

**(A₂)-Conditions and
Carleson Inequalities**

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(A_2) -Conditions and Carleson Inequalities

in

Bergman Spaces

by

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and

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Dedicated to Professor Mitsuru Nakai on his sixtieth birthday

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Abstract. Let ν and μ be finite positive measures on the open unit disk D . We say that ν and μ satisfy the (ν, μ) -Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 d\mu$$

for all analytic polynomials f . In this paper, we study the necessary and sufficient condition for the (ν, μ) -Carleson inequality. We establish it when ν or μ is an absolutely continuous measure with respect to the Lebesgue area measure which satisfies the (A_2) -condition. Moreover, many concrete examples of such measures are given.

§ 1. Introduction

Let D denote the open unit disk in the complex plane. For $1 \leq p \leq \infty$, let L^p denote the Lebesgue space on D with respect to the normalized Lebesgue area measure m , and $\|\cdot\|_p$ represents the usual L^p -norm. For $1 \leq p < \infty$, let L^p_a be the collection of analytic functions f on D such that $\|f\|_p$ is finite, which are so called the Bergman spaces. For any z in D , let ϕ_z be the Möbius function on D , that is

$$\phi_z(w) = \frac{z-w}{1-\bar{z}w} \quad (w \in D),$$

and put

$$\beta(z, w) = 1/2 \log(1 + |\phi_z(w)|)(1 - |\phi_z(w)|)^{-1} \quad (z, w \in D).$$

For $0 < r < \infty$ and z in D , set

$$D_r(z) = \{w \in D; \beta(z, w) < r\}$$

be the Bergman disk with "center" z and "radius" r , and we define an average of a finite positive measure μ on $D_r(a)$ by

$$\hat{\mu}_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} d\mu \quad (a \in D),$$

and if there exists a non-negative function u in L^1 such that $d\mu = u dm$, then we may write it \hat{u}_r instead of $\hat{\mu}_r$.

Let ν and μ be finite positive measures on D , and let P be the set of all analytic polynomials. We say that ν and μ satisfy the (ν, μ) -Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 d\mu$$

for all f in P . Our purpose of this paper is to study conditions on ν and μ so that the (ν, μ) -Carleson inequality is satisfied. If $\nu \leq C\mu$

on D , then the (ν, μ) -Carleson inequality is true. However it is clear that this sufficient condition for the (ν, μ) -Carleson inequality is too strong. A reasonable and natural condition is the following: there exist $r > 0$ and $\gamma > 0$ such that

$$(*) \quad \widehat{\nu}_r(a) \leq \gamma \widehat{\mu}_r(a) \quad (a \in D).$$

The averages $\widehat{\mu}_r(a)$ are sometimes computable. If $\mu = m$, then $\widehat{\mu}_r(a) = 1$ on D . If $d\mu = (1 - |z|^2)^\alpha d m$ for $\alpha > -1$, then $\widehat{\mu}_r(a)$ is equivalent to $(1 - |a|^2)^\alpha$ on D .

When $d\mu = (1 - |z|^2)^\alpha d m$ for $\alpha > -1$, Oleinik-Pavlov [7], Hastings [2], or Stegenga [8] showed that ν and μ satisfy the Carleson inequality if and only if they satisfy (*). In §3 of this paper, when $d\mu = u d m$ and u satisfies the $(A_2)_\partial$ -condition (the definition is in § 3), we obtain that the (ν, μ) -Carleson inequality is satisfied if and only if they satisfy (*). We show that if both u and u^{-1} are in BMO_∂ (see [9;p127]), then u satisfies the $(A_2)_\partial$ -condition. We give some concrete examples which satisfy the $(A_2)_\partial$ -condition.

When $\nu = m$ and $d\mu = \chi_G d m$, where χ_G is a characteristic function of a measurable subset G of D , Luecking [4] showed the equivalence between the (ν, μ) -Carleson inequality and the condition (*). If we do not put any hypotheses on μ , the problem is very difficult. The equivalence between the (ν, μ) -Carleson inequality and the condition (*) is not known even if $\nu = m$. Luecking [5] showed the following:

(1) If there exists $\gamma > 0$ such that $\widehat{m}_r(a) \leq \gamma \widehat{\mu}_r(a)$ for all $r > 0$ and a in D , then the (m, μ) -Carleson inequality is satisfied.

(2) Suppose the (μ, m) -Carleson inequality is valid (equivalently $\widehat{\mu}_r$ is bounded on D). Then the (m, μ) -Carleson inequality implies the condition (*).

In §2 of this paper, we give a sufficient condition (close to that of (1)) for the (ν, μ) -Carleson inequality when ν is not necessarily m . Moreover, using the idea of Luecking's proof of (2), a generalization of (2) is given. In §4, when $d\nu = v d m$ and v satisfies the (A_2) -condition (the definition is in §3), we establish a more natural

extension of (2) under some condition of a quantity $\varepsilon_r(\nu)$ (the definition is in §2), that is, $\varepsilon_r(\nu) \rightarrow 0$ as $r \rightarrow \infty$. The (A_2) -condition is weaker than the $(A_2)_0$ -condition. We give some concrete examples which satisfy the (A_2) -condition or the above condition of $\varepsilon_r(\nu)$.

§ 2. (ν, μ) -Carleson inequality

Let G be a measurable subset of D and u be a non-negative function in L^1 , and put

$$(\widehat{u^{-1}})_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} u^{-1} \chi_G \, d m.$$

Particularly, when $G = D$, we will omit the letter D in the above notation. The following proposition 1 gives a general sufficient condition on ν and μ which satisfy the (ν, μ) -Carleson inequality. In order to prove it we use ideas in [5] and [9;p109]. Since $(\widehat{u^{-1}})_r(a)^{-1} \leq \widehat{u}_r(a)$ for all a in D , proposition 1 is also related with (1) of §1 (cf.[5; Theorem 4.2]).

Proposition 1. Suppose that $d\mu = u \, d m$. Put $E_r = \{z \in D; \text{there is a } w \in \text{supp } \nu \text{ such that } \beta(z, w) < r/2\}$. If there exist $r > 0$ and $\gamma > 0$ such that $u > 0$ a.e. on $E = E_r$, and $\widehat{\nu}_r(a) \times (\widehat{u^{-1}})_r(a) \leq \gamma$ for all a in D , then there is a constant $C > 0$ such that

$$\int_D |f|^2 \, d\nu \leq C \int_E |f|^2 \, d\mu$$

for all f in P .

