NOTES ON VERSAL DEFORMATION OF FIRST ORDER PDE AND WEB STRUCTURE

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Series ♯223. December 1993

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

NOTES ON VERSAL DEFORMATION OF FIRST ORDER PDE AND WEB STRUCTURE

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ABSTRACT. We apply Thom-Mather theory to the diagram of smooth map germs associated to first order partial differential equations. This reduces the problem of function moduli of infinite dimension for PDE as well as divergent diagrams of map germs to the theory of deformation of functions on singular varieties.

INTRODUCTION

A *first order partial differential equation* (PDE) (or *an implicit differential equation*) on $\mathbb{R}^n$ is a subvariety V of the projectivised cotangent space $PT^*\mathbb{R}^n$ (or $PT^*\mathbb{C}_n$) furnished with the canonical contact structure [4.9.24]. The subject of this note is to study the local topological structure of the PDE at the singular points of the projection of the variety to the base space $\mathbb{R}^n$. This subject was studied by many authors Thom [33], Dara [13], Takens, Arnol'd [1-6] (see the reference of the papers. A historical remark is seen in [4]). Recently Hayakawa-Ishikawa-Izumiya-Yamaguchi [21] found a new link of the first order PDE's with complete integral and the singularity theory of the so-called generating functions of legendrian submanifolds, and classified generic PDE with complete integral on the plane. The singularity theory of generating functions is in our setting coherent to the theory of deformation of functions on singular varieties. Izumiya [23] classified some more PDE's using the classification of functions on varieties by Goryunov [19]. In this note we explain this link clearly and develop the unfolding theory of PDE. We consider only the case dim $V = n$ and assume the V as well as the various spaces and mappings are C^∞ smooth (In the complex analytic case most arguments remain valid). Darboux's theorem asserts that contact structure on manifold is locally unique up to contact diffeomorphism. In this note $PT^*\mathbb{R}^n$ is replaced by the 1-jet bundle $J^1(\mathbb{R}^{n-1}, \mathbb{R})$ with the contact form $\omega = p\ dx\ -\ dy$, where x, y are respectively the coordinates of $\mathbb{R}^{n-1}, \mathbb{R}$ and p is the coordinate of $\mathbb{R}^{n-1*}$, the fibre of the projection $ev : J^1(\mathbb{R}^{n-1}, \mathbb{R}) \to \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$. Assume $V = \mathbb{R}^n$, $ev \circ \mathcal{I}(0) = 0 \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathcal{I} : \mathbb{R}^n \to J^1(\mathbb{R}^{n-1}, \mathbb{R})$ is an immersion.

Key words and phrases. First order defferential equiation, Hamilton-Jacobi equation, Web structure, Unfolding theory.

The direct image of the pull back $\mathcal{I}^*\omega$ under the projection $ev \circ \mathcal{I}$ defines a multi valued 1-form (implicit differential equation) on the *configuration space* : the base space $\mathbb{R}^{n-1} \times \mathbb{R}$. A *complete integral* is a germ of non singular smooth function $\lambda : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ such that $d\lambda \wedge \mathcal{I}^*\omega$ vanishes identically on a neighbourhood of 0. In this note we call a first order differential equation with complete integral a PDE for simplicity. Then the images $D_t = ev \circ \mathcal{I}(\lambda^{-1}(t)), t \in \mathbb{R}$, constitute the integral submanifolds (possibly singular) of the equation on a neighbourhood of $0 \in \mathbb{R}^n$. We call D_t a *solution* of the equation and call the family $\mathcal{W}_\mathcal{I} = \{D_t, t \in \mathbb{R}\}$ the *solution web* (or *web*) of the equation $\mathcal{I}$. Conversely, assuming the nonsingular points of $ev \circ \mathcal{I}$ and λ are dense, the submanifold V is the closure of the set of those $(x, y, p) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1*}$, p being conormal to a D_t at a smooth point (x, y). The subject of this note is to study the geometric structure of the solution webs as well as to classify the webs in terms of Thom-Mather theory.

We call two webs $\mathcal{W}_\mathcal{I} = \{D_t\}, \mathcal{W}_\mathcal{J} = \{D'_t\}$ of PDE's $\mathcal{I}, \mathcal{J}$ are C^∞ (resp.*topologically*) *equivalent* if there exists a germ of diffeomorphisms (resp. homeomorphisms) $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\psi(D_t) = D'_{k(t)}$ for t in a neighbourhood of $0 \in \mathbb{R}$. In other words $\mathcal{W}_\mathcal{I}, \mathcal{W}_\mathcal{J}$ are C^∞ equivalent if there exists a germ of a contact diffeomorphism $d\psi$ of the Legendre fibration $J^1(\mathbb{R}^{n-1}, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ sending the image of $\mathcal{I}$ to that of $\mathcal{J}$ [21,23]. When k is the identity, we say $\mathcal{I}, \mathcal{J}$ are *strictly* C^∞ (resp.*strictly topologically*) *equivalent*. We say that the solution web $\mathcal{W}_\mathcal{I}$ is a *non singular d-web at* $q \in \mathbb{R}^n$ if the number of t for which D_t passes through q is d, those solutions $D_{t_i}, i = 1, ..., d$, are smooth and meet in general position at q and D_t forms a non-singular foliation at q as t varies nearby t_i for each $i = 1, ..., d$. We call the maximum of such d for q nearby $0 \in \mathbb{R}^n$ the *web number*. In other words the web number is the topological degree of $ev \circ \mathcal{I}$ for generic $\mathcal{I}$ (Proposition 9.5). The singular locus $\mathrm{Sing}(\mathcal{W}_\mathcal{I})$ of $\mathcal{W}_\mathcal{I}$ is the set of those q where $\mathcal{W}_\mathcal{I}$ is not a nonsingular d-web, d being the web number. Dufour [15] proved that nonsingular d-webs, $n + 1 \leq d$ are topologically equivalent if and only if they are C^∞ equivalent. By an easy calculation we see that the C^∞ equivalence classes of d-webs, $n + 1 \leq d$, form subsets of infinite codimension in the jet space of d-tuples of level functions defined at $q \in \mathbb{R}^n$ (Theorem 10.1). So even topological classification fails for the solution webs. In fact Arnol'd [6], Carneiro [10], Dufour [15] and Hayakawa-Ishikawa-Izumiya-Yamaguchi [21] showed that C^∞ classes of some PDE have moduli of infinite dimension called the *function moduli*, which are parametrized by the space of smooth functions defined on the configuration space $\mathbb{R}^n$ at 0.

In §4 we develope the Thom-Mather theory for PDE with complete integral to understand the structure of function moduli, and we give a "*versal* PDE" $\mathcal{I}'$ (versal unfolding) defined on $\mathbb{R}^{n+s}$ for a generic PDE $\mathcal{I}$, which has the following properties.

