Buchsbaum complexes with
linear resolutions

Takayuki Hibi

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Buchsbaum complexes with linear resolutions

Takayuki Hibi *
Department of Mathematics
Faculty of Science
Hokkaido University
Kita-ku, Sapporo 060, Japan
E-mail:hibi@math.hokudai.ac.jp

Abstract

We study the Betti numbers which appear in a minimal free resolution of the Stanley-Reisner ring $k[\Delta]$ of a Buchsbaum complex Δ and find a characterization of Buchsbaum complexes Δ for which a minimal free resolution of $k[\Delta]$ is linear.

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1 Background and results

(1.1) A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \sigma$. Each element σ of Δ is called a *face* of Δ. Let $\sharp(\sigma)$ denote the cardinality of a finite set σ. We set $d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$. We say that Δ is *pure* if $\sharp(\sigma) = d$ for every maximal face σ of Δ.

Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. On the other hand, if σ is a face of Δ, then we define the subcomplexes $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{star}_\Delta(\sigma)$ to be

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$$

$$\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta\}.$$

Thus, in particular, $\mathrm{link}_\Delta(\emptyset) = \mathrm{star}_\Delta(\emptyset) = \Delta$.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

(1.2) Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k. Consult [Bru–Her], [H_1], [Hoc], [Sta] for the detailed information about Stanley-Reisner rings.

In what follows, we consider A to be the graded ring $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ as a graded module over A with the quotient grading.

Let $A(j)$, $j \in \mathbf{Z}$, denote the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Here $\mathbf{Z}$ is the set of integers.

(1.3) A graded *finite free resolution* of $k[\Delta]$ over A is an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^{\beta_h} A(-a_{h_j}) \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j=1}^{\beta_1} A(-a_{1_j}) \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \qquad (1)$$

of graded modules over A, where $\bigoplus_{j=1}^{\beta_i} A(-a_{i_j})$ with each $a_{i_j} \in \mathbf{Z}$, $1 \leq i \leq h$, is a graded free module of rank β_i $(< \infty)$, and where every φ_i is degree-preserving. The *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the minimal h possible in (1). It is known that $v - d \leq \mathrm{hd}_A(k[\Delta]) \leq v$. The second inequality is Hilbert's syzygy theorem ([Bru–Her, Corollary (2.2.14)]), and the first inequality follows from Auslander-Buchsbaum formula ([Bru–Her, Theorem (1.3.3)]). A finite free resolution (1) is called *minimal* if each β_i is smallest possible. A minimal free resolution of $k[\Delta]$ over A exists and is essentially unique. See, e.g., [Bru–Her, p. 35] and [H_3, Section 5].

We say that a simplicial complex Δ is *Cohen-Macaulay* over a field k if $\mathrm{hd}_A(k[\Delta]) = v - d$. On the other hand, a simplicial complex Δ is called *Buchsbaum* over k if Δ is 'locally' Cohen-Macaulay (i.e., $\mathrm{link}_\Delta(\{x_i\})$ is Cohen-Macaulay for every $1 \leq i \leq v$) over k and is pure.

(1.4) Suppose that a finite free resolution (1) is minimal with $h = \mathrm{hd}_A(k[\Delta])$. We say that $\beta_i = \beta_i^A(k[\Delta])$ is the *i-th Betti number* of $k[\Delta]$ over A. Hochster's formula [Hoc, Theorem (5.1)] guarantees that

$$\bigoplus_{j=1}^{\beta_i} A(-a_{i_j}) \cong \bigoplus_{W \subset V} A(-\sharp(W))^{\dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W;k)} \qquad (2)$$

as graded modules over A. Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k) \qquad (3)$$

for every $1 \leq i \leq h$. Hence Δ is Cohen-Macaulay over k (i.e., $\beta_i = 0$ for every $i > v - d$) if and only if $\tilde{H}_{i-\sharp(W)}(\Delta_{V-W}; k) = 0$ for every subset $W \subset V$ and for each $i < d - 1$.

A minimal free resolution (1) is called *q-linear* if $a_{i_j} = (q-1) + i$ for every $1 \leq i \leq h$ and $1 \leq j \leq \beta_i$. We say that $k[\Delta]$ has a q-linear resolution if a graded minimal free resolution of $k[\Delta]$ over A is q-linear. Thus, in particular, if $k[\Delta] = A/I_\Delta$ has a q-linear resolution, then I_Δ is generated by square-free monomials of degree q. We say that a simplicial complex Δ has a q-linear resolution over k if $k[\Delta]$ has a q-linear resolution.