Proof. - Suppose that $\widehat{\nu}_{2r}(a) \times (\widehat{u^{-1}})_{2r}(a) \leq \gamma$ for all a in D , and put $E = \{z \in D; \text{there is a } w \in \text{supp } \nu \text{ such that } \beta(z, w) < r\}$. By an elementary theory for Bergman disks, there is a positive integer $N = N_r$,

such that there exists $\{\lambda_n\} \subset D$ satisfying that $D = \cup D_r(\lambda_n)$ and any z in D belongs to at most N of the sets $D_{2r}(\lambda_n)$ (cf. [9;p62]), therefore

$$\begin{aligned} & \int_{\text{supp } \nu} |f|^2 d\nu \\ & \leq \sum \int_{D_r(\lambda_n) \cap \text{supp } \nu} |f|^2 d\nu \\ & \leq \sum \nu(D_r(\lambda_n)) \times \sup\{|f(z)|^2; z \in D_r(\lambda_n) \cap \text{supp } \nu\}. \end{aligned}$$

By proposition 4.3.8 in [9;p62], there is a constant $C = C_r > 0$ such that

$$|f(z)| \leq \frac{C}{m(D_r(z))} \int_{D_r(z)} |f(w)| dm(w)$$

for all f analytic, z in D . If z in $D_r(\lambda_n) \cap \text{supp } \nu$, then $D_r(z)$ is contained in $D_{2r}(\lambda_n) \cap E$, and there exists a constant $K = K_r > 0$ such that $m(D_{2r}(\lambda_n)) \leq K m(D_r(z))$ for all $n \geq 1$ (cf. [9;p61]). Hence the Cauchy-Schwarz's inequality implies that

$$\begin{aligned} & \int_D |f|^2 d\nu \\ & \leq \sum \nu(D_r(\lambda_n)) \times \left(\frac{KC}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n) \cap E} |f| dm \right)^2 \\ & \leq \sum \nu(D_r(\lambda_n)) \times K^2 C^2 \\ & \quad \times \left(\frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} |f|^2 u \chi_E dm \right) \\ & \quad \times \left(\frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} u^{-1} \chi_E dm \right) \\ & \leq K^2 C^2 \sum \widehat{\nu}_{2r}(\lambda_n) \times (u_E^{-1})_{2r}(\lambda_n) \\ & \quad \times \left(\int_{D_{2r}(\lambda_n) \cap E} |f|^2 u dm \right). \end{aligned}$$

By the hypothesis and a choice of disks, it follows that

$$\int_D |f|^2 d\nu \leq K^2 C^2 \gamma N \int_E |f|^2 d\mu.$$

This completes the proof. ■

Let μ be a finite nonzero positive measure on D . For any a in D , put

$$k_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2 \quad (z \in D),$$

and a function $\tilde{\mu}$ on D is defined by

$$\tilde{\mu}(a) = \int_D |k_a|^2 d\mu.$$

Moreover, for any fixed $r < \infty$, put

$$\varepsilon_r(\mu) = \sup_{a \in D} \left(\int_{D \setminus D_r(a)} |k_a|^2 d\mu \right) \times \left(\int_D |k_a|^2 d\mu \right)^{-1}.$$

If there exists a non-negative function u in L^1 such that $d\mu = u dm$, then making a change of variable, it is easy to see that

$$\varepsilon_r(\mu) = \sup_{a \in D} \left(\int_{D \setminus D_r(0)} u \circ \phi_a dm \right) \times \left(\int_D u \circ \phi_a dm \right)^{-1}.$$

In general $0 < \varepsilon_r(\mu) \leq 1$. In this section and §4, this quantity ε_r is important. The following proposition 2 gives two general necessary conditions on ν and μ which satisfy the (ν, μ) -Carleson inequality. In order to prove (2) of proposition 2 we use ideas in [5; Theorem 4.3]. Since $\varepsilon_r(\mu) < 1$ and $\varepsilon_r(\mu) \rightarrow 0$ ($r \rightarrow \infty$), (2) of proposition 2 is related with (2) of §1.

Lemma 1. Let μ be a finite positive measure on D and $0 < r < \infty$, then the following (1)~(3) are equivalent.

(1) $\varepsilon_r(\mu) < 1$.

(2) There is a $\delta = \delta_r < \infty$ such that

$$\int_{D \setminus D_r(a)} |k_a|^2 d\mu \leq \delta \int_{D_r(a)} |k_a|^2 d\mu$$

for all a in D .

(3) There is a $\rho = \rho_r < \infty$ such that

for all a in D . $\tilde{\mu}(a) \leq \rho \tilde{\mu}_r(a)$

Proof. The implication (1) \Rightarrow (2) is trivial. (2) \Rightarrow (3) and (3) \Rightarrow (1) follow from lemma 4.3.3 in [9;p60]. In fact, by lemma 4.3.3, there exist $L = L_r > 0$ and $M = M_r > 0$ such that

$$L \leq m(D_r(a)) \times \inf\{|k_a(z)|^2; z \in D_r(a)\}$$

and

$$m(D_r(a)) \times \sup\{|k_a(z)|^2; z \in D_r(a)\} \leq M$$

for all a in D . Thus remainder implications are obtained. ■

Proposition 2. Suppose that ν and μ satisfy the (ν, μ) -Carleson inequality, then the following are true.

(1) If there exists $r < \infty$ such that $\varepsilon_r(\mu) < 1$, then there exists $\gamma > 0$ such that $\widehat{\nu}_r(a) \leq \gamma \widehat{\mu}_r(a)$ for all a in D .

(2) If $d\nu = \nu dm$, $\nu > 0$ a.e. on D , $\varepsilon_t(\nu) \rightarrow 0$ ($t \rightarrow \infty$), and there are $l > 0$ and $\gamma' > 0$ such that $\widehat{\mu}_l(a) \times (\nu^{-1})_l(a) \leq \gamma'$ for all a in D , then there are $r > 0$ and $\gamma = \gamma_r > 0$ such that $\widehat{\nu}_r(a) \leq \gamma \widehat{\mu}_r(a)$ for all a in D .

Proof. Since $k_a(z)$ is uniformly approximated by polynomials, the inequality is valid for $f = k_a$, that is

$$\int_D |k_a|^2 d\nu \leq C \int_D |k_a|^2 d\mu.$$

Firstly, we show that (1) is true. The above inequality and lemma 1 imply that

$$\begin{aligned} \widetilde{\nu}(a) &\leq C \widetilde{\mu}(a) \\ &\leq C \rho \widehat{\mu}_r(a) \end{aligned}$$

for all a in D . Moreover, by lemma 4.3.3 in [9;p60], there exists a constant $L > 0$ such that

$$\widehat{\nu}_r(a) \leq L^{-1} \widetilde{\nu}(a)$$

for all a in D . Hence we have that

$$\widehat{\nu}_r(a) \leq C \rho L^{-1} \widehat{\mu}_r(a)$$

for all a in D .

