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# Computation of Betti numbers of monomial ideals associated with stacked polytopes

Naoki Terai Takayuki Hibi

**Abstract**

Let $P(v,d)$ be a stacked $d$-polytope with $v$ vertices, $\Delta(P(v,d))$ the boundary complex of $P(v,d)$, and $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ the Stanley–Reisner ring of $\Delta(P(v,d))$ over a field $k$. We compute the Betti numbers which appear in a minimal free resolution of $k[\Delta(P(v,d))]$ over $A$, and show that every Betti number depends only on $v$ and $d$ and is independent of the base field $k$.

## Introduction

Let $\Delta$ be a simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$. Thus $\Delta$ is a collection of subsets of $V$ such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element $\sigma$ of $\Delta$ is called a *face* of $\Delta$. Set $d = \max\{\sharp(\sigma); \sigma \in \Delta\}$ and define the dimension of $\Delta$ to be $\dim \Delta = d - 1$. Here $\sharp(\sigma)$ is the cardinality of a finite set $\sigma$. A maximal face of $\Delta$ is also called a *facet* of $\Delta$.

Let $A = k[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in $v$-variables over a field $k$ with the standard grading, i.e., each $\deg x_i = 1$. We identify each $x_i \in V$ with the indeterminate $x_i$ of $A$. Define $I_\Delta$ to be the ideal of $A$ which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. The quotient algebra $k[\Delta] := A/I_\Delta$ is called the *Stanley–Reisner ring* of $\Delta$ over $k$. We may regard $k[\Delta] = \bigoplus_{n \geq 0}(k[\Delta])_n$ as a graded module over $A$ with the quotient grading. We refer the reader to [Bru–Her], [H], [Hoc], [Sta] for the detailed information about Stanley–Reisner rings.

We are interested in a minimal free resolution of $k[\Delta]$ over $A$. Let $A(j)$, $j \in \mathbf{Z}$, denote the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}}[A(j)]_n$ over $A$ with $[A(j)]_n := A_{n+j}$. Here $\mathbf{Z}$ is the set of integers.

A graded *finite free resolution* of $k[\Delta]$ over $A$ is an exact sequence

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (1)$$

of graded modules over $A$, where each $\bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{i_j}}$ is a graded free module of rank $\sum_{j \in \mathbf{Z}} \beta_{i_j}$ ($< \infty$), and where every $\varphi_i$ is degree-preserving. The *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over $A$ is the minimal $h$ possible in (1). It is known (see, e.g., [Bru–Her, Theorem (1.3.3), Corollary (2.2.14)]) that $v - d \leq \mathrm{hd}_A(k[\Delta]) \leq v$. A finite free resolution (1) is called *minimal* if each $\beta_{i_j}$ is smallest possible. A minimal free resolution of $k[\Delta]$ over $A$ exists and is essentially unique. See, e.g., [Bru–Her, p. 35]. When a finite free resolution (1) is minimal with $h = \mathrm{hd}_A(k[\Delta])$, we say that $\beta_{i_j} = \beta_{i_j}^A(k[\Delta])$ is the *$i_j$-th Betti number* of $k[\Delta]$ over $A$. Let $\beta_i = \beta_i^A(k[\Delta])$ denote the rank of the $i$-th free module which appears in a minimal free resolution of $k[\Delta]$ over $A$; viz, $\beta_i = \sum_{j \in \mathbf{Z}} \beta_{i_j}$.

In the paper [T–H], we give a combinatorial formula to compute the Betti numbers of the Stanley–Reisner ring of the boundary complex of the cyclic $d$-polytope $C(v, d)$ with $v$ vertices and show that these Betti numbers do not depend on the base field. The purpose of the present paper is to study the same problem for a stacked $d$-polytope $P(v, d)$ with $v$ vertices. In the combinatorial theory of convex polytopes, the cyclic polytope appears in the upper bound theorem and the stacked polytope appears in the lower bound theorem. See, e.g., [Bay–Lee] for background information about cyclic polytopes and stacked polytopes.

## §1. Stacked polytopes and Betti numbers

We refer the reader to, e.g., [Brø] for foundations on convex polytopes. Starting with a $d$-simplex, one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex $d$-polytope with $v$ vertices, called a *stacked polytope*. Recall that the boundary complex $\Delta(P)$ of a simplicial $d$-polytope $P \subset \mathbf{R}^N$ with the vertex set $V$ is the simplicial complex on $V$ of dimension $d - 1$ whose faces are those subsets $\sigma$ of $V$ such that the convex hull of $\sigma$ in $\mathbf{R}^N$ is a face of $P$.

