

**Stanley-Reisner rings
whose Betti numbers are
independent of the base field**

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Series #276. January 1995

HOKKAIDO UNIVERSITY
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Stanley–Reisner rings whose Betti numbers are independent of the base field

Naoki Terai Takayuki Hibi

Abstract

We study the Betti numbers which appear in a minimal free resolution of the Stanley–Reisner ring $k[\Delta] = A/I_\Delta$ of a simplicial complex Δ over a field k . It is known that the second Betti number of $k[\Delta]$ is independent of the base field k . We show that, when the ideal I_Δ is generated by square-free monomials of degree two, the third and fourth Betti numbers are also independent of k . On the other hand, we prove that, if the geometric realization of Δ is homeomorphic to either the 3-sphere or the 3-ball, then all the Betti numbers of $k[\Delta]$ are independent of the base field k .

Introduction

Let $A = k[x_1, x_2, \dots, x_v]$ denote the polynomial ring in v -variables over a field k , which will be considered to be the graded algebra $A = \bigoplus_{n \geq 0} A_n$ over k with the standard grading, i.e., each $\deg x_i = 1$. Let \mathbf{Z} (resp. \mathbf{Q} , \mathbf{N}) denote the set of integers (resp. rational numbers, non-negative integers). We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Let I be an ideal of A generated by homogeneous polynomials and R the quotient algebra A/I . When R is regarded as a graded module over A with the quotient grading, it has a graded *finite free resolution*

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{hj}} \xrightarrow{\varphi_h} \dots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} R \longrightarrow 0; \quad (1)$$

where each $\bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{ij}}$, $1 \leq i \leq h$, is a graded free module of rank $0 \neq \sum_{j \in \mathbf{Z}} \beta_{ij} < \infty$, and where every φ_i is degree-preserving. Moreover, there exists a unique such resolution which minimizes each $\beta_{i,j}$; such a resolution is called *minimal*. If a finite free resolution (1) is minimal, then the *homological*

dimension $\text{hd}_A(R)$ of R over A is the non-negative integer h and $\beta_i = \beta_i^A(R) := \sum_{j \in \mathbf{Z}} \beta_{i,j}$ is called the i -th *Betti number* of R over A .

In this paper, we study the Betti numbers of $R = A/I$ over A when an ideal I is generated by square-free monomials, i.e., R is the Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ of a simplicial complex Δ ([Sta₁], [Rei]). The first Betti numbers of $k[\Delta]$ is equal to the number of minimal generators of the ideal I_Δ , i.e., the number of minimal non-faces of Δ . Hence, the first Betti number of $k[\Delta]$ is independent of the base field k . See [T-H₅] for numerical study on the number of minimal non-faces of a simplicial complex. On the other hand, Bruns and Herzog ([Bru-Her₂]) proved that, by a ring-theoretical technique, the second Betti number of $k[\Delta]$ is independent of the base field k ; while, based on Alexander duality theorem of topology, a short proof of their result is obtained in [T-H₁].

This paper is organized as follows. In Section 1, we recall some notation on simplicial complex and Hochster's topological formula on Betti numbers of Stanley-Reisner rings. In Section 2, we show that both the third and fourth Betti numbers of $k[\Delta]$ are independent of the base field k , when the ideal I_Δ is generated by square-free monomials of degree two (e.g., Δ is the order complex of a finite partially ordered set). On the other hand, even though it is possible to define the Stanley-Reisner ring $\mathbf{Z}[\Delta] = A/I_\Delta$ of Δ over the commutative ring \mathbf{Z} , a minimal free resolution of $\mathbf{Z}[\Delta]$ over the polynomial ring $A = \mathbf{Z}[x_1, x_2, \dots, x_v]$ does not necessarily exist. Moreover, there exists a minimal free resolution of $\mathbf{Z}[\Delta]$ over A if and only if all the Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k . Thus, it might be of interest to find a natural class of simplicial complexes Δ for which all the Betti numbers $\beta_i^A(k[\Delta])$ are independent of k . In Section 3, we prove that all the Betti numbers of $k[\Delta]$ are independent of the base field k if the geometric realization of Δ is either the 3-sphere or the 3-ball. See also [T-H₂] and [T-H₃].