(1.5) We here summarize some fundamental material on Cohen-Macaulay complexes. Let Δ be a simplicial complex of dimension $d-1$ and $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ, i.e., f_i is the number of faces σ of Δ with $\sharp(\sigma) = i+1$. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$$

with $f_{-1} := 1$. Recall that the *Hilbert series* of $k[\Delta] = \bigoplus_{n\geq 0}(k[\Delta])_n$ is the formal power series $F(k[\Delta], \lambda) = \sum_{n=0}^{\infty}(\dim_k(k[\Delta])_n)\lambda^n$ in λ. Then

$$F(k[\Delta], \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^d}.$$

The h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of a Cohen-Macaulay complex Δ is non-negative, i.e., each $h_i \geq 0$. Every Cohen-Macaulay complex is pure. Moreover, for each face σ of a Cohen-Macaulay complex Δ, $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{star}_\Delta(\sigma)$ are Cohen-Macaulay.

If Δ is a Cohen-Macaulay complex of dimension $d-1$ with the h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$, then the non-positive integer

$$a(\Delta) := -d + \max\{i \mid h_i \neq 0\}$$

is called the *a-invariant* of Δ. In other words, the a-invariant $a(\Delta)$ of Δ is just the degree of the rational function $F(k[\Delta], \lambda)$ in λ.

It follows from (1.4) that a Cohen-Macaulay complex Δ over k of dimension $d-1$ satisfies $\tilde{H}_i(\Delta; k) = 0$ for every $i < d-1$. Moreover, a Buchsbaum complex Δ of dimension $d-1$ is Cohen-Macaulay if and only if $\tilde{H}_i(\Delta; k) = 0$ for every $i < d-1$. On the other hand, given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_{d-1}) \in \mathbf{Z}^d$ with each $\delta_i \geq 0$, there exists a simplicial complex Δ of dimension $d-1$ such that, for an arbitrary field k, Δ is Buchsbaum over k with $\dim_k \tilde{H}_i(\Delta; k) = \delta_i$ for every $0 \leq i \leq d-1$ ([Bjö–H]).

Again, we refer the reader to [Bru–Her], [H_1], [Hoc], [Sta] for further developments on Cohen-Macaulay complexes.

We are now in the position to state our main results in the paper.

(1.6) THEOREM. *Fix integers t and d with $1 \leq t < d$. Let Δ be a simplicial complex of dimension $d-1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ and $k[\Delta] = A/I_\Delta$ the Stanley-Reisner ring of Δ. Suppose that Δ is Buchsbaum, but not Cohen-Macaulay, and that I_Δ is generated by square-free monomials of degree $t+1$. Then $k[\Delta]$ has a $(t+1)$-linear resolution if and only if the following conditions are satisfied:*

(i) $\tilde{H}_i(\Delta; k) = 0$ *if* $i \neq t-1$;

(ii) $a(\mathrm{link}_\Delta(\{x_i\})) \leq -(d-t-1)$ *for every* $1 \leq i \leq v$.

(1.7) THEOREM. *Let Δ be a Buchsbaum complex of dimension $d-1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ and $k[\Delta] = A/I_\Delta$ the Stanley-Reisner ring of Δ. Then, for every integer i with $v - d < i \leq v$, we have*

$$\beta_i^A(k[\Delta]) = \sum_{j=0}^{v} \binom{v}{j} \dim_k \tilde{H}_{v-i-j-1}(\Delta; k).$$

It follows immediately from Theorem (1.7) (and, in fact, is well known) that

$$\mathrm{hd}_A(k[\Delta]) = v - \min[\{i \mid \tilde{H}_i(\Delta; k) \neq 0\} \bigcup \{d-1\}] - 1$$

for an arbitrary Buchsbaum complex Δ of dimension $d-1$.

After summarizing some fundamental techniques in Section 2, our proofs of Theorems (1.6) and (1.7) are given in Sections 3 and 4. We conclude this paper with making a collection of examples in Section 5. See also [Bru–H] for a related topic on Cohen-Macaulay partially ordered sets.

