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Abstract

Let Δ be a simplicial complex of dimension $d-1$ on the vertex set V and write ξ_i for the number of "minimal" subsets $\sigma \subset V$ with $\sharp(\sigma) = i+1$ and $\sigma \notin \Delta$. We discuss what can be said about the combinatorial sequence $(\xi_1, \xi_2, \ldots, \xi_d)$ associated with Δ.

§1. Background and results

(1.1) A simplicial complex Δ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \le i \le v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. Set $d := \max\{\sharp(\sigma); \sigma \in \Delta\}$ and define the dimension of Δ to be $\dim \Delta = d-1$. Here $\sharp(\sigma)$ is the cardinality of a finite set σ.

Let $A = k[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley–Reisner ring* of Δ over k. We refer the reader to, e.g., [Bru–Her], [H], [Hoc] and [Sta] for the detailed information about algebra and combinatorics on Stanley–Reisner rings of simplicial complexes.

Let $\mathcal{A}_\Delta$ denote the set of all subsets σ of V such that (i) $\sigma \notin \Delta$ and (ii) $\sigma - \{x\} \in \Delta$ for each $x \in \sigma$. Then, it follows immediately that the set of square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$ with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \in \mathcal{A}_\Delta$ is a minimal system of generators of the ideal I_Δ. We write $\xi_i = \xi_i(\Delta)$ for the number of $\sigma \in \mathcal{A}_\Delta$ with $\sharp(\sigma) = i+1$. Thus $\xi_i = 0$ if $i > d$. In the present paper, we are interested in the combinatorial sequence $\xi(\Delta) := (\xi_1, \xi_2, \ldots, \xi_d)$ associated with a simplicial complex Δ of dimension $d-1$.

If $d = 2$, then we easily see that there exist simplicial complexes Δ and Δ' with $\xi(\Delta) = (1,2)$ and $\xi(\Delta') = (2,1)$, while there exists no simplicial complex Δ with $\xi(\Delta) = (1,1)$.

(1.2) In what follows, we consider A to be the graded ring $A = \bigoplus_{n\geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n\geq 0}(k[\Delta])_n$ as a graded module over A with the quotient grading. Let $A(j)$, $j \in \mathbf{Z}$, denote the graded module $A(j) = \bigoplus_{n\in\mathbf{Z}}[A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Here $\mathbf{Z}$ is the set of integers.

A graded *finite free resolution* of $k[\Delta]$ over A is an exact sequence

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (1)$$

of graded modules over A, where each $\bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{i_j}}$ is a graded free module of rank $\sum_{j\in\mathbf{Z}} \beta_{i_j}$ $(< \infty)$, and where every φ_i is degree-preserving. The *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the minimal h possible in (1). It is known that $v - d \leq \mathrm{hd}_A(k[\Delta]) \leq v$. The second inequality is Hilbert's syzygy theorem ([Bru–Her, Corollary (2.2.14)]), and the first inequality follows from Auslander-Buchsbaum formula ([Bru–Her, Theorem (1.3.3)]). A finite free resolution (1) is called *minimal* if each β_{i_j} is smallest possible. A minimal free resolution of $k[\Delta]$ over A exists and is essentially unique. See, e.g., [Bru–Her, p. 35].

Suppose that a finite free resolution (1) is minimal with $h = \mathrm{hd}_A(k[\Delta])$. We say that $\beta_{i_j} = \beta_{i_j}^A(k[\Delta])$ is the *i_j-th Betti number* of $k[\Delta]$ over A. Since the kernel of φ_0 is equal to the ideal I_Δ, each $\beta_{1_{j+1}}$ coincides with $\xi_j(\Delta)$. Thus, discussing the sequence $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$ of Δ is our starting point of combinatorial study on the doubly indexed Betti number sequence $\{\beta_{i_j}\}_{(i,j)}$ of $k[\Delta]$ over A. Note that, in general, β_{i_j} may depend on the base field k if $i \geq 3$.