Our main result in this paper is to present a combinatorial formula for the computation of the Betti numbers of the Stanley–Reisner ring associated with the boundary complex of a stacked $d$-polytope $P(v, d)$ with $v$ vertices.

(1.1) THEOREM. *Fix $v > d \geq 3$. Let $P(v, d)$ be a stacked $d$-polytope with $v$ vertices and $\Delta(P(v, d))$ its boundary complex. Then, a minimal free* *resolution of the Stanley–Reisner ring* $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ *over* $A$ *is of the form*

$$\begin{aligned}
0 &\longrightarrow A(-v) \longrightarrow A(-v+d)^{b_{v-d-1}} \bigoplus A(-v+2)^{b_1} \\
&\longrightarrow A(-v+d+1)^{b_{v-d-2}} \bigoplus A(-v+3)^{b_2} \longrightarrow \cdots \\
&\longrightarrow A(-3)^{b_2} \bigoplus A(-d-1)^{b_{v-d-2}} \\
&\longrightarrow A(-2)^{b_1} \bigoplus A(-d)^{b_{v-d-1}} \longrightarrow A \longrightarrow k[\Delta(P(v,d))] \longrightarrow 0,
\end{aligned}$$

*where*

$$b_i = (v-d-1)\binom{v-d-1}{i} - \sum_{j=i}^{v-d-2} \binom{j}{i}$$

*for each* $1 \leq i \leq v-d-1$.

When $d = 2$, $\Delta(P(v,2))$ is the cycle $C_v$ with $v$ vertices. A minimal free resolution of $k[C_v] = A/I_{C_v}$ over $A$ is a *pure resolution*, which is discussed in, e.g., [B–H$_1$] and [B–H$_2$].

(1.2) COROLLARY. *Every Betti number of* $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ *over* $A$ *is independent of the base field* $k$ *and of the combinatorial type of* $P(v,d)$.

## §2. Proof of Theorem (1.1)

**(2.1)** Given a subset $W$ of the vertex set $V$ of a simplicial complex $\Delta$, the subcomplex $\Delta_W$ is defined to be $\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. Let $\tilde{H}_i(\Delta; k)$ denote the $i$-th reduced simplicial homology group of $\Delta$ with the coefficient field $k$. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Suppose that a graded finite free resolution (1) of $k[\Delta]$ over $A$ is minimal with $h = \mathrm{hd}_A(k[\Delta])$. Then, Hochster's formula [Hoc, Theorem (5.1)] guarantees that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W) = j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \qquad (2)$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k).$$

(**2.2**) Let $P = P(v, d)$ be a stacked $d$-polytope with the vertex set $V$, $\sharp(V) = v$, $\Delta = \Delta(P)$ the boundary complex of $P$, and $\mathcal{F}$ a facet of $P$ with the vertex set $X$. Let $P'$ denote a stacked $d$-polytope with $(v+1)$-vertices which is obtained by building a shallow pyramid over $\mathcal{F}$ with a new vertex $\alpha$, and $\Delta'$ the boundary complex of $P'$. Let $V' = V \bigcup \{\alpha\}$ be the vertex set of $\Delta'$ and $W$ a subset of $V'$. We fix a base field $k$.

(2.2.1) LEMMA (a) *If* $\alpha \notin W$ *and* $X \not\subset W$, *then*

$$\Delta'_W = \Delta_W.$$

(b) *If* $\alpha \notin W$, $W \neq V$ *and* $X \subset W$, *then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_W; k) & (i \neq d-2) \\ \dim_k \tilde{H}_i(\Delta_W; k) + 1 & (i = d-2). \end{cases}$$

(c) *If* $\alpha \in W$ *and* $X \bigcap W \neq \emptyset$, *then, for each* $i$, *we have*

$$\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W-\{\alpha\}}; k).$$

(d) *If* $\alpha \in W$, $W \neq \{\alpha\}$ *and* $X \bigcap W = \emptyset$, *then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k) & (i \neq 0) \\ \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k) + 1 & (i = 0). \end{cases}$$

*Proof.* (a) In general, $\Delta' = (\Delta - \{X\}) \bigcup \{\sigma \subset V' \mid \sigma \subset X \bigcup \{\alpha\}, \sigma \neq X\}$. Hence, we have $\Delta'_W = \Delta_W$ if $\alpha \notin W$ and $X \not\subset W$.