(1) The solution web $\mathcal{W}_\mathcal{I}$ is induced from $\mathcal{W}_{\mathcal{I}'}$ by the natural imbedding $\mathbb{R}^n \to \mathbb{R}^{n+s}$ transverse to the solutions of $\mathcal{I}'$,

(2) For a deformation $\mathcal{I}_t$ of $\mathcal{I}$, there exists a family of imbeddings $i_t : \mathbb{R}^n \to \mathbb{R}^{n+s}$ transverse to the solutions of $\mathcal{I}'$ by which the solution web $\mathcal{W}_{\mathcal{I}_t}$ is induced from $\mathcal{W}_{\mathcal{I}'}$,

(3) $\mathcal{W}_{\mathcal{I}'}$ has the web number $\leq n + s$ and any deformation is trivial

(For an example of a versal PDE, see Example 2.3.)

Two PDE's are S-*equivalent* if their versal unfoldings are C^∞ equivalent. In §6 it is shown that the versal unfolding of a PDE $\mathcal{I}$ is determined by a *function on a variety* associated to $\mathcal{I}$, and two PDE's are S-*equivalent* if the associated functions on varieties are C^∞-equivalent (Corollary 6.4). This is an analogy of those results due to Mather [25] for stable unfoldings and to the author [29,30] for composite map germs. A result by Matsuoka [27] applies here to reduce the classification of functions on singular varieties to classifying morphisms of $\mathbb{R}$-algebras of finite dimension. Those results suggest that S-equivalence classes have moduli generically of finite dimension. (Recall that almost all smooth function germs on a smoth variety have moduli of finite dimension [1].) We say a PDE $\mathcal{I}$ is *simple* if the function associated to $\mathcal{I}$ is simple in the sense of the singularity theory.

In the complex analytic case the singular locus $\mathrm{Sing}(\mathcal{W}_\mathcal{I})$ of a versal and simple PDE is a hypersurface, of which the complement is a $K(\pi,1)$-space, where π is a subgroup of the braid group $B(d)$ of finite index and d is the web number of $\mathcal{W}_\mathcal{I}$ (Theorem 9.8).

To study the hierarchy of the singularities of $\mathcal{W}_\mathcal{I}$ the aouthor suggests the notion of *equisingularity* for PDE's as follows. Two germs of webs $\mathcal{W}_\mathcal{I}, \mathcal{W}_\mathcal{J}$ are *equisingular* if their versal unfoldings are topologically equivalent as composite map germs (see §10 for the precise definition). We say two webs have the same *contour* if their germs are equisingular at each q on a neighbourhod of $0 \in \mathbb{R}^n$, and the webs have the same *contour type* if there exists a germ of homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h(\mathcal{W}_\mathcal{I})$ and $\mathcal{W}_\mathcal{J}$ have the same contour. In the final section we prove that the contour type is generically determined by finite jet of the imbedding $\mathcal{I}$ and stable under deformation within the set of first order PDE's with complete integral (Theorem 11.1).

1. Preliminaries

In this section we recall the method to construct Legendre submanifolds of $J^1(\mathbb{R}^n, \mathbb{R})$ formulated by Zakalyukin [36] and ascribed to Arnold and Hörmander [22], and we apply the method to the PDE's with complete integrals following the idea due to Hayakawa-Ishikawa-Izumiya-Yamaguchi [21]. We begin with a brief exposition on the construction method of Legendrian submanifolds as follows. For a subvariety $X \subset \mathbb{R}^{n+1}$ of codimension 1, Nash modification $\tilde{X}$ of X is defined by the closure of the set of those pairs $(q, p) \in \mathbb{R}^{n+1} \times P^{n*} = PT^*\mathbb{R}^{n+1}$ of smooth points $q \in X$ and p conormal to the tangent space TX_q of X at q. It is easy to see that $\tilde{X}$ is a Legendre variety: the contact form vanishes identically on the smooth part of $\tilde{X}$. When X is presented as a discriminant set (critical value set) of a generic smooth map germ $h : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0$, $\tilde{X}$ is determined by the restriction of h to the critical point set $\Sigma(h)$. Roughly stating if h has corank 1 i.e. corank $dh(0) = 1$, then $\Sigma(h)$ is smooth and $\tilde{X}$ is a germ of Legendre submanifold.

From now on replace the projectivised cotangent space with the 1-jet space of function germs on $\mathbb{R}^n$, $J^1(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n*}$. Let $h : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0$ a germ of smooth map of corank 1. Then there exist germs of diffeomorphisms $\phi : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+k}, 0$, $\psi : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+1}, 0$ such that $h' = \psi \circ h \circ \phi$ is of the form $h'(z, x) = (x, h_x(z)), z \in \mathbb{R}^k, x \in \mathbb{R}^n$. Assume h is of this form and $(\partial h_x/\partial z_1, \ldots, \partial h_x/\partial z_k)|\mathbb{R}^n \times 0$ is non singular. Then $\Sigma(h)$ is a smooth submanifold of dimension n, on which h restricts to a finite-to-one and generically immersive mapping to the discriminant set $D(h)$ assuming a certain generic condition and the natural projection $D(h) = h(\Sigma(h))$ onto the x-space $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is diffeomorphic at a generic point. The tangent plane of $D(h)$ at such a generic point

$(x, h_x(z)) \in D(h)$ is given by the graph of the linear function

$$(\partial h_x/\partial x_1(z), \ldots, \partial h_x/\partial x_n(z)) \in \mathbb{R}^{n*}.$$

Therefore the Legendre submanifold $\tilde{X}$ associated to $D(h)$ is the image of the map

$$(x,\ h_x(z),\ \partial h_x/\partial x_1(z), \ldots, \partial h_x/\partial x_n(z))$$

of $\Sigma(h)$ into $J^1(\mathbb{R}^n, \mathbb{R})$. The family of functions h_x is called the *generating function.* On the other hand it is known to Zakalyukin that all germs of Legendre submanifolds of $J^1(\mathbb{R}^n, \mathbb{R})$ are obtained in this way (see e.g. [29,36]). We denote the Legendre submanifold above constructed by L_h. Two germs of Legendre submanifolds $L, L' \subset J^1(\mathbb{R}^n, \mathbb{R})$ are *contact equivalent* if there exists a germ of diffeomorphism $\phi : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+1}, 0$ such that $d\phi(L) = L'$, where $d\phi$ is the diffeomorphism of neighbourhoods of L, L' in the jet space induced from ϕ.

