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Riesz's Functions In Weighted Hardy And Bergman Spaces

by

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and

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Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

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Abstract. Let μ be a finite positive Borel measure on the closed unit disc $\bar{D}$. For each a in $\bar{D}$, put

$$S(a) = inf \int_{\bar{D}} | f |^p \, d\mu$$

where f ranges over all analytic polynomials with $f(a) = 1$. This upper semicontinuous function $S(a)$ is called a Riesz's function and studied in detail. Moreover several applications are given to weighted Bergman and Hardy spaces.

§1. Introduction

Let D be the open unit disc in the complex plane $\mathbf{C}$. P denotes a set of all analytic polynomials and H denotes a set of all analytic functions on D. Suppose $0 < p < \infty$. When μ is a finite positive Borel measure on $\bar{D}$ and $a \in \bar{D}$, put

$$S(\mu, a) = S(\mu, p, a) = inf\{\int_{\bar{D}} \mid f \mid^p d\mu \ ; \ f \in P \text{ and } f(a) = 1\}$$

and

$$R(\mu, a) = R(\mu, p, a) = sup\{\mid f(a) \mid^p \ ; \ f \in P \text{ and } \int_{\bar{D}} \mid f \mid^p d\mu \leq 1\}.$$

When μ is a finite positive Borel measure on D and $a \in D$, put

$$s(\mu, a) = s(\mu, p, a) = inf\{\int_D \mid f \mid^p d\mu \ ; \ f \in H \text{ and } f(a) = 1\}$$

and

$$r(\mu, a) = r(\mu, p, a) = sup\{\mid f(a) \mid^p \ ; \ f \in H \text{ and } \int_D \mid f \mid^p d\mu \leq 1\}.$$

The four functions S, R, s and r are called Riesz's functions. In this paper we study these four Riesz's functions. M.Riesz used such functions to solve the moment problem on the real line (cf. [6, Chapter 5]). T.Kriete and T.Trent [7] also investigated the relationship between μ and $R(\mu, 2, a)$. In the investigations of Riesz's functions, the most fundamental and important result is the following theorem by G.Szegő (cf. [5, Chapter 3]). He proved it only when $p = 2$ but it can be proved for arbitrary p.

Szegő's Theorem. Suppose $0 < p < \infty$, μ is a finite positive Borel measure on $\bar{D}$ with $supp\ \mu \subseteq \partial D$ and $d\mu/(d\theta/2\pi) = w(e^{i\theta})$.

Then,

$$S(\mu, p, a) = (1- \mid a \mid^2) exp(logw)^\wedge(a) \qquad (a \in D)$$

where $(logw)^\wedge(a) = \int_0^{2\pi} logw(e^{i\theta}) \frac{1- \mid a \mid^2}{\mid 1 - \bar{a}e^{i\theta} \mid^2} d\theta/2\pi.$

It is most desirable to describe $S(\mu, p, a)$ using μ as in Szegő's Theorem, when μ is an arbitrary measure on $\bar{D}$. However such a problem is very difficult except for some special measures μ. In Section 2, we study the behaviour of $S(\mu, p, a)$ as $\mid a \mid \rightarrow 1$ for an arbitrary measure on $\bar{D}$. Moreover we note that $S(\mu, p, a)\ R(\mu, p, a) = 1 \quad (a \in \bar{D})$. Thus we need to know only S or R. In this paper, the results and the proofs about s and r are very similar to those about S and R. Hence we concentrate on only S or R in

Sections 2, 3 and 4. Let m be the normalized area measure on D, that is, $dm = rdrd\theta/\pi$. In Section 3, we give the several lower estimates of S using $d\mu/dm$. It is more difficult to give the upper estimates of S. We do it only in very special cases. In Section 4, we show that $R(\mu, p, a)$ is not in $L^1(\mu)$ if *supp* μ is not a finite set

Suppose $0 < p < \infty$. $H^p(\mu)$ denotes the closure of P in $L^p(\mu)$ when μ is a finite positive Borel measure on $\bar{D}$. $H^p(\mu)$ is called a weighted Hardy space. If $d\mu = d\theta/2\pi, H^p(\mu) = H^p$ is the classical Hardy space. When μ is a finite positive Borel measure on D, then one defines $L^p_a(\mu) = H \cap L^p(\mu)$. $L^p_a(\mu)$ is called a weighted Bergman space. If $\mu = m, L^p_a(\mu) = L^p_a$ is the usual Bergman space. H^p can be embedded in H. $L^p_a = H^p(m)$ and hence L^p_a is closed. We are interested in the following questions : (1) When $H^p(\mu)$ can be embedded in H ? (2) When $L^p_a(\mu)$ is closed ? (3) When $H^p(\mu)$ can be embedded in $L^p_a(\mu)$? Of course it is very interesting to know when $L^p_a(\mu) = H^p(\mu)$, where μ is a measure on D. This problem is classical and important (cf. [2]). However, in this paper we are not going to consider this problem. The problem (2) was studied by M.Yamada [13]. If μ is a measure on D, the problem (1) is equivalent to (3). Note that the measure μ for (2) satisfies (3). In Section 5, we study the three problems above. For example, for some compact set K in D, if $\int_{\bar{D}\setminus K} logW dm > -\infty$ then $H^p(\mu)$ can be embedded in H where $W = d\mu/dm$. This result follows from the lower estimate of $S(\mu, p, a)$ in Section 3.

In this paper, we will use the following notations. For each $a \in D$, *let* ϕ_a be the Möbius function on D, that is,

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \qquad (z \in D),$$

and put

$$\beta(a, z) = \frac{1}{2}\, log \frac{1 + |\phi_a(z)|}{1 - |\phi_a(z)|} \qquad (a, z \in D).$$

For $0 < r \leq \infty$ and $a \in D$,

$$D_r(a) = \{z \in D \ ; \ \beta(a, z) < r\}$$

be the Bergman disc with 'center' a and 'radius' r. For $u \in L^1(m)$,

$$\tilde{u}(a) = \int_D u \circ \phi_a(z) dm(z) \qquad (a \in D).$$

Then $\tilde{u}$ may be bounded on D even if u is not bounded on D.

§2. Riesz's function

If $\mu = m$, then for $0 < p < \infty$ $S(m,p,a) = (1-|a|^2)^2$. Hence $\mu = m$ or $supp\ \mu \subseteq \partial D$, by Szegö's Theorem $\lim_{r \to 1-} S(\mu, p, re^{i\theta}) = 0$ $a.e.\theta$. In this section, we show that this is true in general. In particular, R is not bounded on D. In fact, for arbitrary μ, we show that $\lim_{r\to 1-} S(\mu,p,re^{i\theta}) = 0$ except a countable set of θ.