(b) Let $\Gamma$ denote the set of all subsets of $X$ and set $\partial\Gamma = \Gamma - \{X\}$. Then $\Delta'_W \bigcup \Gamma = \Delta_W$ and $\Delta'_W \bigcap \Gamma = \partial\Gamma$. Since $\Gamma$ is a simplicial $(d-1)$-ball, $\partial\Gamma$ is a simplicial $(d-2)$-sphere and $\tilde{H}_{d-1}(\Delta_W; k) = 0$, the required equalities follow from the reduced Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_i(\partial\Gamma; k) &\longrightarrow \tilde{H}_i(\Gamma; k) \bigoplus \tilde{H}_i(\Delta'_W; k) \longrightarrow \tilde{H}_i(\Delta_W; k) \\ \longrightarrow \tilde{H}_{i-1}(\partial\Gamma; k) &\longrightarrow \tilde{H}_{i-1}(\Gamma; k) \bigoplus \tilde{H}_{i-1}(\Delta'_W; k) \longrightarrow \tilde{H}_{i-1}(\Delta_W; k) \\ \longrightarrow \cdots & \end{aligned}$$

(c) If $X \subset W$, then the geometric realization of $\Delta'_W$ is homeomorphic to that of $\Delta_{W-\{\alpha\}}$. Thus $\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$ for each $i$. On the other hand, if $X \bigcap W \neq X$, then $\Delta_{W-\{\alpha\}} \bigcup \Delta'_{W\cap(\{\alpha\}\cup X)} = \Delta'_W$ and $\Delta_{W-\{\alpha\}} \bigcap \Delta'_{W\cap(\{\alpha\}\cup X)} = \Delta_{W\cap X}$. Since both $\Delta'_{W\cap(\{\alpha\}\cup X)}$ and $\Delta_{W\cap X}$ are contractible, again the reduced Mayer–Vietoris exact sequence guarantees the desired equalities.

(d) Since $\Delta'_W$ is the disjoint union of $\Delta_{W-\{\alpha\}}$ and one point $\{\alpha\}$, we immediately have the required equalities. Q. E. D.

(2.2.2) COROLLARY. *Let $\Delta = \Delta(P)$ denote the boundary complex of a stacked d-polytope $P = P(v,d)$ with the vertex set $V$, $\sharp(V) = v$. Then, for every non-empty subset $W$ of $V$ with $W \neq V$ and for each $i \neq 0, d-2$, we have*

$$\tilde{H}_i(\Delta_W; k) = 0.$$

*Proof.* If $v = d+1$, i.e., $P$ is a $d$-simplex, then $\Delta_W$ is contractible. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ for each $i$. We now work with the same situation as in the above Lemma (2.2.1) and suppose that $\tilde{H}_i(\Delta_W; k) = 0$ for every non-empty subset $W$ of $V$ with $W \neq V$ and for each $i \neq 0, d-2$. Let $W$ be a non-empty subset of $V'$ with $W \neq V'$. If $W = V' - \{\alpha\}$, then $\Delta'_W$ is a simplicial $(d-1)$-ball. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ for each $i$. Moreover, if $W = \{\alpha\}$, then $\tilde{H}_i(\Delta_W; k) = 0$ for each $i$. On the other hand, if $W$ is a non-empty subset of $V'$ with $W \neq V'$ such that $W \neq V$ and $W \neq \{\alpha\}$, and if $i \neq 0, d-2$, then $\dim_k \tilde{H}_i(\Delta'_W; k) = \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$ by Lemma (2.2.1). Hence, $\tilde{H}_i(\Delta_W; k) = 0$ as desired. Q. E. D.

**(2.3)** Fix $d \geq 3$, and keep the notation $P$, $P'$, $\Delta$ and $\Delta'$ in (2.2). Let $\beta_{i_j}$ be the $i_j$-th Betti number of $k[\Delta]$ and $\beta'_{i_j}$ the $i_j$-th Betti number of $k[\Delta']$.

(2.3.1) LEMMA. *For each $i \geq 1$ we have*

$$\beta'_{i_{i+1}} = \beta_{i_{i+1}} + \beta_{i-1_i} + \binom{v-d}{i}.$$

*Proof.* By virtue of Eq. (2) as well as Lemma (2.2.1), we have

$$\begin{aligned}
\beta'_{i_{i+1}} &= \sum_{W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\
&= \sum_{\alpha \notin W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) + \sum_{\alpha \in W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\
&= \sum_{W \subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W; k) + \sum_{W \subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W; k) + \binom{v-d}{i} \\
&= \beta_{i_{i+1}} + \beta_{i-1_i} + \binom{v-d}{i}
\end{aligned}$$

as desired. Q. E. D.

