty) \quad in \quad L^2(U). \tag{3.1.11}$$

These proofs are given in [9].

3.2 Pre-indicator Function

In this section we define the pre-indicator function. Then, based on this pre-indicator function, we will define in sections 3.3 and 3.4 the so called indicator function which is the mathematical testing machine to detect the inclusion.

For $(y,s),(y',s') \in \Omega_T \setminus \overline{D}$ such that $(y,s) \neq (y',s')$, let $G_{(y,s)}(x,t)$ and $G^*_{(y',s')}(x,t)$ be

$$G_{(y,s)}(x,t) := \begin{cases} \dfrac{1}{[4\pi(t-s)]^{n/2}} \exp\left[-\dfrac{|x-y|^2}{4(t-s)}\right] & (t > s) \\ \\ 0 \qquad (t \geq s), \end{cases} \tag{3.2.1}$$

$$G^*_{(y',s')}(x,t) := \begin{cases} 0 \qquad (s' \geq t) \\ \\ \dfrac{1}{[4\pi(s'-t)]^{n/2}} \exp\left[-\dfrac{|x-y'|^2}{4(s'-t)}\right] & (t < s'). \end{cases} \tag{3.2.2}$$

By Runge's approximation theorem (Theorem 3.1.3, 3.1.4), we can select sequences $\{v^j_{(y,s)}\}$, $\{\varphi^j_{(y',s')}\} \subset H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ such that

$$\begin{cases} P_\phi v^j_{(y,s)} = 0 \ \text{ in } \Omega_{(-\varepsilon,T+\varepsilon)}, \ \ v^j_{(y,s)}\big|_{(-\varepsilon,0]} = 0, \\ v^j_{(y,s)} \to G_{(y,s)} \ (j \to \infty) \ \text{ in } L^2(U) \end{cases} \tag{3.2.3}$$

and

$$\begin{cases} P_\phi^* \varphi^j_{(y',s')} = 0 \ \text{ in } \Omega_{(-\varepsilon,T+\varepsilon)}, \ \ \varphi^j_{(y',s')}\big|_{[T,T+\varepsilon)} = 0, \\ \varphi^j_{(y',s')} \to G^*_{(y',s')} \ (j \to \infty) \ \text{ in } L^2(U) \end{cases} \tag{3.2.4}$$

for each open set $U \subset \Omega_{(-\varepsilon,T+\varepsilon)}$ satisfying $U\big|_{[0,T]} \supset \overline{D}$, $U \not\ni (y,s)$, (y',s') of the type given in Runge's approximation theorems. Here $U\big|_{[0,T]}$ denote the restriction of the set U to $\mathbb{R}^n \times [0,T]$. We call these $\{v^j_{(y,s)}\}, \{\varphi^j_{(y',s')}\}$ Runge's approximation functions.

Remark 3.2.1. Later we will move (y,s) along some non-selfintersecting continuous curve $\mathcal{C}^s := \{(y(\lambda),s); 0 \le \lambda \le 1\}$ for fixed $s \in (0,T)$ such that $y(0), y(1) \in \partial\Omega, y(\lambda) \in \Omega$ $(0 < \lambda < 1)$. Since the approximation domain U must avoid (y,s) and has to contain $\overline{D}$, we want to have $\Omega_{(-\varepsilon,T+\varepsilon)} \setminus \overline{U}$ small and narrow as much as possible. Hence, for fixed $j \in \mathbb{N}$ and $(y,s) = (y(\lambda),s) \in \mathcal{C}^s \setminus \overline{D}$ with $0 < \lambda < 1$, we take $U = U_j$ and $\{v^j_{(y,s)}\}$ in (3.2.3) as follows.

(i) Each U_j satisfies (3.1.5).

(ii) $\{U_j\}$ satisfies $\overline{U_j} \subset U_{j+1}$ $(j \in \mathbb{N})$ and $\displaystyle\bigcup_{j=1}^{\infty} U_j = \Omega_{(-\varepsilon,T+\varepsilon)} \setminus \mathcal{C}^s_\lambda$,

where $\mathcal{C}^s_\lambda := \{(y(\lambda'),s); 0 \le \lambda' \le \lambda\}$.

(iii) Each $v^j_{(y,s)}$ satisfies

$$\|v^j_{(y,s)} - G_{(y,s)}\|_{L^2(U_j)} < \frac{1}{j} \quad (j = 1,2,\cdots). \tag{3.2.5}$$

In the same way, we take $U = U_j$ and $\{\varphi^j_{(y',s')}\}$ in (3.2.4) by making the replacement $(y,s) = (y',s')$.

Using these Runge's approximation functions $\{v^j_{(y,s)}\}, \{\varphi^j_{(y',s')}\}$, we define the pre-indicator function as follow.

Definition 3.2.1 (Pre-indicator Function).

Let $(y,s),(y',s') \in \Omega_T \setminus \overline{D}$ such that $(y,s) \ne (y',s')$, and $\{v^j_{(y,s)}\}, \{\varphi^j_{(y',s')}\} \subset H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ be Runge's approximation functions as above. We define the pre-indicator function $I(y',s';y,s)$ by

$$I(y',s';y,s) := \lim_{j\to\infty} \int_{\Gamma_T} \{\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T} \varphi^j_{(y',s')}\big|_{\Gamma_T} - \Pi_D(\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T})\partial_\nu \varphi^j_{(y',s')}\big|_{\Gamma_T}\}\, d\sigma\, dt. \tag{3.2.6}$$

Next we analyze the behavior of the pre-indicator function. To do it, we have to represent the pre-indicator function in terms of the reflected solution which is defined as follow.

Theorem 3.2.1 ([39]) (Unique Solvability of Parabolic Equation 2).

For given $F \in L^2((0,T);(H^1(\Omega))^*)$, there exists a unique solution $\upsilon = \upsilon(F) \in W(\Omega_T)$ to

$$\begin{cases} P_D\upsilon = F \ \text{ in } \Omega_T \\ \partial_\nu \upsilon = 0 \ \text{ on } \Gamma_T, \ \ \upsilon(x,0) = 0. \end{cases} \tag{3.2.7}$$

Lemma 3.2.2 (Reflected Solution).

For Runge's approximation function $\{v^j_{(y,s)}\} \subset H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$, let $u^j_{(y,s)} := u\big(\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T}\big)$ and $w^j_{(y,s)} := u^j_{(y,s)} - v^j_{(y,s)}$, then $w^j_{(y,s)}$ has a limit $w_{(y,s)} \in W(\Omega_T)$ satisfying

$$\begin{cases} P_D w_{(y,s)} = (k-1)\,\mathrm{div}_x(\chi_{D(t)}\nabla_x G_{(y,s)}) \ \text{ in } \Omega_T \\ \partial_\nu w_{(y,s)} = 0 \ \text{ on } \Gamma_T, \ \ w_{(y,s)}(x,0) = 0\,. \end{cases} \tag{3.2.8}$$

We call $w_{(y,s)}$ the reflected solution.

Proof $w^j_{(y,s)}$ satisfies

$$\begin{cases} P_D w^j_{(y,s)} = (k-1)\,\mathrm{div}_x(\chi_{D(t)}\nabla_x v^j_{(y,s)}) \text{ in } \Omega_T \\ \partial_\nu w^j_{(y,s)} = 0 \text{ on } \Gamma_T, \ \ w^j_{(y,s)}(x,0) = 0 \end{cases} \tag{3.2.9}$$

i.e

$$\int_{\Omega_T} (-w^j_{(y,s)}\partial_t\varphi + \gamma(x,t)\nabla_x w^j_{(y,s)} \cdot \nabla_x\varphi)\, dx\, dt = -(k-1)\int_D \nabla_x v^j_{(y,s)} \cdot \nabla_x\varphi\, dx\, dt \tag{3.2.10}$$

for all $\varphi \in W(\Omega_T)$ with $\varphi = 0$ at $t = T$.

From this relation, we have

$$\begin{aligned}
\|w^j_{(y,s)}\|_{W(\Omega_T)} &\le C\Big(\int_0^T \|\operatorname{div}(\chi_{D(t)}\nabla_x v^j_{(y,s)})\|^2_{(H^1(\Omega))^*}\, dt\Big)^{\frac{1}{2}} \\
&\le C''\Big(\sup_{\varphi\in H^1(\Omega);\|\varphi\|_{H^1(\Omega)}=1}\int_0^T \|\nabla_x v^j_{(y,s)}\|_{L^2(D(t))}\|\nabla_x\varphi\|_{L^2(\Omega)}\, dt\Big)^{\frac{1}{2}} \\
&\le C''\|\nabla_x v^j_{(y,s)}\|_{L^2(D)}.
\end{aligned} \tag{3.2.11}$$

(see [39]). Therefore

$$\|w^{j_m}_{(y,s)} - w^{j_n}_{(y,s)}\|_{W(\Omega_T)} \le C''\|\nabla_x v^{j_m}_{(y,s)} - \nabla_x v^{j_n}_{(y,s)}\|_{L^2(D)} \to 0 \text{ as } m, n \to \infty. \tag{3.2.12}$$

On the other hand, taking the limit in (3.2.10), we obtain

$$\int_{\Omega_T} (-w_{(y,s)}\partial_t\varphi + \gamma(x,t)\nabla_x w_{(y,s)} \cdot \nabla_x\varphi)\, dx\, dt = -(k-1)\int_D \nabla_x G_{(y,s)} \cdot \nabla_x\varphi\, dx\, dt \tag{3.2.13}$$

for all $\varphi \in W(\Omega_T)$ with $\varphi = 0$ at $t = T$. This shows $w_{(y,s)}$ satisfies (3.2.8). Here we used the well known interior Schauder-type estimate (cf.[28]) in $D \subset U$. for taking the limit in (3.2.10) and (3.2.12).

Proposition 3.2.3. *For $(y,s), (y',s') \in \Omega_T \setminus \overline{D}$ such that $(y,s) \neq (y',s')$, then*

$$I(y',s';y,s) = (k-1)\int_D \nabla_x(G_{(y,s)} + w_{(y,s)}) \cdot \nabla_x G^*_{(y',s')}\, d\sigma\, dt. \tag{3.2.14}$$

Proof

$$u^j_{(y,s)} = v^j_{(y,s)} + w^j_{(y,s)} \to G_{(y,s)} + w_{(y,s)} \ (j \to \infty) \ \text{ in } L^2(U) \tag{3.2.15}$$

where $U \subset \Omega_{(-\varepsilon,T+\varepsilon)}$ is open set satisfying $U \not\ni (y,s)$ and $\overline{D} \subset U\big|_{(0,T)}$.

From Green's theorem and (3.2.15),

$$\begin{aligned}
&\int_{\Gamma_T} \{\partial_\nu v^j_{(y,s)}\varphi^j_{(y',s')} - \Pi_D(\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T})\partial_\nu\varphi^j_{(y',s')}\}\, d\sigma\, dt \\
&= \int_{\Omega_T} (k-1)\chi_{D(t)}\nabla_x u^j_{(y,s)} \cdot \nabla_x\varphi^j_{(y',s')}\, dx\, dt \\
&\to (k-1)\int_D \nabla_x(G_{(y,s)} + w_{(y,s)}) \cdot \nabla_x G^*_{(y',s')}\, dx\, dt
\end{aligned} \tag{3.2.16}$$

due to $(y,s),(y',s')\notin\overline{D}$. Here we also used the well known interior Schauder-type estimate in $D\subset U$ for taking the limit in (3.2.16).

Proposition 3.2.4. *For $(y,s),(y',s')\in\Omega_T\setminus\overline{D}$ such that $(y,s)\neq(y',s')$, then*

$$I(y',s';y,s)=-\int_{\Omega_{s'}}(\partial_t w_{(y,s)}G^*_{(y',s')}+\nabla_x w_{(y,s)}\cdot\nabla_x G^*_{(y',s')})\,dx\,dt. \tag{3.2.17}$$

Proof From (3.2.13), $w_{(y,s)}$ satisfies

$$\int_{\Omega_T}(-w_{(y,s)}\partial_t\varphi+\nabla_x w_{(y,s)}\cdot\nabla_x\varphi)\,dx\,dt=-(k-1)\int_D\nabla_x(w_{(y,s)}+G_{(y,s)})\cdot\nabla_x\varphi\,dx\,dt \tag{3.2.18}$$

for all $\varphi\in W(\Omega_T)$ with $\varphi=0$ at $t=T$.

Let $B_r(y,s)$ be an open ball in $\mathbb{R}^{n+1}$ whose radius is $r>0$ centered at (y,s) and we simply write $B_r=B_r(0,0)$ and set $\eta\in C_0^\infty(\mathbb{R}^{n+1})$ such that

$$0\le\eta\le1,\quad \eta=\begin{cases}1 \text{ on } B_r\\ 0 \text{ on } \mathbb{R}^{n+1}\setminus\overline{B_R}\end{cases}\quad\text{and } |\nabla_x\eta|\le\frac{C}{R-r} \tag{3.2.19}$$

for $0<r<R<\infty$. For such η, we define

$$\eta_\varepsilon(x,t):=\eta\left(\frac{x-y'}{\varepsilon},\frac{s'-t}{\varepsilon^2}\right),\quad \widetilde{\eta}_\varepsilon(x,t):=1-\eta_\varepsilon(x,t) \tag{3.2.20}$$

for $0<\varepsilon<\varepsilon_0$ with small ε_0 such that $\operatorname{supp}\eta_\varepsilon\subset E\subset\overline{E}\subset\Omega_T\setminus\overline{D}$ with some relatively compact open set E and $\operatorname{supp}\eta_\varepsilon\not\ni(y,s)$ for $0<\varepsilon<\varepsilon_0$.