Proposition 1. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. Then the following are valid for $R(a) = R(\mu,p,a)$ and $S(a) = S(\mu,p,a)$.

(1) $R(\mu,p,a)\ S(\mu,p,a) = 1$ for $a \in \bar{D}$, assuming $\infty \times 0 = 1$.

(2) $R(\mu)$ is lower semicontinuous on $(0,\infty) \times D$, and $S(\mu)$ is upper semicontinuous on the same set. Moreover $R(\mu,p,a) \geq 1/\mu(\bar{D})$ and $S(\mu,p,a) \leq \mu(\bar{D})$.

(3) If $logR$ or R is in $L^1(m)$, then for $a \in D$

$$R(a) \leq exp(logR)^{\sim}(a) \leq \tilde{R}(a).$$

(4) If $r < \infty$, then for $a \in D$

$$logR(a) \leq \left(\frac{1+s\mid a\mid}{1-s\mid a\mid}\right)^2 \frac{1}{m(D_r(a))}\int_{D_r(a))} logRdm$$

where $s = \tanh r$. Hence for $a \in D$

$$logR(a) \leq \left(\frac{1+\mid a\mid}{1-\mid a\mid}\right)^2 \int_D logRdm.$$

These inequalities are also valid for R instead of $logR$.

(5) For $a \in D$,

$$S(\mu,p,a) \geq S(S(\mu)dm,p,a).$$

(6) R is not bounded on D and $\bar{D}$.

Proof. (1) It is easy to see that $1 \leq R(a)S(a)$ for $a \in \bar{D}$. If $1 < R(a)S(a)$, then there exists a positive constant γ such that $1 \leq \gamma\ S(a)$ and $\gamma < R(a)$. Hence $1 \leq \gamma \int \mid g \mid^p d\mu$ for any $g \in P$ with $g(a) = 1$ and so

$$\mid f(a)\mid^p \leq \gamma \int_{\bar{D}} \mid f\mid^p d\mu \text{ for any } f \in P.$$

This implies $\gamma \geq R(a)$. This contradiction shows that $1 = R(a)\ S(a)$. (2) is clear by (1). (3) If $f \in P$, then $log \mid f \mid$ is subharmonic on D and hence for any $a \in D$,

$$log \mid f(a)\mid^p \leq \int_D log \mid f(z)\mid^p \frac{(1-\mid a\mid^2)^2}{\mid 1-\bar{a}z\mid^4} dm(z).$$

Assuming $\int | f |^p \, d\mu \leq 1$, by definition on R

$$logR(a) \leq \int_D logR(z) \, \frac{(1-| a |^2)^2}{| 1 - \bar{a}z |^4} dm(z).$$

This implies $R(a) \leq exp(logR)^{\sim}(a) \leq \tilde{R}(a)$. (4) If $0 < r < \infty$, for any $a \in D_r(0)$ and any $f \in P$,

$$log \mid f(a) \mid^p \leq \frac{1}{m(D_r(0))} \int_{D_r(a)} log \mid f(z) \mid^p \frac{(1-| a |^2)^2}{| 1 - \bar{a}z |^4} dm(z)$$

and hence

$$log \mid f(a) \mid^p \leq \frac{1}{m(D_r(a))} \left(\frac{1+s \mid a \mid}{1 - s \mid a \mid}\right)^2 \int_{D_r(a)} log \mid f \mid^p dm$$

where $s = \tanh r$. This proof is same to that of [14, Proposition 4.3.8.]. Assuming $\int | f |^p \, d\mu \leq 1$, we get (4) as in (3). (5) By (1),

$$\int | f |^p \, d\mu \geq S(\mu, z) \mid f(z) \mid^p \qquad (z \in D).$$

and hence $\int | f |^p \, d\mu \geq \int | f |^p \, S(\mu) dm$. Assuming $f(a) = 1$ and $a \in D$, we get $S(\mu, a) \geq S(S(\mu)dm, a)$. (6) If $R(\mu, p, a)$ is bounded on $\bar{D}$, then $H^p(\mu) \subset L^\infty(\mu)$. By [11, Theorem 5.2], $H^p(\mu)$ is finitely dimensional. It is easy to see that $supp\ \mu$ is a finite set. Then trivially $R(\mu, p, a) = \infty$ except $supp\ \mu$. The proof of the statement for D is same to that for $\bar{D}$, assuming $\mu = \mu \mid D$.

Even if v is not bounded, $\tilde{v}$ may be bounded. However (3) and (6) of Proposition 1 show that $\tilde{R}$ is also not bounded. The following theorem gives a stronger result.

Theorem 2. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. If $a \in \partial D$, then the following are valid.

(1) $\mu(\{a\}) = 0$ if and only if $S(\mu, p, a) = 0$.

(2) $\lim_{r \to 1-} S(\mu, p, ra) = 0$ except a countable set of a in ∂D.

(3) If $\mu(\{a\}) = 0$ and $\{a_n\}$ is a sequence in D with $lim\ a_n = a$, then $\lim_{n \to \infty} S(\mu, p, a_n) = 0$.

(4) If $\mu(\{a\}) > 0$, then for each n, the set $\{z \in D \ ; \ | z - a |< 1/n\} \cap \{z \in D \ ; \ S(\mu, p, z) < 1/n\}$ is a nonempty open set.

(5) If $b < c$ and $E = \{z \in D \ ; \ z = re^{i\theta}, 0 \leq r < 1 \text{ and } b \leq \theta \leq c\}$, then R is not bounded on E.

Proof. We may assume $a = 1$. (1) If $\mu(\{1\}) > 0$, then $| f(1) |^p \leq \int | f |^p \, d\mu/\mu(\{1\})$ and so $R(\mu, p, 1) \leq 1/\mu(\{1\})$. (1) of Proposition 1 implies $S(\mu, p, 1) > 0$. Conversely suppose $\mu(\{1\}) = 0$. If $z \in \bar{D}$ and $z \neq 1$, then $\lim_{t \to 1+} | (1-t)/(z-t) |= 0$ and

$$\left| \frac{z-1}{z-t} - 1 \right| = \left| \frac{1-t}{z-t} \right| < 1 \qquad (t > 1).$$

For any $t > 1$,

$$S(\mu, p, 1) \leq \int_{\bar{D}} \left|1 - \frac{z-1}{z-t}\right|^p d\mu(z) = \int_{\bar{D}\setminus\{1\}} \left|\frac{1-t}{z-t}\right|^p d\mu(z).$$