§1. Simplicial complexes and Stanley-Reisner rings

We first recall some notation on simplicial complexes and Hochster's topological formula on Betti numbers of Stanley-Reisner rings. We refer the reader to, e.g., [Bru-Her₁], [H₁], [Hoc] and [Sta₁] for the detailed information about combinatorial and algebraic background.

(1.1) A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \dots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element σ of Δ is called a *face* of Δ . Let

$\#\sigma$ denote the cardinality of a finite set σ . We set $d = \max\{\#\sigma \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$.

Given a subset W of V , the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ . In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. On the other hand, if σ is a face of Δ , then we define the subcomplexes $\text{link}_\Delta(\sigma)$ and $\text{star}_\Delta(\sigma)$ to be

$$\text{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\};$$

$$\text{star}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta\}.$$

Thus, in particular, $\text{link}_\Delta(\emptyset) = \text{star}_\Delta(\emptyset) = \Delta$.

Let $\tilde{H}_i(\Delta; k)$ denote the i -th *reduced simplicial homology group* of Δ with the coefficient field k . Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

(1.2) Let $A = k[x_1, x_2, \dots, x_v]$ be the polynomial ring in v -variables over a field k . Here, we identify each $x_i \in V$ with the indeterminate x_i of A . Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k . In what follows, we consider A to be the graded algebra $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ as a graded module over A with the quotient grading.

(1.3) Let $h = \text{hd}_A(k[\Delta])$ denote the homological dimension of $k[\Delta]$ over A and consider a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_j} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{1j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (2)$$

of $k[\Delta]$ over A . It is known that $v - d \leq h \leq v$. Hochster's formula [Hoc, Theorem (5.1)] guarantees that

$$\beta_{ij} = \sum_{W \subset V, \#\sigma(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \quad (3)$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\#\sigma(W)-i-1}(\Delta_W; k).$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. Munkres [Mun] proved that $v - \text{hd}_A(k[\Delta])$ depends only on the geometric realization of Δ . Moreover, if Δ is the order complex of a modular lattice, then the last Betti number of $k[\Delta]$ can be computed by means of the Möbius function of the lattice ([H₂], [H₃]). See also [Bac], [B-H₁], [B-H₂], [Frö], [H₄] and [T-H₄] for related topics and results.

§2. Ideals I_Δ generated by monomials of degree two

The purpose of this section is to show that the third and fourth Betti numbers of a Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ are independent of the base field k when the ideal I_Δ is generated by square-free monomials of degree two. For example, the ideal I_Δ associated with a simplicial complex Δ is generated by square-free monomials of degree two when, e.g., Δ is the order complex ([Sta₂, p.120]) of a finite partially ordered set.

Let Δ (resp. Δ') be a simplicial complex on the vertex set V (resp. V') and suppose that $V \cap V' = \emptyset$. Recall that the *simplicial join* $\Delta * \Delta'$ of Δ and Δ' is the simplicial complex on the vertex set $V \cup V'$ which consists of all subsets of $V \cup V'$ of the form $\sigma \cup \tau$ with $\sigma \in \Delta$ and $\tau \in \Delta'$.

(2.1) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = v$ and suppose that the ideal I_Δ is generated by square-free monomials of degree two. Then $\tilde{H}_n(\Delta; k) = 0$ if $v < 2(n+1)$. Moreover, if $v = 2(n+1)$, then $\tilde{H}_n(\Delta; k) \neq 0$ if and only if Δ is the simplicial join of $n+1$ copies of the 0-sphere $S^0 (= \bullet \bullet)$.*

Proof. We first show that $\tilde{H}_n(\Delta; k) = 0$ if $v < 2(n+1)$. Suppose that $I_\Delta \neq (0)$ and $x, y \in V$ with $xy \in I_\Delta$. We set $\Delta_1 = \text{star}_\Delta(\{x\})$ and $\Delta_2 = \Delta_{V-\{x\}}$. Then $\Delta_1 \cup \Delta_2 = \Delta$ and $\Delta_1 \cap \Delta_2 = \text{link}_\Delta(\{x\})$. Note that the ideals $I_{\Delta_1}, I_{\Delta_2}, I_{\Delta_1 \cap \Delta_2}$ are generated by square-free monomials of degree two. On the other hand, since $\{y\} \notin \Delta_1, \{x\} \notin \Delta_2$ and $\{x\}, \{y\} \notin \Delta_1 \cap \Delta_2$, we may assume that $\tilde{H}_n(\Delta_1; k) = 0, \tilde{H}_n(\Delta_2; k) = 0$ and $\tilde{H}_{n-1}(\Delta_1 \cap \Delta_2; k) = 0$. Hence, thanks to the reduced Mayer-Vietoris exact sequence, we have $\tilde{H}_n(\Delta; k) = 0$ as desired.