We take $\varphi=\widetilde{\eta}_\varepsilon G^*_{(y',s')}$ in (3.2.18). From (3.2.14), and $\eta_\varepsilon=0$ on D,

$$\begin{aligned}(\text{RHS of (3.2.18)})&=-(k-1)\int_D\nabla_x(w_{(y,s)}+G_{(y,s)})\cdot\nabla_x G^*_{(y',s')}\,dx\,dt\\&=-I(y',s';y,s).\end{aligned} \tag{3.2.21}$$

On the other hand, by $G^*_{(y',s')}=0$ on $t>s'$,

$$\begin{aligned}(\text{LHS of (3.2.18)})=&\int_{\Omega_{s'}}\partial_t w_{(y,s)}G^*_{(y',s')}\,dx\,dt-\int_{\Omega_{s'}}\partial_t w_{(y,s)}G^*_{(y',s')}\eta_\varepsilon\,dx\,dt\\&+\int_{\Omega_{s'}}\nabla_x w_{(y,s)}\cdot\nabla_x G^*_{(y',s')}\,dx\,dt-\int_{\Omega_{s'}}\nabla_x w_{(y,s)}\cdot\nabla_x G^*_{(y',s')}\eta_\varepsilon\,dx\,dt\\&-\int_{\Omega_{s'}}\nabla_x w_{(y,s)}\cdot\nabla_x\eta_\varepsilon G^*_{(y',s')}\,dx\,dt.\end{aligned} \tag{3.2.22}$$

Note that $G^*_{(y',s')},\nabla_x G^*_{(y',s')}\in L^1(\Omega_T)$ and $w_{(y,s)}\in C^\infty(E)$, we obtain

$$\int_{\Omega_{s'}}\partial_t w_{(y,s)}G^*_{(y',s')}\eta_\varepsilon\,dx\,dt=\int_{E\cap\Omega_{s'}}\partial_t w_{(y,s)}G^*_{(y',s')}\eta_\varepsilon\,dx\,dt\to0, \tag{3.2.23}$$

$$\int_{\Omega_{s'}} \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s')} \eta_\varepsilon \, dx\, dt = \int_{E\cap\Omega_{s'}} \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s')} \eta_\varepsilon \, dx\, dt \to 0 \tag{3.2.24}$$

as $\varepsilon \to 0$ and set $Q_r(y',s') := \left\{(x,t) \in \mathbb{R}^{n+1}; \frac{|x-y'|^2}{r^2} + \frac{(s'-t)^2}{r^4} \le 1\right\}$,

$$\begin{aligned}
&\int_{\Omega_{s'}} \nabla_x w_{(y,s)} \cdot \nabla_x \eta_\varepsilon G^*_{(y',s')} \, dx\, dt = \int_{E\cap\Omega_{s'}} \nabla_x w_{(y,s)} \cdot \nabla_x \eta_\varepsilon G^*_{(y',s')} \, dx\, dt \\
&\le \sup_{(x,t)\in E} |\nabla_x w_{(y,s)}| \int_{(Q_{\varepsilon R}(y',s')\setminus Q_{\varepsilon r}(y',s'))\cap\Omega_{s'}} |\nabla_x \eta_\varepsilon| G^*_{(y',s')} \, dx\, dt \\
&\le \frac{C}{\varepsilon(R-r)} \int_{(Q_{\varepsilon R}(y',s')\setminus Q_{\varepsilon r}(y',s'))\cap\Omega_{s'}} G^*_{(y',s')} \, dx\, dt \\
&\le C'\varepsilon^{-1} \int_{(B_R\setminus B_r)\cap\{\tau>0\}} \frac{1}{(4\pi\varepsilon^2\tau)^{\frac{n}{2}}} \exp\left[-\frac{|\xi|^2}{4\tau}\right] \varepsilon^{n+2} \, d\xi\, d\tau \le C''\varepsilon \to 0
\end{aligned} \tag{3.2.25}$$

as $\varepsilon \to 0$. These yield (3.2.17).

Proposition 3.2.5. *Let $(y,s),(y',s') \in \Omega_T \setminus \overline{D}$ such that $(y,s) \neq (y',s')$. Then,*

$$\int_{\Omega_{s'}} (\partial_t w_{(y,s)} G^*_{(y',s')} + \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s')}) \, dx\, dt = w_{(y,s)}(y',s') + \int_{\partial\Omega_{s'}} w_{(y,s)} \partial_\nu G^*_{(y',s')} \, d\sigma\, dt. \tag{3.2.26}$$

Proof To prove this proposition, we show the following Claim .

Claim

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega_{s'}} (\partial_t w_{(y,s)} G^*_{(y',s'+\varepsilon)} + \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s'+\varepsilon)}) \, dx\, dt \\
&= \int_{\Omega_{s'}} (\partial_t w_{(y,s)} G^*_{(y',s')} + \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s')}) \, dx\, dt
\end{aligned} \tag{3.2.27}$$

(Proof of Claim) For some small $\delta > 0$,

$$\begin{aligned}
&\int_{\Omega_{s'}} [(\partial_t w_{(y,s)} G^*_{(y',s'+\varepsilon)} + \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s'+\varepsilon)}) \\
&\qquad - (\partial_t w_{(y,s)} G^*_{(y',s')} + \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s')})] \, dx\, dt \\
&= \left[\int_{\Omega_{s'}\setminus(B_\delta(y,s)\cup B_\delta(y',s'))} + \int_{B_\delta(y,s)} + \int_{B_\delta(y',s')\cap\Omega_{s'}}\right] \\
&\qquad [(\partial_t w_{(y,s)} (G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')}) + \nabla_x w_{(y,s)} \cdot \nabla_x (G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})] \, dx\, dt
\end{aligned} \tag{3.2.28}$$

Obviously, (1st term of (3.2.28)) $\to 0$ as $\varepsilon \to 0$.

For the 2nd term of (3.2.28), $G^*_{(y',s')} \in C^\infty(B_\delta(y,s))$ we have

$$\begin{aligned}
&\int_{B_\delta(y,s)} |\partial_t w_{(y,s)}(G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})|\,dx\,dt \\
&\le \int_{B_\delta(y)_{(s-\delta,s+\delta)}} |\partial_t w_{(y,s)}(G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})|\,dx\,dt \\
&\le \|\partial_t w_{(y,s)}\|_{L^2((s-\delta,s+\delta);(H^1(B_\delta(y)))^*)}\|G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')}\|_{L^2((s-\delta,s+\delta);H^1(B_\delta(y)))} \to 0
\end{aligned} \tag{3.2.29}$$

and

$$\begin{aligned}
&\int_{B_\delta(y,s)} |\nabla_x w_{(y,s)}||\nabla_x(G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})|\,dx\,dt \\
&\le \|\nabla_x w_{(y,s)}\|_{L^2(B_\delta(y,s))}\|\nabla_x(G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})\|_{L^2(B_\delta(y,s))} \to 0
\end{aligned} \tag{3.2.30}$$

as $\varepsilon \to 0$, where $B_r(y) := \{x \in \mathbb{R}^n; |x-y| \le r\}$.

The 3rd terms can be estimated as follows.

$$\begin{aligned}
&\int_{B_\delta(y',s')\cap\Omega_{s'}} \partial_t w_{(y,s)}(G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})\,dx\,dt \\
&= \int_{B_\delta(y',s'-\varepsilon)\cap\Omega_{s'-\varepsilon}} \partial_t w_{(y,s)}(x,t+\varepsilon)G^*_{(y',s')}(x,t)\,dx\,dt - \int_{B_\delta(y',s')\cap\Omega_{s'}} \partial_t w_{(y,s)}G^*_{(y',s')}\,dx\,dt \\
&= \int_{U_1} \partial_t(w_{(y,s)}(x,t+\varepsilon) - w_{(y,s)}(x,t))G^*_{(y',s')}(x,t)\,dx\,dt \\
&\quad + \int_{U_2} \partial_t w_{(y,s)}(x,t+\varepsilon)G^*_{(y',s')}(x,t)\,dx\,dt - \int_{U_3} \partial_t w_{(y,s)}(x,t)G^*_{(y',s')}(x,t)\,dx\,dt.
\end{aligned} \tag{3.2.31}$$

where

$$\begin{aligned}
U_1 &:= \{(x,t) \in B_\delta(y',s') \cap B_\delta(y',s'-\varepsilon); t < s'-\varepsilon\}, \\
U_2 &:= \{(x,t) \in B_\delta(y',s'-\varepsilon); t < s'-\varepsilon, (x,t) \notin U_1\}, \\
U_3 &:= \{(x,t) \in B_\delta(y',s'); s'-\varepsilon < t < s'\}.
\end{aligned} \tag{3.2.32}$$

Here note that $w_{(y,s)} \in C^\infty(B_\delta(y',s'))$ and $G^*_{(y',s')} \in L^1(\Omega_T)$, then

$$|(\text{1st term of (3.2.31)})| \le \sup_{(x,t)\in B_\delta(y',s')\cap B_\delta(y',s'-\varepsilon)} |\partial_t(w_{(y,s)}(x,t+\varepsilon) - w_{(y,s)}(x,t))| \to 0 \tag{3.2.33}$$

as $\varepsilon \to 0$, and

$$|(\text{2nd term of (3.2.31)})| \le C|B_\delta(y',s'-\varepsilon) \setminus B_\delta(y',s')| \to 0, \tag{3.2.34}$$

$$|(\text{3rd term of (3.2.31)})| \le C\varepsilon \to 0 \tag{3.2.35}$$

as $\varepsilon \to 0$. In the same way using $\nabla_x(G_{(y',x'+\varepsilon)} - G^*_{(y',s')})$ is dominated by some $L^1(B_\delta(y',s'))$ independent of ε, we can show

$$\int_{B_\delta(y',s')} \nabla_x w_{(y,s)} \cdot \nabla_x(G^*_{(y',s'+\varepsilon)} - G^*_{(y',s')})\, dx\, dt \to 0 \tag{3.2.36}$$

as $\varepsilon \to 0$. Hence we have proven Claim .

Now we return to the proof of proposition 3.2.5. From Claim ,

$$\begin{aligned}
&(\text{LHS of } (3.2.26)) = \lim_{\varepsilon\to 0} \int_{\Omega_{s'}} (\partial_t w_{(y,s)} G^*_{(y',s'+\varepsilon)} + \nabla_x w_{(y,s)} \cdot \nabla_x G^*_{(y',s'+\varepsilon)})\, dx\, dt \\
&= \lim_{\varepsilon\to 0}\left[\int_{\Gamma_{s'}} w_{(y,s)} \partial_\nu G^*_{(y',s'+\varepsilon)}\, d\sigma\, dt + \int_\Omega w_{(y,s)}(x,s') G^*_{(y',s'+\varepsilon)}(x,s')\, dx\right] \\
&= \lim_{\varepsilon\to 0}\left[\int_{\Gamma_{s'}} w_{(y,s)} \partial_\nu G^*_{(y',s'+\varepsilon)}\, d\sigma\, dt + \int_\Omega w_{(y,s)}(x,s') G_{(y',0)}(x,\varepsilon)\, dx\right] \\
&= \int_{\Gamma_{s'}} w_{(y,s)} \partial_\nu G^*_{(y',s')}\, d\sigma\, dt + \lim_{\varepsilon\to 0} \int_\Omega w_{(y,s)}(x,s') G_{(y',0)}(x,\varepsilon)\, dx.
\end{aligned} \tag{3.2.37}$$

Then, the proof will be finished if we remind the well known fact.

Fact

$$\lim_{\varepsilon\to 0} \int_\Omega w_{(y,s)}(x,s') G_{(y',0)}(x,\varepsilon)\, dx = w_{(y,s)}(y',s'). \tag{3.2.38}$$

From these propositions and $G_{(y',s')} = 0$ on $t > s'$, we obtain an important representation formula for the pre-indicator function.

Theorem 3.2.6 (Representation Formula).
For $(y,s), (y',s') \in \Omega_T \setminus \overline{D}$ such that $(y,s) \neq (y',s')$,

$$I(y',s';y,s) = -w_{(y,s)}(y',s') - \int_{\Gamma_T} w_{(y,s)} \partial_\nu G^*_{(y',s')}\, d\sigma\, dt. \tag{3.2.39}$$

3.3 1 Dimensional and Time-independent Case

We assume that $\Omega = (a_1, a_0), D(t) = (d_1, d_0)$ $(a_1 < d_1 < d_0 < a_0)$ and set $w_{+,0} := w_{(y,s)}\big|_{(d_0,a_0)_T}$, $w_- := w_{(y,s)}\big|_{(d_1,d_0)_T}$ and $w_{+,1} := w_{(y,s)}\big|_{(a_1,d_1)_T}$.