Next, we prove that (2) is true. For any a in D and $r \geq l$, put $d\mu_{a,r} = (1 - \chi_{D_r(a)})d\mu$. By the latter half of the hypothesis in (2), we have that

$$(\mu_{a,r})_i(\lambda) \times (\nu^{-1})_i(\lambda) \leq \gamma'$$

for all a, λ in D , and $r \geq l$. Set $E_{a,r,l} = \{z \in D; \text{there is a } w \text{ in } \text{supp } \mu_{a,r} \text{ such that } \beta(z,w) < l/2\}$. By proposition 1, there exists a constant $C' > 0$ such that

$$\int_{D \setminus D_r(a)} |f|^2 d\mu \leq C' \int_{E_{a,r,l}} |f|^2 d\nu$$

for all a in D , $r \geq l$ and f in P . Here we claim that $E_{a,r,l}$ is contained in $D \setminus D_{r/2}(a)$. In fact, since $D \setminus D_r(a)$ contains $\text{supp } \mu_{a,r}$ and $r \geq l$, if z belongs to $E_{a,r,l}$ then there exists w in D such that $\beta(w,a) \geq r$ and $\beta(w,z) < r/2$. Therefore,

$$\begin{aligned} r &\leq \beta(w,a) \\ &\leq \beta(w,z) + \beta(z,a) \\ &< r/2 + \beta(z,a), \end{aligned}$$

thus we have that z is contained in $D \setminus D_{r/2}(a)$. Particularly put $f = k_a$ in the above inequality, then

$$\int_{D \setminus D_r(a)} |k_a|^2 d\mu \leq C' \int_{D \setminus D_{r/2}(a)} |k_a|^2 d\nu$$

for all a in D and $r \geq l$. It follows that

$$\begin{aligned} &\int_{D_r(a)} |k_a|^2 d\mu \\ &= \int_D |k_a|^2 d\mu - \int_{D \setminus D_r(a)} |k_a|^2 d\mu \\ &\geq C^{-1} \int_D |k_a|^2 d\nu - C' \int_{D \setminus D_{r/2}(a)} |k_a|^2 d\nu. \end{aligned}$$

By the definition of $\varepsilon_r(\nu)$, the above inequality implies that

$$\begin{aligned} & \int_{D_r(a)} |k_a|^2 d\mu \\ & \geq (C^{-1} - C' \varepsilon_{r/2}(\nu)) \int_D |k_a|^2 d\nu \end{aligned}$$

for all a in D and $r \geq l$. Here let r be sufficiently large, then by the hypothesis on $\varepsilon_r(\nu)$, $C^{-1} - C' \varepsilon_{r/2}(\nu) > 0$, and by lemma 4.3.3 in [9;p60], we conclude that

$$\hat{\mu}_r(a) \geq [M^{-1}(C^{-1} - C' \varepsilon_{r/2}(\nu))L] \hat{\nu}_r(a)$$

for all a in D . ■

§ 3. (A_2) -condition

For a complex measure μ on D , recall that a function $\tilde{\mu}$ on D is defined by

$$\tilde{\mu}(a) = \int_D |k_a|^2 d\mu.$$

Particularly, if there exists a complex valued L^1 -function u such that $d\mu = u dm$, then we denote the function by \tilde{u} instead of $\tilde{\mu}$, and say that \tilde{u} is the Berezin transform of the function u .

Let ν and u be non-negative functions in L^1 , put $d\nu = \nu dm$ and $d\mu = u dm$. Suppose that there is a constant $\gamma > 0$ such that

$$\tilde{\nu}(a) \times (u^{-1})\tilde{\nu}(a) \leq \gamma$$

for all a in D , then lemma 4.3.3 in [9;p60] implies that there exist $r > 0$ and $\gamma' > 0$ such that

$$\hat{\nu}_r(a) \times (u^{-1})\hat{\nu}_r(a) \leq \gamma'$$

for all a in D , and hence by proposition 1, we obtain that the (ν, μ) -Carleson inequality is satisfied. In the above two inequalities,

if we put $u = v$, then such a function u is interesting for us.

A non-negative function u in L^1 is said to satisfy a $(A_2)_\partial$ -condition, if there exists a constant $A > 0$ such that

$$\tilde{u}(a) \times (u^{-1})^\sim(a) \leq A$$

for all a in D . If there exist $r > 0$ and $A_r > 0$ such that

$$\hat{u}_r(a) \times (u^{-1})_r^\sim(a) \leq A_r$$

for all a in D , then we say that u satisfies a (A_2) -condition. In [6], the (A_2) -condition is called Condition C_2 . It is known that u satisfies the (A_2) -condition for some $0 < r < \infty$ if and only if u satisfies the (A_2) -condition for all $0 < r < \infty$ [6]. Hence it shows that the definition of the (A_2) -condition is independent of r . In general, lemma 4.3.3 in [9;p60] and the familiar inequality between the harmonic and arithmetic means imply that for any $0 < r < \infty$ there exists a constant $M = M_r > 0$ such that $M^{-1}(u^{-1})^{\sim -1} \leq (u^{-1})_r^{\sim -1} \leq \hat{u}_r \leq M \tilde{u}$. Therefore, if u satisfies the $(A_2)_\partial$ -condition, then $(u^{-1})^{\sim -1}$, $(u^{-1})_r^{\sim -1}$, \hat{u}_r , and \tilde{u} are equivalent. Similarly, if u satisfies the (A_2) -condition, then $(u^{-1})_r^{\sim -1}$, and \hat{u}_r are equivalent. When u is in $L^1(\partial D)$ (L^1 is a usual Lebesgue space on the unit circle and $k_a(z)$ is a normalized reproducing kernel of a Hardy space), the $(A_2)_\partial$ -condition has been studied in [3; (c) of Theorem2].

The following theorem 3 gives a necessary and sufficient condition in order to satisfy the (ν, μ) -Carleson inequality when $d\mu = u dm$ and u satisfies the $(A_2)_\partial$ -condition.

Theorem 3. Suppose that u satisfies the $(A_2)_\partial$ -condition, then the following are equivalent.

(1) There is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 u dm$$

for all f in P .

(2) There exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \leq \gamma \hat{u}_r(a)$$

for all a in D .

(3) For any $r > 0$, there exists $\gamma = \gamma_r > 0$ such that

$$\hat{\nu}_r(a) \leq \gamma \hat{u}_r(a)$$

for all a in D .

Proof. Suppose that (1) holds. Since u satisfies the $(A_2)_\partial$ -condition, by (1) of proposition 8, u satisfies a relation in (3) of lemma 1 for all $r > 0$. Therefore, (3) follows from (1) of proposition 2. The implication (3) \Rightarrow (2) is obvious. We will show that (2) \Rightarrow (1). Since u satisfies the $(A_2)_\partial$ -condition, u^{-1} is integrable, hence $u > 0$ a.e. on D . Moreover, by (5) of proposition 4, u satisfies the (A_2) -condition for all $r > 0$ and therefore (2) implies that

$$\widehat{v}_r(a) \times (u^{-1})_r(a) \leq A_r \gamma$$

for all a in D . In the statement of proposition 1, put $E = D$, then the above fact shows that the inequality in (1) is satisfied. This completes the proof. ■

For any u in L^2 , a in D , we put

$$MO(u)(a) = \{ |u|^2 \widetilde{}(a) - |\widetilde{u}(a)|^2 \}^{1/2},$$

and let BMO_∂ be the space of functions u such that $MO(u)(a)$ is bounded on D (cf.[9;p127]). We give several simple sufficient conditions.