(2.3.2) COROLLARY. *Let $\Delta = \Delta(P)$ denote the boundary complex of a stacked d-polytope $P = P(v,d)$, $d \geq 3$, with $v$ vertices. Then, for each $1 \leq i \leq v-d-1$, the $i_{i+1}$-th Betti number of $k[\Delta] = A/I_\Delta$ over $A$ is*

$$\beta^A_{i_{i+1}}(k[\Delta]) = (v-d-1)\binom{v-d-1}{i} - \sum_{j=i}^{v-d-2}\binom{j}{i}.$$

*Proof.* In this proof, we set $\binom{a}{0} = 0$ for every integer $a \geq 0$. Thanks to Lemma (2.3.1), we have

$$\begin{aligned}
\beta^A_{i_{i+1}}(k[\Delta]) &= (v-d-2)\binom{v-d-2}{i} - \sum_{j=i}^{v-d-3}\binom{j}{i} \\
&\quad +(v-d-2)\binom{v-d-2}{i-1} - \sum_{j=i-1}^{v-d-3}\binom{j}{i-1} + \binom{v-d-1}{i} \\
&= (v-d-2)(\binom{v-d-2}{i} + \binom{v-d-2}{i-1}) + \binom{v-d-1}{i} \\
&\quad -(\binom{i-1}{i-1} + \sum_{j=i}^{v-d-3}(\binom{j}{i} + \binom{j}{i-1})) \\
&= (v-d-2)\binom{v-d-1}{i} + \binom{v-d-1}{i} \\
&\quad -(\binom{i}{i} + \sum_{j=i}^{v-d-3}\binom{j+1}{i}) \\
&= (v-d-1)\binom{v-d-1}{i} - \sum_{j=i}^{v-d-2}\binom{j}{i}
\end{aligned}$$

as required. Q. E. D.

(2.4) We are now in the position to give a proof of Theorem (1.1). Since $\Delta = \Delta(P(v,d))$ is a simplicial $(d-1)$-sphere with $v$ vertices, we know that the homological dimension of $k[\Delta] = A/I_\Delta$ over $A$ is $\text{hd}_A(k[\Delta]) = v-d$ and that $\beta^A_{i_j}(k[\Delta]) = \beta^A_{v-d-i_{v-j}}(k[\Delta])$ for every $i$ and $j$. By Corollary (2.2.2), we have $\beta^A_{i_j}(k[\Delta]) = 0$ for each $1 \leq i \leq v-d-1$ and for each $j \neq i+1, i+d-1$. On the other hand, Corollary (2.3.2) enables us to compute $b_i = \beta^A_{i_{i+1}}(k[\Delta]) = \beta^A_{v-d-i_{v-i-1}}(k[\Delta])$ for each $1 \leq i \leq v-d-1$. Hence, we obtain a desired minimal free resolution of $k[\Delta]$ over $A$.

## §3. Unimodality of Betti number sequences

Let $\beta_i$, $0 \leq i \leq v-d$, denote the rank of the $i$-th free module which appears in a minimal free resolution of $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ over $A$. Then, $\beta_0 = \beta_{v-d} = 1$ and $\beta_i = b_i + b_{v-d-i}$ for each $1 \leq i \leq v-d-1$ with the notation of Theorem (1.1). The sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ is called the *Betti number sequence* of $k[\Delta(P(v,d))]$ over $A$. This sequence is *symmetric*, i.e., $\beta_i = \beta_{v-d-i}$ for every $0 \leq i \leq v-d$. We now show that the symmetric sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ is *unimodal*, i.e., $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{[(v-d)/2]}$.

The following Lemma (3.1) follows from a simple combinatorial argument based on Lemma (2.3.1).

(3.1) LEMMA. (a) *If $v-d$ is even, then*

$$b_{\frac{v-d}{2}} \geq b_{\frac{v-d}{2}-1} \geq b_{\frac{v-d}{2}+1} \geq \cdots \geq b_1 \geq b_{v-d-1}.$$

(b) *If $v-d$ is odd, then*

$$b_{\frac{v-d-1}{2}} \geq b_{\frac{v-d-1}{2}+1} \geq b_{\frac{v-d-1}{2}-1} \geq \cdots \geq b_1 \geq b_{v-d-1}.$$

(3.2) COROLLARY. *Fix $v > d \geq 3$. Let $P = P(v,d)$ a stacked $d$-polytope with $v$ vertices and $\Delta = \Delta(P)$ its boundary complex. Then, the Betti number sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ of $k[\Delta] = A/I_\Delta$ over $A$ is unimodal.*

*Proof.* By Lemma (3.1), we have

$$1 \leq b_1 \leq b_2 \leq \cdots \leq b_{[\frac{v-d}{2}]} \geq b_{[\frac{v-d}{2}]+1} \geq \cdots \geq b_{v-d-1} \geq 1.$$

Hence, the Betti number sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ of $k[\Delta] = A/I_\Delta$ over $A$ is unimodal. Q. E. D.

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