As $t \to 1$, by the Lebesgue's dominated convergence theorem, $S(\mu, p, 1) = 0$. (2) Suppose $\mu(\{1\}) = 0$. If there exist a sequence $\{r_n\}$ and a positive constant ε such that $0 < r_n < 1$ with $r_n \to 1$ and $S(\mu, p, r_n) \geq \varepsilon > 0$, then

$$| f(r_n) |^p \leq \frac{1}{\varepsilon} \int_{\bar{D}} | f |^p \, d\mu \text{ and so } | f(1) |^p \leq \frac{1}{\varepsilon} \int_{\bar{D}} | f |^p \, d\mu.$$

This implies $S(\mu, p, 1) > 0$ and contradicts (1). Hence if $\mu(\{1\}) = 0$, then $\lim_{r \to 1-} S(\mu, p, r) = 0$. This implies (2) because $\{a \in \partial D \ ; \ \mu(\{a\}) > 0\}$ is a countable set. (3) is clear by the proof of (2). (4) Suppose $\mu(\{1\}) > 0$ and for each n, put

$$G_n = \{z \in \bar{D} \ ; \ | z - 1 | < \frac{1}{n}\} \cap \{z \in \bar{D} \ ; \ S(\mu, p, z) < \frac{1}{n}\}.$$

Since $\{z \in \partial D \ ; \ \mu(\{z\}) > 0\}$ is a countable set, for each n there exists $b_n \in \{z \in \partial D \ ; \ | z - 1 | < \frac{1}{n}\}$ with $\mu(\{b_n\}) = 0$. Then $S(\mu, p, b_n) = 0$ by (1) and hence G_n is not empty. G_n is a relatively open set in $\bar{D}$ by (2) of Proposition 1 and so $G_n \cap D$ is a nonempty open set. (5) follows from (2).

If $R(\mu, 2, a) < \infty$, then there exists k_a in $H^2(\mu)$ such that $f(a) = \int f(z)\overline{k_a(z)} d\mu(z)$ for any f in $H^2(\mu)$ and hence $R(\mu, 2, a) = \int | k_a(z) |^2 \, d\mu(z)$. Thus the results in this section give the informations about the reproducing kernel k_a.

§3. Estimate of Riesz's function

In this section we give upper and lower estimates of S. The lower ones will be used later. The following proposition is a generalization of Szegő's Theorem in Introduction. In fact, if $\mu \mid D$ is a zero measure, then it gives Szegő's Theorem.

Proposition 3. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure such that $(d\mu \mid \partial D)/(d\theta/2\pi) = w(e^{i\theta}), \mu \mid D = \sum a_j \delta_{z_j}$ and $\sum(1- \mid z_j \mid) < \infty$. Let b be a Blaschke product of $\{z_\ell\}$ and b_j a Blaschke product of $\{z_\ell\}_{\ell \neq j}$. Then for all $a \in D, (1- \mid a \mid^2) exp(log w)^\wedge(a) \leq S(\mu, p, a)$. If $a \in D \setminus \{z_\ell\}$, then

$$S(\mu, p, a) \leq \mid b(a) \mid^{-p} (1- \mid a \mid^2) exp(log w)^\wedge(a).$$

If $a = z_j$, then

$$S(\mu, p, a) \leq \mid b_j(a) \mid^{-p} (1- \mid a \mid^2) exp(log w)^\wedge(a) + a_j.$$

In particular, $S(\mu, p, a) > 0$ if and only if $log w \in L^1(d\theta)$.

Proof. Since $S(\mu, p, a) \geq S(wd\theta/2\pi, p, a)$ for all $a \in D$, by Szegő's Theorem $(1- \mid a \mid^2) exp(log w)^\wedge(a) \leq S(\mu, p, a)$ for all $a \in D$. Let B_n be a finite Blaschke product of $\{z_1, z_2, \cdots, z_n\}$. If $a \in D \setminus \{z_\ell\}$, then

$$\begin{aligned} & S(\mu, p, a) \\ \leq \quad & inf\{\int \mid \frac{B_n}{B_n(a)} g \mid^p d\mu \mid \partial D + \sum_{j=1}^{\infty} a_j \mid \frac{B_n(z_j)}{B_n(a)} g(z_j) \mid^p ; \ g \in P \text{ and } g(a) = 1\} \\ = \quad & \frac{1}{\mid B_n(a) \mid^p} inf\{\int \mid B_n g \mid^p d\mu \mid \partial D + \sum_{j=n+1}^{\infty} a_j \mid B_n(z_j) \mid^p \mid g(z_j) \mid^p ; \ g \in P \\ \text{and} \quad & g(a) = 1\}. \end{aligned}$$

As $n \to \infty$,

$$S(\mu, p, a) \leq \frac{1}{\mid b(a) \mid^p} inf\{\int \mid g \mid^p d\mu \mid \partial D ; \ g \in P \ \text{ and } \ g(a) = 1\}.$$

Now by Szegő's Theorem, for each $a \in D$ $S(\mu, p, a) \leq \mid b(a) \mid^{-p} (1- \mid a \mid^2) exp(log w)^\wedge(a)$. Let $B_{j,n}$ be a finite Blaschke product of $\{z_1, z_2, \cdots, z_n\} \setminus \{z_j\}$. If $a = z_j$ and $n > j$, then

$$\begin{aligned} & S(\mu, p, a) \\ \leq \quad & inf\{\int \mid \frac{B_{j,n}}{B_{j,n}(a)} g \mid^p d\mu ; \ g \in P \text{ and } g(a) = 1\} \\ = \quad & \frac{1}{\mid B_{j,n}(a) \mid^p} inf\{\int \mid B_{j,n} g \mid^p d\mu \mid \partial D + a_j \mid B_{j,n}(a) \mid^p \\ + \quad & \sum_{\ell \geq n+1} a_\ell \mid B_{j,n}(z_\ell) \mid^p \mid g(z_\ell) \mid^p ; \ g \in P \text{ and } g(a) = 1\}. \end{aligned}$$

As $n \to \infty$, by Szegő's Theorem, for $a = z_j$,

$$S(\mu, p, a) \leq \mid b_j(a) \mid^{-p} (1- \mid a \mid^2) exp(log w)^\wedge(a) + a_j.$$

The following proposition is related to Theorem 2 in this paper and Theorem in [7]. In fact, if $\tilde{W}$ is bounded on D, then $(1- \mid a \mid^2)^{-2} S(W dm, p, a)$ is bounded on D. Moreover if W is continuous on $\bar{D}$, then for all $e^{i\theta}$

$$\lim_{a \to e^{i\theta}} (1-|a|^2)^2 R(Wdm, p, a) = 1/W(e^{i\theta}).$$

Proposition 4. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on $\bar{D}$.