Proposition 4. Let u be a non-negative function in L^1 , then the following are true.

(1) If both \widetilde{u} and $(u^{-1})_r$ are in L^∞ , then u satisfies the $(A_2)_\partial$ -condition.

(2) If both u and u^{-1} are in BMO_∂ , then u satisfies the $(A_2)_\partial$ -condition.

(3) Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. If u_1^p and u_2^q satisfy the $(A_2)_\partial$ -condition, then $u = u_1 u_2$ satisfies the $(A_2)_\partial$ -condition.

(4) Suppose that f is a complex valued function in L^1 such that $f \neq 0$ on D , f^{-1} is in L^1 , $\widetilde{f} \times (f^{-1})_r$ is in L^∞ , and $|\arg f| \leq \pi/2 - \varepsilon$ for some $\varepsilon > 0$. If $u = |f|$, then u satisfies the $(A_2)_\partial$ -condition.

(5) If u satisfies the $(A_2)_\partial$ -condition, then u satisfies the (A_2) -condition.

Proof. (1) is trivial. By proposition 6.1.7 in [9;p108], we have that

$$\widetilde{u}(a) \times (u^{-1})_r(a) \leq MO(u)(a) \times MO(u^{-1})(a) + 1.$$

This implies that (2) is true. The Hölder's inequality implies that (3) is true. (5) follows from lemma 4.3.3 in [9;p60].

We show that (4) is true. Suppose that $u = |f|$ and there exists $\varepsilon > 0$ such that $|\arg f| \leq \pi/2 - \varepsilon$ on D . Since $|\arg f| \leq \pi/2 - \varepsilon$ on D , there exists $\delta > 0$ such that $\cos(\arg f) \geq \delta$ on D . Therefore, we have that

$$\begin{aligned} \operatorname{Re} f &= |f| \times \cos(\arg f) \geq |f| \cdot \delta \\ &= \delta u. \end{aligned}$$

For any a in D , it follows that

$$\delta \tilde{u}(a) \leq \int \operatorname{Re} f \cdot |k_a|^2 dm \leq |\tilde{f}(a)|.$$

Similarly, we have that

$$\delta (u^{-1})^\sim(a) \leq |(f^{-1})^\sim(a)|.$$

Thus,

$$\tilde{u}(a) \times (u^{-1})^\sim(a) \leq \delta^{-2} \times |\tilde{f}(a)| \times |(f^{-1})^\sim(a)|$$

for all a in D , and hence (4) follows. ■

We exhibit some concrete examples which satisfy the $(A_2)_\partial$ -condition.

Proposition 5. If u is a function that is given by (1), (2), or (3), then u satisfies the $(A_2)_\partial$ -condition.

(1) For any $-1 < \alpha < 1$, put $u(z) = (1 - |z|^2)^\alpha$.

(2) Let $\{b_j\}$ be a finite sequence of complex numbers in $D \cup \partial D$ with $b_i \neq b_j (i \neq j)$, and let $0 \leq \alpha(j) < 2$ for all j or $-2 < \alpha(j) \leq 0$ for all j . Put $u = \prod p_j^{\alpha(j)}$ where $p_j(z) = |z - b_j|$.

(3) Let $\{b_j\}, \{p_j\}$ as in (2) and $-1 < \alpha(j) < 1$ for all j . Put $u = \prod p_j^{\alpha(j)}$.

Proof. We suppose that u has the form of (1). For any a in D , making a change of variable, we have that

$$\tilde{u}(a) \times (u^{-1})^\sim(a) = \int (1 - |a|^2)^\alpha (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} dm(z)$$

$$\begin{aligned}
& \times \int (1-|a|^2)^{-\alpha}(1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha} dm(z) \\
& = \int (1-|z|^2)^{\alpha}|1-\bar{a}z|^{-2\alpha} dm(z) \\
& \quad \times \int (1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha} dm(z).
\end{aligned}$$

Since $-1 < \alpha < 1$, Rudin's lemma(cf.[9;p53]) implies that both factors of the right hand side in the above equality are bounded. Hence u satisfies the $(A_2)_{\partial}$ -condition.

We show that u satisfies the $(A_2)_{\partial}$ -condition when u has the form of (2). Let α be a real number such that $0 < \alpha < 2$. For any fixed b in D , put $p(z) = |z - b|$. Firstly, we show that the Berezin transform of $p^{-\alpha}$ is bounded. In fact, making a change of variable, elementary calculations show that

$$(p^{-\alpha})^{\sim}(a) \leq \|1 - \bar{a}b\|^{-\alpha} \|1 - \bar{a}z\|_{\infty}^{\alpha} \times \int |\phi_a(b) - z|^{-\alpha} dm(z).$$

Since $\phi_a(b) - z$ lies in $2D = \{2z; z \in D\}$ for any a, z in D and an area measure is translation invariant, we have that

$$(p^{-\alpha})^{\sim}(a) \leq (1-|b|)^{-\alpha} \|1 - \bar{a}z\|_{\infty}^{\alpha} \times \int_{2D} |w|^{-\alpha} dm(w)$$

for all a in D . Hence we obtain that the Berezin transform of $p^{-\alpha}$ is bounded. Next, let b be in ∂D and put $p(z) = |z - b|$. Then, as in the proof of the above case, we have that

$$(p^{\alpha})^{\sim}(a) \leq |a - b|^{\alpha} \cdot \|\phi_a(b) - z\|_{\infty}^{\alpha} \times \int \|1 - \bar{a}z\|^{-\alpha} dm(z),$$

and

$$(p^{-\alpha})^{\sim}(a) \leq |a - b|^{-\alpha} \|1 - \bar{a}z\|_{\infty}^{\alpha} \times \int_{2D} |w|^{-\alpha} dm(w).$$

Therefore, Rudin's lemma implies that p^{α} satisfies the $(A_2)_{\partial}$ -condition. For any b_1 in D and b_2 in ∂D , put $p_1(z) = |z - b_1|$ and $p_2(z) = |z - b_2|$. Fix $0 < \alpha(j) < 2$ for $j = 1, 2$ and $\varepsilon > 0$. Because