Two map germs $g = (x, g_x), h = (x, h_x) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}$ of corank 1 of the above form are *equivalent as families of subvarieties* $V_{x,y} = \{g_x = y\}, W_{x,y} = \{h_x = y\} \subset \mathbb{R}^n$ if there exist a family of germs of diffeomorphisms $\psi_{x,y}$ of $\mathbb{R}^k$, a germ of diffeomorphism $\phi = (\psi, \chi)$ of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with $\psi_{0,0}(0) = 0$, $\psi(0) = 0$ and a family of function germs $U_{x,y}$ on $\mathbb{R}^k$ with $U_{0,0}(0) \neq 0$ such that $U_{x,y}(h_x - y) = g_{\psi(x,y)} \circ \psi_{x,y} - \chi(x, y)$ for (x, y) on a neighbourhood of $0 \in \mathbb{R}^{n+k}$. Two map germs $g = (x, g_x) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, h = (x, h_x) : \mathbb{R}^{n+l}, 0 \to \mathbb{R}^{n+1}$ are *stably equivalent* if there exist non degenerate quadratic functions Q_1 Q_2 of respectively (m-k)-valuables and (m-l)-valuables such that the germs $(x, g_x + Q_1), (x, h_x + Q_2) : \mathbb{R}^{n+m}, 0 \to \mathbb{R}^{n+1}, 0$ are equivalent in the above sence.

Theorem 1.1 (Zakalykin[36]). *Two Lagrangians $L_h, L_k \subset J^1(\mathbb{R}^n, \mathbb{R})$ are contact equivalent if and only if h, k are stablly equivalent.*

Construction of Legendre submanifold using complete integral.

Let $\mathcal{I} = (\mathcal{I}_x, \mathcal{I}_y, \mathcal{I}_p) : \mathbb{R}^n \to J^1(\mathbb{R}^{n-1}, \mathbb{R}) = \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1*}$ be a germ of imbedding at $0 \in \mathbb{R}^n$ transverse to the contact elements and assume it admits a complete integral λ. Define the germ of Legendre imbedding

$$\tilde{\mathcal{I}} = (\mathcal{I}_x, \lambda, \mathcal{I}_y, \mathcal{I}_p, \alpha) : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n*}$$

with a function α in $\mathbb{R}^n$ satisfying $\tilde{\mathcal{I}}^*_{\tilde{\omega}} = \mathcal{I}^*\omega + \alpha\ d\lambda = 0$, where $\omega, \tilde{\omega}$ are respactively the canonical contact forms of the jet spaces $J^1(\mathbb{R}^{n-1}, \mathbb{R}), J^1(\mathbb{R}^n, \mathbb{R})$. Then by the above result the image of the $\tilde{\mathcal{I}}$ is identified with the image of a map

$$(x_1, \ldots, x_n, h_x, \partial h_x/\partial x_1, \ldots, \partial h_x/\partial x_n) : \Sigma(x, h_x) \to J^1(\mathbb{R}^n, \mathbb{R})$$

for an $h = (x, h_x) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0$ and a positive integer k. By this identification the divergent diagram

$$\mathbb{R} \xleftarrow{\ \lambda\ } \mathbb{R}^n \xrightarrow{ev \circ \mathcal{I} = (\mathcal{I}_x, \mathcal{I}_y)} \mathbb{R}^{n-1} \times \mathbb{R}$$

is identified with the restriction of the divergent diagram (which is called the *integral diagram* by Izumiya in [23])

$$(*) \qquad \mathbb{R} \xleftarrow{\;x_n\;} \Sigma(x,h_x) \xrightarrow{\;f=(x_1,\dots,x_{n-1},h_x)\;} \mathbb{R}^{n-1}\times\mathbb{R},$$

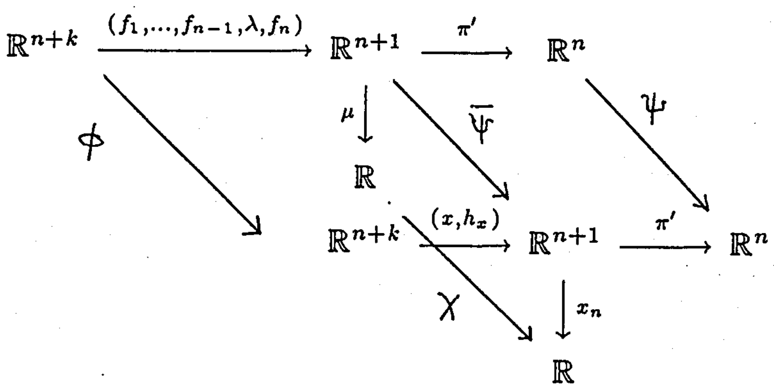
which is the restriction of the following divergent diagram to $\Sigma(x,h_x) \subset \mathbb{R}^{n+k}$

$$(**) \qquad \begin{array}{ccccc} \mathbb{R}^{n+k} & \xrightarrow{(x,h_x)} & \mathbb{R}^{n+1} & \xrightarrow{\pi'} & \mathbb{R}^{n-1}\times\mathbb{R} \\ & \searrow^{\lambda} & \downarrow{x_n} & & \\ & & \mathbb{R} & & \end{array}$$

where π' is the projection forgetting the n-th factors. Then the integral curve $\lambda^{-1}(t) \subset \Sigma(x,h_x)$ is identified with the critical point set of the restriction of f, $(x_1,\dots,x_{n-1},h_x) : \mathbb{R}^{n-1}\times t\times\mathbb{R}^k \to \mathbb{R}^{n-1}\times\mathbb{R}$. The solution D_t is the discriminant (critical value) set of the restriction, which is the transverse intersection of the discriminant $D(x,h_x)$ with $\mathbb{R}^{n-1}\times t\times\mathbb{R}$.

From PDE to divergent diagrams and families of functions on varieties.

Let $\lambda : \mathbb{R}^{n+k},0 \to \mathbb{R},0$, $f : \mathbb{R}^{n+k},0 \to \mathbb{R}^n,0$ be smooth map germs and assume that λ is nonsingular and corank $(f,\lambda) \le 1$. Permutating the entries of $f = (f_1,\dots,f_n)$ if necessary there exist germs of diffeomorphisms $\phi, \tilde{\psi}, \psi$ and χ such that the following diagram commutes



where h_x is a smooth family of functions. So we are led to the study of the divergent diagrams $(***)$ with corank $(f,x_n) = 1$,

$$(***) \qquad \mathbb{R} \xleftarrow{\;\lambda=x_n\;} \mathbb{R}^{n+k} \xrightarrow{\;f\;} \mathbb{R}^{n-1}\times\mathbb{R}.$$

Define the *solution* by $D_t = D(f,\lambda)\cap\mathbb{R}^n\times t$: the discriminant set of the restriction $f : \lambda^{-1}(t) \to \mathbb{R}^n$. We denote the family of the solutions $D_t, t\in\mathbb{R}$ by $\mathcal{W}_{f,\lambda}$ and call the *solution web* of (f,λ).