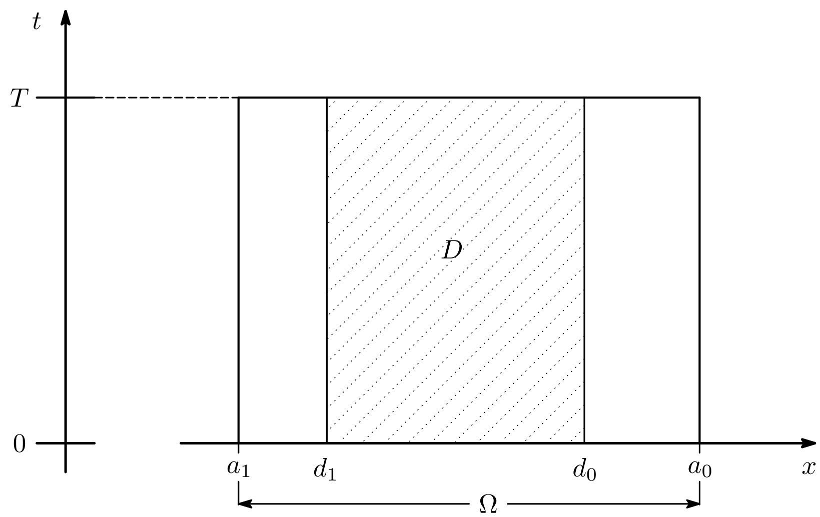


Figure 3.2: 1 dimensional and time-independent case

(3.2.8) is equivalent to the following transmission boundary problem (3.3.1)-(3.3.3):

$$\begin{cases} \partial_t w_{+,0} - \partial_x^2 w_{+,0} = 0 \quad \text{in } (d_0, a_0)_T \\ \partial_t w_- - k\partial_x^2 w_- = (k-1)\partial_x^2 G_{(y,s)} \quad \text{in } (d_1, d_0)_T \\ \partial_t w_{+,1} - \partial_x^2 w_{+,1} = 0 \quad \text{in } (a_1, d_1)_T \end{cases} \tag{3.3.1}$$

$$w_{+,0}(x,0) = 0 \text{ on } (d_0, a_0), \quad w_-(x,0) = 0 \text{ on } (d_0, d_1), \quad w_{+,1}(x,0) = 0 \quad \text{on } (d_0, a_0) \tag{3.3.2}$$

$$\begin{cases} \partial_x w_{+,0}(a_0, t) = 0, \quad \partial_x w_{+,1}(a_1, t) = 0 \quad \text{on } (0,T) \\ \partial_x w_{+,0}(d_0, t) - k\partial_x w_-(d_0, t) = (k-1)\partial_x G_{(y,s)}(d_0, t) \quad \text{on } (0,T) \\ \partial_x w_{+,1}(d_1, t) - k\partial_x w_-(d_1, t) = (k-1)\partial_x G_{(y,s)}(d_1, t) \quad \text{on } (0,T) \\ w_{+,0}(d_0, t) = w_-(d_0, t), \quad w_-(d_1, t) = w_{+,1}(d_1, t) \text{ on } (0,T). \end{cases} \tag{3.3.3}$$

Taking the Laplace transform with respect to t, when $y \in (d_0, a_0)$, we obtain the following ordinary differential equations:

$$\begin{cases} \tau\mathcal{L}(w_{+,0}) - \partial_x^2\mathcal{L}(w_{+,0}) = 0 \quad \text{in } (d_0, a_0) \\ \tau\mathcal{L}(w_-) - k\partial_x^2\mathcal{L}(w_-) = (k-1)\dfrac{\exp(-st-(y-x)\sqrt{\tau})}{2}\sqrt{\tau} \quad \text{in } (d_1, d_0) \\ \tau\mathcal{L}(w_{+,1}) - \partial_x^2\mathcal{L}(w_{+,1}) = 0 \quad \text{in } (a_1, d_1) \end{cases} \tag{3.3.4}$$

and the boundary conditions

$$\begin{cases} \partial_x\mathcal{L}(w_{+,0})(a_0,\tau) = 0, \quad \partial_x\mathcal{L}(w_{+,1})(a_1,\tau) = 0 \\ \partial_x\mathcal{L}(w_{+,0})(d_0,\tau) - k\partial_x\mathcal{L}(w_-)(d_0,\tau) = (k-1)\dfrac{\exp(-s\tau-(y-d_0)\sqrt{\tau})}{2} \\ \partial_x\mathcal{L}(w_{+,1})(d_1,\tau) - k\partial_x\mathcal{L}(w_-)(d_1,\tau) = (k-1)\dfrac{\exp(-s\tau-(y-d_1)\sqrt{\tau})}{2} \\ \mathcal{L}(w_{+,0})(d_0,\tau) = \mathcal{L}(w_-)(d_0,\tau), \quad \mathcal{L}(w_-)(d_1,\tau) = \mathcal{L}(w_{+,1})(d_1,\tau) \end{cases} \tag{3.3.5}$$

Here the Laplace transform $\mathcal{L}(u)(x,\tau)$ of $u(x,t)$ is defined by

$$\mathcal{L}(u)(x,\tau) := \int_0^\infty e^{-t\tau}u(x,t)\,dt \tag{3.3.6}$$

and we have used

$$\begin{aligned} \mathcal{L}(\partial_x G_{(y,s)}) &= -\frac{x-y}{2|x-y|}\exp(-s\tau-|x-y|\sqrt{\tau}), \\ \mathcal{L}(\partial_x^2 G_{(y,s)}) &= \frac{\sqrt{\tau}}{2}\exp(-s\tau-|x-y|\sqrt{\tau}). \end{aligned} \tag{3.3.7}$$

Solving (3.3.4), we get

$$\begin{aligned} &\mathcal{L}(w_{+,0}) = C^1_{+,0}\exp(\sqrt{\tau}x) + C^2_{+,0}\exp(-\sqrt{\tau}x), \\ &\mathcal{L}(w_{+,1}) = C^1_{+,1}\exp(\sqrt{\tau}x) + C^2_{+,1}\exp(-\sqrt{\tau}x), \\ &\mathcal{L}(w_-) = -\frac{\exp(-s\tau-(y-x)\sqrt{\tau})}{2\sqrt{\tau}} + \left\{C^1_-\exp\left(\sqrt{\frac{\tau}{k}}x\right) + C^2_-\exp\left(-\sqrt{\frac{\tau}{k}}x\right)\right\}. \end{aligned} \tag{3.3.8}$$

for some constant $C^1_{+,0}, C^2_{+,0}, C^1_{+,1}, C^2_{+,1}, C^1_-, C^2_- \in \mathbb{C}$. Let's find these coefficients.

From boundary condition, we obtain

$$C^2_{+,0} = \exp(2a_0\sqrt{\tau})C^1_{+,0}, \quad C^2_{+,1} = \exp(2a_1\sqrt{\tau})C^1_{+,1}. \tag{3.3.9}$$

Plug in (3.3.9) to (3.3.5),

$$\begin{cases} (e^{d_0\sqrt{\tau}} + e^{(2a_0-d_0)\sqrt{\tau}})C^1_{+,0} - e^{d_0\sqrt{\frac{\tau}{k}}}C^1_- - e^{-d_0\sqrt{\frac{\tau}{k}}}C^2_- = -\dfrac{e^{-s\tau-(y-d_0)\sqrt{\tau}}}{2\sqrt{\tau}} \\ (e^{d_1\sqrt{\tau}} + e^{(2a_1-d_1)\sqrt{\tau}})C^1_{+,1} - e^{d_1\sqrt{\frac{\tau}{k}}}C^1_- - e^{-d_1\sqrt{\frac{\tau}{k}}}C^2_- = -\dfrac{e^{-s\tau-(y-d_1)\sqrt{\tau}}}{2\sqrt{\tau}} \\ (e^{d_0\sqrt{\tau}} - e^{(2a_0-d_0)\sqrt{\tau}})C^1_{+,0} - \sqrt{k}(e^{d_0\sqrt{\frac{\tau}{k}}}C^1_- - e^{-d_0\sqrt{\frac{\tau}{k}}}C^2_-) = -\dfrac{e^{-s\tau-(y-d_0)\sqrt{\tau}}}{2\sqrt{\tau}} \\ (e^{d_1\sqrt{\tau}} - e^{(2a_1-d_1)\sqrt{\tau}})C^1_{+,1} - \sqrt{k}(e^{d_1\sqrt{\frac{\tau}{k}}}C^1_- - e^{-d_1\sqrt{\frac{\tau}{k}}}C^2_-) = -\dfrac{e^{-s\tau-(y-d_1)\sqrt{\tau}}}{2\sqrt{\tau}} \end{cases} \tag{3.3.10}$$

$\Updownarrow$

$$\begin{bmatrix} e^{d_0\sqrt{\tau}}+e^{(2a_0-d_0)\sqrt{\tau}} & 0 & -e^{d_0\sqrt{\frac{\tau}{k}}} & -e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}}+e^{(2a_1-d_1)\sqrt{\tau}} & -e^{d_1\sqrt{\frac{\tau}{k}}} & -e^{-d_1\sqrt{\frac{\tau}{k}}} \\ e^{d_0\sqrt{\tau}}-e^{(2a_0-d_0)\sqrt{\tau}} & 0 & -\sqrt{k}e^{d_0\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}}-e^{(2a_1-d_1)\sqrt{\tau}} & -\sqrt{k}e^{d_1\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_1\sqrt{\frac{\tau}{k}}} \end{bmatrix} \begin{bmatrix} C^1_{+,0} \\ C^1_{+,1} \\ C^1_- \\ C^2_- \end{bmatrix} = -\frac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}} \begin{bmatrix} e^{d_0\sqrt{\tau}} \\ e^{d_1\sqrt{\tau}} \\ e^{d_0\sqrt{\tau}} \\ e^{d_1\sqrt{\tau}} \end{bmatrix}. \tag{3.3.11}$$

By Crámer's theorem, we obtain

$$\begin{cases} C^1_{+,0} = -\dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}\dfrac{\mathbb{D}_1}{\mathbb{D}_0},\ C^1_{+,1} = -\dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}\dfrac{\mathbb{D}_2}{\mathbb{D}_0} \\ C^1_- = -\dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}\dfrac{\mathbb{D}_3}{\mathbb{D}_0},\ C^2_- = -\dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}\dfrac{\mathbb{D}_4}{\mathbb{D}_0} \end{cases} \tag{3.3.12}$$

i.e.

$$\begin{cases} \mathcal{L}(w_{+,0}) = -\dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}(\exp(\sqrt{\tau}x)+\exp(2a_0-\sqrt{\tau}x))\dfrac{\mathbb{D}_1}{\mathbb{D}_0} \\ \mathcal{L}(w_{+,1}) = -\dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}(\exp(\sqrt{\tau}x)+\exp(2a_1-\sqrt{\tau}x))\dfrac{\mathbb{D}_2}{\mathbb{D}_0} \\ \mathcal{L}(w_-) = -\dfrac{\exp(-s\tau-(y-x)\sqrt{\tau})}{2\sqrt{\tau}} - \dfrac{e^{-s\tau-y\sqrt{\tau}}}{2\sqrt{\tau}}\left\{\dfrac{\mathbb{D}_3}{\mathbb{D}_0}\exp\left(\sqrt{\dfrac{\tau}{k}}x\right)+\dfrac{\mathbb{D}_4}{\mathbb{D}_0}\exp\left(-\sqrt{\dfrac{\tau}{k}}x\right)\right\} \end{cases} \tag{3.3.13}$$

where

$$\mathbb{D}_0 := \begin{vmatrix} e^{d_0\sqrt{\tau}}+e^{(2a_0-d_0)\sqrt{\tau}} & 0 & -e^{d_0\sqrt{\frac{\tau}{k}}} & -e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}}+e^{(2a_1-d_1)\sqrt{\tau}} & -e^{d_1\sqrt{\frac{\tau}{k}}} & -e^{-d_1\sqrt{\frac{\tau}{k}}} \\ e^{d_0\sqrt{\tau}}-e^{(2a_0-d_0)\sqrt{\tau}} & 0 & -\sqrt{k}e^{d_0\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}}-e^{(2a_1-d_1)\sqrt{\tau}} & -\sqrt{k}e^{d_1\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_1\sqrt{\frac{\tau}{k}}} \end{vmatrix},$$

$$\mathbb{D}_1 := \begin{vmatrix} e^{d_0\sqrt{\tau}} & 0 & -e^{d_0\sqrt{\frac{\tau}{k}}} & -e^{-d_0\sqrt{\frac{\tau}{k}}} \\ e^{d_1\sqrt{\tau}} & e^{d_1\sqrt{\tau}}+e^{(2a_1-d_1)\sqrt{\tau}} & -e^{d_1\sqrt{\frac{\tau}{k}}} & -e^{-d_1\sqrt{\frac{\tau}{k}}} \\ e^{d_0\sqrt{\tau}} & 0 & -\sqrt{k}e^{d_0\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_0\sqrt{\frac{\tau}{k}}} \\ e^{d_1\sqrt{\tau}} & e^{d_1\sqrt{\tau}}-e^{(2a_1-d_1)\sqrt{\tau}} & -\sqrt{k}e^{d_1\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_1\sqrt{\frac{\tau}{k}}} \end{vmatrix},$$

$$\mathbb{D}_2 := \begin{vmatrix} e^{d_0\sqrt{\tau}}+e^{(2a_0-d_0)\sqrt{\tau}} & e^{d_0\sqrt{\tau}} & -e^{d_0\sqrt{\frac{\tau}{k}}} & -e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}} & -e^{d_1\sqrt{\frac{\tau}{k}}} & -e^{-d_1\sqrt{\frac{\tau}{k}}} \\ e^{d_0\sqrt{\tau}}-e^{(2a_0-d_0)\sqrt{\tau}} & e^{d_0\sqrt{\tau}} & -\sqrt{k}e^{d_0\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}} & -\sqrt{k}e^{d_1\sqrt{\frac{\tau}{k}}} & \sqrt{k}e^{-d_1\sqrt{\frac{\tau}{k}}} \end{vmatrix},$$

$$\mathbb{D}_3 := \begin{vmatrix} e^{d_0\sqrt{\tau}} + e^{(2a_0-d_0)\sqrt{\tau}} & 0 & e^{d_0\sqrt{\tau}} & -e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}} + e^{(2a_1-d_1)\sqrt{\tau}} & e^{d_1\sqrt{\tau}} & -e^{-d_1\sqrt{\frac{\tau}{k}}} \\ e^{d_0\sqrt{\tau}} - e^{(2a_0-d_0)\sqrt{\tau}} & 0 & e^{d_0\sqrt{\tau}} & \sqrt{k}e^{-d_0\sqrt{\frac{\tau}{k}}} \\ 0 & e^{d_1\sqrt{\tau}} - e^{(2a_1-d_1)\sqrt{\tau}} & e^{d_1\sqrt{\tau}} & \sqrt{k}e^{-d_1\sqrt{\frac{\tau}{k}}} \end{vmatrix}$$

and

$$\mathbb{D}_4 := \begin{vmatrix} e^{d_0\sqrt{\tau}} + e^{(2a_0-d_0)\sqrt{\tau}} & 0 & -e^{d_0\sqrt{\frac{\tau}{k}}} & e^{d_0\sqrt{\tau}} \\ 0 & e^{d_1\sqrt{\tau}} + e^{(2a_1-d_1)\sqrt{\tau}} & -e^{d_1\sqrt{\frac{\tau}{k}}} & e^{d_1\sqrt{\tau}} \\ e^{d_0\sqrt{\tau}} - e^{(2a_0-d_0)\sqrt{\tau}} & 0 & -\sqrt{k}e^{d_0\sqrt{\frac{\tau}{k}}} & e^{d_0\sqrt{\tau}} \\ 0 & e^{d_1\sqrt{\tau}} - e^{(2a_1-d_1)\sqrt{\tau}} & -\sqrt{k}e^{d_1\sqrt{\frac{\tau}{k}}} & e^{d_1\sqrt{\tau}} \end{vmatrix}.$$