$b_1 \neq b_2$, there exist measurable subsets B_j of D such that $B_1 \cap B_2 = \emptyset$ and $p_j \geq \varepsilon$ on B_j^c for $j=1, 2$. Set $B_0 = D \setminus B_1 \cup B_2$, then

$$\begin{aligned} & (p_1^{\alpha(1)} \cdot p_1^{\alpha(2)}) \tilde{\sim}(a) \times (p_1^{-\alpha(1)} \cdot p_2^{-\alpha(2)}) \tilde{\sim}(a) \\ & \leq (p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}) \tilde{\sim}(a) \times \left(\varepsilon^{-\alpha(1)-\alpha(2)} \int_{B_0} |k_a|^2 dm \right. \\ & \quad \left. + \varepsilon^{-\alpha(2)} \int_{B_1} p_1^{-\alpha(1)} |k_a|^2 dm + \varepsilon^{-\alpha(1)} \int_{B_2} p_2^{-\alpha(2)} |k_a|^2 dm \right) \\ & \leq M_0 \times \varepsilon^{-\alpha(1)-\alpha(2)} + M_0 \times \varepsilon^{-\alpha(2)} \cdot (p_1^{-\alpha(1)}) \tilde{\sim}(a) \\ & \quad + M_1 \times \varepsilon^{-\alpha(1)} \cdot (p_2^{\alpha(2)}) \tilde{\sim}(a) \cdot (p_2^{-\alpha(2)}) \tilde{\sim}(a), \end{aligned}$$

where $M_0 = \|p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}\|_\infty$ and $M_1 = \|p_1^{-\alpha(1)}\|_\infty$. Hence we have that $p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}$ satisfies the $(A_2)_\emptyset$ -condition. If u has the form of (2), then applying the same argument for finitely many factors of u and u^{-1} , we obtain that u satisfies $(A_2)_\emptyset$ -condition.

Apparently, (3) follows from (2) of this proposition and (3) of proposition 4. In fact, we let $-1 < \alpha(j) < 1$ for all j , and set $j(+)=\{j; \alpha(j) \geq 0\}$, $j(-)=\{j; \alpha(j) < 0\}$. Put $u_1 = \prod_{j(+)} p_j^{\alpha(j)}$ and $u_2 = \prod_{j(-)} p_j^{\alpha(j)}$, then u_1^2 and u_2^2 satisfy the $(A_2)_\emptyset$ -condition. Hence, (3) of proposition 4 implies that $u = u_1 \times u_2$ satisfies the $(A_2)_\emptyset$ -condition. ■

Corollary 1 is a partial result of [2], [7] and [8].

Corollary 1. (Oleinik-Pavlov-Hastings-Stegenga) Let ν be a finite positive measure on D . For any $-1 < \alpha < 1$, there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 (1-|z|^2)^\alpha dm$$

for all f in P if and only if there exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \leq \gamma(1-|a|^2)^\alpha$$

for all a in D .

Proof. Since $[(1-|z|^2)^{\alpha}]^{\wedge}(a)$ is comparable to $(1-|a|^2)^{\alpha}$, by theorem 3 and (1) of proposition 5 the corollary follows. ■

Lemma 2. Let $\{b_j\}$ be a finite sequence of complex numbers in $D \cup \partial D$ with $b_i \neq b_j (i \neq j)$, and let $\{\alpha(j)\}$ be a finite sequence of real numbers such that $-2 < \alpha(j)$ when j is in Λ (the definition of Λ is below). Put $p_j(z) = |z - b_j|$ and $u = \prod p_j^{\alpha(j)}$, and let $0 < r < \infty$, then there are constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\gamma_1 \hat{u}_r(a) \leq \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma_2 \hat{u}_r(a)$$

for all a in D , here $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

Proof. For any fixed $0 < r < \infty$, in general, lemma 4.3.3 in [9;p60] implies that there are constants $L > 0$ and $M > 0$ such that

$$L \hat{u}_r(a) \leq \int_{D_r(0)} u \circ \phi_a \, dm \leq M \hat{u}_r(a)$$

for all a in D , where u is a non-negative integrable function on D . Let $u = \prod |z - b_j|^{\alpha(j)}$, $\{b_j\} \subset D \cup \partial D$, $b_i \neq b_j (i \neq j)$, and $\alpha(j)$ be real numbers. Then, by the same calculations in the proof of (2) of proposition 5, we have that

$$\begin{aligned} & \int_{D_r(0)} u \circ \phi_a \, dm \\ &= \prod |1 - \bar{a} b_j|^{\alpha(j)} \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} \cdot |1 - \bar{a} z|^{-\sum \alpha(j)} \, dm(z). \end{aligned}$$

Put

$$I(a) = \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} \, dm(z),$$

then it is easy to see that $\int_{D_r(0)} u \circ \phi_a \, dm$ is equivalent to $I(a) \times \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$.

Firstly, we show that the lemma is true when $0 \leq \alpha(j)$ for all j . By the above facts, it is enough to prove that the integration

$$I(a) = \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} \, dm(z)$$

is bounded below for all a in D , because $0 \leq \alpha(j)$. Conversely, suppose that there exists $\{a_n\} \subset D$ such that $I(a_n) < 1/n$. Here we can choose a subsequence $\{a_k\} \subset \{a_n\}$ such that $a_k \rightarrow a' (k \rightarrow \infty)$, where a' may be in $D \cup \partial D$. Therefore, Fatou's lemma implies that $I(a') = 0$, thus it follows that $\prod |\phi_{a'}(b_j) - z|^{\alpha(j)} = 0$ on $D_r(0)$. This contradiction implies that the assertion is true when $0 \leq \alpha(j)$ for all j .

Next, we prove that the lemma is true when $-2 < \alpha(j) < 0$ for all j in Λ^c and $-\infty < \alpha(j) < 0$ for all j in Λ . In fact, we claim that $I(a)$ is bounded for all a in D . If j is in Λ , then $|\phi_a(b_j)| = 1$ for all a in D , therefore $|\phi_a(b_j) - z|^{-1}$ is bounded, because z belongs to $D_r(0)$. Analogously, if j is in Λ^c , then $|\phi_a(b_j)| \rightarrow 1$ ($|a| \rightarrow 1$), therefore $|\phi_a(b_j) - z|^{-1}$ is bounded when a is nearby ∂D , because z belongs to $D_r(0)$. Thus, it is sufficient to prove that

$$J(a) = \int_{D_r(0)} \prod_{j \in \Lambda^c} |\phi_a(b_j) - z|^{\alpha(j)} dm(z)$$

is bounded for all a in $U_\eta(0) = \{a \in D; |a| \leq \eta\}$, where $0 < \eta < 1$ is a constant which is close to 1. Put

$$\Phi_{i,j}(a) = |\phi_a(b_i) - \phi_a(b_j)| \quad (i, j \in \Lambda^c, a \in U_\eta(0)).$$