Secondly, let us assume $v = 2(n+1)$. If Δ is the simplicial join of $n+1$ copies of S^0 , then the geometric realization of Δ is the n -sphere S^n . Thus $\tilde{H}_n(\Delta; k) \neq 0$. On the other hand, suppose that $\tilde{H}_n(\Delta; k) \neq 0$. Then $I_\Delta \neq (0)$. Let $xy \in I_\Delta$ and $\Delta_1 = \text{star}_\Delta(\{x\}), \Delta_2 = \Delta_{V-\{x\}}$ as above. Since $\tilde{H}_n(\Delta_1; k) = 0, \tilde{H}_n(\Delta_2; k) = 0$ and $\tilde{H}_n(\Delta; k) \neq 0$, the reduced Mayer-Vietoris exact sequence guarantees that $\tilde{H}_{n-1}(\text{link}_\Delta(\{x\}); k) \neq 0$. Let v'

be the number of vertices of $\text{link}_\Delta(\{x\})$. Then $v' \leq v - 2 = 2n$ since $\{x\}, \{y\} \notin \text{link}_\Delta(\{x\})$, while $v' \geq 2n$ since $\tilde{H}_{n-1}(\text{link}_\Delta(\{x\}); k) \neq 0$. Hence $v' = 2n$. Thus, we may assume that $\text{link}_\Delta(\{x\})$ is the simplicial join of n copies of S^0 . Let $z \in V$ be an arbitrary vertex of Δ with $z \neq x$ and $z \neq y$. Then, since $v' = 2n$, $\{z\} \in \text{link}_\Delta(\{x\})$. Hence, there exists an element $w \in V - \{x, y\}$ such that $zw \in I_{\text{link}_\Delta(\{x\})}$. Since I_Δ is generated by square-free monomials of degree two, we have $zw \in I_\Delta$. Consequently, for an arbitrary element $\alpha \in V$, there exists a unique element $\beta \in V$ such that $\{\alpha, \beta\} \notin \Delta$. Hence, Δ is the simplicial join of $n + 1$ copies of the 0-sphere S^0 as required. Q. E. D.

(2.2) COROLLARY. *Suppose that the ideal I_Δ is generated by square-free monomials of degree two and that a finite free resolution (2) of $k[\Delta] = A/I_\Delta$ over A is minimal. Then, $\beta_{i,j} = 0$ for all i and j with $j > 2i$.*

Proof. By Lemma (2.1), we have $\tilde{H}_{\sharp(W)-i-1}(\Delta_W; k) = 0$ if $\sharp(W) < 2(\sharp(W) - i)$, i.e., $\sharp(W) > 2i$. Hence, thanks to Hochster's formula (3), $\beta_{i,j} = 0$ for all i and j with $j > 2i$. Q. E. D.

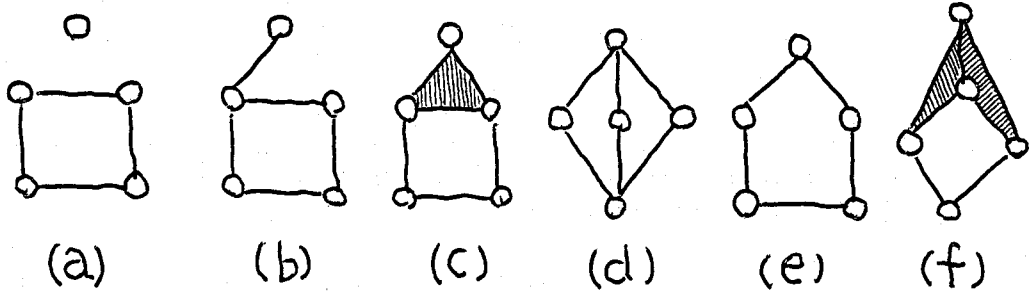
Taylor [Tay] constructed an explicit (not necessarily minimal) finite free resolution of $k[\Delta] = A/I_\Delta$ over A . The above Corollary (2.2) also follows immediately from Taylor resolutions.