The diagram (***) is oftain regarded as a family of the restrictions of λ to the fibres of f,

$$\lambda_q : f^{-1}(q) \longrightarrow \mathbb{R}$$

with the parameter space $\mathbb{R}^n$, as in the theory of mappings a smooth map germ $h : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ is regarded as a family of the varieties $h^{-1}(q), q \in \mathbb{R}^n$. In the paper by Goryunov [19], λ_q is regarded as the one parameter family of the varieties $V_t = f^{-1}(q) \cap \lambda^{-1}(t), t \in \mathbb{R}$. (See [28] for the various results on the functions on varieties). By definition we obtain

Proposition 1.2. *Let $q \in \mathbb{R}^n - D(f)$. Then $q \in D_t$ if and only if λ_q has the critical value t.*

This suggests a relation of our unfolding theory to the stationary phase method. Two diagrams $(f, \lambda), (g, \mu)$ are *algebraically S-equivalent* if there esist an $\mathbb{R}$-algebra isomorphism $\phi* : \epsilon(n+k)/g \to \epsilon(n+k)/f$ and a germ of diffeomorphism χ of $\mathbb{R}, 0$ such that $\phi^*(\mu) = \chi \circ \mu$. Then there exists a germ of "diffeomorphism" ϕ of $\mathbb{R}^n, 0$ such that $\phi(f^{-1}(0)) = g^{-1}(0)$, ϕ induces ϕ^* and the following diagrams commutes

$$\begin{array}{ccc} \lambda_0 : f^{-1}(0) & \longrightarrow & \mathbb{R} \\ \phi \downarrow & & \downarrow \chi \\ \mu_0 : g^{-1}(0) & \longrightarrow & \mathbb{R} \end{array}$$

(see the proof of Theorem 6.3). Theorems 6.3, 7.1 assert that the classification of the divergent diagrams (***) reduces to that of the restrictions λ_0.

2. Examples

Example 2.1.

Consider the differential equation

$$(y')^2 = 4y, \quad y' = \frac{dy}{dx_1}$$

in the x_1y-plane $\mathbb{R} \times \mathbb{R}$. This defines a nonsingular variety V in $J^1(\mathbb{R}, \mathbb{R})$ by $p^2 = 4y$, on which the projection ev to the base space (x_1y-plane) restricts to the fold map. V is not transverse to the contact elements at the singular locus of the projection. It is easy to see that this equation admits the following family of algebraic solutions D_{x_2} in x_1y-plane

$$y = (x_1 - x_2)^2, \; y' = 2(x_1 - x_2)$$

with the parameter $x_2 \in \mathbb{R}$. The variety V is the image of the imbedding (lift) $\mathcal{I} : \mathbb{R}^2 \to J^1(\mathbb{R}, \mathbb{R})$ defined by

$$\mathcal{I}(x_1, x_2) = (x_1, (x_1 - x_2)^2, 2(x_1 - x_2))$$

and the complete integral is given by $\lambda(x_1, x_2) = x_2$. In the manner of the previous section, this admits the Legendre imbedding

$$\begin{aligned} \tilde{\mathcal{I}}(x_1, x_2) &= (x_1, x_2, h_x, \partial h_x/\partial x_1, \partial h_x/\partial x_2) \\ &= (x_1, x_2, (x_1 - x_2)^2, 2(x_1 - x_2), -2(x_1 - x_2)) \in J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \end{aligned}$$

which is given by the generating function

$$h_x(z) = \epsilon z^2 + (x_1 - x_2)^2, \quad z \in \mathbb{R}, \ \epsilon = \pm 1,$$

with the singular locus $\Sigma(x, h_x) = x_1 x_2 -$ plane. The level curves of the critical value of $\lambda = x_2$ restricted to the fibres of f constitute the solutions D_{x_2}. The S-equivalence of the differential equation at a $q = (x_1, y) \in \mathbb{R}^2$ is determined by the restriction of the complete integral $\lambda = x_2$ to the fibre of the map $f = (x_1, h_x) : \mathbb{R}^3 \to \mathbb{R}^2$ over q. The S-eqivalence class at each $q \in \mathbb{R}^2$ is displayed in the following figure.