For any fixed $i, j \in \Lambda^c$, since $\Phi_{i,j}$ is a continuous function on $U_\eta(0)$ and Möbius functions are one-to-one correspondence on D , there exists $\varepsilon(i, j) > 0$ such that $\Phi_{i,j}(a) \geq \varepsilon(i, j)$ for all a in $U_\eta(0)$ when $i \neq j$. Put $\varepsilon = \min\{\varepsilon(i, j)/2; i, j \in \Lambda^c \text{ such that } i \neq j\}$, $B_j(a) = \{z \in D_r(0); |\phi_a(b_j) - z| < \varepsilon\}$ and $B_0(a) = D_r(0) \setminus \cup B_j(a)$. For any j in $\Lambda^c \cup \{0\}$, since $|\phi_a(b_i) - z| \geq \varepsilon$ when z belongs to $B_j(a)$ and i belongs to Λ^c such that $i \neq j$, therefore we have that

$$J(a) \leq \sum_{j \in \Lambda^c} \varepsilon^{\alpha(j)} \int_{B_j(a)} |\phi_a(b_j) - z|^{\alpha(j)} dm(z) \\ + \varepsilon^\alpha \int_{B_0(a)} dm(z)$$

$$\leq \sum_{j \in \Lambda^c} \varepsilon^{\alpha - \alpha(j)} \int_{2D} |w|^{\alpha(j)} dm(w) + \varepsilon^\alpha,$$

where

$$\alpha = \sum_{j \in \Lambda^c} \alpha(j).$$

Therefore, J is bounded on $D_r(0)$, and hence we obtain that I is bounded on D .

Using the above facts, we can show that the assertion is true when u has the general form of the statement of this lemma. Let $\{\alpha(j)\}$ be a finite sequence of real numbers such that $-2 < \alpha(j) < \infty$ when j is in Λ^c and $-\infty < \alpha(j) < \infty$ when j is in Λ . As in the proof of proposition 5, set $j(+)=\{j; \alpha(j) \geq 0\}$ and $j(-)=\{j; \alpha(j) < 0\}$, then we have that

$$I(a) \leq 2^{\sum_{j \in (+)} \alpha(j)} \int_{D_r(0)} \prod_{j \in (-)} |\phi_a(b_j) - z|^{\alpha(j)} dm(z)$$

and

$$I(a) \geq 2^{\sum_{j \in (-)} \alpha(j)} \int_{D_r(0)} \prod_{j \in (+)} |\phi_a(b_j) - z|^{\alpha(j)} dm(z).$$

Therefore, we obtain that I is bounded and bounded below on D . Hence, this completes the proof. ■

Corollary 2. Let u be a non-negative function in L^1 that is given by (2), or (3) of proposition 5 and ν be a finite positive measure on D , then there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 u dm$$

for all f in P if and only if there exist $r > 0$ and $\gamma = \gamma_r > 0$ such that

$$\hat{\nu}_r(a) \leq \gamma \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$$

for all a in D , here $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

Proof. The corollary follows from theorem 3, proposition 5 and lemma 2. ■

We give a characterization of u which satisfies the (A_2) -condition or the $(A_2)_\partial$ -condition when u is a modulus of a rational function or a modulus of a polynomial, respectively. Let u be a non-negative integrable function on D , then it is easy to see that if u satisfies the $(A_2)_\partial$ -condition then u^{-1} is integrable on D . But, we claim that the converse is true, when u is a modulus of a polynomial. As the result, we show that the $(A_2)_\partial$ -condition is properly contained in the (A_2) -condition. The essential part of the following theorem is proved in proposition 5 and lemma 2.

Theorem 6. Let $\{b_j\}$ be a finite sequence of complex numbers such that $b_i \neq b_j (i \neq j)$ and $\{\alpha(j)\}$ be a finite sequence of real numbers. Put $p_j(z) = |z - b_j|$ and $u = \prod p_j^{\alpha(j)}$, then the following are true.

(1) If $\alpha(j) \geq 0$ for all j or $\alpha(j) \leq 0$ for all j , then u satisfies the $(A_2)_\partial$ -condition if and only if $\alpha(j) < 2$ or $\alpha(j) > -2$ when b_j is in $D \cup \partial D$, respectively.

(2) u satisfies the (A_2) -condition if and only if $-2 < \alpha(j) < 2$ when b_j is in D .

Proof. (1) By (2) of proposition 5 and the remark above this theorem, it is enough to prove that u^{-1} is not integrable on D when $\alpha(j) \geq 2$ for some b_j in $D \cup \partial D$. Suppose that there is a j such that b_j in $D \cup \partial D$ and $\alpha(j) \geq 2$, then there exists a L^∞ -function h such that $u(z) = |z - b_j|^2 \cdot h(z)$. It is easy to see that u^{-1} is not integrable on $U = \{z \in D; |z - b_j| < \text{dist}(b_j, \partial D)\}$ when b_j is in D , therefore we consider the case when $b_j = 1$. Put $M_2 = \|h\|_\infty$, then

$$\int u^{-1} dm \geq M_2^{-1} \int_0^1 2r \int_0^{2\pi} |1 - r e^{i\theta}|^{-2} d\theta / 2\pi dr$$

$$= M_2^{-1} \int_0^1 2r(1-r^2)^{-1} dr$$

$$= M_2^{-1} \int_0^1 t^{-1} dt.$$

Hence we obtain that u^{-1} is not integrable.

(2) Suppose that $-2 < \alpha(j) < 2$ when b_j is in D , then apparently lemma 2 implies that u satisfies the (A_2) -condition. Conversely, suppose that there exist $r > 0$ and $A_r > 0$ such that

$$\widehat{u}_r(a) \times (u^{-1})_r(a) \leq A_r$$

for all a in D . Since \widehat{u}_r is non-zero on D , therefore $(u^{-1})_r(a) < \infty$ for all a in D . By the same argument in (1), we have that $\alpha(j)$ must be less than 2 when b_j is in D . In fact, if $\alpha(j) \geq 2$ for some b_j in D , then there exists a function h such that $u(z) = |z - b_j|^2 \cdot h(z)$. Put

$$\varepsilon = \min\{\text{dist}(b_i, b_j)/2; i \neq j\}$$

and

$$U(j) = \{z \in D; |z - b_j| < \varepsilon\},$$

then obviously h is bounded on $U(j)$. Since there exists a_j such that a center of the Bergman disk $D_r(a_j)$ is just equal to b_j , therefore we have that u^{-1} is not integrable on $D_r(a_j) \cap U(j)$, and thus, it follows that the average of u^{-1} on $D_r(a_j)$ is infinite. This contradicts the above fact. Consequently, we obtain that $\alpha(j)$ must lie in $(-\infty, 2)$ when b_j is in D . Applying the same argument to u^{-1} , we have that $\alpha(j)$ must lie in $(-2, \infty)$ when b_j is in D . Therefore, we conclude that $-2 < \alpha(j) < 2$ when b_j is in D . ■

§ 4. Uniformly absolutely continuous

Recall that

$$\varepsilon_r(\mu) = \sup_{a \in D} \left(\int_{D \setminus D_r(a)} |k_a|^2 d\mu \right) \times \left(\int_D |k_a|^2 d\mu \right)^{-1}$$

where μ is a finite positive measure on D (see lemma 1 and proposition 2). Using the quantity ε_r , we give a necessary condition on ν and μ which satisfy the (ν, μ) -Carleson inequality.