(1) $\tilde{\mu}(a) \geq (S(\mu))^{\sim}(a) \quad (a \in D)$.
(2) If $d\mu = Wdm$ and $a \in D$, then

$$(1-|a|^2)^2 exp(logW)^{\sim}(a) \leq S(\mu, p, a) \leq (1-|a|^2)^2 \tilde{W}(a).$$

(3) $S(Wdm, a) = (1-|a|^2)^2 S(W \circ \phi_a dm, 0)$ for $a \in D$.

Proof. (1) For all $z \in D$

$$\int |f|^p d\mu \geq |f(z)|^p S(z) \text{ and so } \int |f|^p d\mu \geq \int |f|^p Sdm.$$

Assuming $f(z) = \{(1-|a|^2)/(1-\bar{a}z)^2\}^{2/p}$ for $a \in D, \tilde{\mu}(a) \geq \tilde{S}(a)$. (2) If $logW \in L^1(m)$, then

$$S(Wdm, p, a) = inf\{\int |f|^p Wdm \; ; \; f \in P \text{ and } f(a) = 1\}$$

$$= inf\{\int |g|^p W \circ \phi_a \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm \; ; \; g \in H^p(W \circ \phi_a dm) \text{ and } g(0) = 1\}$$

$$= (1-|a|^2)^2 inf\{\int |k|^p W \circ \phi_a dm \; ; \; k \in H^p(W \circ \phi_a dm) \text{ and } k(0) = 1\}$$

$$\geq (1-|a|^2)^2 exp \int (logW) \circ \phi_a dm = (1-|a|^2)^2 exp(logW)^{\sim}(a).$$

The inequality above is proved by two Jensen's inequalities. (3) is clear by the proof of (2).

In (2) of Proposition 4, we can get estimates of $S(\mu, p, a)$ as in Proposition 3 when $d\mu = Wdm + \sum_{j=1}^{\infty} a_j \delta_{z_j}$, $\{z_j\} \subset D$ and $\sum (1-|z_j|) < \infty$. The following theorem is important in this paper and the following lemma is used to prove it.

Lemma 1. Let $\Delta_s(a)$ be the set $\{z \in D \; ; \; |(a-z)/(1-\bar{a}z)| < s\}$ where $a \in D$ and $s \in (0,1)$. If $t \in (0,1)$ and $1 - s^2 = (1-|a|^2)(1-t^2)/5$, then $\overline{\Delta_t(0)} \subset \Delta_s(a)$.

Proof. The Euclidean center and radius of $\Delta_s(a)$ are

$$C = \frac{1-s^2}{1-s^2 \mid a \mid^2} a \ , \ R = \frac{1- \mid a \mid^2}{1-s^2 \mid a \mid^2} s$$

respectively. Hence to prove $\overline{\Delta_t(0)} \subset \Delta_s(a)$, it is sufficient to show that

$$t + \frac{1-s^2}{1-s^2 \mid a \mid^2} \mid a \mid \leq \frac{1- \mid a \mid^2}{1-s^2 \mid a \mid^2} s.$$

If $1 - s^2 = (1- \mid a \mid^2)(1-t^2)/5$, then

$$1 - s^2 \leq \frac{(1- \mid a \mid^2)(1-t^2)}{5- \mid a \mid^2}$$

and hence $s^2 \geq \{4 + (1- \mid a \mid^2)t^2\}/(5- \mid a \mid^2)$. The last inequality is equivalent to

$$1 - s^2 \leq \frac{(1- \mid a \mid^2)(s^2 - t^2)}{4}.$$

Then

$$1 - s^2 \leq \frac{(1- \mid a \mid^2)(s-t)}{2} \frac{s+t}{2} \leq \frac{(1- \mid a \mid^2)(s-t)}{\mid a \mid (t \mid a \mid +1)}$$

because $s + t \leq 2$ and $\mid a \mid (t \mid a \mid +1) \leq 2$. This implies that

$$t + \frac{1-s^2}{1-s^2 \mid a \mid^2} \mid a \mid \leq \frac{1- \mid a \mid^2}{1-s^2 \mid a \mid^2} s.$$

Theorem 5. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on $\bar{D}$. Let $d\mu/dm = Wdm, K$ an arbitrary compact set in D and $t = max\{\mid z \mid ; \ z \in K\}$. Then, for $a \in D$

$$S(\mu, p, a) \geq \frac{(1- \mid a \mid^2)^3(1-t^2)}{5} exp \frac{2^4 \cdot 5}{(1- \mid a \mid^2)^3(1-t^2)} \int_{K^c} log(W \wedge 1) dm.$$

If $1 \leq p < \infty$ and $a \in D$, then

$$S(\mu, p, a) \geq \frac{(1- \mid a \mid^2)^{3(2-\frac{1}{p})}(1-t^2)^{2-\frac{1}{p}}}{2^{4(1-\frac{1}{p})} \cdot 5^{2-\frac{1}{p}}} \left(\int_{K^c} W^{-\frac{1}{p-1}} dm \right)^{\frac{1}{p}-1}.$$

Proof. By two Jensen's inequalities, for $a \in D$

$$\begin{aligned} & S(\mu, p, a) \geq S(Wdm, p, a) \\ = \ & inf\{\int \mid g \mid^p W \circ \phi_a \frac{(1- \mid a \mid^2)^2}{\mid 1 - \bar{a}z \mid^4} dm \ ; \ g(0) = 1\} \\ = \ & (1- \mid a \mid^2)^2 inf\{\int \mid k \mid^p W \circ \phi_a dm \ ; \ k(0) = 1\} \end{aligned}$$

$$\begin{aligned}
&\geq (1-|a|^2)^2\int_0^1 2rdr\ exp\int_0^{2\pi} logW\circ\phi_a d\theta/2\pi \\
&\geq (1-|a|^2)^2(1-s^2)\int_s^1 \frac{2r}{1-s^2}dr\ exp\int_0^{2\pi} logW\circ\phi_a d\theta/2\pi \\
&\geq (1-|a|^2)^2(1-s^2)\ exp\frac{1}{1-s^2}\int_s^1 2dr\int_0^{2\pi} logW\circ\phi_a d\theta/2\pi \\
&= (1-|a|^2)^2(1-s^2)\ exp\frac{1}{1-s^2}\int_{D\backslash\Delta_s(0)} logW\circ\phi_a dm \\
&= (1-|a|^2)^2(1-s^2)\ exp\frac{1}{1-s^2}\int_{D\backslash\Delta_s(a)} logW\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}dm \\
&\geq (1-|a|^2)^2(1-s^2)\ exp\frac{(1-|a|^2)^2}{(1-|a|)^4}\ \frac{1}{1-\ s^2}\int_{D\backslash\Delta_s(a)} log(W\wedge 1)dm
\end{aligned}$$

where $s\in(0,1)$ and $\Delta_s(a)=\{z\in D\ ;\ |(a-z)/(1-\bar{a}z)|<s\}$. For each compact set $K\subset D$, if $t=max\{|z|\ ;\ z\in K\}$ and $1-s^2=(1-|a|^2)(1-t^2)/5$, then by Lemma 1 $\overline{\Delta_t(0)}\subset\Delta_s(a)$. Hence $K\subset\Delta_s(a)$ and so $K^c\supset D\backslash\Delta_s(a)$. Thus, if $1-s^2=(1-|a|^2)(1-t^2)/5$, then

$$\frac{(1-|a|^2)^2}{(1-|a|)^4}\ \frac{1}{1-s^2}=\frac{(1+|a|)^4}{(1-|a|^2)^2(1-s^2)}\leq\frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}$$

and hence for all $a\in D$

$$S(\mu,p,a)\geq\frac{(1-|a|^2)^3(1-t^2)}{5}\ exp\frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}\int_{K^c} log(W\wedge 1)dm.$$