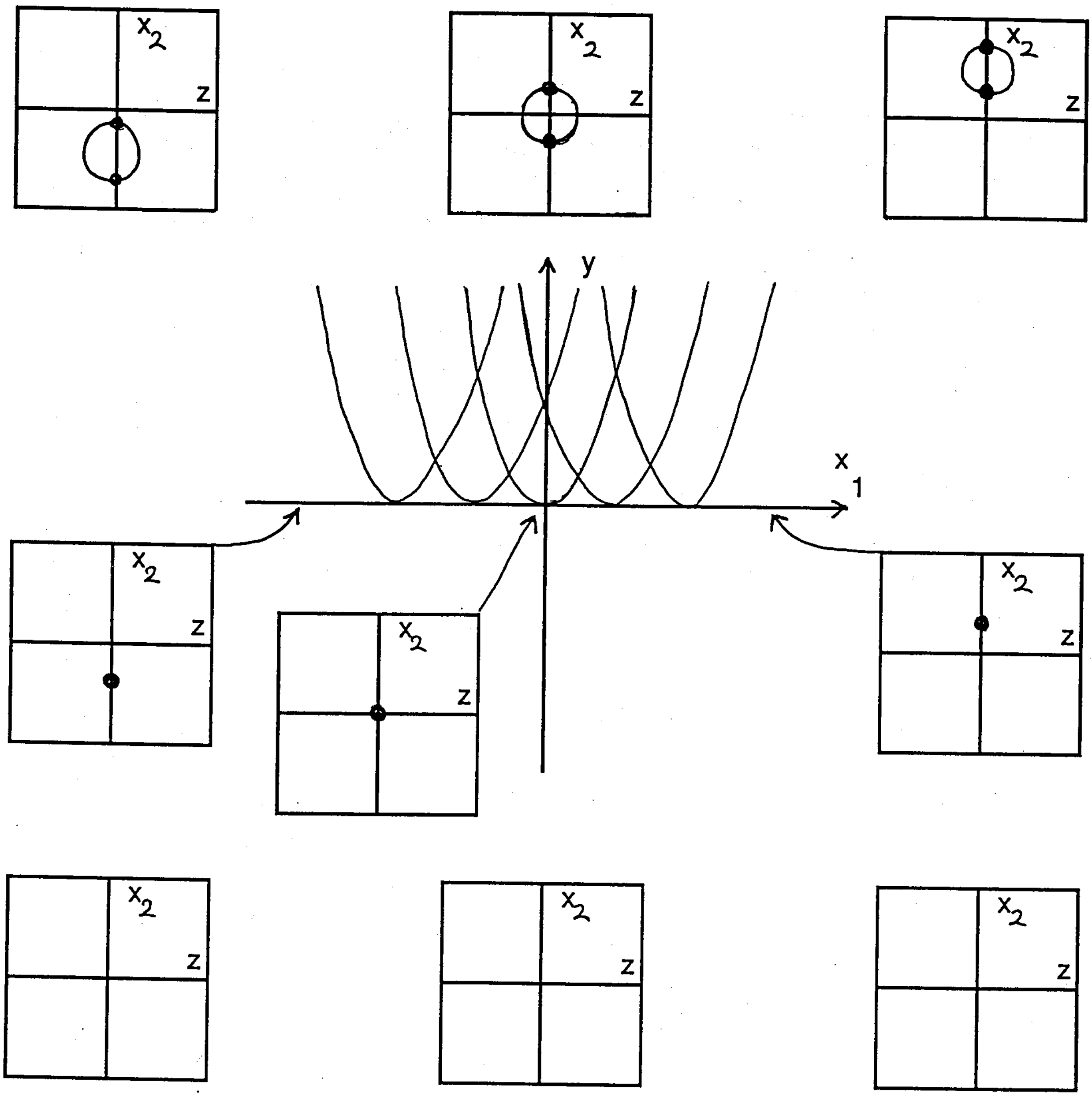


Fig.1a, ε=1

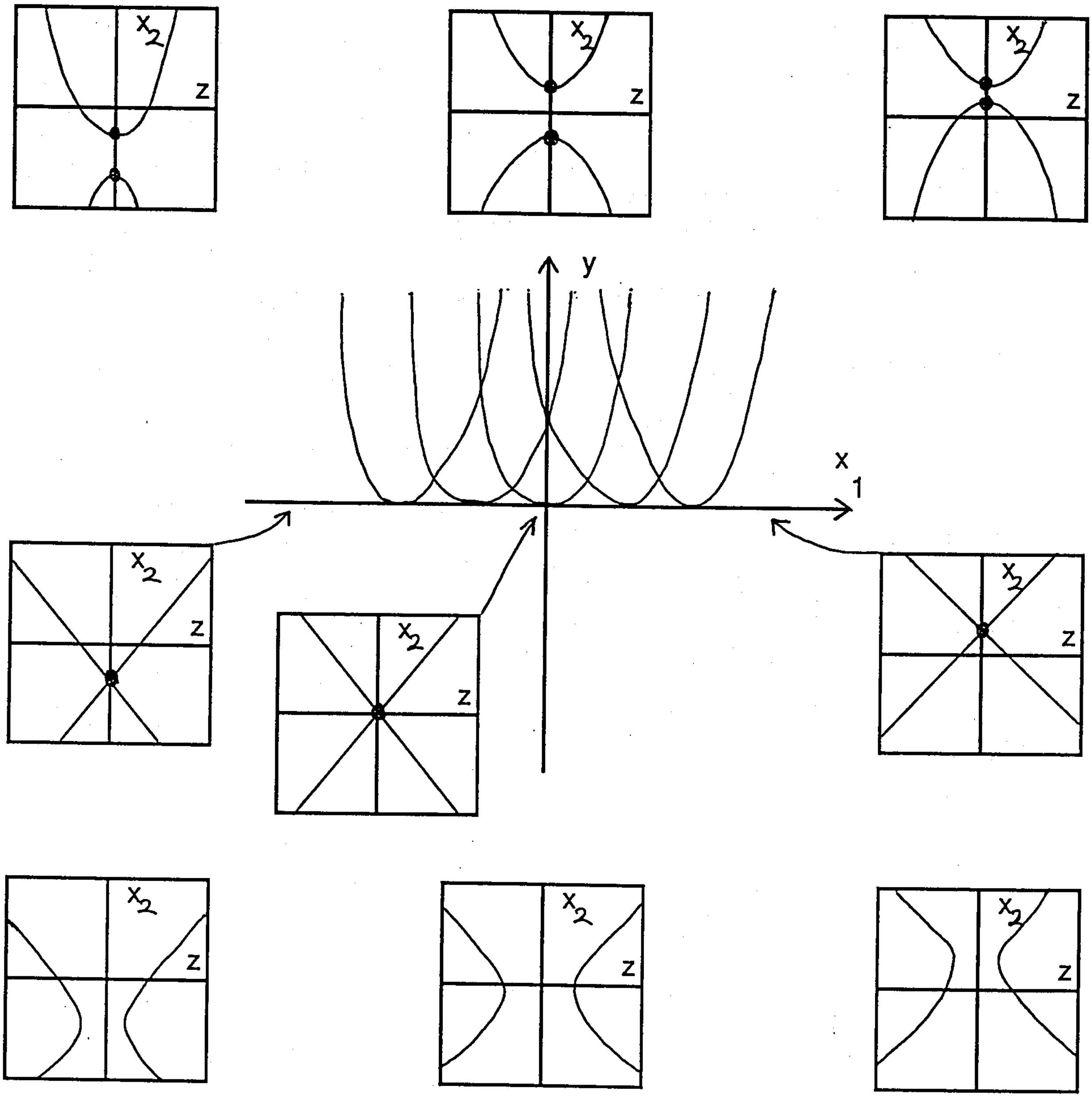


Fig.1b,ε=-1

Example 2.2.

Consider the following differential equation in the x_1y-plane

$$3(y')^2 = x_1.$$

This defines a nonsingular variety $V = \{3p^2 = x_1\} \subset J^1(\mathbb{R}, \mathbb{R})$ transverse to the contact elements, on which the projection ev to the base space restricts to the fold mapping. This equation admits the family of algebraic solutions

$$x_1 = 3z^2, \; y = x_2 - 2z^3, \; y' - z, \; z \in \mathbb{R}.$$

with the parameter $x_2 \in \mathbb{R}$. The variety V is the image of the imbedding

$$\mathcal{I}(z, x_2) = (3z^2, x_2 - 2z^3, -z)$$

and the complete integral is given by $\lambda = x_2$. This admits the Legendre imbedding into $J^1(\mathbb{R}, \mathbb{R})$ defined by

$$\tilde{\mathcal{I}}(z, x_2) = (3z^2, x_2, x_2 - 2z^3, -z, 1),$$

which is given by the generating function

$$h_x(z) = z^3 - x_1 z + x_2.$$

Example 2.3.