Calculating $\mathbb{D}_j$ $(j = 0, 1, 2, 3, 4)$ more carefully, then we obtain

$$\begin{aligned} \mathbb{D}_0 = {} & e^{d_0\sqrt{\tau}} e^{2(a_0-d_0)\sqrt{\tau}} e^{d_1\sqrt{\tau}} e^{(d_0-d_1)\sqrt{\frac{\tau}{k}}} \\ & \times [-(1 + e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{k(1 + e^{-2(d_1-a_1)\sqrt{\tau}})(1 - e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) + \sqrt{k}(1 - e^{-2(d_1-a_1)\sqrt{\tau}})(1 + e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}})\} \\ & - (1 - e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{\sqrt{k}(1 + e^{-2(d_1-a_1)\sqrt{\tau}})(1 + e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) + (1 - e^{-2(d_1-a_1)\sqrt{\tau}})(1 - e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}})\}], \end{aligned} \tag{3.3.14}$$

$$\begin{aligned} \mathbb{D}_1 = {} & e^{d_0\sqrt{\tau}} e^{(d_0-d_1)\sqrt{\frac{\tau}{k}}} e^{d_1\sqrt{\tau}} \\ & \times [(1 + e^{-2(d_1-a_1)\sqrt{\tau}}) \\ & \quad \times \{-k(1 - e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) + \sqrt{k}(1 + e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) - 2\sqrt{k}e^{-(d_0-d_1)(1+\frac{1}{\sqrt{k}})\sqrt{\tau}}\} \\ & + (1 - e^{-2(d_1-a_1)\sqrt{\tau}}) \\ & \quad \times \{-\sqrt{k}(1 + e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) + 2\sqrt{k}e^{-(d_0-d_1)(1+\frac{1}{\sqrt{k}})\sqrt{\tau}} + (1 - e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}})\}], \end{aligned} \tag{3.3.15}$$

$$\begin{aligned} \mathbb{D}_2 = {} & e^{d_0\sqrt{\tau}} e^{d_1\sqrt{\tau}} e^{2(a_0-d_0)\sqrt{\tau}} e^{(d_0-d_1)\sqrt{\frac{\tau}{k}}} \\ & \times [(1 + e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{2\sqrt{k}e^{(d_0-d_1)(1-\frac{1}{\sqrt{k}})\sqrt{\tau}} - \sqrt{k}(1 + e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) - k(1 - e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}})\} \\ & - (1 - e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{-2\sqrt{k}e^{(d_0-d_1)(1-\frac{1}{\sqrt{k}})\sqrt{\tau}} + \sqrt{k}(1 + e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}}) + (1 - e^{-2(d_0-d_1)\sqrt{\frac{\tau}{k}}})\}], \end{aligned} \tag{3.3.16}$$

$$\begin{aligned}\mathbb{D}_3 = {} & e^{2d_0\sqrt{\tau}}e^{2(a_0-d_0)\sqrt{\tau}}e^{d_1\sqrt{\tau}}e^{-d_1\sqrt{\frac{\tau}{k}}} \\ & \times [(1+e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{\sqrt{k}(1+e^{-2(d_1-a_1)\sqrt{\tau}}) - 2\sqrt{k}e^{-\{(d_0-d_1)(1+\frac{1}{\sqrt{k}})+2(d_1-a_1)\}\sqrt{\tau}} + (1-e^{-2(d_1-a_1)\sqrt{\tau}})\} \\ & + (1-e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{\sqrt{k}(1+e^{-2(d_1-a_1)\sqrt{\tau}}) + (1-e^{-2(d_1-a_1)\sqrt{\tau}}) + 2e^{-\{(d_0-d_1)(1+\frac{1}{\sqrt{k}})+2(d_1-a_1)\}\sqrt{\tau}}\}]\end{aligned} \tag{3.3.17}$$

and

$$\begin{aligned}\mathbb{D}_4 = {} & e^{2d_0\sqrt{\tau}}e^{2(a_0-d_0)\sqrt{\tau}}e^{d_1\sqrt{\tau}}e^{d_1\sqrt{\frac{\tau}{k}}} \\ & \times [(1+e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{\sqrt{k}(1+e^{-2(d_1-a_1)\sqrt{\tau}}) - (1-e^{-2(d_1-a_1)\sqrt{\tau}}) - 2\sqrt{k}e^{-(d_0-d_1)(1-\frac{1}{\sqrt{k}})\sqrt{\tau}}e^{-2(d_1-a_1)\sqrt{\tau}})\} \\ & - (1-e^{-2(a_0-d_0)\sqrt{\tau}}) \\ & \quad \times \{(1-e^{-2(d_1-a_1)\sqrt{\tau}}) - \sqrt{k}(1+e^{-2(d_1-a_1)\sqrt{\tau}}) + 2e^{-(d_0-d_1)(1-\frac{1}{\sqrt{k}})\sqrt{\tau}}e^{-2(d_1-a_1)\sqrt{\tau}}\}].\end{aligned} \tag{3.3.18}$$

Firstly, let's consider about $\mathcal{L}(w_{+,0})$. Since (3.3.14) and (3.3.15), $\mathcal{L}(w_{+,0})$ is represented as

$$\begin{aligned}\mathcal{L}(w_{+,0}) &= -\frac{1}{2\sqrt{\tau}}(e^{-s\tau-(y-x)\sqrt{\tau}} + e^{-s\tau-(y+x-2a_0)\sqrt{\tau}}) \\ &\quad \times \frac{((-k+\sqrt{k})+(-\sqrt{k}+1)) + O(e^{-2L_1\sqrt{\tau}})}{e^{2(a_0-d_0)\sqrt{\tau}}((-(k+\sqrt{k})-(\sqrt{k}+1)) + O(e^{-2L_0\sqrt{\tau}}))} \\ &= -\frac{1}{2\sqrt{\tau}}(e^{-s\tau-(y-x+2(a_0-d_0))\sqrt{\tau}} + e^{-s\tau-(y+x-2d_0)\sqrt{\tau}}) \\ &\quad \times \left(\frac{1-k}{-(1+\sqrt{k})^2} + O(e^{-2L_0'\sqrt{\tau}})\right) \quad (\Re\tau > 0)\end{aligned} \tag{3.3.19}$$

where

$$\begin{cases} L_0 := \min\left\{a_0-d_0, \dfrac{d_0-d_1}{\sqrt{k}}, d_1-a_1\right\} \\ L_1 := \min\left\{\dfrac{d_0-d_1}{\sqrt{k}}, \dfrac{d_0-d_1}{2}\left(1+\dfrac{1}{\sqrt{k}}\right), d_1-a_1\right\} \\ L_0' = \min\{L_0, L_1\}. \end{cases} \tag{3.3.20}$$

Here, we know

$$\frac{e^{-s\tau-(y-x+2(a_0-d_0))\sqrt{\tau}}}{2\sqrt{\tau}} = \mathcal{L}(G_{(y,s)}(x-2(a_0-d_0),\cdot)), \frac{e^{-s\tau-(y+x-2d_0)\sqrt{\tau}}}{2\sqrt{\tau}} = \mathcal{L}(G_{(y,s)}(-x+2d_0,\cdot)), \tag{3.3.21}$$

then

$$\begin{aligned}\mathcal{L}(w_{+,0}) &= \mathcal{L}(G_{(y,s)}(x-2(a_0-d_0),\cdot)+G_{(y,s)}(-x+2d_0,\cdot))\left(\frac{1-\sqrt{k}}{1+\sqrt{k}}+O(e^{-2L_0'\sqrt{\tau}})\right)\\ &= \frac{1-\sqrt{k}}{1+\sqrt{k}}\mathcal{L}(G_{(y,s)}(-x+2d_0,\cdot))+O\left(\frac{e^{-s\tau-C_0\sqrt{\tau}}}{\sqrt{\tau}}\right)\end{aligned} \tag{3.3.22}$$

where $C_0 := \min\{2L_0', a_0-d_0\}$ due to

$$y+x-2d_0+2L_0' > 2L_0', \quad y-x+2(a_0-d_0) > a_0-d_0 \text{ if } y,x\in(d_0,a_0). \tag{3.3.23}$$

We set

$$\mathcal{R}_{(y,s)}(x,\tau) := \mathcal{L}(w_{+,0}-w_{+,0;p})(x,\tau) \tag{3.3.24}$$

with

$$w_{+,0;p}(x,t) := \frac{1-\sqrt{k}}{1+\sqrt{k}}G_{(y,s)}(-x+2d_0,t), \tag{3.3.25}$$

then $\mathcal{R}_{(y,s)}(x,\tau)$ is analytic in τ for $\Re\tau > 0$.

$$\int_{-\infty}^{\infty}|\mathcal{R}_{(y,s)}(x,\sigma+i\omega)|\,d\omega \le M\int_{-\infty}^{\infty}\frac{e^{-s\sigma-\frac{C_1'}{\sqrt{2}}[(\sigma^2+\omega^2)^{\frac{1}{2}}+\sigma]^{\frac{1}{2}}}}{(\sigma^2+\omega^2)^{\frac{1}{4}}}\,d\omega < \infty \quad (\sigma>0). \tag{3.3.26}$$

Also we have to show next Claim

Claim

$$R_{(y,s)}(x,0) = 0, \tag{3.3.27}$$

where

$$R_{(y,s)}(x,t) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{t\tau}\mathcal{R}_{(y,s)}(x,\tau)\,d\tau \tag{3.3.28}$$

with $\sigma > 0$.

(Proof of Claim) By Cauchy's theorem,

$$R^N_{(y,s)}(x,0) = 0, \tag{3.3.29}$$

where $R^N_{(y,s)}(x,t) := \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}^N} e^{t\tau}\mathcal{R}_{(y,s)}(x,\tau)\,d\tau$ with a close curve

$$\mathcal{C}^N := \{\sigma+Ne^{i\psi}; -\pi/2\le\psi\le\pi/2\}\cup[\sigma-iN,\sigma+iN]$$

for $N \in \mathbb{N}$.

Hence,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\sigma-iN}^{\sigma+iN}\mathcal{R}_{(y,s)}(x,\tau)\,d\tau\right| &\le \frac{M}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-s(\sigma+N\cos\psi)-\frac{C_1'}{\sqrt{2}}[(\sigma^2+2N\sigma\cos\psi+N^2)^{\frac{1}{2}}+(\sigma+N\cos\psi)]^{\frac{1}{2}}}}{(\sigma^2+2N\sigma\cos\psi+N^2)^{\frac{1}{4}}}\,d\psi\\ &\le \frac{M}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\psi}{(\sigma^2+N^2)^{\frac{1}{4}}} = \frac{M}{2(\sigma^2+N^2)^{\frac{1}{4}}}\to 0\end{aligned} \tag{3.3.30}$$

as $N \uparrow \infty$. Therefore the proof of Claim is complete. □

By (3.3.26), Claim, and by taking inverse Laplace transform with respect to τ in (3.3.24), we obtain

$$R_{(y,s)}(x,t) = w_{+,0}(x,t) - w_{+,0;p}(x,t) \tag{3.3.31}$$

(cf.[25]). Moreover, we obtain

$$\begin{aligned}|R_{(y,s)}(x,t)| = \left|\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{t\tau}\mathcal{R}_{(y,s)}(x,\tau)\,d\tau\right| &\le \frac{M}{2\pi}\int_{-\infty}^{\infty}\frac{e^{(t-s)\sigma}\cdot e^{-\frac{C_1'}{\sqrt{2}}[(\sigma^2+\rho^2)^{\frac{1}{2}}+\sigma]^{\frac{1}{2}}}}{(\sigma^2+\rho^2)^{\frac{1}{4}}}\,d\rho\\ &\le \frac{Me^{(t-s)\sigma}}{2\pi}\int_{-\infty}^{\infty}\frac{d\rho}{(\sigma^2+\rho^2)^{\frac{1}{4}}} < \infty\end{aligned} \tag{3.3.32}$$

for $\sigma > 0$.

From same idea as this, by (3.3.14) and (3.3.16), we can rewrite $\mathcal{L}(w_{+,1})$ as

$$\begin{aligned}\mathcal{L}(w_{+,1}) &= -\frac{1}{2\sqrt{\tau}}(e^{-s\tau-(y-x)\sqrt{\tau}} + e^{-s\tau-(y+x-2a_1)\sqrt{\tau}})\\ &\quad\times\frac{(2\sqrt{k}-(-2\sqrt{k}))e^{(d_0-d_1)(1-\frac{1}{\sqrt{k}})\sqrt{\tau}} + O(e^{-L_2\sqrt{\tau}})}{(-(k+\sqrt{k})-(\sqrt{k}+1)) + O(e^{-2L_0\sqrt{\tau}})}\\ &= -\frac{4\sqrt{k}}{(1-\sqrt{k})^2}\frac{e^{-s\tau-(y-(x+(d_0-d_1)(1-\frac{1}{\sqrt{k}})))\sqrt{\tau}}}{2\sqrt{\tau}} + O\left(\frac{e^{-L_2'\sqrt{\tau}}}{\sqrt{\tau}}\right)\end{aligned} \tag{3.3.33}$$

for $k > 1$ and

$$\begin{aligned}\mathcal{L}(w_{+,1}) &= -\frac{1}{2\sqrt{\tau}}(e^{-s\tau-(y-x)\sqrt{\tau}} + e^{-s\tau-(y+x-2a_1)\sqrt{\tau}})\\ &\quad\times\frac{(-k+\sqrt{k})-(\sqrt{k}+1) + O(e^{-L_2''\sqrt{\tau}})}{(-(k+\sqrt{k})-(\sqrt{k}+1)) + O(e^{-2L_0\sqrt{\tau}})}\\ &= \frac{e^{-s\tau-(y-x)\sqrt{\tau}}}{2\sqrt{\tau}} + O\left(\frac{e^{-s\tau-L_2'''\sqrt{\tau}}}{\sqrt{\tau}}\right)\end{aligned} \tag{3.3.34}$$

for $0 < k < 1$ due to $(0 <)y - x < y + x - 2a_1$ when $x \in (a_1, d_1), y \in (d_0, a_0)$. Therefore

$$\mathcal{L}(w_{+,1}) = \begin{cases} -\dfrac{4}{(1+\sqrt{k})^2}\mathcal{L}(G_{(y,s)}\Big(x + (d_0 - d_1)\Big(1 - \dfrac{1}{\sqrt{k}}\Big), \cdot\Big)) + O\Big(\dfrac{e^{-s\tau - L_2'\sqrt{\tau}}}{\sqrt{\tau}}\Big) & \text{if } k > 1 \\ \dfrac{1-\sqrt{k}}{1+\sqrt{k}}\mathcal{L}(G_{(y,s)}(x, \cdot)) + O\Big(\dfrac{e^{-s\tau - L_2'''\sqrt{\tau}}}{\sqrt{\tau}}\Big) & \text{if } 0 < k < 1 \end{cases} \tag{3.3.35}$$

Finally we consider about for $\mathcal{L}(w_-)$. Firstly we obtain

$$\frac{\mathbb{D}_3}{\mathbb{D}_0}\exp\Big(\sqrt{\frac{\tau}{k}}x\Big) = \frac{2(1+\sqrt{k}) + O(e^{-L_3\sqrt{\tau}})}{-(1+\sqrt{k})^2 + O(e^{-2L_0\sqrt{\tau}})}e^{(\frac{x}{\sqrt{k}} + (1 - \frac{1}{\sqrt{k}})d_0)\sqrt{\tau}} \tag{3.3.36}$$

due to (3.3.14) and (3.3.17).