2 Fundamental techniques

Some basic techniques for the proofs of Theorems (1.6) and (1.7) are summarized in this section.

(2.1) LEMMA. *Let Δ be a simplicial complex on the vertex set V. Fix $x \in V$ and set $\Delta' = \mathrm{link}_\Delta(\{x\})$. Then, there exists a long exact sequence of vector spaces over a field k as follows:*

$$\begin{array}{ll} 0 & \longrightarrow \tilde{H}_{-2}(\Delta';k) \longrightarrow \tilde{H}_{-1}(\Delta;k) \longrightarrow \tilde{H}_{-1}(\Delta_{V-\{x\}};k) \\ & \longrightarrow \tilde{H}_{-1}(\Delta';k) \longrightarrow \tilde{H}_{0}(\Delta;k) \longrightarrow \tilde{H}_{0}(\Delta_{V-\{x\}};k) \\ & \longrightarrow \cdots \\ & \longrightarrow \tilde{H}_{i-1}(\Delta';k) \longrightarrow \tilde{H}_{i}(\Delta;k) \longrightarrow \tilde{H}_{i}(\Delta_{V-\{x\}};k) \\ & \longrightarrow \cdots \end{array}$$

Proof. The above long exact sequence is just the degree $(0,0,\ldots,0) \in \mathbf{Z}^v$ part, $v = \sharp(V)$, of the long exact sequence of local cohomology modules [H_3, Corollary (14.2)]. Q. E. D.

The above Lemma (2.1) plays an essential role for the computation of the Cohen-Macaulay type (i.e., the $(v-d)$-th Betti number) of a Cohen-Macaulay complex, see [H_2].

(2.2) LEMMA. *Let Δ be a Cohen-Macaulay complex of dimension $d-1$ on the vertex set V. Then $\tilde{H}_i(\Delta_{V-W};k) = 0$ for every subset W of V and for each $i < d-1-\sharp(W)$.*

Proof. Thanks to (1.4), we know $\tilde{H}_{i-\sharp(W)}(\Delta_{V-W};k) = 0$ for every subset W of V and for each $i < d-1$. Hence, $\tilde{H}_j(\Delta_{V-W};k) = 0$ for every subset W of V and for each $j < d-1-\sharp(W)$ as desired. Q. E. D.

The proof of the following Lemma (2.3) contains some standard techniques in the study of minimal free resolutions of Stanley-Reisner rings of Cohen-Macaulay complexes.

(2.3) LEMMA. *Suppose that Δ is a Cohen-Macaulay complex of dimension $d-1$ on the vertex set V. Fix $0 \le \rho \in \mathbf{Z}$ and let $a(\Delta)$ be the a-invariant of Δ defined in (1.5). Then $a(\Delta) \le -\rho$ if and only if $\tilde{H}_j(\Delta_W;k) = 0$ for every subset W of V and for each $j > d-\rho-1$.*

Proof. Let $v = \sharp(V)$ and suppose that a finite free resolution (1) with $h = v - d$ is a minimal free resolution of $k[\Delta]$ over A. We define $c_i = \min\{a_{i_j} \mid 1 \leq j \leq \beta_i\}$ and $e_i = \max\{a_{i_j} \mid 1 \leq j \leq \beta_i\}$ for each $1 \leq i \leq h$. Then $0 < c_1 < c_2 < \cdots < c_h$ since (1) is minimal. Moreover, it is known, e.g., [E–G, p. 128] that $0 < e_1 < e_2 < \cdots < e_h$ since Δ is Cohen-Macaulay over k. Hence

$$(1-\lambda)^v F(k[\Delta], \lambda) = 1 + \sum_{i=1}^{h} \sum_{j=1}^{\beta_i} (-1)^i \lambda^{a_{i_j}}$$

(cf. [Bru–Her, Lemma (4.1.3)]) is a polynomial in λ of degree e_h. Here, $F(k[\Delta], \lambda)$ is the Hilbert series of $k[\Delta]$. Thus $a(\Delta) = -v + e_h$. Hence $a(\Delta) \leq -\rho$ if and onply if $e_h \leq v - \rho$. Now, suppose that $\tilde{H}_j(\Delta_W; k) = 0$ for every subset W of V and for each $j > d - \rho - 1$. Thus, in particular, $\tilde{H}_{\sharp(W)-(v-d)-1}(\Delta_W; k) = 0$ if $\sharp(W) > v - \rho$. Hence, by Eq. (2), $e_h \leq v - \rho$. On the other hand, if $e_h \leq v - \rho$, then $e_i \leq d - \rho + i$. Hence, again by Eq. (2), $\tilde{H}_{\sharp(W)-i-1}(\Delta_W; k) = 0$ if $\sharp(W) > d - \rho + i$. Thus $\tilde{H}_j(\Delta_W; k) = 0$ for every subset W of V and for each $j > d - \rho - 1$ as required. Q. E. D.