(2.3) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = 7$. Suppose that I_Δ is generated by square-free monomials of degree two and that $\tilde{H}_2(\Delta; k) \neq 0$. Then, one of the following conditions (i) and (ii) is satisfied:*

- (i) Δ is the simplicial join of the cycle of length 5 and 0-sphere S^0 ;
- (ii) there exists $x \in V$ such that $\Delta_{V-\{x\}} = S^0 * S^0 * S^0$.

Proof. Suppose that there exists no $x \in V$ with $\Delta_{V-\{x\}} = S^0 * S^0 * S^0$. Let $x \in V$ and set $\Delta_1 = \text{star}_\Delta(\{x\})$, $\Delta_2 = \Delta_{V-\{x\}}$. Then $\Delta = \Delta_1 \cup \Delta_2$ and $\text{link}_\Delta(\{x\}) = \Delta_1 \cap \Delta_2$. Since Δ_1 is contractible, we have $\tilde{H}_2(\Delta_1; k) = 0$. On the other hand, since $\Delta_2 \neq S^0 * S^0 * S^0$, we have $\tilde{H}_2(\Delta_2; k) = 0$ by Lemma (2.1). Thus, thanks to the reduced Mayer-Vietoris exact sequence, we have $\tilde{H}_1(\text{link}_\Delta(\{x\}); k) \neq 0$ since $\tilde{H}_2(\Delta; k) \neq 0$. Let V' denote the vertex set of $\text{link}_\Delta(\{x\})$. Then $\sharp(V') \geq 4$ by Lemma (2.1). Moreover, again by Lemma (2.1), if $\sharp(V') = 4$, then $\text{link}_\Delta(\{x\})$ is the cycle of length 4. If the number of vertices of $\text{link}_\Delta(\{y\})$ is equal to 4 for every $y \in V$, then $\dim \Delta = 2$ and the number of faces σ of Δ with $\sharp(\sigma) = 3$ is $(4 \times \sharp(V)) \div 3 = \frac{28}{3}$, a contradiction.

Hence, there exists $z \in V$ such that the number of vertices of $\text{link}_\Delta(\{z\})$ is greater than or equal to 5. If the number of vertices of $\text{link}_\Delta(\{z\})$ is equal to 6, then $\Delta = \text{star}_\Delta(\{z\})$ and, therefore, Δ is contractible, which contradicts $\tilde{H}_2(\Delta; k) \neq 0$. Thus, the number of vertices of $\text{link}_\Delta(\{z\})$ is equal to 5. Since $\tilde{H}_1(\text{link}_\Delta(\{z\}); k) \neq 0$ and the ideal $I_{\text{link}_\Delta(\{z\})}$ is generated by square-free monomials of degree two, it follows easily that $\text{link}_\Delta(\{z\})$ is one of the following figures:



If $\text{link}_\Delta(\{z\})$ is one of the above figures (a), (b), (c), (d), (f) and if $\tilde{H}_2(\Delta; k) \neq 0$, then there exists $x \in V$ with $\Delta_{V-\{x\}} = \mathbf{S}^0 * \mathbf{S}^0 * \mathbf{S}^0$ (the routine details should be omitted). On the other hand, if $\text{link}_\Delta(\{z\})$ is the graph of figure (e) and if $\tilde{H}_2(\Delta; k) \neq 0$, then Δ is the simplicial join of the cycle of length 5 and 0-sphere \mathbf{S}^0 as required. Q. E. D.

We are now in the position to state the main result of this section.

(2.4) THEOREM. *Let Δ be a simplicial complex and suppose that the ideal I_Δ is generated by square-free monomials of degree two. Then, both the third Betti number $\beta_3^A(k[\Delta])$ and the fourth Betti number $\beta_4^A(k[\Delta])$ of $k[\Delta] = A/I_\Delta$ over A are independent of the base field k .*