Also we obtain

$$\frac{\mathbb{D}_4}{\mathbb{D}_0}\exp\Big(-\sqrt{\frac{\tau}{k}}x\Big) = \frac{-2(1-\sqrt{k}) + O(e^{-L_4\sqrt{\tau}})}{-(1+\sqrt{k})^2 + O(e^{-2L_0\sqrt{\tau}})}e^{(-\frac{x}{\sqrt{k}} + (1 - \frac{1}{\sqrt{k}})d_0 + \frac{2}{\sqrt{k}}d_1)\sqrt{\tau}} \tag{3.3.37}$$

for $k > \Big(1 + \dfrac{2(d_1 - a_1)}{d_0 - d_1}\Big)^{-2}$ $(k \neq 1)$,

$$\frac{\mathbb{D}_4}{\mathbb{D}_0}\exp\Big(-\sqrt{\frac{\tau}{k}}x\Big) = \frac{-2(1+\sqrt{k}) + O(e^{-L_4'\sqrt{\tau}})}{-(1+\sqrt{k})^2 + O(e^{-2L_0\sqrt{\tau}})}e^{(-\frac{x}{\sqrt{k}} + (1 + \frac{1}{\sqrt{k}})d_1 - 2(d_1 - a_1))\sqrt{\tau}} \tag{3.3.38}$$

for $0 < k \leq \Big(1 + \dfrac{2(d_1 - a_1)}{d_0 - d_1}\Big)^{-2}$. So we conclude

$$\begin{aligned} \mathcal{L}(w_-) &= -\frac{e^{-s\tau - (y-x)\sqrt{\tau}}}{2\sqrt{\tau}} + \frac{2}{1+\sqrt{k}}\frac{e^{-s\tau - (y - (\frac{x}{\sqrt{k}} + (1 - \frac{1}{\sqrt{k}})d_0)\sqrt{\tau})}}{2\sqrt{\tau}} + O\Big(\frac{e^{-s\tau - L_4''\sqrt{\tau}}}{\sqrt{\tau}}\Big) \\ &= \mathcal{L}(-G_{(y,s)}(x, \cdot) + \frac{2}{1+\sqrt{k}}G_{(y,s)}\Big(\frac{x}{\sqrt{k}} + \Big(1 - \frac{1}{\sqrt{k}}\Big)d_0, \cdot\Big)) + O\Big(\frac{e^{-s\tau - L_4''\sqrt{\tau}}}{\sqrt{\tau}}\Big) \end{aligned} \tag{3.3.39}$$

because for $k > \Big(1 + \dfrac{2(d_1 - a_1)}{d_0 - d_1}\Big)^{-2}$ $(k \neq 1)$,

$$-y - \frac{x}{\sqrt{k}} + \Big(1 - \frac{1}{\sqrt{k}}\Big)d_0 + \frac{2}{\sqrt{k}}d_1 < -\frac{d_0 - d_1}{\sqrt{k}}(< 0) \tag{3.3.40}$$

and for $0 < k \leq \Big(1 + \dfrac{2(d_1 - a_1)}{d_0 - d_1}\Big)^{-2}$,

$$-y - \frac{x}{\sqrt{k}} + \Big(1 + \frac{1}{\sqrt{k}}\Big)d_1 - 2(d_1 - a_1) < -d_0 - \frac{d_1}{\sqrt{k}} + \Big(1 + \frac{1}{\sqrt{k}}\Big)d_1 - 2(d_1 - a_1) < -2(d_1 - a_1)(< 0) \tag{3.3.41}$$

when $y \in (d_0, a_0)$, $x \in (d_1, d_0)$.

Next, we define indicator function for the time-independent case $\mathcal{I}_{\text{ind}}(y,s;\varepsilon)$ for identifying d_0 as follow.

Definition 3.3.1 (Indicator function for the time-independent case). For $(y,s) \in \Omega_T \setminus \overline{D}$, $\varepsilon > 0$, we define indicator function for the time-independent case $\mathcal{I}_{\text{ind}}(y,s;\varepsilon)$ by

$$\mathcal{I}_{\text{ind}}(y,s;\varepsilon) := |I(y+\varepsilon, s+\varepsilon^2; y, s)|. \tag{3.3.42}$$

Theorem 3.3.1. *Let $(y,s) \in \Omega_T$. If $(y,s) \in \partial D$, then $\mathcal{I}_{\text{ind}}(y,s;\varepsilon) \to \infty$ as $\varepsilon \to 0$, and if $y > d_0$, then $\mathcal{I}_{\text{ind}}(y,s;\varepsilon)$ is bounded as $\varepsilon \to 0$.*

Proof The integrand of the 2nd term of (3.2.39) has no singularity on Γ_T. Hence 2nd term of (3.2.39) is bounded as $(y',s') \to (y,s)$. Then, the conclusion follows by observing

$$w_{+,0;p}(y+\varepsilon, s+\varepsilon^2) = \frac{C}{\varepsilon} \exp\left[-\frac{(2y+\varepsilon-2d_0)^2}{4\varepsilon^2}\right]. \tag{3.3.43}$$

Theorem 3.3.2. *For any $0 < s < T$,*

$$(1) \quad \lim_{\varepsilon \to 0} \mathcal{I}_{\text{ind}}(y,s;\varepsilon) = \infty \text{ when } y = d_0, \tag{3.3.44}$$

$$(2) \quad \lim_{\varepsilon \to 0} \mathcal{I}_{\text{ind}}(y,s;\varepsilon) \le C \text{ when } y \ne d_0. \tag{3.3.45}$$

Therefore d_0 is given by

$$d_0 = \inf\{y < a_0; \lim_{\varepsilon \to 0} \mathcal{I}_{\text{ind}}(y',s;\varepsilon) < \infty \text{ for any } y' \in (y, a_0)\}. \tag{3.3.46}$$

Remark 3.3.1. A similar identification can be done for d_1.

3.4 1 Dimensional and Time-dependent Case

We assume $\Omega = (a_1, a_0), D(t) = (d_1(t), d_0(t))$ $(a_1 < d_1(t) < d_0(t) < a_0)$ for each $0 \le t \le T$ with $d_0, d_1 \in C^1([0,T])$.

For fixed $t = \theta$, let $w^{\theta}_{(y,s)}$ be the reflected solution satisfying (3.2.8) with $D = D(\theta)_T$.

Set $w^{\theta}_{+,0} := w^{\theta}_{(y,s)}\big|_{(d_0(\theta),a_0)_T}$, $w^{\theta}_{-} := w^{\theta}_{(y,s)}\big|_{(d_1(\theta),d_0(\theta))_T}$ and $w^{\theta}_{+,1} := w^{\theta}_{(y,s)}\big|_{(a_1,d_1(\theta))_T}$.

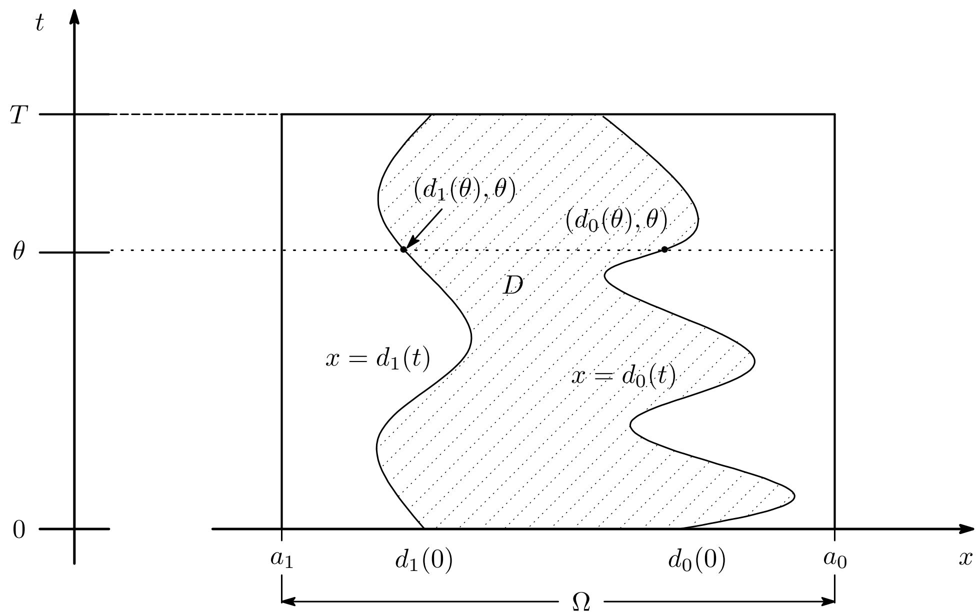


Figure 3.3: 1 dimensional and time-dependent case

From the discussion for the 1 dimensional and time-independent case, when $y \in \Omega_T \setminus \overline{D}$, we obtain

$$\mathcal{L}(w^{\theta}_{+,0}) = \frac{1-\sqrt{k}}{1+\sqrt{k}}\mathcal{L}(G_{(y,s)}(-x + 2d_0(\theta), \cdot)) + \mathcal{L}(w^{\theta}_{+,0;r}) \tag{3.4.1}$$

and

$$\begin{aligned}\mathcal{L}(w^{\theta}_{-}) = & -\mathcal{L}(G_{(y,s)}(x,\cdot)) + \frac{2}{1+\sqrt{k}}\mathcal{L}(G_{(y,s)}\left(\frac{x}{\sqrt{k}} + \left(1 - \frac{1}{\sqrt{k}}\right)d_0(\theta), \cdot\right)) \\ & + \mathcal{L}(w^{\theta}_{-;r}).\end{aligned} \tag{3.4.2}$$

Let

$$w_{+,0;p}(x,t) := \frac{1-\sqrt{k}}{1+\sqrt{k}} G_{(y,s)}(-x+2d_0(t),t), \tag{3.4.3}$$

$$w_{-;p}(x,t) := -G_{(y,s)}(x,t) + \frac{2}{1+\sqrt{k}} G_{(y,s)}\left(\frac{x}{\sqrt{k}} + \left(1-\frac{1}{\sqrt{k}}\right)d_0(t),t\right) \tag{3.4.4}$$

and

$$w_p := \begin{cases} w_{+,0;p} & \text{on } (d_0(t),a_0) \\ w_{-;p} & \text{on } (d_1(t),d_0(t)) \end{cases} \quad \text{for each } 0 \le t \le T. \tag{3.4.5}$$

Lemma 3.4.1. *$w_{(y,s)} - w_p$ is bounded in $H^{1,0}(\Omega_T)$ as (y,s) tends to $x = d_0(t)$.*

Proof To show this lemma, it is sufficient to prove $P_D(w_{(y,s)} - w_p)$ is bounded in $L^2((0,T);H^{-1}(\Omega))$ with $H^{-1}(\Omega) := (H_0^1(\Omega))^*$ norm, because we have the following unique solvability:

Theorem 3.4.2 ([39]) (Unique solvability 3).
For given $f \in L^2((0,T);(H^{\frac{1}{2}}(\Gamma))^*)$ and $F \in L^2((0,T);H^{-1}(\Omega))$, there exists a solution $\upsilon = \upsilon(f,F) \in W(\Omega_T)$ to

$$\begin{cases} P_D\upsilon = F \quad in \quad \Omega_T \\ \partial_\nu \upsilon = f \;\; on \;\; \partial\Omega_T, \quad \upsilon(x,0) = 0. \end{cases} \tag{3.4.6}$$

For any $\varphi \in W(\Omega_T)$ with $\operatorname{supp}\varphi \subset \widetilde{\Omega}_T := \{(d,a_0)_T; d \in \mathbb{R}$ such that $d_1(t) < d < d_0(t)$ for each $t \in [0,T]\}$ and $\varphi = 0$ at $t = T$,

$$\begin{aligned}
&\langle P_D w_p, \varphi\rangle \\
=&\frac{2(1-\sqrt{k})}{1+\sqrt{k}}\Big[\int_0^T\int_{d_0(t)}^{a_0} d'_0(t)(\partial_x G_{(y,s)})(-x+2d_0(t),t)\varphi\,dx\,dt \\
&-\frac{1}{\sqrt{k}}\int_0^T\int_d^{d_0(t)} d'_0(t)(\partial_x G_{(y,s)})\left(\frac{x}{\sqrt{k}}+\left(1-\frac{1}{\sqrt{k}}\right)d_0(t),t\right)\varphi\,dx\,dt\Big] \\
&-(k-1)\int_0^T\int_d^{d_0(t)}(\partial_x G_{(y,s)})\partial_x\varphi\,dx\,dt.
\end{aligned} \tag{3.4.7}$$

From (3.2.8),

$$\langle P_D w_{(y,s)}, \varphi\rangle = -(k-1)\int_0^T\int_d^{d_0(t)}(\partial_x G_{(y,s)})\partial_x\varphi\,dx\,dt \tag{3.4.8}$$

for any $\varphi \in W(\Omega_T)$ with $\operatorname{supp}\varphi \subset \widetilde{\Omega}_T$ and $\varphi = 0$ at $t = T$, then

$$\begin{aligned}
&\langle P_D(w_{(y,s)} - w_p), \varphi\rangle \\
=&-\frac{2(1-\sqrt{k})}{1+\sqrt{k}}\Big[\int_0^T\int_{d_0(t)}^{a_0} d'_0(t)(\partial_x G_{(y,s)})(-x+2d_0(t),t)\varphi\,dx\,dt \\
&-\frac{1}{\sqrt{k}}\int_0^T\int_d^{d_0(t)} d'_0(t)(\partial_x G_{(y,s)})\left(\frac{x}{\sqrt{k}}+\left(1-\frac{1}{\sqrt{k}}\right)d_0(t),t\right)\varphi\,dx\,dt\Big]
\end{aligned} \tag{3.4.9}$$

for any $\varphi \in W(\Omega_T)$ with $\operatorname{supp}\varphi \subset \widetilde{\Omega}_T$ and $\varphi = 0$ at $t = T$.