3 Proof of Theorem (1.6)

First, we recall our situation of Theorem (1.6). Fix integers t and d with $1 \leq t < d$. Let Δ be a simplicial complex of dimension $d - 1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ and $k[\Delta] = A/I_\Delta$ the Stanley-Reisner ring of Δ. Suppose that Δ is a Buchsbaum complex which is not Cohen-Macaulay and that the ideal I_Δ is generated by square-free monomials of degree $t + 1$.

By virtue of Eq. (2), the Stanley-Reisner ring $k[\Delta]$ has a $(t+1)$-linear resolution if and only if, for every subset W of the vertex set V, we have $\tilde{H}_i(\Delta_W; k) = 0$ if $i \neq t - 1$. Thus, in particular, if $k[\Delta]$ has a $(t+1)$-linear resolution, then $\tilde{H}_i(\Delta; k) = 0$ if $i \neq t - 1$. Moreover, $\tilde{H}_{t-1}(\Delta; k) \neq 0$ since Δ is Buchsbaum but not Cohen-Macaulay.

(3.1) LEMMA. *Fix $x \in V$ and let Δ' denote* $\operatorname{link}_\Delta(\{x\})$. *Then $a(\Delta') \leq -(d - t - 1)$ if and only if $\tilde{H}_i(\Delta'_W; k) = 0$ for every subset W of V and for each i with $i \neq t - 2$, $i \neq t - 1$.*

Proof. Since the subcomplex Δ' is Cohen-Macaulay of dimension $d - 2$, thanks to Lemma (2.3), $a(\Delta') \leq -(d - t - 1)$ if and only if $\tilde{H}_i(\Delta'_W; k) = 0$

for every subset W of V and for each $i > (d-1)-(d-t-1)-1 = t-1$. On the other hand, every subset τ of $V-\{x\}$ with $\sharp(\tau) \leq t-1$ belongs to Δ', thus $\tilde{H}_i(\Delta'_W;k) = 0$ if $W \subset V$ and $i < t-2$. Hence $a(\Delta') \leq -(d-t-1)$ if and only if $\tilde{H}_i(\Delta'_W;k) = 0$ for every subset W of V and for each i with $i \neq t-2$, $i \neq t-1$ as required. Q. E. D.

Now, suppose that $k[\Delta]$ has a $(t+1)$-linear resolution. Fix $x \in V$ and let Δ' denote the subcomplex $\text{link}_\Delta(\{x\})$. Since $\tilde{H}_i(\Delta_W;k) = 0$ for every subset W of V and for each $i \neq t-1$, replacing Δ with $\Delta_{W\cup\{x\}}$ in the long exact sequence of Lemma (2.1), we obtain $\tilde{H}_i(\Delta'_W;k) = 0$ for every subset W of V and for each i with $i \neq t-2$, $i \neq t-1$. Hence, by Lemma (3.1), $a(\Delta') \leq -(d-t-1)$ as desired.

On the other hand, suppose that $a(\text{link}_\Delta(\{x\})) \leq -(d-t-1)$ for every $x \in V$ and that $\tilde{H}_i(\Delta;k) = 0$ if $i \neq t-1$. We show that $\tilde{H}_i(\Delta_W;k) = 0$ for each $i \neq t-1$ and for every subset W of V. Let W be a subset of V with $W \neq V$ and suppose that $\tilde{H}_i(\Delta_{W\cup\{x\}};k) = 0$ for every $x \in V-W$ and for each $i \neq t-1$. Let $x \in V-W$ and $\Delta' = \text{link}_\Delta(\{x\})$. Since $\tilde{H}_i(\Delta'_W;k) = 0$ if $i \neq t-2$, $i \neq t-1$ by Lemma (3.1) (and noting that $\text{link}_{\Delta_{W\cup\{x\}}}(\{x\}) = \Delta'_{W\cup\{x\}} = \Delta'_W$), thanks to Lemma (2.1), we obtain $\tilde{H}_i(\Delta_W;k) = 0$ if $i \neq t-2$, $i \neq t-1$. Moreover, since every subset σ of W with $\sharp(\sigma) \leq t$ is a face of Δ_W, we have $\tilde{H}_i(\Delta_W;k) = 0$ for each $i < t-1$. Thus $\tilde{H}_i(\Delta_W;k) = 0$ if $i \neq t-1$. Hence $k[\Delta]$ has a $(t+1)$-linear resolution.