We are now in the position to give the main result in this paper. We say that a simplicial complex Δ of dimension $d-1$ on the vertex set V with $\sharp(V) = v$ is *Cohen-Macaulay* over a field k if $\mathrm{hd}_A(k[\Delta]) = v - d$.

(1.3) THEOREM. *Fix an integer $d \geq 2$ and suppose that a sequence $(\eta_1, \eta_2, \ldots, \eta_d) \in \mathbf{Z}^d$ with each $\eta_i \geq 0$ is given.*

(a) *There exists a Cohen-Macaulay complex Δ of dimension $d-1$ with $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$ such that $\xi_i = \eta_i$ for each $2 \leq i \leq d$ and $\xi_1 \geq \eta_1$.*

(b) *Set $q := \max(\{i; \eta_i \neq 0\} \cup \{0\})$. Then, there exists a Cohen-Macaulay complex Δ of dimension $d-1$ with $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$ such that $\xi_i = \eta_i$ for each $i \neq q$ and (if $q > 0$ then) $\xi_q \geq \eta_q$.*

In Section 2, after summarizing some fundamental results on Cohen–Macaulay complexes, our proof of Theorem (1.3) will be given. We conclude this paper with making a collection of examples in Section 3.

§2. Proof of Theorem (1.3)

First, we recall some basic results on Cohen–Macaulay complexes. Again, we refer the reader to, e.g., [Bru–Her], [H], [Hoc] and [Sta] for further information about Cohen–Macaulay complexes.

(2.1) Let $\tilde{H}_i(\Delta; k)$ denote the i-th reduced simplicial homology group of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

If σ belongs to Δ, then we define the subcomplexes $\mathrm{link}_\Delta(\sigma)$ to be

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta; \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}.$$

Reisner's criterion [Rei] guarantees that a simplicial complex Δ is Cohen–Macaulay over k if and only if $\tilde{H}_i(\mathrm{link}_\Delta(\sigma); k) = 0$ for each $\sigma \in \Delta$ (possibly $\sigma = \emptyset$) and for every $i \neq \dim \mathrm{link}_\Delta(\sigma)$.

The following Lemma (2.2) follows from a simple observation (e.g., [H, p. 57 – 58]) based on the reduced Mayer–Vietoris exact sequence together with Reisner's criterion.

(2.2) LEMMA. *Let Δ be a simplicial complex of dimension $d-1$ on the vertex set V, and let Δ', Δ'' be subcomplexes of Δ. Suppose that Δ' and Δ'' are Cohen–Macaulay complexes of dimension $d-1$ and that $\Delta' \cap \Delta''$ consists of all subset of a face σ of Δ with $\sharp(\sigma) = d$. Then $\Delta' \cup \Delta''$ is Cohen–Macaulay.*

Let Δ' (resp. Δ'') be a simplicial complex on the vertex set V' (resp. V'') with $V' \cap V'' = \emptyset$. Then, the *simplicial join* $\Delta' * \Delta''$ of Δ' and Δ'' is the simplicial complex $\{\sigma' \cup \sigma''; \sigma' \in \Delta', \sigma'' \in \Delta''\}$ on the vertex set $V' \cup V''$. Note that $\dim \Delta' * \Delta'' = \dim \Delta' + \dim \Delta'' + 1$

(2.3) LEMMA. *If Δ' and Δ'' are Cohen–Macaulay, then $\Delta' * \Delta''$ is Cohen–Macaulay.*

The *i-skeleton* of a simplicial complex Δ is defined to be the subcomplex $\Delta^{(i)} := \{\sigma \in \Delta; \sharp(\sigma) \leq i+1\}$ of Δ, where $0 \leq i \in \mathbf{Z}$.