By integration by parts with respect to x, we have

$$\begin{aligned}\langle P_D(w_{(y,s)} - w_p), \varphi\rangle =& \frac{2(1-\sqrt{k})}{1+\sqrt{k}} \int_0^T d_0'(t)\Big(\int_{d_0(t)}^{a_0} G_{(y,s)}(-x + 2d_0(t), t)\partial_x\varphi\, dx \\ &+ \int_d^{d_0(t)} G_{(y,s)}\Big(\frac{x}{\sqrt{k}} + \Big(1 - \frac{1}{\sqrt{k}}\Big)d_0(t), t\Big)\partial_x\varphi\, dx\Big)dt.\end{aligned} \tag{3.4.10}$$

Since $G_{(y,s)} \in L^2(\Omega_T)$ in 1 dimension, we can easily see

$$P_D(w_{(y,s)} - w_p) \in L^2((0,T); H^{-1}(\Omega)) \tag{3.4.11}$$

i.e

$$w_{(y,s)} - w_p \in H^{1,0}(\Omega_T). \tag{3.4.12}$$

Now we define $\mathcal{S}_{r;\alpha}(y,s)$ as an open sector of small radius r, small angle 2α with vertex (y,s) and parallel to the x axis which figure is given below.

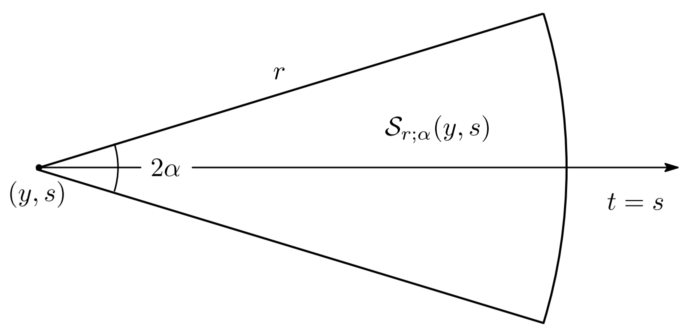


Figure 3.4: The sector $\mathcal{S}_{r;\alpha}(y,s)$

Geometric assumption for Theorem 3.1.4

We can take small $r > 0, \alpha > 0$ such that for each $0 < s < T$, $\mathcal{S}_{r;\alpha}(y,s)$ touches $\partial_x D$ at only one point $(d_0(s), s)$ as $y \downarrow d_0(s)$ along the line $t = s$.

Proposition 3.4.3. *For each* $0 < s < T$,

$$\|\nabla_x w_p\|_{L^2(\mathcal{S}_{r;\alpha}(y,s))} \to \infty \text{ as } y \downarrow d_0(s). \tag{3.4.13}$$

Proof From Fatou's lemma,

$$\begin{aligned}
&\liminf_{y\to d_0(s)} \|\nabla_x w_p\|^2_{L^2(\mathcal{S}_{r;\alpha}(y,s))} = \liminf_{y\to d_0(s)} \|\nabla_x w_{+,0;p}\|^2_{L^2(\mathcal{S}_{r;\alpha}(y,s))} \\
&\geq C\int_s^{s+r\sin\alpha}\int_{d_0(s)+\frac{t-s}{\tan\alpha}}^{d_0(s)+r\cos\alpha} \frac{(x+d_0(s)-2d_0(t))^2}{(t-s)^3}\exp\Big[-\frac{(x+d_0(s)-2d_0(t))^2}{2(t-s)}\Big]\,dx\,dt \\
&\geq C\int_s^{s+r\sin\alpha}(t-s)^{-\frac{3}{2}}\int_{\frac{\sqrt{2}(d_0(s)-d_0(t))}{\sqrt{t-s}}+\frac{\sqrt{t-s}}{\sqrt{2}\tan\alpha}}^{\frac{\sqrt{2}(d_0(s)-d_0(t))}{\sqrt{t-s}}+\frac{r\cos\alpha}{\sqrt{2(t-s)}}} \xi^2 e^{-\xi^2}\,d\xi\,dt \\
&\geq C\int_s^{s+r\sin\alpha}(t-s)^{-\frac{3}{2}}\int_{-\frac{\sqrt{2}|d_0(s)-d_0(t)|}{\sqrt{t-s}}+\frac{\sqrt{t-s}}{\sqrt{2}\tan\alpha}}^{-\frac{\sqrt{2}|d_0(s)-d_0(t)|}{\sqrt{t-s}}+\frac{r\cos\alpha}{\sqrt{2(t-s)}}} \xi^2 e^{-\xi^2}\,d\xi\,dt.
\end{aligned} \tag{3.4.14}$$

By $d_0 \in C^1([0,T])$, for $(0<)s<t(<T)$,

$$|d_0(s)-d_0(t)| < \beta(t-s) \tag{3.4.15}$$

for some constant $\beta > 0$. Therefore

$$\liminf_{y\to d_0(s)} \|\nabla_x w_p\|^2_{L^2(\mathcal{S}_{r;\alpha}(y,s))} \geq C\int_s^{s+r\sin\alpha}(t-s)^{-\frac{3}{2}}\int_{-\sqrt{2}\beta\sqrt{t-s}+\frac{\sqrt{t-s}}{\sqrt{2}\tan\alpha}}^{-\sqrt{2}\beta\sqrt{t-s}+\frac{r\cos\alpha}{\sqrt{2(t-s)}}} \xi^2 e^{-\xi^2}\,d\xi\,dt. \tag{3.4.16}$$

Here

$$\begin{aligned}
\int_{-\sqrt{2}\beta\sqrt{t-s}+\frac{\sqrt{t-s}}{\sqrt{2}\tan\alpha}}^{-\sqrt{2}\beta\sqrt{t-s}+\frac{r\cos\alpha}{\sqrt{2(t-s)}}} \xi^2 e^{-\xi^2}\,d\xi &\geq \frac{1}{2}\int_{-\sqrt{2}\beta\sqrt{t-s}+\frac{\sqrt{t-s}}{\sqrt{2}\tan\alpha}}^{-\sqrt{2}\beta\sqrt{t-s}+\frac{r\cos\alpha}{\sqrt{2(t-s)}}} e^{-\xi^2}\,d\xi \\
&\geq \frac{e^{-C'(t-s)}}{2\sqrt{2}}\left(\frac{r\cos\alpha}{\sqrt{t-s}}-\frac{\sqrt{t-s}}{\tan\alpha}\right).
\end{aligned} \tag{3.4.17}$$

Therefore

$$\begin{aligned}
\liminf_{y\to d_0(s)} \|\nabla_x w_p\|^2_{L^2(\mathcal{S}_{r;\alpha}(y,s))} &\geq C\int_s^{s+r\sin\alpha}(t-s)^{-\frac{3}{2}}\left(\frac{r\cos\alpha}{\sqrt{t-s}}-\frac{\sqrt{t-s}}{\tan\alpha}\right)dt \\
&\geq C\int_s^{s+\frac{r\sin\alpha}{2}}(t-s)^{-\frac{3}{2}}\left(\frac{r\cos\alpha}{\sqrt{t-s}}-\frac{\sqrt{t-s}}{\tan\alpha}\right)dt \\
&\geq C'\int_s^{s+\frac{r\sin\alpha}{2}}(t-s)^{-\frac{3}{2}}\,dt = \infty.
\end{aligned} \tag{3.4.18}$$

Definition 3.4.1 (Indicator Function for the Time-dependent Case).
For $(y,s) \in \Omega_T \setminus \overline{D}$, we define indicator function for the time-dependent case $\mathcal{I}_{\mathrm{dep}}(y,s)$ by

$$\mathcal{I}_{\mathrm{dep}}(y,s) := \|\nabla_{y'} I(\cdot,\cdot;y,s)\|_{L^2(\mathcal{S}_{r;\alpha}(y,s))}. \tag{3.4.19}$$

Theorem 3.4.4. *For each $0<s<T$, $\mathcal{I}_{\mathrm{dep}}(y,s)$ is finite for each $d_0(s)<y<a_0$, but $\mathcal{I}_{\mathrm{dep}}(y,s)\to\infty$ as $y \downarrow d_0(s)$.*

Hence we know $d_0(s)$ based on $\mathcal{I}_{\mathrm{dep}}(y,s)\to\infty$ as $y\downarrow d_0(s)$.

3.5 Proof of Runge's Approximation Theorem Based on Hahn-Banach Theorem

In this section, we prove Runge's approximation theorem based on Hahn-Banach theorem. Another proof of Runge's approximation theorem based on the single layer potential is given Section 4.2, which is more constructive. To prove Runge's approximation theorem based on Hahn-Banach theorem, we need the following well known unique solvability result. We need the following well known unique solvability result.

Theorem 3.5.1 ([26]) (Unique solvability 4).
For $F \in L^2(\Omega_{(T_0',T_1')})$, there exists a unique solution $w = w(F) \in H^{2,1}(\Omega_{(T_0',T_1')})$ to

$$\begin{cases} P_\phi^* w = F \ \ in\ \Omega_{(T_0',T_1')} \\ w = 0 \ \ on\ \Gamma_{(T_0',T_1')},\ \ w(x,T_1') = 0. \end{cases} \tag{3.5.1}$$

We only give the proof of Theorem 3.1.3 (cf.[27]). Let

$$X := \{v\,\big|_U; v \in H^{2,1}(V), P_\phi v = 0 \text{ in } V,\ \ v\big|_{(T_0',T_0]} = 0\} \tag{3.5.2}$$

and

$$Y := \{u\,\big|_U; u \in H^{2,1}(\Omega_{(T_0',T_1')}), P_\phi u = 0 \text{ in } \Omega_{(T_0',T_1')},\ \ u\big|_{(T_0',T_0]} = 0\}. \tag{3.5.3}$$

We want to prove that if $f_0 \in L^2(U)$ satisfies $(f_0, u)_{L^2(U)} = 0\ \ (u\big|_U \in Y)$, then $(f_0, v)_{L^2(U)} = 0\ \ (v\big|_U \in X)$.

For such $f_0 \in L^2(U)$, by Theorem 3.5.1, there exists a unique $w_0 \in H^{2,1}(\Omega_{(T_0',T_1')})$ satisfying

$$\begin{cases} P_\phi^* w_0 = F_0 \quad \text{in} \quad \Omega_{(T_0',T_1')} \\ w_0 = 0 \quad \text{on}\ \Gamma_{(T_0',T_1')},\ \ w_0(x,T_1') = 0. \end{cases} \tag{3.5.4}$$

where

$$F_0 := \begin{cases} f_0 \quad \text{in} \quad U \\ 0 \quad \text{in} \quad \Omega_{(T_0',T_1')} \setminus \overline{U}. \end{cases} \tag{3.5.5}$$

For any $u\big|_U \in Y$,

$$\begin{aligned} 0 &= \int_U u f_0 \, dx\, dt = \int_{\Omega_{(T_0',T_1')}} u P_\phi^* w_0 \, dx\, dt = \int_{\Omega_{(T_0',T_1')}} (u P_\phi^* w_0 - P_\phi u w_0)\, dx\, dt \\ &= \int_{\Gamma_{(T_0',T_1')}} (\partial_\nu u w_0 - u \partial_\nu w_0)\, d\sigma\, dt = -\int_{\Gamma_{(T_0',T_1')}} u \partial_\nu w_0 \, d\sigma\, dt. \end{aligned} \tag{3.5.6}$$

Then we obtain $\partial_\nu w_0\big|_{\partial\Omega_{(T_0',T_1')}} = 0$, since $u\big|_{\partial\Omega_{(T_0',T_1')}} \in L^2((T_0',T_1'); H^{\frac{1}{2}}(\Gamma))$ can be taken arbitrarily.

Then using the unique continuation property (cf.[20]),

$$w_0 = 0 \quad \text{on} \quad \overline{\Omega_{(T_0',T_1')} \setminus \overline{U}}. \tag{3.5.7}$$

Let $v\big|_U \in X$, *i.e.*

$$v \in H^{2,1}(V); P_\phi v = 0 \quad \text{in } V, \quad v\big|_{(T_0',T_0]} = 0. \tag{3.5.8}$$

Then, by (3.5.7),

$$(v, f_0)_{L^2(U)} = \int_U (v P_\phi^* w_0 - w_0 P_\phi v)\, dx\, dt = \int_{\partial_x U} (w_0 \partial_\nu v - v \partial_\nu w_0)\, d\sigma\, dt = 0. \tag{3.5.9}$$

□

Chapter 4

Numerical Realization

4.1 Realization in 1 Dimensional Time-independent Case

In this section, we consider the case of 1 dimensional time-independent case for simplicity. *i.e* $\Omega = (a_1, a_0)$ and $D(t) = (d_1, d_0)$ with $a_1 < d_1 < d_0 < a_0$.

And we assume $y \in (d_0, a_0)$ and $s, s' \in (0, T)$ meeting $s' > s$. Now we will construct open set $V = V(y, s; y', s') \subset \Omega_T$ of Runge's approximation theorem (Theorem 3.1.3 and 3.1.4) in a very simple way. $V(y, s; y', s')$ needs to satisfy these following conditions.