4 Proof of Theorem (1.7)

Let Δ be a Buchsbaum complex of dimension $d-1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ and $k[\Delta] = A/I_\Delta$ the Stanley-Reisner ring of Δ over k. Our Theorem (1.7) follows immediately from Lemma (4.1) below, which is a result among the various consequences of Lemma (2.1), together with the formula (3).

(4.1) LEMMA. *$\tilde{H}_i(\Delta;k) \cong \tilde{H}_i(\Delta_{V-W};k)$ for every subset W of V and for each $i < d-1-\sharp(W)$.*

Proof. First, fix $x \in V$ and let $\Delta' = \text{link}_\Delta(\{x\})$. Since Δ' is Cohen-Macaulay of dimension $d-2$, we have $\tilde{H}_i(\Delta';k) = 0$ for every $i < d-2$. Thus, the long exact sequence of Lemma (2.1) guarantees $\tilde{H}_i(\Delta; k) \cong \tilde{H}_i(\Delta_{V-\{x\}}; k)$ for every $i < d-2$. Now, let $\sharp(W) > 1$ and fix $y \in W$. Set $W' = W - \{y\}$, and we may assume that $\tilde{H}_i(\Delta; k) \cong \tilde{H}_i(\Delta_{V-W'}; k)$ for each $i < d - 1 - \sharp(W') = d - \sharp(W)$. Let $\Delta' = \mathrm{link}_\Delta(\{y\})$. Then, by Lemma (2.2), $\tilde{H}_i(\Delta'_{V-W'}; k) = 0$ for each $i < d-2-\sharp(W') = d-1-\sharp(W)$. Thus, if we replace Δ with $\Delta_{V-W'}$ and x with y in Lemma (2.1), then $\tilde{H}_i(\Delta_{V-W'}; k) \cong \tilde{H}_i(\Delta_{V-W}; k)$ for each $i < d-1-\sharp(W)$. Hence $\tilde{H}_i(\Delta; k) \cong \tilde{H}_i(\Delta_{V-W}; k)$ for each $i < d-1-\sharp(W)$ as required. Q. E. D.

We now turn to the proof of Theorem (1.7). By virtue of the formula (3), we have

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{v-\sharp(W)-i-1}(\Delta_{V-W}; k).$$

Moreover, if $i > v - d$ then

$$v - \sharp(W) - i - 1 < d - 1 - \sharp(W),$$

thus

$$\tilde{H}_{v-\sharp(W)-i-1}(\Delta_{V-W}; k) \cong \tilde{H}_{v-\sharp(W)-i-1}(\Delta; k)$$

by Lemma (4.1). Hence, for each $i > v - d$, we obtain

$$\begin{aligned} \beta_i^A(k[\Delta]) &= \sum_{W \subset V} \dim_k \tilde{H}_{v-\sharp(W)-i-1}(\Delta; k) \\ &= \sum_{j=0}^{v} \binom{v}{j} \dim_k \tilde{H}_{v-i-j-1}(\Delta; k). \end{aligned}$$

as desired.

5 Some examples

(5.1) Let Δ be a Cohen-Macaulay complex of dimension $d-1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ and $k[\Delta] = A/I_\Delta$ the Stanley-Reisner ring of Δ. Then $k[\Delta]$ has a q-linear resolution if and only if every subset σ of V with $\sharp(\sigma) < q$ belongs to Δ and $a(\Delta) = -(d-q+1)$. Let $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ be the f-vector and $h(\Delta) = (h_0, h_1, \ldots, h_d)$ the h-vector of Δ defined in (1.5). Then $k[\Delta]$ has a q-linear resolution if and only if $f_i = \binom{v}{i+1}$ for every

$i \leq q-2$ and $h_{q-1} \neq 0$, $h_q = \cdots = h_d = 0$. In other words, $k[\Delta]$ has a q-linear resolution if and only if $h_i = \binom{v-d+i-1}{i}$ for every $1 \leq i \leq q-1$ and $h_q = \cdots = h_d = 0$. Thus, in particular, $k[\Delta]$ has a 2-linear resolution if and only if $h(\Delta) = (1, v-d, 0, \ldots, 0)$.