Proof. First, we study the third Betti number $\beta_3^A(k[\Delta])$ of $k[\Delta]$ over A . Let V be the vertex set of Δ . Thanks to Corollary (2.2), what we must prove is that β_3 is independent of the base field k for every $j \leq 6$. Thus, by virtue of Hochster's formula (3), what we must prove is that $\dim \tilde{H}_{\sharp(W)-4}(\Delta_W; k)$ is independent of k for every $W \subset V$ with $\sharp(W) \leq 6$. If $\sharp(W) = 5$, then $\tilde{H}_i(\Delta_W; k) = 0$ for every $i \geq 2$ by Lemma (2.1). Thus, since the reduced Euler characteristic $\tilde{\chi}(\Delta)$ and $\dim_k \tilde{H}_0(\Delta_W; k)$ are independent of k , it follows from Euler-Poincaré formula that $\dim \tilde{H}_1(\Delta_W; k)$ is independent of k . On the other hand, if $\sharp(W) = 6$, then $\dim \tilde{H}_2(\Delta_W; k) = 0$ unless Δ_W is the simplicial join of three copies of the 0-sphere by Lemma (2.1). Moreover, if Δ_W is the simplicial join of three copies of the 0-sphere, then $\dim \tilde{H}_2(\Delta_W; k) = 1$

for an arbitrary field k .

Secondly, we show that the fourth Betti number $\beta_4^A(k[\Delta])$ of $k[\Delta]$ over A is independent of the base field k . We must prove that $\dim \tilde{H}_{\sharp(W)-5}(\Delta_W; k)$ is independent of k for every $W \subset V$ with $\sharp(W) \leq 8$. If either $\sharp(W) = 6$ or $\sharp(W) = 8$, then we can show that $\dim \tilde{H}_{\sharp(W)-5}(\Delta_W; k)$ is independent of k by the similar technique with Lemma (2.1) as above. Let $\sharp(W) = 7$ and suppose that $\tilde{H}_2(\Delta_W; k) \neq 0$. Then, by Lemma (2.3), we easily see that Δ_W has the homotopy type of one of the following spaces: (i) the 2-sphere; (ii) the disjoint union of the 2-sphere and a single point; (iii) the space $X \cup Y$, where X is the 2-sphere and Y is either the 1-sphere or the 2-sphere, such that $X \cap Y$ consists of a single point. Hence, $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of the base field k as desired. Q. E. D.

We do not know if there exists a simplicial complex Δ such that I_Δ is generated by square-free monomials of degree two and that the fifth Betti number $\beta_5^A(k[\Delta])$ of $k[\Delta] = A/I_\Delta$ over A does depend on the base field k .

On the other hand, let B_N denote the Boolean lattice of rank $N > 0$ and $\Delta(B_N)$ its order complex. If $N \gg 0$, then there exists $j \geq 5$ such that both $\beta_j^A(k[\Delta(B_N)])$ and $\beta_{(2^N - N - 1) - j}^A(k[\Delta(B_N)])$ depend on the base field k . See Example (3.3).

§3. Finite free resolutions of the n -sphere

In general, it is possible to define the Stanley–Reisner ring $\mathbf{Z}[\Delta] = A/I_\Delta$ of Δ over the commutative ring \mathbf{Z} . However, a minimal free resolution of $\mathbf{Z}[\Delta]$ over the polynomial ring $A = \mathbf{Z}[x_1, x_2, \dots, x_v]$ does not necessarily exist. On the other hand, there exists a minimal free resolution of $\mathbf{Z}[\Delta]$ over A if and only if all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k (see, e.g., [Has–Kur]). Thus, it might be of interest to find a natural class of simplicial complexes Δ for which all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k . The main purpose of this section is to show that if $|\Delta|$ is the n -sphere \mathbf{S}^n (or the n -ball \mathbf{B}^n) with $n \leq 3$, then all Betti numbers $\beta_i^A(k[\Delta])$ of $k[\Delta]$ are independent of k . Moreover, we construct a shellable simplicial complex Δ with $|\Delta| = \mathbf{S}^4$ such that some Betti number $\beta_i^A(k[\Delta])$ does depend on the base field k .

(3.1) PROPOSITION. (a) *Let Δ be a simplicial complex and suppose that the geometric realization $|\Delta|$ of Δ is a connected 3-manifold without boundary. Then, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k if $|\Delta|$ is orientable and $\tilde{H}_1(\Delta; \mathbf{Z}) = 0$.*

(b) Let Δ be a simplicial complex such that $|\Delta|$ is a connected 2-manifold without boundary. Then, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k if and only if $|\Delta|$ is orientable.

Proof. By virtue of Hochster's formula, in order for all Betti numbers $\beta_i^A(k[\Delta])$ to be independent of the base field k , it is necessary and sufficient that $\dim_k \tilde{H}_j(\Delta_W; k)$ is independent of k for every subset W of the vertex set V and for each integer $j \geq -1$ (since $\dim_k \tilde{H}_j(\Delta_W; k) \geq \dim_{\mathbb{Q}} \tilde{H}_j(\Delta_W; \mathbb{Q})$ for an arbitrary field k).