Now we will prove the second inequality. Instead of two Jensen's inequalities, we will use Kolmogoroff's inequality (cf.[12, Theorem 4.3.1]). For $a\in D$, if $1\leq p<\infty$ and $1/p+1/q=1$,

$$\begin{aligned}
&S(\mu,p,a) \\
&\geq (1-|a|^2)^2\int_0^1 2rdr\left(\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}d\theta/2\pi\right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)\int_s^1\frac{2r}{1-s^2}dr\left(\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}d\theta/2\pi\right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)\left(\frac{1}{1-s^2}\int_s^1 2rdr\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}d\theta/2\pi\right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\left(\int_{D\backslash\Delta_s(0)}(W\circ\phi_a)^{-\frac{1}{p-1}}dm\right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\left(\int_{D\backslash\Delta_s(a)}W^{-\frac{1}{p-1}}\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}dm\right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\left\{\frac{(1-|a|^2)^2}{(1-|a|)^4}\int_{D\backslash\Delta_s(a)}W^{-\frac{1}{p-1}}dm\right\}^{-\frac{1}{q}}
\end{aligned}$$

$$\geq \frac{(1-|a|^2)^{2(1+\frac{1}{q})}(1-s^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}}\left(\int_{D\backslash\Delta_s(a)} W^{-\frac{1}{p-1}}dm\right)^{-\frac{1}{q}}$$

where $s \in (0,1)$. As in the proof of the first inequality, for each compact set $K \subset D$, if $t = max\{|z| \; ; \; z \in K\}$ and $1-s^2 = (1-|a|^2)(1-t^2)/5$, then $K^c \supset D\backslash\Delta_s(a)$. Thus, if $1-s^2 = (1-|a|^2)(1-t^2)/5$, then for all $a \in D$

$$S(\mu,p,a) \geq \frac{(1-|a|^2)^{3(1+\frac{1}{q})}(1-t^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}\cdot 5^{1+\frac{1}{q}}}\left(\int_{K^c} W^{-\frac{1}{p-1}}dm\right)^{-\frac{1}{q}}.$$

The second inequality of Theorem 5 implies $S(\mu,1,a) \geq (1-|a|^2)^3 \times (1-t^2)(1/5)ess.inf\{W(x) \; ; \; x \in K^c\}$. Let σ be a finite positive Borel measure on [0,1]. $\mu(re^{i\theta}) = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ is more general than $Wdm = 2rdr \times W(re^{i\theta})d\theta/2\pi$. If $\sigma(r)$ is singular to the Lebesgue measure on [0, 1], then μ is singular to m. However we can give an interesting lower estimate. It is different from that of Theorem 5 in case of $\mu = Wdm$.

Theorem 6. Suppose $0 < p < \infty$ and $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ where $\sigma(r)$ is a finite positive Borel measure on [0, 1]. If $\mathbf{W}(e^{i\theta}) = \sup_r W(re^{i\theta})$ and $W_r(e^{i\theta}) = W(re^{i\theta})$, then for $a \in D$

$$\begin{aligned}
&(1-|a|^2)\int_{|a|}^1 exp(logW_r)^\wedge(a)d\sigma(r) \leq S(\mu,p,a) \\
&\leq \sigma([0,1])\, inf\left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&\leq \sigma([0,1])\, inf\left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, \mathbf{W}(e^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\}.
\end{aligned}$$

Proof. For $a \in D$,

$$\begin{aligned}
S(\mu,p,a) &= inf\left\{\int_0^1 d\sigma(r)\int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&\geq \int_0^1 d\sigma(r)\, inf\left\{\int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&= \int_{|a|}^1 d\sigma(r)\, inf\left\{\int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&= \int_{|a|}^1 (1-|a|^2)\, exp(logW_r)^\wedge(a)d\sigma(r).
\end{aligned}$$

We used Szegö's Theorem in the last equality. The upper estimates are trivial.

Corollary 1. Let $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ as in Theorem 6 and $0 < p < \infty$.
(1) If $W(re^{i\theta}) \equiv 1$, then for $a \in D$

$$(1-|a|^2)\ \sigma([|a|,1]) \le S(\mu,p,a) \le (1-|a|^2)\sigma([0,1]).$$

In particular, $S(\mu,p,0)=\sigma([0,1])$.

(2) If $W(re^{i\theta}) = |h(re^{i\theta})|$ for some outer function h in $H^1(d\theta)$, then for $a \in D$

$$(1-|a|^2)\int_{|a|}^{1} W(ra)d\sigma(r) \le S(\mu,p,a) \le (1-|a|^2)W(a)\sigma([0,1]).$$

(3) If $1<p<\infty$ and $\mathbf{W}(e^{i\theta}) = \sup_r W(re^{i\theta})$ satisfies the A_p condition, then there exists a positive constant γ such that for $a \in D$

$$S(\mu,p,a) \le \gamma(1-|a|^2)\ exp(log\mathbf{W})^{\wedge}(a)\sigma([0,1]).$$

Proof. (1) is a special case of (2). (2) Since h is an outer function in H^1, for $a \in D$

$$exp(logW_r)^{\wedge}(a) = exp(log|h_r|)^{\wedge}(a) = |h(ra)| = W(ra)$$

and

$$\inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\ W(re^{i\theta})d\theta/2\pi \right\}$$
$$= \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p |h(e^{i\theta})|\ d\theta/2\pi = (1-|a|^2)\ |h(a)| = (1-|a|^2)W(a).$$

Now Theorem 6 implies (2). (3) By a theorem of M.Rosenblum (cf. [10] and [9, Theorem 2.2]), there exists a positive constant γ such that for any $f \in P$

$$\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\ \mathbf{W}(e^{i\theta})d\theta/2\pi \le \gamma \int_0^{2\pi} |f(e^{i\theta})|^p\ \mathbf{W}(e^{i\theta})d\theta/2\pi$$

because $\mathbf{W} \in A_p$. By Theorem 6 and Szegő's Theorem, for $a \in D$

$$\inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\ \mathbf{W}(e^{i\theta})d\theta/2\pi \right\} \le \gamma \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p\ \mathbf{W}(e^{i\theta})d\theta/2\pi$$
$$= \gamma(1-|a|^2)exp(log\mathbf{W})^{\wedge}(a).$$

This implies (3).