Consider the following (non versal) differential equation in the x_1y-plane

$$y = x_1 y' + (y')^3.$$

This defines a nonsingular variety $V = \{y = x_1 p + p^3\} \subset J^1(\mathbb{R}, \mathbb{R})$, on which the projection ev to the base space restricts to the Whitney cusp mapping. The variety V is not transverse to the contact elements at the singular locus. This equation admits the family of algebraic solutions

$$y = x_1 x_2 + x_2^3, \tag{1}$$

with the parameter $x_2 \in \mathbb{R}$. The variety V is the image of the imbedding

$$\mathcal{I}(x_1, x_2) = (x_1, x_1 x_2 + x_2^3, x_2)$$

and the complete integral is given by $\lambda = x_2$. This admits the Legendre imbedding into $J^1(\mathbb{R}^2, \mathbb{R})$ defined by

$$\tilde{\mathcal{I}}(x_1, x_2) = (x_1, x_2, x_1 x_2 + x_2^3, x_2, x_1 + 3x_2^2),$$

which is given by the generating function

$$h_x(z) = z^2 + x_1 x_2 + x_2^3.$$

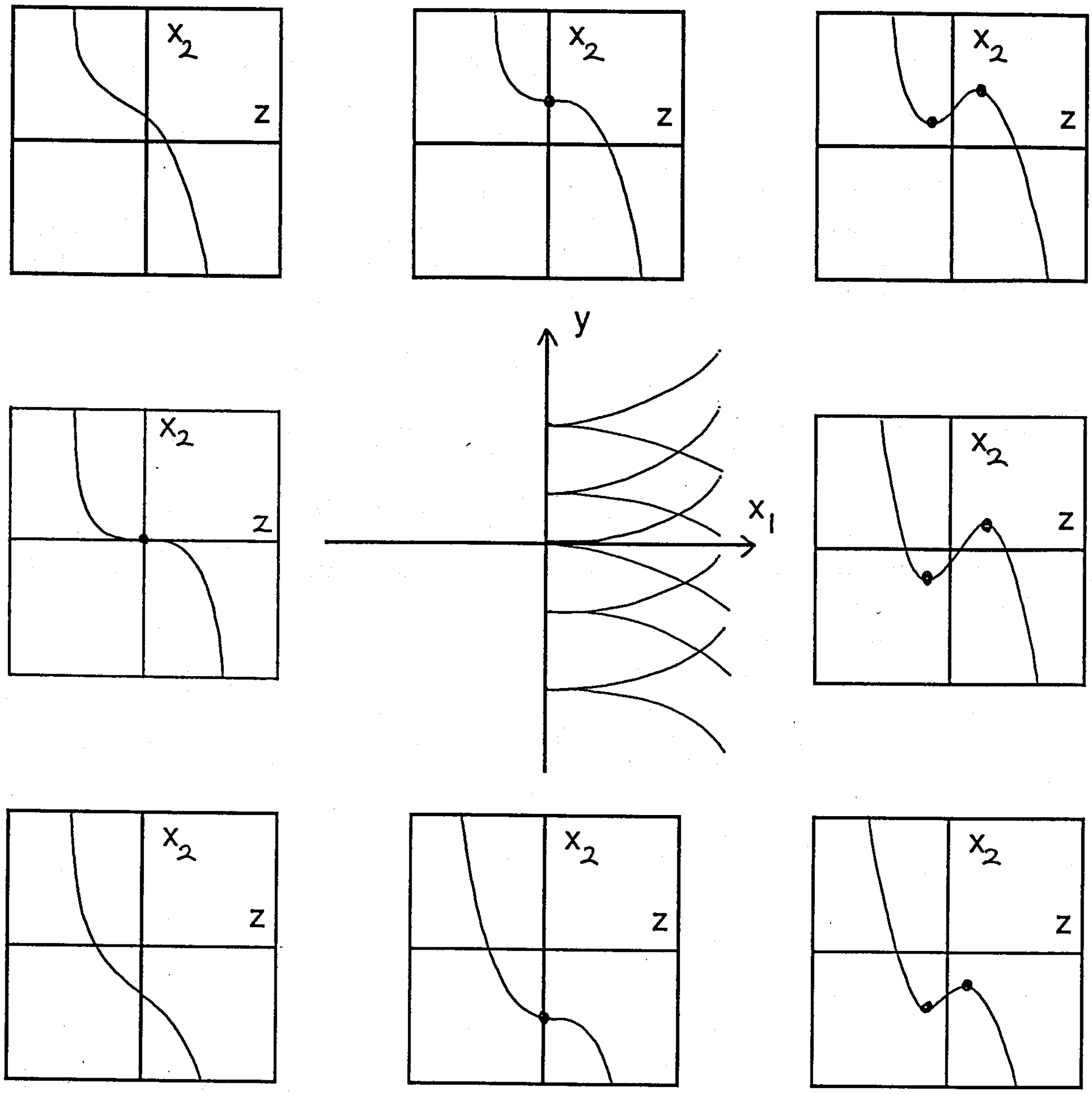


Fig. 2

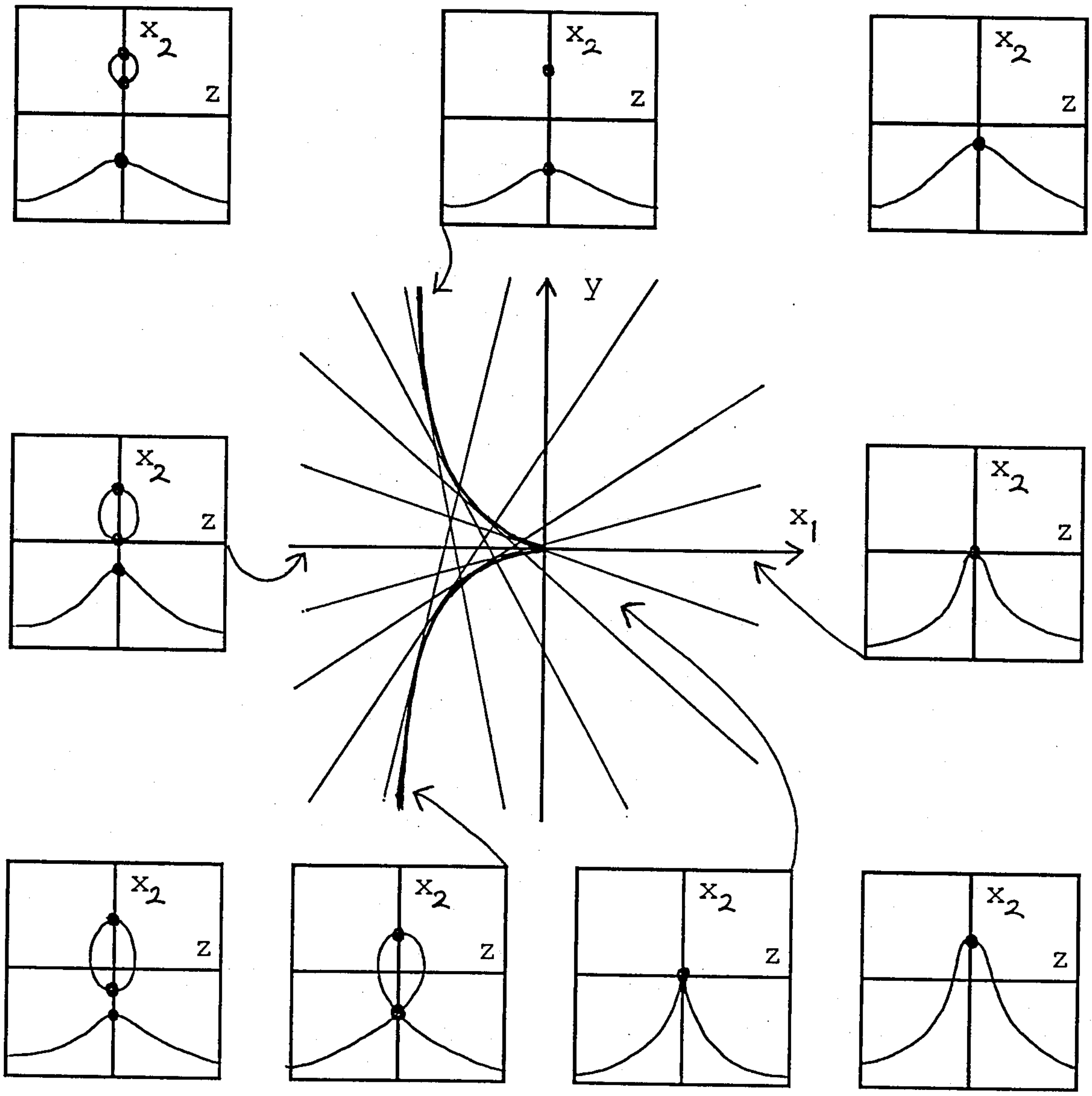


Fig. 3

Versal unfolding. Now we will construct the versal unfolding of Example 2.3. The divergent diagram

$$\mathbb{R} \xleftarrow{\lambda=x_2} \mathbb{R}^3 \xrightarrow{f=(x_1,h_x)} \mathbb{R}^2$$

is not strictly stable i.e. a deformation is not strictly equivalent to the trivial unfolding (see §3 for the definition). By Theorem 5.1, this admits the stable unfolding with one parameter

$$\mathbb{R} \xleftarrow{\Lambda=x_2} \mathbb{R}^4 \xrightarrow{F=(x_1,u,h_{x,u})} \mathbb{R}^3,$$

where

$$h_{x,u}(z) = z^2 + x_1(x_2-u) + (x_2-u)^3.$$

The singular locus $\Sigma(F,\Lambda)$ of the map $(F,\Lambda) : \mathbb{R}^4 \to \mathbb{R}^3 \times \mathbb{R}$ is defined by $z=0$ and identified with the x_1x_2u-space, on which the above divergent diagram restricts to the integral diagram