(a) Suppose that $|\Delta|$ is orientable and that $\tilde{H}_1(\Delta; \mathbb{Z}) = 0$. Let $W = V$, i.e., $\Delta_W = \Delta$. Obviously, $\dim_k \tilde{H}_0(\Delta; k) = 0$. Since $\tilde{H}_1(\Delta; \mathbb{Z}) = 0$, it follows that $\dim_k \tilde{H}_1(\Delta; k) = 0$. Moreover, by Poincaré duality, $\dim_k \tilde{H}_3(\Delta; k) = 1$ and $\dim_k \tilde{H}_2(\Delta; k) = 0$. Let W denote an arbitrary non-empty subset of V with $W \neq V$. Since $|\Delta|$ is orientable, by Alexander duality, we have $(\tilde{H}_2(\Delta_W; k) \cong) \tilde{H}^2(\Delta_W; k) \cong \tilde{H}^0(|\Delta| - |\Delta_W|; k)$. Hence, $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of k . Thus, since $\tilde{H}_3(\Delta_W; \mathbb{Z})$ is torsion-free, it follows that $\dim_k \tilde{H}_3(\Delta_W; k)$ is independent of k . Moreover, since $\tilde{\chi}(\Delta_W)$ is independent of k , $\dim_k \tilde{H}_1(\Delta_W; k)$ is also independent of the base field k .

(b) First, suppose that $|\Delta|$ is non-orientable. Then $\tilde{H}_2(\Delta; \mathbb{Q}) = 0$. Since $H_2(\Delta; \mathbb{Z}/2\mathbb{Z}) \cong H^0(\Delta; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ by Poincaré duality, it follows that $\dim_k \tilde{H}_2(\Delta; k)$ depends on the base field k . On the other hand, let us assume that $|\Delta|$ is orientable. By Poincaré duality, we have $\dim_k \tilde{H}_2(\Delta; k) = 1$. Moreover, if W is a subset of V with $W \neq V$ and if Δ_W is of dimension two, then Δ_W possesses non-empty boundary. Hence Δ_W has the homotopy type of the geometric realization of a one-dimensional simplicial complex; in particular $\dim_k \tilde{H}_2(\Delta_W; k) = 0$. Consequently, for every subset W of V , $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of k . Since $\tilde{\chi}(\Delta_W)$ is independent of k , $\dim_k \tilde{H}_1(\Delta_W; k)$ is also independent of k . Q. E. D.

On the other hand, it follows easily that, for a simplicial complex Δ on the vertex set V , all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k if one of the following conditions is satisfied: (i) $\dim \Delta \leq 1$; (ii) Δ is a 2-manifold with non-empty boundary; (iii) $\sharp(V) \leq 5$.

(3.2) THEOREM. Let Δ be a simplicial complex and suppose that the geometric realization $|\Delta|$ of Δ is the n -sphere \mathbb{S}^n (or the n -ball \mathbb{B}^n) with $n \leq 3$. Then, the Betti number $\beta_i^A(k[\Delta])$ is independent of the base field k for every $i \geq 0$.

Proof. If $|\Delta| = \mathbb{S}^n$, then the above Proposition (3.1) guarantees that all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k .

On the other hand, suppose that $|\Delta| = \mathbf{B}^n$ and define Δ' to be the simplicial complex $\Delta \cup (\partial\Delta * \{\text{a single point}\})$. Thus, $|\Delta'| = \mathbf{S}^n$. Let V denote the vertex set of Δ . Then $\Delta'_V = \Delta$. Hence, it follows that, for every subset W of V and for each integer $j \geq -1$, $\dim_k \tilde{H}_j(\Delta_W; k)$ is independent of the base field k as required. Q. E. D.