(H1): $\overline{D} \subset V(y, s; y', s')$.

(H2): $(y, s), (y', s') \in \Omega_T \setminus \overline{V(y, s; y', s')}$.

From these conditions, we can choose $V(y, s; y', s')$ as

$$V(y) := (v_1, v_0(y))_T \text{ satisfying } v_1 < d_1 < d_0 < v_0(y) < y \tag{4.1.1}$$

since we will set $y' = y + \varepsilon, s' = s + \varepsilon^2$ for small $\varepsilon > 0$ in our argument. The configuration of $V(y)$ depends only on y. Moreover, when we approximate d_0 by $y \in (v_0(y), a_0)$ with $v_0(y)$ independent of s.

Firstly, we consider the Runge's approximation function $v^j_{(y,s)}(x, t)$ to $G_{(y,s)}(x, t) = G(x, t; y, s)$ in $V(y)$.

For this purpose the potential expression

$$\begin{aligned} &\int_0^t \int_\Gamma G(x, t; \xi, \tau) g(\xi, \tau)\, ds(\xi)\, d\tau \\ &= \int_0^t G(x, t; a_0, \tau) g(a_0, \tau)\, d\tau + \int_0^t G(x, t; a_1, \tau) g(a_1, \tau)\, d\tau =: \mathcal{H}(\vec{g})(x, t) \end{aligned} \tag{4.1.2}$$

for $(x,t)\in\Omega_T$, where $\vec{g}(\tau):=(g_{a_0}(\tau),g_{a_1}(\tau))^T$ and $g_{a_0}(\tau),g_{a_1}(\tau)$ are density functions in $x=a_0,a_1$, respectively.

It is easy to see that $\mathcal{H}(\vec{g})(x,t)$ for any continuous density $\vec{g}$ satisfies $P_\phi\cdot\ =0$ in Ω_T and $G_{(y,s)}(x,t)$ solves heat equation in $V(y)$. If we require that

$$\mathcal{H}(\vec{g})(x,t)\approx G_{(y,s)}(x,t),\quad (x,t)\in V(y), \tag{4.1.3}$$

then the well-posedness of direct heat problem will generate

$$\mathcal{H}(\vec{g})(x,t)\approx G_{(y,s)}(x,t),\quad (x,t)\in V(y). \tag{4.1.4}$$

Noticing $G_{(y,s)}(x,t)\equiv 0$ for $t\in[0,s]$, so we choose the density functions

$$g_{a_0}(\tau)\equiv g_{a_1}(\tau)\equiv 0,\quad \tau\in[0,s], \tag{4.1.5}$$

then (4.1.3) holds for $t\in[0,s]$ obviously. Therefore it is enough to require that

$$\mathcal{H}(\vec{g})(x,t)\approx G_{(y,s)}(x,t),\quad (x,t)\in\{v_1,v_0(y)\}_{(s,T)} \tag{4.1.6}$$

with the expression

$$\mathcal{H}(\vec{g})(x,t)=\int_s^t G(x,t;a_0,\tau)g_{a_0}(\tau)\,d\tau+\int_s^t G(x,t;a_1,\tau)g_{a_1}(\tau)\,d\tau,\quad t\in(s,T) \tag{4.1.7}$$

due to (4.1.2) and (4.1.6). We can write (4.1.7) explicitly as the following matrix form

$$\mathbb{A}(\vec{g})(t):=\int_s^t \mathbf{A}(t-\tau)\vec{g}(\tau)\,d\tau\approx\vec{b}(t-s),\quad t\in(s,T) \tag{4.1.8}$$

with

$$\mathbf{A}(t):=\frac{1}{\sqrt{t}}\begin{pmatrix} e^{-\frac{(a_0-v_0(y))^2}{4t}} & e^{-\frac{(v_0(y)-a_1)^2}{4t}} \\ e^{-\frac{(a_0-v_1)^2}{4t}} & e^{-\frac{(v_1-a_1)^2}{4t}}\end{pmatrix},\quad \vec{b}(t):=\frac{1}{\sqrt{t}}\left(e^{-\frac{(y-v_0(y))^2}{4t}},e^{-\frac{(y-v_1(y))^2}{4t}}\right)^T$$

due to (4.1.7) for given $v_0(y),v_1,y,s$.

Once upon we determine $g_{a_0}(\tau),g_{a_1}(\tau)$ for $\tau\in(s,T)$ from (4.1.8), we can approximate $G_{(y,s)}(x,t)$ in $V(y)$ by

$$v(x,t):=\mathcal{H}(\vec{g})(x,t),\quad (x,t)\in\Omega_T.$$

Noticing (4.1.6), we in fact have $\mathcal{H}(\vec{g})(x,t)\equiv G_{(y,s)}(x,t)\equiv 0$ for $t\in(0,s)$ and $\mathcal{H}(\vec{g})(x,t)\approx G_{(y,s)}(x,t)$ for $t\in(s,T)$. Noticing the infinite differentiability of $\mathcal{H}(\vec{g})$, it of course meets the heat equation in $V(y)$.

One possible choice of $\vec{g}$ such that (4.1.8) holds is the minimum norm solution to

$$\mathbb{A}(\vec{g})(t) = \vec{b}(t-s), \quad t \in (s,T) \tag{4.1.9}$$

with discrepancy $1/j$. That is, there exists a unique $\vec{g}^j(t)$ defined in (s,T) such that

$$\|\vec{g}^j\| = \inf\left\{\|\vec{g}\| : \|\mathbb{A}(\vec{g})(\cdot) - \vec{b}(\cdot - s) \leq \frac{1}{j}\right\},$$

which can be determined uniquely by

$$\begin{cases} \|\mathbb{A}(\vec{g})(\cdot) - \vec{b}(\cdot - s)\| = \frac{1}{j}, \\ \alpha\vec{g} + \mathbb{A}^*\mathbb{A}(\vec{g}) = \mathbb{A}^*(\vec{b}), \end{cases} \tag{4.1.10}$$

where both the norm and the adjoint operator $\mathbb{A}^*$ depend on the choice of function space. The theoretical issue is we should prove the denseness of $Range(\mathbb{A})$ so that the approximation in the boundary is possible.

Now we denote by

$$v^j(x,t) := v^j_{(y,s)}(x,t) = \mathcal{H}(\vec{g}^j)(x,t), \quad (x,t) \in \Omega_T \tag{4.1.11}$$

clearly to express the dependence on (y,s), noticing $v^j(x,t) \to G_{(y,s)}(x,t)$ in $V(y)$. Now we can write the boundary value of v^j is

$$\begin{pmatrix} v^j_{(y,s)}(x,t)\big|_{x=a_0} \\ v^j_{(y,s)}(x,t)\big|_{x=a_1} \end{pmatrix} = \int_s^t \mathbf{B}(t-\tau)\vec{g}^j(\tau)\,d\tau \tag{4.1.12}$$

with the matrix

$$\mathbf{B}(t) := \frac{1}{\sqrt{4\pi t}} \begin{pmatrix} 1 & e^{-\frac{(a_0-a_1)^2}{4t}} \\ e^{-\frac{(a_0-a_1)^2}{4t}} & 1 \end{pmatrix},$$

where we define $\mathbf{B}(0) = 0$. To compute the indicator function, it follows that the boundary derivatives $\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T}$, is also needed, excepted for $v^j_{(y,s)}\big|_{\Gamma_T}$. We compute $\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T}$ from (4.1.12) directly using the density function, rather than by the differential procedure from $v^j_{(y,s)}$ itself, due the ill-posedness of differential computation from discrete data. A simple computation from (4.1.8) and (4.1.12) generates that

$$\begin{pmatrix} \partial_\nu v^j_{(y,s)}(x,t)\big|_{x=a_0} \\ \partial_\nu v^j_{(y,s)}(x,t)\big|_{x=a_1} \end{pmatrix} = \int_s^t \mathbf{B}_\partial(t-\tau)\vec{g}^j(\tau)\,d\tau \tag{4.1.13}$$

with the matrix

$$\mathbf{B}_\partial(t) := -\frac{a_0 - a_1}{4\sqrt{\pi}t^{\frac{3}{2}}} e^{-\frac{(a_0-a_1)^2}{4t}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We also define $\mathbf{B}_\partial(0) = 0$.

On the other hand, it is obvious that

$$G^*_{(y',s')}(x,t) = G_{(y',T-s')}(x, T-t).$$

From this relation, if we construct the same approximate domain for two points $(y,s), (y',s')$, as done in this paper, we get that the Runge approximation function $\phi^j_{(y',s')}$ can be chosen easily, that is,

$$\phi^j_{(y',s')}(x,t) = v^j_{(y',T-s')}(x, T-t) \tag{4.1.14}$$

is an approximation function to $G^*_{(y',s')}$. In this way, the Runge's approximation functions $v^j_{(y,s)}(x,t), \varphi^j_{(y',s')}$ are constructed. Finally we use (4.1.14) to determine the Cauchy data $\varphi^j_{(y',s')}\big|_{\Gamma_T}, \partial_\nu \varphi^j_{(y',s')}\big|_{\Gamma_T}$, that is,

$$\begin{cases} \varphi^j_{(y',s')}(x,t)\big|_{\Gamma_T} = v^j_{(y,T-s')}(x,T-t)\big|_{\Gamma_T}, \\ \partial_\nu \varphi^j_{(y',s')}(x,t)\big|_{\Gamma_T} = \partial_\nu v^j_{(y,T-s')}(x,T-t)\big|_{\Gamma_T}. \end{cases} \tag{4.1.15}$$

Finally we conclude at the end of this section that (4.1.10)-(4.1.13) provide a implementable way to the construction of Runge approximation functions $\{v^j_{(y,s)}, \varphi^j_{(y',s')}\}$. In this way, we can compute the indicator function, since the map Π_D is given. In our realization, we simulate $\Pi_D\big(\partial_\nu v^j_{(y,s)}\big|_{\Gamma_T}\big)$ by solving direct problem. In this way the right-hand boundary, that is the value $x = d_0$ can be identify by taking $y \to d_0$.

4.2 Discrete Scheme for Indicator Function

In this section, we give the details for the discretization of computing pre-indicator function $I(y',s';y,s)$. As stated previously, we begin with (4.1.10)-(4.1.13). It is easy to see that $\mathbb{A} : L^2(s,T) \times L^2(s,T) \to L^2(s,T) \times L^2(s,T)$ defined in (4.1.8) mapping $\mathbb{A}(\vec{g}) = \vec{\psi}$ has the adjoint operator

$$\mathbb{A}^*(\vec{\psi})(t) = \int_t^T \mathbf{A}^T(\tau - t)\vec{\psi}(\tau)\, d\tau. \tag{4.2.1}$$

Therefore the regularizing equation in (4.1.11) for given $\alpha > 0$ becomes

$$\alpha\vec{g}(t) + \int_t^T \mathbf{A}^T(\tau - t)\int_s^\tau \mathbf{A}(\tau-\varsigma)\vec{g}(\varsigma)\,d\varsigma\,d\tau = \int_t^T \mathbf{A}^T(\tau-t)\vec{b}(\tau-s)\,d\tau, \quad t\in(s,T), \tag{4.2.2}$$

which can be written as

$$\alpha\vec{g}(t) + \int_s^T \mathbf{G}(t,\varsigma)\vec{g}(\varsigma)\,d\varsigma = \int_t^T \mathbf{A}^T(\tau-t)\vec{b}(\tau-s)\,d\tau, \quad t\in(s,T), \tag{4.2.3}$$

with the kernel

$$\mathbf{G}(t,\varsigma) := \begin{cases} \displaystyle\int_t^T \mathbf{A}^T(\tau-t)\mathbf{A}(\tau-\varsigma)\,d\tau, & s<\varsigma<t, \\ \displaystyle\int_\varsigma^T \mathbf{A}^T(\tau-t)\mathbf{A}(\tau-\varsigma)d\tau & t\le\varsigma<T. \end{cases} \tag{4.2.4}$$

Denote by $\vec{g}_\alpha(t)$ the solution to this equation, then for $\alpha > 0$ satisfying

$$\|\mathbb{A}(\vec{g})_\alpha(\cdot) - \vec{b}(\cdot - s)\|_{L^2(s,T)\times L^2(s,T)} = \frac{1}{j}, \tag{4.2.5}$$

we obtain the corresponding density function $\vec{g}_j(\tau) := \vec{g}_{\alpha(j)}(\tau) \in L^2(s,T)\times L^2(s,T)$. The existence of this solution is standard if $Range(\mathbb{A})$ is dense in $L^2(s,T)\times L^2(s,T)$. We give the denseness by the following result.

Theorem 4.2.1. *The range of operator $\mathbb{A}$ is dense in $L^2(s,T)\times L^2(s,T)$.*

Proof Let $\widetilde{E}\subset\Omega_{(-\infty,T)}$ be an approximation domain with C^2 lateral boundary $\partial_x\widetilde{E}$. For safety we assume $\overline{\widetilde{E}}\cap\{t=0\}$ and $\overline{\widetilde{E}}\cap\{t=T\}$ are bounded domains with C^2 boundary if $n\ge 2$. Moreover, for any $t'\in[0,T]$, $\overline{\widetilde{E}}\cap\{t'\}\subset\Omega\times\{t'\}$.

We define the single layer potential $S\varphi$ with density $\varphi\in L^2(\partial_x\widetilde{E})$ by

$$(S\varphi)(x,t) := \int_{\Gamma_T} G(x,t;y,s)\varphi(y,s)\,d\sigma(y)\,ds. \tag{4.2.6}$$

Claim

$$\text{If } (y,s)\notin\overline{\widetilde{E}}, \text{ then } G(x,t;y,s)\in\overline{R(S)}^{L^2(\partial_x\widetilde{E})} \tag{4.2.7}$$

Proof $L^2(\partial_x\widetilde{E}) = \overline{R(S)}\oplus N(S^*)$ with $N(S^*) := \{\psi\in L^2(\partial_x\widetilde{E}); S^*\psi = 0\}$.

Hence it is enough to prove $G(x,t;y,s)\in N(S^*)^\perp$.