$$\mathbb{R} \xleftarrow{\Lambda=x_2} \Sigma(F,\Lambda) = \mathbb{R}^3 \xrightarrow{F} \mathbb{R}^3,$$

and the restriction of F is given by

$$F(x_1,x_2,u) = (x_1,u,x_1(x_2-u)+(x_2-u)^3).$$

The level surfaces of the complete integral $\lambda = x_2$ in $\mathbb{R}^3$ pjoject by F to the solutions in the x_1yu-space $\mathbb{R}^3$, which satisfy the following versal PDE

$$(2) \qquad \left\{ \begin{array}{l} y = x_1y_{x_1} + (y_{x_1})^3, \\ y_u = -x_1 - 3(y_{x_1})^2 \end{array} \right\}$$

The variety definig this PDE is the image of the imbedding

$$\tilde{\mathcal{I}}' = (x_1,u,x_1(x_2-u)+(x_2-u)^3, x_2-u, -x_1-3(x_2-u)^2).$$

By definition $F^{-1}(x_1,u,y) \subset \mathbb{R}\times u \times \mathbb{R} = \mathbb{R}^2$ is the parallel translation of $f^{-1}(x_1,y) \subset \mathbb{R}^2$ by $(u,0) \in \mathbb{R}^2$. Identifying $F^{-1}(x_1,u,y)$ with $f^{-1}(x_1,y)$ naturally, the restrictions $\lambda_{x_1,y}, \Lambda_{x_1,u,y}$ of the complete integral x_2 satisfy $\lambda_{x_1,y}+u = \Lambda_{x_1,u,y}$. Let i be the imbedding of x_1y-space into x_1uy-space defined by $i(x_1,y) = (x_1,\phi(x_1,y),y)$ and j the imbedding of x_1x_2z-space into x_1x_2uz-space defined by $j(x_1,x_2,z) = (x_1,x_2,\ \phi(x_1,h_x(z)),z)$. These imbeddings induce the divergent diagram (g,μ) by $F\circ j = i \circ g$ and $\mu = \Lambda \circ j$. Clearly $f = g$. By definition $\Lambda_{x_1,\phi(x_1,y),y} = \lambda_{x_1,y} + \phi(x_1,y) : f^{-1}(x_1,y) \to \mathbb{R}$. This shows that μ restricts to $\lambda + \phi(f)$ on $\Sigma(g,\mu)$. The second term $\phi(f)$ is called the *function moduli.* The solutions of the equation (1) are the transverse intersections of the solutions of this equation (2) with the x_1y-plane naturally imbedded in x_1yu-space, and any deformation of (1) is obtained by a suitable deformation of the natural imbedding by Theorem 7.1.

Example 2.4.