(2.4) LEMMA (e.g., [Bru–Her, Exercise (5.1.23)]). *Every skeleton of a Cohen–Macaulay complex is Cohen–Macaulay.*

(2.5) Let Δ' and Δ'' be simplicial complexes of dimension $d-1$ with $\xi(\Delta') = (\xi'_1, \xi'_2, \ldots, \xi'_d)$ and $\xi(\Delta'') = (\xi''_1, \xi''_2, \ldots, \xi''_d)$. We choose any face σ of Δ' and any face τ of Δ'' with $\sharp(\sigma) = \sharp(\tau) = d$. Identifying σ with τ yields a new complex Δ of dimension $d-1$. If $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$, then $\xi_i = \xi'_i + \xi''_i$ for every $2 \leq i \leq d$ and $\xi_1 \geq \xi'_1 + \xi''_1$. Moreover, thanks to Lemma (2.2), if both Δ' and Δ'' are Cohen–Macaulay, then Δ is Cohen–Macaulay.

(2.6) Let Δ' and Δ'' be simplicial complexes with $\xi(\Delta') = (\xi'_1, \xi'_2, \ldots)$ and $\xi(\Delta'') = (\xi''_1, \xi''_2, \ldots)$. Then $\xi_i(\Delta' * \Delta'') = \xi_i(\Delta') + \xi_i(\Delta'')$ for every $i \geq 1$.

Let 2^{j+1} denote the set which consists of all subsets of a $(j+1)$-element set $\{x_1, x_2, \ldots, x_{j+1}\}$ and define $\Gamma_j(1)$ to be the simplicial complex $2^{j+1} - \{x_1, x_2, \ldots, x_{j+1}\}$ of dimension $j-1$. Moreover, for each integer $n \geq 1$, we define $\Gamma_j(n+1)$ to be the simplicial join $\Gamma_j(n) * \Gamma_j(1)$ of $\Gamma_j(n)$ and $\Gamma_j(1)$. Then, by Lemma (2.3), every $\Gamma_j(n)$ is Cohen–Macaulay. If $\xi(\Gamma_j(n)) = (\xi_1, \xi_2, \ldots)$, then $\xi_i = 0$ for each $i \neq j$ and $\xi_j = n$. Also, set $\Gamma_j(0) := \emptyset$.

(2.7) Let Δ be a simplicial complex of dimension $d-1$ with $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$ and $\Delta^{(i)}$ the i-skeleton of Δ. If $\xi(\Delta^{(i)}) = (\lambda_1, \lambda_2, \ldots, \lambda_{i+1})$, then $\lambda_j = \xi_j$ for every $1 \leq j \leq i$ and $\lambda_{i+1} = \xi_{i+1} + f_{i+1}$, where f_{i+1} is the number of faces σ of Δ with $\sharp(\sigma) = i+2$.

Proof of Theorem (1.3). (a) First, note that $\xi(2^d) = (0, 0, \ldots, 0)$, where 2^d is a Cohen–Macaulay complex of dimension $d-1$ which consists of all subsets of a d-element set. We now show that if there exists a Cohen–Macaulay complex Δ of dimension $d-1$ with $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$ and if $2 \leq i \leq d$, then there exists a Cohen–Macaulay complex Δ' of dimension $d-1$ with $\xi(\Delta') = (\xi'_1, \xi'_2, \ldots, \xi'_d)$ such that $\xi'_j = \xi_j$ for each $2 \leq j \neq i$ and $\xi'_i = \xi_i + 1$. For this, choose any face σ of Δ with $\sharp(\sigma) = d$ and any face τ of the simplicial join $\Gamma_i(1) * 2^{d-i}$ with $\sharp(\tau) = d$ (in the notation of (2.6)). Then, thanks to (2.5), identifying σ with τ yields a required complex Δ' of dimension $d-1$. On the other hand, if Δ is a Cohen–Macaulay complex of dimension $d-1$ with $\xi(\Delta) = (\xi_1, \xi_2, \ldots, \xi_d)$, then the similar technique enables us to show that there exists a Cohen–Macaulay complex Δ' of dimension $d-1$ with $\xi(\Delta') = (\xi'_1, \xi'_2, \ldots, \xi'_d)$ such that $\xi'_i = \xi_i$ for each $i \neq 1$ and $\xi'_1 > \xi_1$.