(5.2) Let Δ be the simplicial complex (cf. [Rei]) on the vertex set $V = \{1,2,3,4,5,6\}$ whose maximal faces are $\{1,2,4\}$, $\{1,2,6\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,5,6\}$, $\{2,3,5\}$, $\{2,3,6\}$, $\{2,4,5\}$, $\{3,4,6\}$ and $\{4,5,6\}$. If $\mathrm{char}(k) \neq 2$, then Δ is Cohen-Macaulay over k with $h(\Delta) = (1,3,6,0)$. Hence $k[\Delta] = A/I_\Delta$ has a 3-linear resolution. On the other hand, if $\mathrm{char}(k) = 2$, then Δ is not Cohen-Macaulay (but Buchsbaum), and a minimal free resolution of $k[\Delta]$ over A is not linear.

(5.3) Let Δ be a Cohen-Macaulay complex of dimension $d-1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ and suppose that a finite free resolution (1) of $k[\Delta]$ over A is minimal with $h = v-d$. Let $\mathcal{B}_i^A(k[\Delta])$, $1 \leq i \leq v-d$, denote the set $\{s \in \mathbf{Z} \mid a_{i_j} = s+i \text{ for some } 1 \leq j \leq \beta_i\}$. Then, if follows from the inequalities $c_i < c_{i+1}$ and $e_i < e_{i+1}$, $1 \leq i < h$, in the proof of Lemma (2.3) that $\mathcal{B}_i^A(k[\Delta]) \subset \{1, 2, \ldots, d\}$ for every $1 \leq i \leq v-d$. Thus, in particular, $k[\Delta]$ has a q-linear resolution if and only if $\mathcal{B}_i^A(k[\Delta]) = \{q-1\}$ for every $1 \leq i \leq v-d$.

Let Δ be a Cohen-Macaulay complex of dimension $(d-1)+1 (\geq 2)$ on the vertex set V with $\sharp(V) = v+1$ such that, for each $x \in V$, there exists a maximal face σ of Δ with $x \notin \sigma$. Suppose that $k[\Delta]$ has a 2-linear resolution and let W be a subset of V with $\tilde{H}_{\sharp(W)-(v-d)-1}(\Delta_W; k) \neq 0$. Fix $x \in V - W$ and define $\Delta' = \{\sigma \in \Delta \mid \sharp(\sigma) \leq d \text{ and } x \notin \sigma\}$. It follows from, e.g., [H$_3$, p. 39] that Δ' is a Cohen-Macaulay complex of dimension $d-1$ on the vertex set $V' = V - \{x\}$ with $\sharp(V') = v$. Since $\tilde{H}_{\sharp(W)-i-1}(\Delta_W; k) \cong \tilde{H}_{\sharp(W)-i-1}(\Delta'_W; k)$ for every subset W of V' with $\sharp(W) < d+i$, we have $\mathcal{B}_i^A(k[\Delta']) \bigcap \{1, 2, \ldots, d-1\} = \{1\}$ for every $1 \leq i \leq v-d$. Moreover, since $x \notin \sigma$ for some maximal face σ of Δ, we have $d \in \mathcal{B}_1^A(k[\Delta'])$. Thus $\mathcal{B}_i^A(k[\Delta']) = \{1, d\}$ for every $1 \leq i \leq v-d$.

Fix v and d with $v > d \geq 2$. Then, it might be of interest to find all subsets S of $\{1, 2, \ldots, d\}$ such that there exists a Cohen-Macaulay complex Δ of dimension $d-1$ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ with $\mathcal{B}_i^A(k[\Delta]) = S$

for every $1 \leq i \leq v - d$.