§4. Carleson inequality and Riesz's function

Let ν and μ be finite positive Borel measures on $\bar{D}$ and $1 \le p < \infty$. We say that ν and μ satisfy the (ν,μ,p) - Carleson inequality, if there exists a constant $\gamma > 0$ such that

$$\int_{\bar{D}} |f|^p \, d\nu \le \gamma \int_{\bar{D}} |f|^p \, d\mu$$

for all $f \in P$ (see [8]). ν and μ satisfy the (ν,μ,p) - Carleson inequality if and only if $H^p(\mu) \subset H^p(\nu)$ and the inclusion mapping $i_p : H^p(\mu) \to H^p(\nu)$ is bounded. We say that for $p > 1, \nu$ and μ satisfy the (ν,μ,p)-vanishing Carleson inequality if $H^p(\mu) \subset H^p(\nu)$ and $i_p : H^p(\mu) \to H^p(\nu)$ is compact. We say that for $p = 1$, ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality if i_p is star-compact. We could not prove Theorem 7 for $p = 1$ because we do not know anything about the predual of $H^1(\mu)$. Using Riesz's functions, we will show vanishing Carleson inequalities. As a result, we show that $R(\mu,p) \notin L^1(\mu)$ if *supp* μ is not a finite set. Moreover, from given a measure μ, we will show how to construct a measure ν such that the (ν,μ,p) -vanishing Carleson inequality is valid.

Theorem 7. Suppose $1 < p < \infty$, and ν and μ are finite positive Borel measures on $\bar{D}$.

(1) If $\int R(\mu,p)d\nu < \infty$, then ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality and

$$R(\mu,p,a) \le \left(\int R(\mu,p)d\nu\right) R(\nu,p,a) \qquad (a \in \bar{D}).$$

(2) If V is a Borel function such that $0 \le V \le S$ on $\bar{D}$, then $V \mid g \mid^p$ is bounded on $\bar{D}$ for each g in $H^p(\mu)$, and Vdm and μ satisfy the (Vdm,μ,p)-vanishing Carleson inequality.

Proof. (1) By definition of $R(\mu,p,a)$, for $a \in \bar{D}$,

$$|f(a)|^p \le R(\mu,p,a) \int |f|^p \, d\mu \qquad (f \in P).$$

Hence if $\gamma = \int R(\mu,p)d\nu < \infty$, then $\int |f|^p \, d\nu \le \gamma \int |f|^p \, d\mu$ $(f \in P)$ and so $i_p : H^p(\mu) \to H^p(\nu)$ is bounded. We will show that i_p is compact. If $f_n \to f$ weakly in $H^p(\mu)$, then there exists a finite positive constant γ' such that

$$\int |f_n - f|^p \, d\mu \le \gamma' \quad \text{for all } n.$$

By the hypothesis, $R(\mu,p,a) < \infty \quad \nu - a.e.$ on $\bar{D}$ and so $f_n \to f \quad \nu - a.e.$ on $\bar{D}$ because $f_n \to f$ weakly. Moreover by definition of $R(\mu,p,a), |f_n(a) - f(a)|^p \le \gamma' R(\mu,p,a)$ and by the hypothesis, $R(\mu,p,a) \in L^1(\nu)$. Thus

$$\int |f_n - f|^p \, d\nu \to 0 \quad \text{as } n \to \infty.$$

by the Lebesgue's dominated convergence theorem. This implies i_p is compact. Since $\int |f|^p d\nu \leq \gamma \int |f|^p d\mu$ and $\gamma = \int R(\mu,p)d\nu$, assuming $f(a)=1$, we get $S(\nu,p,a) \leq \gamma\ S(\mu,p,a)$. Now by (1) of Proposition 1, we get the inequality of (1). (2) If $0 \leq V \leq S$, then $VR \leq 1$ and hence $V(a) \mid f(a) \mid^p$ is bounded on $\bar{D}$ by $\int |f|^p d\mu$, for each $f \in H^p(\mu)$. Moreover if $\nu = Vdm$ and $0 \leq V \leq S$, then $\int R(\mu,p)d\nu \leq \int dm = 1$ and hence by (1) ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

Corollary 2. If $0 < p < \infty$ and *supp* μ is not a finite set, then $R(\mu,p) \notin L^1(\mu)$.

Proof. Suppose $1 < p < \infty$. If $R(\mu,p) \in L^1(\mu)$, then the inclusion map $i_p : H^p(\mu) \to H^p(\mu)$ is compact. It is easy to see that i_p is an identity operator. Hence the unit ball of $H^p(\mu)$ is compact with respect to the norm. Therefore $H^p(\mu)$ is finitely dimensional. This contradicts that *supp* μ is not a finite set. This implies that $R(\mu,p) \notin L^1(\mu)$. For $0 < p \leq 1$, the proof is due to the referee. Choose n sufficiently large that $np > 1$. If $g(a) = 1$ then $g^n(a) = 1$ as well, and g^n is a polynomial if g is a polynomial. Thus,

$$\begin{aligned} S(\mu,p,a) &= inf\{\int_{\bar{D}} |f|^p d\mu \ ;\ f \in P, f(a)=1\} \\ &\leq inf\{\int_{\bar{D}} |g^n|^p d\mu \ ;\ g \in P,\ g(a)=1\} = S(\mu,np,a). \end{aligned}$$

This implies that $R(\mu,p) \notin L^1(\mu)$ for $0 < p \leq 1$.

Corollary 3. Suppose $1 < p < \infty$ and $d\mu/dm = W$

(1) If $logW \in L^1(m)$ and $d\nu = (1-|z|^2)^2 exp(logW)^{\sim} dm$, then ν and μ satisfy the (ν,μ,p) - vanishing Carleson inequality.

(2) If $\chi_{K^c}\log(W \wedge 1) \in L^1(m)$ for some compact set K in D, there exist positive constant a and nonpositive constant b such that $d\nu = a(1-|z|^2)^3\{\exp b(1-|z|^2)^{-3}\}dm$ and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

(3) Suppose $\chi_{K^c}W^{-\frac{1}{p-1}} \in L^1(m)$ for some compact set K in D. If $d\nu = c(1-|z|^2)^{3(2-\frac{1}{p})}dm$, then ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

Suppose $1 < p < \infty$ and $d\mu/dm = W$. If $\chi_{K^c}logW \in L^1(m)$ for some compact set K in D, there exist positive constant a and nonpositive constant b such that

$$\{a(1-|z|^2)^3 \exp\ b(1-|z|^2)^{-3}\} \mid f(z) \mid^p$$

is bounded on D for each $f \in H^p(\mu)$. Here a and b do not depend on f. This is a corollary of (2) in Theorem 7.