(b) Fix an integer $d \geq 2$ and suppose that a sequence $(\eta_1, \eta_2, \ldots, \eta_d) \in \mathbf{Z}^d$ with each $\eta_i \geq 0$ is given. We set $q := \max(\{i; \eta_i \neq 0\} \bigcup \{0\})$. Let Γ denote the simplicial join $\Gamma_1(\eta_1) * \Gamma_2(\eta_2) * \cdots * \Gamma_d(\eta_d)$. Then, by Lemma (2.4) together with (2.7), the $(q-1)$-skeleton $\Gamma^{(q-1)}$ is a Cohen–Macaulay complex with $\xi(\Gamma^{(q-1)}) = (\eta_1, \eta_2, \ldots, \eta_{q-1}, \eta_q + f_q)$, where f_q is the number of faces σ of Γ with $\sharp(\sigma) = q + 1$. Thus, the simplicial join $\Gamma^{(q-1)} * 2^{d-q}$ is a desired Cohen–Macaulay complex. Q. E. D.

§3. Some examples

(3.1) Let $d = 3$ and $(\xi_1, \xi_2, \xi_3) = (1, 3, 0)$. Then, there exists a simplicial complex Δ of dimension two with $\xi(\Delta) = (1, 3, 0)$. In fact, if $V = \{a, b, x, y, z\}$ and $I_\Delta = (ab, axy, bxy, xyz)$, then Δ is of dimension two whose maximal faces are $\{a, x, z\}, \{a, y, z\}, \{b, x, z\}$ and $\{b, y, z\}$. On the other hand, we show that there exists no Cohen–Macaulay complex Δ of dimension two with $\xi(\Delta) = (1, 3, 0)$. Suppose that Δ is a Cohen–Macaulay complex of dimension two with $\xi(\Delta) = (1, 3, 0)$ on the vertex set V with $\sharp(V) = v$. Let f_i denote the number of faces σ of Δ with $\sharp(\sigma) = i + 1$. Then, $f_0 = v, f_1 = v(v-1)/2 - 1$ and $f_2 = v(v-1)(v-2)/6 - (v-2) - 3$. Since Δ is Cohen–Macaulay of even dimension, the reduced Euler characteristic $\tilde{\chi}(\Delta) = f_2 - f_1 + f_0 - 1$ is non-negative. Hence, we have $v \geq 6$. Let M_1, M_2, M_3, M_4 be square-free monomials with $\deg M_1 = 2$ and $\deg M_i = 3$ for $2 \leq i \leq 4$ such that $I_\Delta = (M_1, M_2, M_3, M_4)$. Then, there exists $x \in V$ such that $x \mid M_i$ (i.e., x divides M_i) and $x \mid M_j$ for some $1 \leq i, j \leq 4$ with $i \neq j$. Say, $i = 1$ and $j = 2$. If $y \mid M_3$ and $z \mid M_4$ with $y, z \in V$, then $V - \{x, y, z\}$ belongs to Δ. Hence $\dim \Delta \geq v - 4$. Thus $v = 6$. Now, let $V = \{a, b, x_1, x_2, x_3, x_4\}$ and $M_1 = ab$. Then, either a or b divides M_i for some $2 \leq i \leq 4$. Say, $a \mid M_2$. If $a \mid M_3$ and $x_j \mid M_4$, then $V - \{a, x_j\} \in \Delta$ and $\dim \Delta \geq 3$, a contradiction. Thus, $a \in V$ divides neither M_3 nor M_4. Similarly, $b \in V$ does not divides both M_3 and M_4. Hence, there exists $1 \leq j \leq 4$ with $x_j \mid M_3$ and $x_j \mid M_4$. Then, again $V - \{a, x_j\}$ belongs to Δ and we have $\dim \Delta \geq 3$.

(3.2) Let $d = 2$ and Δ a simplicial complex of dimension one with $\xi(\Delta) = (\xi_1, \xi_2) = (1, n)$. Then Δ is equal to a complete graph minus one edge. Hence, we have $n = (v+2)(v-2)(v-3)/6$ for some $3 \leq v \in \mathbf{Z}$.

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