(3.3) EXAMPLE. Let Γ denote the simplicial complex on the vertex set $V = \{1, 2, 3, 4, 5, 6\}$, discussed in, e.g., [B-H₁], [Rei], whose geometric realization $|\Gamma|$ is the real projective plane. Let Δ denote the simplicial complex which consists of all subsets σ of V with $\sigma \neq V$. Thus, $|\Delta|$ is the 4-sphere. We consider Γ to be a subcomplex of Δ in the obvious way. Let $\text{Sd}(\Delta)$ denote the barycentric subdivision of Δ . If W is the vertex set of $\text{Sd}(\Gamma)$, then $\sharp(W) = 31$ and $\text{Sd}(\Delta)_W = \text{Sd}(\Gamma)$. Thus, we have

$$\dim_{\mathbf{Z}/2\mathbf{Z}} \tilde{H}_{31-28-1}(\text{Sd}(\Delta)_W; \mathbf{Z}/2\mathbf{Z}) > \dim_{\mathbf{Q}} \tilde{H}_{31-28-1}(\text{Sd}(\Delta)_W; \mathbf{Q});$$

$$\dim_{\mathbf{Z}/2\mathbf{Z}} \tilde{H}_{31-29-1}(\text{Sd}(\Delta)_W; \mathbf{Z}/2\mathbf{Z}) > \dim_{\mathbf{Q}} \tilde{H}_{31-29-1}(\text{Sd}(\Delta)_W; \mathbf{Q}).$$

Hence

$$\beta_{28}^A((\mathbf{Z}/2\mathbf{Z})[\text{Sd}(\Delta)]) > \beta_{28}^A(\mathbf{Q}[\text{Sd}(\Delta)]);$$

$$\beta_{29}^A((\mathbf{Z}/2\mathbf{Z})[\text{Sd}(\Delta)]) > \beta_{29}^A(\mathbf{Q}[\text{Sd}(\Delta)]).$$

Note that $\text{hd}_A(k[\text{Sd}(\Delta)]) = 57$ and $\beta_{28}^A(k[\text{Sd}(\Delta)]) = \beta_{29}^A(k[\text{Sd}(\Delta)])$. Since Δ is the boundary complex of the 5-simplex, it follows that Δ is shellable (defined in, e.g., [Bru-Man]). Hence, thanks to [Bjö], $\text{Sd}(\Delta)$ is also shellable.

(3.4) EXAMPLE. Let Δ denote the simplicial complex as in Example (3.3) and define Δ' to be $\Delta - \{\{1, 2, 3, 4, 5\}\}$. Then $|\Delta'|$ is the 4-ball. The similar technique as in Example (3.3) enables us to see that some Betti numbers $\beta_i^A(k[\text{Sd}(\Delta')])$ of the Stanley-Reisner ring $k[\text{Sd}(\Delta')]$ of the barycentric subdivision $\text{Sd}(\Delta')$ of Δ' depend on the base field k . The simplicial complex $\text{Sd}(\Delta')$ is also shellable.

The above Examples (3.3) and (3.4) illustrate the following

(3.5) PROPOSITION. Fix an integer $n \geq 4$ and let V denote the finite set $\{1, 2, \dots, n, n+1, n+2\}$. Define Δ_n to be the simplicial complex which consists of all subsets σ of V with $\sigma \neq V$. Moreover, let Δ'_n denote the simplicial complex $\Delta_n - \{\{1, 2, \dots, n+1\}\}$. Then, there exist integers i and j such that $\beta_i^A(k[\text{Sd}(\Delta_n)])$ and $\beta_j^A(k[\text{Sd}(\Delta'_n)])$ depend on the base field k . Note that both $\text{Sd}(\Delta_n)$ and $\text{Sd}(\Delta'_n)$ are shellable with $|\text{Sd}(\Delta_n)| = \mathbf{S}^n$ and $|\text{Sd}(\Delta'_n)| = \mathbf{B}^n$.

It would, of course, be of interest, for every fixed integer $n \geq 4$, to find an interesting class of simplicial complexes Δ with $|\Delta| = S^n$ such that all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k .

(3.6) THEOREM ([T-H₂], [T-H₃]). (a) Let $C(v, d)$ be the cyclic d -polytopes ([Brø, p. 85]) with v vertices and $\Delta(C(v, d))$ its boundary complex. Then, the Betti number $\beta_i^A(k[\Delta(C(v, d))])$ is independent of the base field k for every $i \geq 0$.

(b) Let $P(v, d)$ be a stacked d -polytopes ([Brø, p. 129]) with v vertices and $\Delta(P(v, d))$ its boundary complex. Then, the Betti number $\beta_i^A(k[\Delta(P(v, d))])$ is independent of the base field k for every $i \geq 0$.

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