§5. $H^p(\mu)$ and $L_a^p(\mu)$

The following is a result of Theorem 5. If $d\mu/dm = W$ and $logW$ is integrable on the complement K^c of a compact set in D, then $H^p(\mu) \subseteq L_a^p(\mu)$. In this section, we show that if $logW$ is locally integrable on K^c, then the same result is true. We give a necessary and sufficient condition for $H^p(\mu) \subset L_a^p(\mu)$ using Riesz's function. Theorem 8 is a joint work with K.Takahashi. A subset E of D is a uniqueness set if E satisfies the following : If f in H is zero on E, then $f \equiv 0$ on D.

Lemma 2. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on D. Then, the following (1) $\sim$ (3) are equivalent.

(1) $\sup_{a\in K} R(\mu,p,a) < \infty$ for all compact set K in D.

(2) $\int_K R(\mu,p)dm < \infty$ for all compact set K in D.

(3) $\int_K log\ R(\mu,p)dm < \infty$ for all compact set K in D.

Proof. Both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We will show (3) $\Rightarrow$ (1). We may assume that $\mu(D) = 1$. For any $f \in P$,

$$log \mid f(0) \mid^p \leq \frac{1}{m(D_r(0))}\int_{D_r(0)} log \mid f \mid^p dm.$$

If $a \in D_r(0)$, then for all $f \in P$

$$log \mid f(a) \mid^p \leq \frac{1}{m(D_r(0))}\int_{D_r(a)} log \mid f \mid^p \frac{(1-\mid a \mid^2)^2}{\mid 1-\bar{a}z \mid^4} dm.$$

Assuming $\int \mid f \mid^p d\mu \leq 1$, we get

$$log\ R(\mu,p,a) \leq \frac{1}{m(D_r(0))} \frac{(1+\mid a \mid)^2}{(1-\mid a \mid)^2}\int_{D_r(a)} log\ R(\mu,p)dm.$$

Since $D_r(a) \subset D_{2r}(0)$ and $R(\mu,p,a) \geq 1$, for each $a \in D_r(0)$ there exists a finite positive constant γ_r such that

$$log\ R(\mu,p,a) \leq \gamma_r \int_{D_{2r}(0)} log\ R(\mu,p)dm.$$

This implies (1).

Lemma 3. Let X be a Banach space which consists of analytic functions on D and contains 1. Suppose there exists a dense subspace Y of X such that if f in Y, then $(f - f(a))/(z - a)$ belongs to Y for some a in D. If $(z - a)X$ is not dense in X, then the functial $f \to f(a)$ is bounded.

Proof. By the hypothesis, if $f \in Y$, then $f = f(a) + (z - a)g$ for some $g \in Y$. Since $(z - a)X$ is not dense in X, there exists $\phi \in X^*$ such that $< (z - a)h, \phi >= 0$ for all $h \in X$ but $< 1, \phi >\neq 0$. Hence, $< f, \phi >= f(a) < 1, \phi >$. Thus $| f(a) |\leq \gamma \|f\|$ for all $f \in X$ where $\gamma =|< 1, \phi >|^{-1} \|\phi\|_*$.

Theorem 8. Suppose $1 \leq p < \infty$ and μ is a finite positive Borel measure on D such that $(supp\ \mu) \cap D$ is a uniqueness set for H.

(1) $L^p_a(\mu)$ is closed if and only if for all compact set K in D

$$\int_K log\ r(\mu, p)dm < \infty \text{ or } \int_K log\ s(\mu, p)dm > -\infty.$$

(2) $H^p(\mu) \subset L^p_a(\mu)$ if and only if for all compact set K in D

$$\int_K log\ R(\mu, p)dm < \infty \text{ or } \int_K log\ S(\mu, p)dm > -\infty.$$

Proof. (1) If $f \in L^p_a(\mu)$, then $(f - f(0))/z$ belongs to H. Since $(f - f(0))/z$ is bounded on $| z |\leq t < 1$ and $1/z$ is bounded on $| z |\geq t, (f - f(0))/z$ belongs to $L^p_a(\mu)$. This implies that $\{f \in L^p_a(\mu)\ ;\ f(0) = 0\} = zL^p_a(\mu)$ and hence $L^p_a(\mu) = \mathbf{C} \oplus zL^p_a(\mu)$. If $Af = zf$ for $f \in L^p_a(\mu)$, then A is a bounded operator on $L^p_a(\mu)$ and the range of A is complemented in $L^p_a(\mu)$ by what was just proved. By [4, Part III, Corollary 2.3], the range of A is closed and hence $zL^p_a(\mu)$ is not dense in $L^p_a(\mu)$. Applying Lemma 3 with $X = Y = L^p_a(\mu)$ and $a = 0, r(\mu, p, 0) < \infty$ follows. The same argument is true for all $a \in D\backslash\{0\}$ and hence $r(\mu, p, a) < \infty$ for all $a \in D$. By the boundedness of holomorphic functions on compact sets and the uniform boundedness principle, $\sup_{a\in K} r(\mu, p, a) < \infty$ for all compact set K in D. As Lemma 2 is also for $r(\mu, p, a)$,

$$\int_K log\ r(\mu, p)dm < \infty \quad \text{or} \quad \int_K log\ s(\mu, p)dm > -\infty.$$

Conversely, suppose $\int_K log\ r(\mu, p)dm < \infty$ for any compact set K. Then by the above lemma, $sup_k\ r(\mu, p) < \infty$ for any compact set K. If f is in the $L^p(\mu)$-norm closure of $L^p_a(\mu)$, then there exists a sequence $\{f_n\}$ in $L^p_a(\mu)$ such that $\int | f - f_n |^p d\mu \to 0$. Then by hypothesis on $r(\mu, p), sup\{| f_n(z) |\ ;\ z \in D_r(0)\} < \infty$ for each $r < \infty$, Hence, for each $r < \infty$ there exists a subsequence $\{f_{n_j}\}$ in $L^p_a(\mu)$ and an analytic function g_r on $D_r(0)$ such that $f_{n_j} \to g_r$ uniformly on $D_r(0)$. This implies that $f = g_r\ \mu - a.e.$ on $D_r(0)$ for all $r < \infty$. Thus $g = lim_r\ g_r$ is analytic on D and $f = g\ \mu - a.e.$ on D.