Theorem 7. Suppose that $d\nu = v dm$, $\varepsilon_t(\nu) \rightarrow 0 (t \rightarrow \infty)$, and that ν satisfies the (A_2) -condition, furthermore μ and ν satisfy the (μ, ν) -Carleson inequality. If there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 d\mu$$

for all f in \mathcal{P} , then there exist $r > 0$ and $\gamma > 0$ such that

$$\widehat{\nu}_r(a) \leq \gamma \widehat{\mu}_r(a)$$

for all a in D .

Proof. By hypotheses on ν and lemma 1, there exist $t > 0$, $\rho > 0$ and $A > 0$ such that

$$\widetilde{\nu} \leq \rho \cdot \widehat{\nu}_t \leq A \rho \cdot (v^{-1})_t^{-1}.$$

Moreover, lemma 4.3.3 in [9;p60] and the (μ, ν) -Carleson inequality imply that there exist $L > 0$ and $C' > 0$ such that

$$L \cdot \widehat{\mu}_t \leq \widetilde{\mu} \leq C' \cdot \widetilde{\nu}.$$

Thus, a desired result follows from (2) of proposition 2. ■

Luecking [5] shows the above theorem when ν is the Lebesgue area measure m . It is clear that $\varepsilon_r(m) \rightarrow 0 (r \rightarrow \infty)$ and m satisfies the (A_2) -condition. Now, we are interested in measures μ such that $\varepsilon_r(\mu) < 1$ or $\varepsilon_r(\mu) \rightarrow 0 (r \rightarrow \infty)$.

Proposition 8. Suppose that $d\mu = u dm$, and u is a non-negative function in L^1 . If u is the function such that (1) or (2), then there exists $0 < r < \infty$ such that $\varepsilon_r(\mu) < 1$.

(1) u satisfies the $(A_2)_\partial$ -condition.

(2) $u(z) = (1 - |z|^2)^\alpha$ for some $1 \leq \alpha < 2$.

Proof. If u has the property in (1), then by the remark above theorem 3, for any $r > 0$ there is a positive constant $\rho = \rho_r$, such that $\tilde{\mu}(a) \leq \rho \hat{\mu}_r(a)$ for all a in D and hence $\varepsilon_r(\mu) < 1$ by lemma 1. Suppose that u has the form of (2). For any fixed $1 \leq \alpha < 2$, put $u(z) = (1 - |z|^2)^\alpha$. Then, Rudin's lemma (cf.[9;p53]) shows that

$$\begin{aligned} \tilde{u}(a) &= (1 - |a|^2)^\alpha \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} dm(z) \\ &\leq \gamma (1 - |a|^2)^\alpha, \end{aligned}$$

where $\gamma > 0$ is finite. On the other hand, lemma 4.3.3 in [9;p60] implies that

$$\begin{aligned} \hat{u}_r(a) &\geq M^{-1} \times (1 - |a|^2)^\alpha \int_{D_r(0)} (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} dm(z) \\ &\geq M^{-1} \times (1 - |a|^2)^\alpha (1 - \tanh^2 r)^\alpha \times 2^{-2\alpha}, \end{aligned}$$

therefore, by (3) of lemma 1, we obtain that $\varepsilon_r(\mu) < 1$. ■

Proposition 9. Suppose that $d\mu = u dm$, and u is a non-negative function in L^1 . If u is one of the following functions (1)~(7), then $\varepsilon_r(\mu) \rightarrow 0$ ($r \rightarrow \infty$).

(1) There exists $\varepsilon_0 > 0$ such that $\tilde{u} \geq \varepsilon_0$ on D , and $\{u \circ \phi_a dm; a \in D\}$ is uniformly absolutely continuous with respect to the Lebesgue area measure m .

(2) There exists $\varepsilon_0 > 0$ such that $\tilde{u} \geq \varepsilon_0$ on D , and there is a constant $C > 0$ such that $(u^{1+\beta})^\sim \leq C$ on D for some $\beta > 0$.

(3) u is in L^∞ , and there exist $r > 0$ and $\delta > 0$ such that $u \geq \delta$ on $D \setminus D_r(0)$.

(4) $u = |p|$, where p is an analytic polynomial which has no zeros on ∂D .

(5) $u(z) = (1 - |z|^2)^\alpha$ for some $-1 < \alpha \leq 1$.

(6) $u = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $0 < \alpha(j) < 2$ for b_j in $D \cup \partial D$, or $-2 < \alpha(j) < 0$ for b_j in $D \cup \partial D$.

(7) $u = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $-1 < \alpha(j) < 1$ for b_j in $D \cup \partial D$.

Proof. Firstly, we show that the assertion is true when u has the property of (1). Since $\{u \circ \phi_a dm; a \in D\}$ is uniformly absolutely continuous, for any $\varepsilon > 0$ there exists $r > 0$ such that $\int_{D_r(0)} u \circ \phi_a dm < \varepsilon_0 \cdot \varepsilon$ for all a in D . Therefore, making a change of variable, let r be sufficiently large, then $\varepsilon_r(\mu) < \varepsilon_0^{-1} \cdot \varepsilon_0 \cdot \varepsilon = \varepsilon$. Hence, we obtain that $\varepsilon_r(\mu) \rightarrow 0 (r \rightarrow \infty)$.

Next, we prove the implications (2) \Rightarrow (1), (3) \Rightarrow (2), and (4) \Rightarrow (3). Then $\varepsilon_r(\mu) \rightarrow 0$ when u is a function such that (2), (3) or (4). In fact, suppose that there exists $\beta > 0$ such that the Berezin transform of the function $u^{1+\beta}$ is bounded, then a set of functions $\{u \circ \phi_a; a \in D\}$ is uniformly integrable (cf. [1; p120]), therefore it follows that $\{u \circ \phi_a dm; a \in D\}$ is uniformly absolutely continuous with respect to m . Hence, (2) implies (1). If there exist $r > 0$ and $\delta > 0$ such that $u \geq \delta$ on $D \setminus D_r(0)$, then

$$\begin{aligned} \tilde{u}(a) &\geq \delta - \delta \int_{D_r(0)} |k_a|^2 dm \\ &= \delta [1 - m(D_r(a))] \\ &\geq \delta (1 - \tanh^2 r) \\ &> 0. \end{aligned}$$

Hence (3) implies (2) because $(u^{1+\beta})^\sim(a) \leq \|u\|_\infty^{1+\beta}$ for all a in D and any $\beta > 0$. Next, let p be an analytic polynomial which has no zeros on ∂D , then there are $r > 0$ and $\delta > 0$ such that $u = |p| \geq \delta$ on $D \setminus D_r(0)$, therefore (4) \Rightarrow (3).