(5.4) Let Δ denote the simplicial complex discussed in (5.2) and set $\Delta' = \Delta - \{4,5,6\}$. Then Δ' is Buchsbaum over an arbitrary field k with $h(\Delta') = (1,3,6,-1)$. We easily see that $\tilde{H}_i(\Delta'; k) = 0$ for $i = -1, 0, 2$ and $\tilde{H}_1(\Delta'; k) \cong k$. Moreover, $a(\text{link}_{\Delta'}(\{j\}) = 0$ for $j = 1,2,3$, and $a(\text{link}_{\Delta'}(\{j\}) = -1$ for $j = 4,5,6$. Since the ideal $I_{\Delta'}$ is generated by square-free monomials of degree 3, thanks to Theorem (1.6), $k[\Delta'] = A/I_{\Delta'}$ has a 3-linear resolution with the Betti numbers $\beta_1 = 11, \beta_2 = 18, \beta_3 = 9$ and $\beta_4 = 1$. Note that $k[\text{link}_{\Delta'}(\{j\})]$ has a 2-linear resolution for $j = 4,5,6$, however, $k[\text{link}_{\Delta'}(\{j\})]$ does not have a linear resolution for $j = 1,2,3$.

(5.5) Let Δ (resp. Δ') denote the simplicial complex on the vertex set $V = \{1,2,3,4,5,6,7,8\}$ whose maximal faces are $\{1,2,4\}$, $\{2,3,4\}$, $\{5,6,8\}$ and $\{6,7,8\}$ (resp. $\{1,2,3\}$, $\{4,5,8\}$, $\{4,7,8\}$, $\{5,6,8\}$ and $\{6,7,8\}$). Then, both Δ and Δ' are Buchsbaum with $\tilde{H}_0(\Delta; k) = \tilde{H}_0(\Delta'; k) \cong k$ and $\tilde{H}_i(\Delta; k) = \tilde{H}_i(\Delta'; k) = 0$ for every $i \neq 0$. Moreover, both I_Δ and $I_{\Delta'}$ are generated by square-free monomials of degree 2. Since $a(\text{link}_\Delta(\{j\}) = -2$ for $j = 1,3,5,7$ and $a(\text{link}_\Delta(\{j\}) = -1$ for $j = 2,4,6,8$, by Theorem (1.6), $k[\Delta]$ has a 2-linear resolution. On the other hand, $k[\Delta']$ does not have a linear resolution since $a(\text{link}_\Delta(\{8\}) = 0$.

(5.6) We give an example of a Buchsbaum complex with a linear resolution for $v = 7$, $d = 4$ and $t = 2$ in the notation of Theorem (1.6).

Let Δ be the simplicial complex on the vertex set $V = \{1,2,3,4,5,6,7\}$ which has the maximal faces $\{1,2,3,4\}$, $\{2,3,4,5\}$, $\{3,4,5,6\}$, $\{4,5,6,7\}$, $\{5,6,7,1\}$, $\{6,7,1,2\}$ and $\{7,1,2,3\}$. Then, the f-vector of Δ is $f(\Delta) = (7,21,21,7)$ and the h-vector of Δ is $h(\Delta) = (1,3,6,-4,1)$. The ideal I_Δ is generated by 125, 236, 347, 451, 562, 673, 714, 135, 246, 357, 461, 572, 613 and 724. The maximal faces of $\Delta' = \text{link}_\Delta(\{1\})$ are $\{2,3,4\}$, $\{2,3,7\}$, $\{2,6,7\}$ and $\{5,6,7\}$. Hence Δ' is Cohen-Macaulay with $h(\Delta') = (1,3,0,0)$ and $a(\Delta') = -2$. (Note that $\text{link}_\Delta(\{j\})$ coincides with Δ' for every $1 < j \leq 7$. Moreover, $k[\Delta']$ has a 2-linear resolution.) Thus Δ is Buchsbaum (but not Cohen-Macaulay since $h_3 < 0$). Let δ_i denote $\dim_k \tilde{H}_i(\Delta; k)$. Then $(\delta_0, \delta_1, \delta_2, \delta_3) = (0,1,0,0)$. Hence, thanks to Theorem (1.6), the Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ has a 3-linear resolution with the Betti numbers $\beta_1 = 14, \beta_2 = 28, \beta_3 = 21, \beta_4 = 7$ and $\beta_5 = 1$.

Given integers d and t with $d > t \geq 1$, it would be of interest to find a Buchsbaum complex of dimension $d-1$ with a $(t+1)$-linear resolution.

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