(2) The 'if' part is same to (1) and hence we will show the 'only if' part. Put $M = \{f \in L^p(\mu)\ ;\ zf \in H^p(\mu)\}$, then M is a closed subspace of $L^p(\mu)$ such that

$$M \supseteq H^p(\mu) \supseteq zM \supseteq H^p(\mu)_0$$

where $H^p(\mu)_0 = \{f \in H^p(\mu)\ ;\ f(0) = 0\}$. $H^p(\mu)_0$ is well defined because $H^p(\mu) \subset L^p_a(\mu)$. Suppose $H^p(\mu) \neq zM$. Then $H^p(\mu) = \mathbf{C} + H^p(\mu)_0 = \mathbf{C} + zM$ and $\mathbf{C} \cap zM = \{0\}$. As in the proof of (1), by [4, Part III, Corollary 2.3], zM is closed and hence $zH^p(\mu)$ is not dense in $H^p(\mu)$. Applying Lemma 3 with $X = H^p(\mu), Y = P$ and $a = 0, R(\mu, p, 0) < \infty$ follows. Suppose $H^p(\mu) = zM$. Then $z^{-1} \in L^p(\mu)$ and hence $\mu(\{0\}) = 0$. If $Af = zf$ for $f \in M$, then A is a one-one bounded operator from M onto $H^p(\mu)$. Therefore A is invertible and hence $A(zM) = zH^p(\mu)$ is closed. Since $H^p(\mu) \subset L^p_a(\mu), zH^p(\mu) \neq H^p(\mu)$ and hence by Lemma 3, $R(\mu, p, 0) < \infty$ follows. The same argument implies that $R(\mu, p, a) < \infty$ for all $a \in D$. Now, as in the proof of (1), Lemma 2 implies the 'only if' part of (2).

Corollary 4. Suppose $1 \leq p < \infty$ and $d\mu/dm = W$. If $logW$ is locally integrable on K_0^c for some compact set K_0 in D, then $L^p_a(\mu)$ is closed and $H^p(\mu) \subseteq L^p_a(\mu)$.

Proof. By (1) of Theorem 8, it is sufficient to prove that for any compact set K in $D, \inf_K\ s(\mu, p) > -\infty$. If $logW$ is integrable on K_0^c, then by the proof of Theorem 5 $inf_K\ s(\mu, p) > -\infty$. For a more general W in this corollary, we have to proceed as the following. Suppose $a \in D$ and $0 < \varepsilon < \delta < 1$. As in the proof of Theorem 5,

$$\begin{aligned} & s(\mu, p, a) \\ \geq\ & (1-\mid a\mid^2)^2 \int_\varepsilon^\delta exp\left(\int_0^{2\pi} logW \circ \phi_a d\theta/2\pi\right) 2rdr \\ \geq\ & (1-\mid a\mid^2)^2(\delta^2 - \varepsilon^2) exp\left(\frac{1}{\delta^2 - \varepsilon^2} \int_{\Delta_\delta(0)\setminus\Delta_\varepsilon(0)} logW \circ \phi_a dm\right) \\ \geq\ & (1-\mid a\mid^2)^2(\delta^2 - \varepsilon^2) exp\left(\frac{2^2}{(1-\mid a\mid)^2(\delta^2 - \varepsilon^2)} \int_{\Delta_\delta(a)\setminus\Delta_\varepsilon(a)} log(W \wedge 1) dm\right). \end{aligned}$$

Suppose K is an arbitrary compact set in D. Put $t = max\{\mid z\mid\ ;\ z \in K_0\}$ and $k = max\{\mid z\mid\ ;\ z \in K\}$. The Euclidean center and radius of $\Delta_\gamma(k)$ $(0 < \gamma < 1)$ are

$$C(\gamma) = \frac{1-\gamma^2}{1-\gamma^2k^2}k, \quad R(\gamma)\frac{1-k^2}{1-\gamma^2k^2}\gamma$$

respectively. Put $\ell = R(\delta) + C(\delta)$ and $s = R(\varepsilon) - C(\varepsilon)$. There exist δ and ε such that

$$\Delta_\ell(0)\setminus\Delta_s(0) \subset D\setminus\Delta_t(0).$$

Then for all $a \in K$

$$\Delta_\delta(a)\setminus\Delta_\varepsilon(a) \subset \Delta_\ell(0)\setminus\Delta_s(0).$$

Hence for all $a \in K$

$$\Delta_\delta(a)\setminus\Delta_\varepsilon(a) \subset K_0^c$$

and so for all $a \in K$

$$s(\mu, p, a) \geq (1-\mid a \mid^2)^2(\delta^2 - \varepsilon^2)exp\left(\frac{2^2}{(1-\mid a \mid^2)^2(\delta^2-\varepsilon^2)}\int_{K_0^c} log(W \wedge 1)dm\right).$$

This shows the corollary.

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References

1. P.S.Bourdon and J.H.Shapiro, Spectral synthesis and common cyclic vectors, Michigan Math. J. 37(1990), 71-90.

2. J.E.Brennan, Weighted polynomial approximation, quasianalyticity and analytic continuation, J.für Mathematik. 357(1984), 23-50.

3. J.B.Conway, Subnormal Operators, Research Notes in Mathematecs, Pitman Advanced Publishing Program.

4. I.Gohberg, S.Goldberg and M.A.Kaashoek, Classes Of Linear Operators Vol.I, Birkhäuser

5. U.Grenander and G.Szegő, Toeplitz Forms And Their Applications, Chelsea Publishing Company.

6. P.Koosis, The Logarithmic Integral I, Cambridge University Press.

7. T.Kriete and T.Trent, Growth near the boundary in $H^2(\mu)$ spaces, Proc. Amer. Math. Soc. 62(1977), 83-88.

8. T.Nakazi and M.Yamada, (A_2)-conditions and Carleson inequalities in Bergman spaces, to appear in Pacific J. Math.

9. R.Rochberg, Toeplitz operators on weighted H^p spaces, Indiana Univ. Math.J. 26(1977), 291-298.

10. M.Rosenblum, Summability of Fourier series in $L^p(d\mu)$, Trans. Amer. Math. Soc. 105(1962), 32-42.

11. W.Rudin, Functional Analysis, McGraw-Hill Book Company.

12. T.P.Srinivasan and J.K.Wang, Weak-* Dirichlet algebras, Proc. Internat. Sympos. on Function Algebras (Tulane Univ., 1965), Scott-Foresman, Chicago,III, 1966, 216-249.

13. M.Yamada, Weighted Bergman space and Szegő's infimum, in preprint.

14. K.Zhu, Operator Theory In Function Spaces, Pure and Applied Mathematics, Marcel Dekker, Inc. New York and Basel, 1990.