We prove that the assertion is true when u has the form of (5). For any fixed $-1 < \alpha \leq 1$, put $u(z) = (1 - |z|^2)^\alpha$ and making a change of variable, then

$$\varepsilon_r(\mu) = \sup \left(\int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} dm(z) \right)^{-1}$$

$$\times \left(\int_{D \setminus D_r(0)} (1-|z|^2)^\alpha |1-\bar{a}z|^{-2\alpha} dm(z) \right).$$

When $0 \leq \alpha \leq 1$, since $0 < 1-|z|^2 \leq 1$, we have that

$$\begin{aligned} & \int_D (1-|z|^2)^\alpha |1-\bar{a}z|^{-2\alpha} dm \\ & \geq 2^{-2\alpha} \int_D (1-|z|^2) dm \\ & = \text{constant.} \end{aligned}$$

If $-1 < \alpha < 0$, then the familiar inequality between the harmonic and arithmetic means shows that

$$\begin{aligned} & \int_D (1-|z|^2)^\alpha |1-\bar{a}z|^{-2\alpha} dm \\ & \geq \left(\int_D (1-|z|^2)^{-\alpha} |1-\bar{a}z|^{2\alpha} dm \right)^{-1} \\ & \geq \text{constant.} \end{aligned}$$

Here, the last inequality follows from Rudin's lemma (cf.[9;p53]). Again using Rudin's lemma, since $-1 < \alpha \leq 1$, there exists $\beta > 0$ such that a set of functions $\{ [(1-|z|^2)^\alpha |1-\bar{a}z|^{-2\alpha}]^{1+\beta}; a \in D \}$ is bounded in L^1 . This implies that the set of these functions are uniformly integrable (cf.[1;p120]), therefore it follows that $\varepsilon_r(\mu) \rightarrow 0 (r \rightarrow \infty)$.

We show that $\varepsilon_r(\mu) \rightarrow 0$ when u has the form of (6). As in the proof of (2) of proposition 5, we only prove that $\varepsilon_r(\mu) \rightarrow 0 (r \rightarrow \infty)$ when $u = p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}$, where $p_1(z) = |z - b_1|$, $p_2(z) = |z - b_2|$, $0 < \alpha(1)$, $\alpha(2) < 2$, and b_1 is in D , b_2 is in ∂D . We suppose that B_j , M_1 , and ε are as in the proof of (2) of proposition 5. By the definition of $\varepsilon_r(\mu)$, we have that

$$\varepsilon_r(\mu) = \sup (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times \tilde{u}(a)^{-1}.$$

Moreover,

$$\begin{aligned}
& (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times \tilde{u}(a)^{-1} \\
& \leq (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times (u^{-1}) \tilde{\sim}(a) \\
& \leq (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times \varepsilon^{-\alpha(1) - \alpha(2)} \int_{B_0} |k_a|^2 dm \\
& \quad + (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times \varepsilon^{-\alpha(2)} \cdot (p_1^{-\alpha(1)}) \tilde{\sim}(a) \\
& \quad + M_1 \times \varepsilon^{-\alpha(1)} \times C \int_{D \setminus D_r(0)} |1 - \bar{a}z|^{-\alpha(2)} dm,
\end{aligned}$$

where

$$C = \|\phi_a(b_2) - z\|_\infty^{\alpha(2)} \times \|1 - \bar{a}z\|_\infty^{\alpha(2)} \times \int_{2D} |w|^{-\alpha(2)} dm.$$

Since u is bounded, therefore $\{u \circ \phi_a; a \in D\}$ is uniformly integrable (cf. [1; p120]), moreover applying the same argument in the proof of this proposition when u has the form of (5), Rudin's lemma implies that a set of functions $\{|1 - \bar{a}z|^{-\alpha(2)}; a \in D\}$ is also uniformly integrable, hence we conclude that $\varepsilon_r(\mu) \rightarrow 0 (r \rightarrow \infty)$. The proof of the latter half of (6) of this proposition is similar that in the above.

If u has the form of (7), then by the similar arguments in the proof of (3) of proposition 5, set $j(+) = \{j; a(j) \geq 0\}$, $j(-) = \{j; a(j) < 0\}$. And put $u_1 = \prod_{j(+)} p_j^{a(j)}$, $u_2 = \prod_{j(-)} p_j^{a(j)}$, then

$$\begin{aligned}
& (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times \tilde{u}(a)^{-1} \\
& \leq (u \chi_{D_r(a)^c}) \tilde{\sim}(a) \times (u^{-1}) \tilde{\sim}(a) \\
& = (u_1 u_2 \chi_{D_r(a)^c}) \tilde{\sim}(a) \times (u_1^{-1} u_2^{-1}) \tilde{\sim}(a).
\end{aligned}$$

Therefore, the desired result follows from the Cauchy-Schwarz's inequality and (6) of this proposition. ■

Corollary 3. Suppose that $d\nu = \nu dm$ and there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 d\mu$$

for all a in D , then the following are true.

(1) If $\nu(z) = (1 - |z|^2)^\alpha$ for some $-1 < \alpha \leq 1$, and there exist $l > 0$ and $\gamma' = \gamma_{l'} > 0$ such that

$$\widehat{\mu}_l(a) \leq \gamma' (1 - |a|^2)^\alpha$$

for all a in D , then there exist $r > 0$ and $\gamma = \gamma_r > 0$ such that

$$(1 - |a|^2)^\alpha \leq \gamma \widehat{\mu}_r(a)$$

for all a in D .

(2) If $\nu = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $0 < \alpha(j) < 2$ for b_j in $D \cup \partial D$ or $-2 < \alpha(j) < 0$ for b_j in $D \cup \partial D$, and if there exist $l > 0$ and $\gamma' = \gamma_{l'} > 0$ such that

$$\widehat{\mu}_l(a) \leq \gamma' \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$$

for all a in D , then there exist $r > 0$ and $\gamma = \gamma_r > 0$ such that

$$\prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma \widehat{\mu}_r(a)$$

for all a in D , where $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

(3) If $\nu = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $-1 < \alpha(j) < 1$ for b_j in $D \cup \partial D$, and if there exist $l > 0$ and $\gamma' = \gamma_{l'} > 0$ such that

$$\widehat{\mu}_l(a) \leq \gamma' \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$$

for all a in D , then there exist $r > 0$ and $\gamma = \gamma_r > 0$ such that

$$\prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma \widehat{\mu}_r(a)$$

for all a in D , where $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

Proof. We show that (1) is true. By the fact in the proof of corollary 1, and the fact that $u(a) = (1 - |z|^2)^\alpha$ satisfies the (A_2) -condition for all $\alpha > -1$ (see[6]), the hypothesis in (1) of the

corollary and proposition 1 imply the (μ, ν) -Carleson inequality. Hence, theorem 7 and proposition 9 show that the assertion is true.

Similarly, (2) and (3) follow from proposition 1, lemma 2, (5) of proposition 4, theorem 6, theorem 7, and proposition 9. ■

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