First of all, we note that S^* is given by

$$S^*(\psi)(y,s) = \int_{\partial_x \widetilde{E}} G(x,t;y,s)\psi(x,t)\,d\sigma(x)\,dt. \tag{4.2.8}$$

Let $\psi = \psi(x,t) \in N(S^*)$ be such that $\psi \in C^0(\overline{\partial_x \widetilde{E}})$ and $\psi\big|_{t=0} = 0$. Note that such ψ's are dense in $N(S^*)$ and $S^* : L^2(\partial_x \widetilde{E}) \to L^2(\Gamma_{(-\infty,T)})$ is a bounded operator. We continuously extend ψ to $t < 0$ and use the same ψ for the extended ψ. (Of course for this, we need to extend $\tilde{E}$ to $t < 0$ without destroying the regularity of its lateral boundary. Let

$$w(y,s) := \int_{\partial_x \widetilde{E}} G(x,t;y,s)\psi(x,t)\,d\sigma(x)\,dt. \tag{4.2.9}$$

Then, we have

$$\begin{cases} P^*_\phi w = 0 \quad \text{in} \quad (\mathbb{R}^n \setminus \overline{\Omega})_{(-\infty,T)} \\ w = 0 \quad \text{on} \quad \Gamma_{(-\infty,T)} \\ w\big|_{s=T} = 0 \end{cases}, \tag{4.2.10}$$

since

$$w(y,s) = \int_0^T dt \int_{\partial_x \widetilde{E}(t)} G(x,t;y,s)\psi(x,t)\,d\sigma(x) = w(y,0) \quad \text{if} \quad s < 0. \tag{4.2.11}$$

where $\partial_x \widetilde{E}(t)$ is cross section of $\partial_x \widetilde{E}$ at t.

For $t > s, x \neq y$,

$$\begin{aligned} G(x,t;y,s) &= \frac{1}{(\sqrt{4\pi(t-s)})^n} \exp\left[-\frac{|x-y|^2}{4(t-s)}\right] \\ &= \pi^{-\frac{n}{2}}|x-y|^{-n}\left(\frac{|x-y|^2}{4(t-s)}\right)^\alpha \tau^{\frac{n}{2}-\alpha}e^{-\tau} \\ &\leq \pi^{-\frac{n}{2}}|x-y|^{-n}\left(\frac{|x-y|^2}{4(t-s)}\right)^\alpha \left(\frac{n}{2}-\alpha\right)^{\frac{n}{2}-\alpha} e^{-(\frac{n}{2}-\alpha)} \end{aligned} \tag{4.2.12}$$

with $0 < \alpha < \dfrac{n}{2}$ (cf. (9.18) in [22]). Therefore

$$G(x,t;y,s) \leq M(t-s)^{-\alpha}|x-y|^{-n+2\alpha} \tag{4.2.13}$$

for some $M > 0$ with $0 < \alpha < \dfrac{n}{2}$. Similary,

$$|\nabla_y G(x,t;y,s)| \leq M'(t-s)^{-\beta}|x-y|^{-n+2\beta} \tag{4.2.14}$$

for some $M' > 0$ with $0 < \beta < 1 + \dfrac{n}{2}$.

Take $0 < \alpha < \dfrac{1}{2} < \beta < 1$, $\alpha + \beta < 1$,

$$\begin{aligned} |w(y,s)| &\le M\|\psi\|_{L^\infty(\partial_x \widetilde{E})}(\mathrm{dist}(\partial_x \widetilde{E}, \Gamma))^{-n+2\alpha}|\Omega| \int_s^T (t-s)^\alpha \, dt \\ &= M\|\psi\|_{L^\infty(\partial_x \widetilde{E})}(\mathrm{dist}(\partial_x \widetilde{E}, \Gamma))^{-n+2\alpha}|\Omega|(1-\alpha)^{-1}(T-s)^{1-\alpha}. \end{aligned} \tag{4.2.15}$$

We clearly see $O(|s|^{-\alpha})$ $(s \to -\infty)$. (Of course we also have $O(|s|^{-\frac{n}{2}})$ $(s \to -\infty)$.)

Taking account of the behavior for large $|y|$, we also have

$$w = \begin{cases} O((T-s)^{1-\alpha})O(|y|^{-n+2\alpha}) \ (s \to T, |y| \gg 1) \\ O(|s|^{-\alpha})O(|y|^{-n+2\alpha}) \ (s \to -\infty, |y| \gg 1). \end{cases} \tag{4.2.16}$$

Therefore, summing up all the behaviors, we have

$$w = \begin{cases} O((T-s)^{1-\alpha}) \ (s \to T), \ O((T-s)^{1-\alpha})O(|y|^{-n+2\alpha}) \ (s \to T, |y| \gg 1), \\ O(|s|^{-\alpha}) \ (s \to -\infty), \ O(|s|^{-\alpha})O(|y|^{-n+2\alpha}) \ (s \to -\infty, |y| \gg 1). \end{cases} \tag{4.2.17}$$

Similarly we have the same estimate for $\nabla_y w$ by replacing α to β. Now, let $K_R := (\mathbb{R}^n \setminus \overline{\Omega}) \cap B_R$ with $B_R := \{|y| < R\}$ for large $R > 0$. Then

$$0 = \int_{K_{R(-\infty,T)}} w P_\phi^* w \, dy \, ds = \int_{K_{R(-\infty,T)}} |\nabla_y w|^2 \, dy \, ds - \int_{\partial B_{R(-\infty,T)}} w \partial_\nu w \, d\sigma(y) \, ds, \tag{4.2.18}$$

because

$$\|w(\,\cdot\,,T)\|^2_{L^2(K_R)} = \|w(\,\cdot\,,-\infty)\|^2_{L^2(K_R)} = 0 \tag{4.2.19}$$

and

$$w = 0 \quad \text{on } \Gamma_{(-\infty,T)}. \tag{4.2.20}$$

Therefore

$$\lim_{R \to \infty} \int_{K_R} |\nabla_y w|^2 \, dy \, ds = 0 \tag{4.2.21}$$

due to

$$\begin{aligned} &\left| \int_{\partial B_{R(-\infty,T)}} w \partial_\nu w \, d\sigma(y) \, ds \right| \\ &\le L \left\{ \int_{\frac{T}{2}}^T (T-s)^{2-(\alpha+\beta)} \, ds + \int_{-\infty}^{\frac{T}{2}} |s|^{-\frac{1}{2}-\beta} \, ds \right\} \int_{S^{n-1}} R^{-2n+2(\alpha+\beta)+n-1} \, dS^{n-1} \\ &= L' R^{-n-1+2(\alpha+\beta)} |S^{n-1}| \to 0 \ (R \to \infty). \end{aligned} \tag{4.2.22}$$

From (4.2.21) and $w = 0$ on $\Gamma_{(-\infty,T)}$, we obtain

$$w = 0 \text{ in } (\mathbb{R}^n \setminus \overline{\Omega})_{(-\infty,T)}. \tag{4.2.23}$$

Therefore, by the unique continuation,

$$w(y,s)=0 \quad i.e. \quad \int_{\partial_x \widetilde{E}} G(x,t;y,s)\psi(x,t)\,dx\,dt=0. \tag{4.2.24}$$

for $\psi \in N(S^*)$. $\square$

The numerical procedure is as follows. For given j, the possible lower and upper bounds for the regularizing parameter can be given based on $\|\mathbb{A}\|, \|\mathbb{A}^*\|, j$. Then we compute the solution to (4.2.3) for different α in this interval respectively and compare their norms. The one with the minimum norm is $\vec{g}_j$.

This is a linear integral equations with unknowns $\vec{g}(t_j)=(g_{a_0}(t_j), g_{a_1}(t_j))^T$ for $j=1,2,\cdots,N$.

Now we state how to simulate Π_D for given Neumann data f is Γ_T by finite element method. For fixed $t\in(s,T)$, we denote $u^h(x,t)$ the approximation to $u(x,t)$. The weak form of equation in (3.1.2) is

$$\frac{d}{dt}\int_\Omega u^h(x,t)\phi(x)\,dx-\int_\Gamma \gamma(x,t)\partial_\nu u^h(x,t)\phi(x)\,dx+\int_\Omega \gamma(x,t)\nabla u^h(x,t)\cdot\nabla\phi(x)\,dx=0$$

for test function $\phi\in H^1(\Omega)$. We require the approximate solution to satisfy the boundary condition, then the above relation is

$$\frac{d}{dt}\int_\Omega u^h(x,t)\phi(x)\,dx-\int_\Gamma f(x,t)\phi(x)\,dx+\int_\Omega \gamma(x,t)\nabla u^h(x,t)\cdot\nabla\phi(x)\,dx=0.$$

Now we expand $u^h(\,\cdot\,,t)$ by base functions $\{\phi_j(x)\}_{j=0}^M$ as

$$u^h(x,t)=\sum_{j=0}^M u_j(t)\phi_j(x) \tag{4.2.25}$$

and take test function as $\phi_j(x)$, we get

$$\begin{cases}\mathbf{C}\dfrac{d\boldsymbol{u}(t)}{dt}+\mathbf{D}(t)\boldsymbol{u}(t)=\boldsymbol{f}(t), t\in(s,T),\\ \boldsymbol{u}(s)=0\end{cases} \tag{4.2.26}$$

with $\mathbf{C}:=(c_{i,j}), \mathbf{D}(t):=(d_{i,j}(t)), \boldsymbol{u}(t):=(u_0(t),\cdots,u_M(t))^T$ and $\boldsymbol{f}(t):=(f_0(t),\cdots,f_M(t))^T$, where the elements are

$$c_{i,j}:=\int_\Omega \phi_i(x)\phi_j(x)\,dx, d_{i,j}(t):=\int_\Omega \gamma(x,t)\nabla\phi_i(x)\nabla\phi_j(x)\,dx, f_i(t):=\int_\Gamma f(x,t)\phi_i(x)\,ds(x)$$

for $i,j=0,\cdots,M$. (4.2.26) can be solved by 1 order implicit scheme recursively from

$$\begin{cases}(\mathbf{C}+\tau\mathbf{D})\boldsymbol{u}(t_l)=\mathbf{C}\boldsymbol{u}(t_{l-1})+\tau\boldsymbol{f}(t_l),\\ \boldsymbol{u}(t_0):=\boldsymbol{u}(s)=0\end{cases} \tag{4.2.27}$$

for $l = 1, \cdots, N$ with step $\tau = \dfrac{T-s}{N}$. In 1 dimensional case, define $[a_1, a_0] = \bigcup_{j=1}^{M} [x_{j-1}, x_j]$, $h = \dfrac{x_M - x_0}{M}$ with $x_0 = a_1, x_M = a_0$ and the base function can be taken as the standard tent-like shape *i.e.*

$$\phi_0(x) = \begin{cases} -\dfrac{x - x_1}{h} & (x \in [x_0, x_1]) \\ 0 & (x \in [x_1, x_M]), \end{cases} \quad \phi_1(x) = \begin{cases} \dfrac{x - x_0}{h} & (x \in [x_0, x_1]) \\ -\dfrac{x - x_1}{h} & (x \in [x_1, x_2]) \\ 0 & (x \in [x_2, x_M]), \end{cases}$$

$$\phi_j(x) = \begin{cases} 0 & (x \in [x_0, x_{j-1}]) \\ \dfrac{x - x_{j-1}}{h} & (x \in [x_{j-1}, x_j]) \\ -\dfrac{x - x_{j+1}}{h} & (x \in [x_j, x_{j+1}]) \\ 0 & (x \in [x_{j+1}, x_M]) \end{cases} \quad (j = 2, \cdots, M-2),$$

$$\phi_{M-1}(x) = \begin{cases} 0 & (x \in [x_0, x_{M-2}]) \\ \dfrac{x - x_{M-2}}{h} & (x \in [x_{M-2}, x_{M-1}]) \\ -\dfrac{x - x_M}{h} & (x \in [x_{M-1}, x_M]) \end{cases} \quad \text{and } \phi_M(x) = \begin{cases} 0 & (x \in [x_0, x_{M-1}]) \\ -\dfrac{x - x_{M-1}}{h} & (x \in [x_{M-1}, x_M]). \end{cases}$$

We also take

$$\gamma_{j-1} := \gamma(x_{j-1}+, t) \text{ on } [x_{j-1}, x_j] \quad j = 1, \cdots, M.$$

Introduce step size $(M+1) \times (M+1)$ standard tridiagonal matrix

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

then we get $\mathbf{C} = \dfrac{1}{6} h\mathbf{J}$. For matrix $\mathbf{D}(t)$, we can also compute to obtain

$$\mathbf{D} = \frac{1}{h} \begin{pmatrix} \gamma_0 & -\gamma_0 & 0 & \cdots & 0 & 0 \\ -\gamma_0 & \gamma_0 + \gamma_1 & -\gamma_1 & \cdots & 0 & 0 \\ 0 & -\gamma_1 & \gamma_1 + \gamma_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \gamma_{M-2} + \gamma_{M-1} & -\gamma_{M-1} \\ 0 & 0 & 0 & \cdots & -\gamma_{M-1} & \gamma_{M-1} \end{pmatrix},$$

here we omit the parameter t. Notice, the function values at nodal are defined as the right limitation of function so that we can treat the discontinuous coefficient $\gamma(x,t)$. Especially, if $\gamma(x,t) \equiv \gamma_0 \neq 1$ for some positive constant, then $\mathbf{D} = \frac{1}{h}\gamma_0 \mathbf{K}$. It is easy to see that (4.2.27) has the form

$$\Big(\mathbf{J} + \frac{6\tau}{h^2} h\mathbf{D}\Big)\boldsymbol{u}(t_l) = \mathbf{J}\boldsymbol{u}(t_{l-1}) + \frac{6\tau}{h}\boldsymbol{u}(t_l)$$

with matrix $\mathbf{J}, h\mathbf{D}$ independent of h, τ. Therefore the equation at each l is solvable for $\frac{\tau}{h^2}$ small enough. From this simulation procedure for Π_D and the construction of $v^j_{(y,s)}, \varphi^j_{(y',s')}$, we can finally compute the indicator function.

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