Consider the following (non versal) differential equation in the x_1y-plane

$$x_1 + 4yy' - 2x_1(y')^2 - 4(y')^3 - 4(y')^5 = 0 \ .$$

This defines a nonsingular variety V transverse to the contact elements in $J^1(\mathbb{R}, \mathbb{R})$, which is the image of the imbedding

$$\mathcal{I}(z, x_2) = (-4(z^3 - 3x_2z), -3(z^4 - 2x_2z^2 + x_2), z),$$

on which the projection ev to the base space restricts to the Whitney cusp mapping. This equation admits the family of algebraic solutions

$$x_1 = -4(z^3 - 3x_2z), \ y = -3(z^4 - 2x_2z^2 + x_2), \ y' = z, \quad z \in \mathbb{R}$$

with the parameter $x_2 \in \mathbb{R}$ and the complete integral on V is given by $\lambda = x_2$. The imbedding $\mathcal{I}$ admits the Legendre imbedding into $J^1(\mathbb{R}^2, \mathbb{R})$ defined by

$$\tilde{\mathcal{I}}(z, x_2) = (-4(z^3 - 3x_2z), x_2, -3(z^4 - 2x_2z^2 + x_2), z, -(6z^2 + 3)),$$

which is defined with the generating function

$$h_x(z) = z^4 - 6x_2z^2 + x_1z - 3x_2.$$

By Theorem 5.1, the divergent diagram $((x_1, h_x), \lambda)$ admits the versal unfolding $(F, \Lambda) = ((x_1, u, h_{x,u}), \lambda)$,

$$h_{x,u} = z^4 - 6(x_2 - u)z^2 + x_1z - 3(x_2 - u).$$

Then $\Sigma(x, u, h_{x,u}) = \{-4z^3 + 12(x_2 - u)z = x_1\}$. Identifying this with x_2uz-space, the restriction (integral diagram) of (F, Λ) to $\Sigma(F, \Lambda)$ is as follows

$$x_2 \xleftarrow{\ \Lambda\ } (x_2, u, z) \xrightarrow{\ F\ } (-4z^3 + 12(x_2 - u)z, u, -3z^4 + 6(x_2 - u)z^2 - 3(x_2 - u)).$$

The images of level surfaces of x_2 by F constitute the solution web and theis conormal vectors satisfies the following versal PDE.

$$\left\{ \begin{array}{l} y_u = -6y_{x_1}^2 + 3, \\ \ 0 = x_1 + 4yy_{x_1} - 2x_1y_{x_1}^2 - 4y_{x_1}^3 - 4y_{x_1}^5. \end{array} \right\}$$

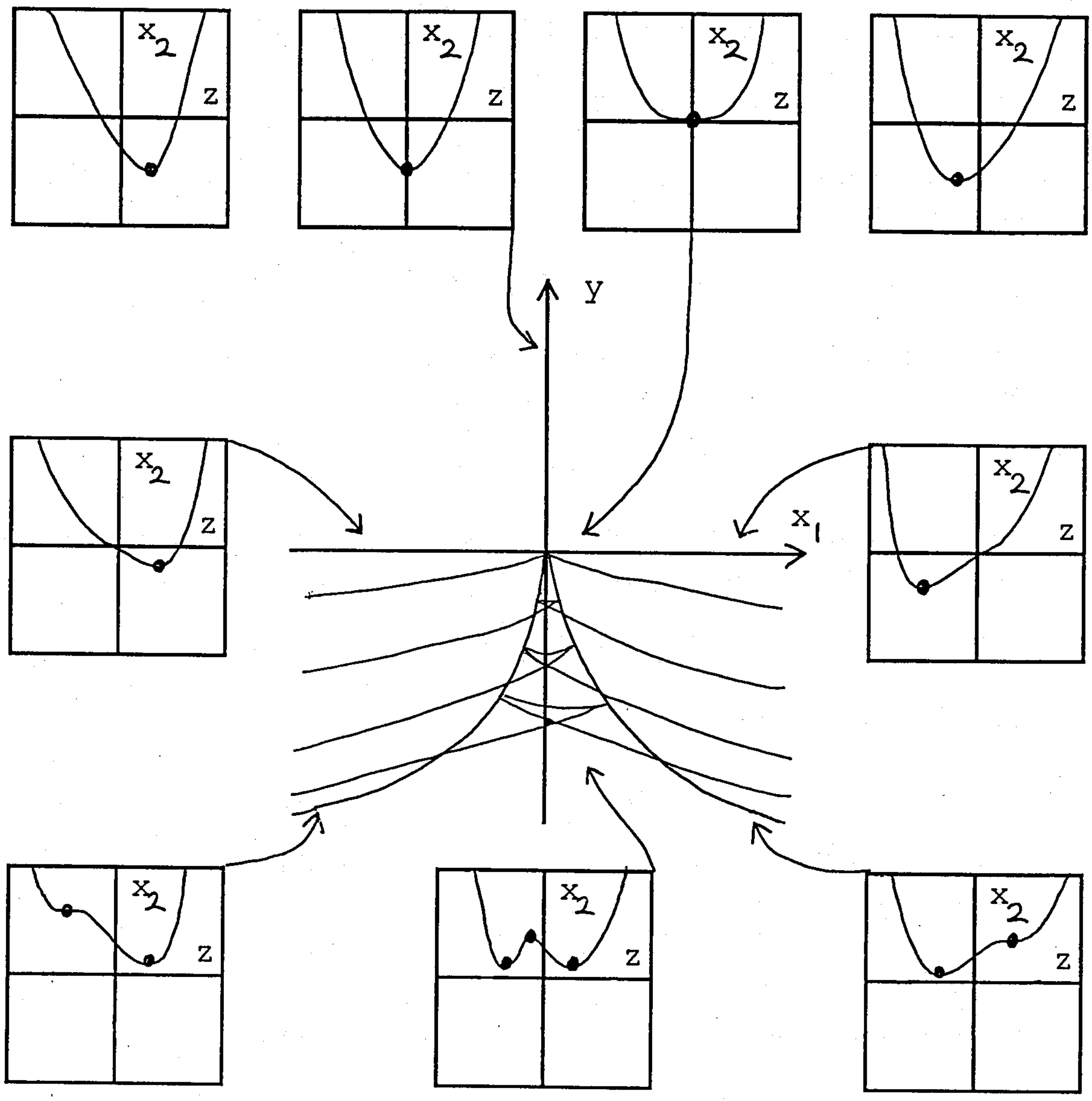


Fig. 4

Example 2.5.

Consider the families of the following algebraic curves

$$x_1 = z^2 + x_2, \; y = 2z^2(z - x_2), \quad z \in \mathbb{R},$$

with the parameter $x_2 \in \mathbb{R}$. Clearly $y' = 3z - 2x_2$. These (x_1, y, y') form a nonsingular variety V in $J^1(\mathbb{R}, \mathbb{R})$, which is the image of the imbedding

$$\mathcal{I}(z, x_2) = (z^2 + x_2, 2z^2(z - x_2), 3z - 2x_2),$$

and the complete integral is given by $\lambda = x_2$. This admits the Legendre imbedding into $J^1(\mathbb{R}^2, \mathbb{R})$ defined by

$$\tilde{\mathcal{I}}(z, x_2) = (z^2 + x_2, x_2, 2z^2(z - x_2), 3z - 2x_2, (2x_2 - 3z - 2z^2)),$$

which is given by the generating function

$$h_x(z) = -z^3 - 3(x_2 - x_1)z + 2x_2(x_2 - x_1).$$

The versal unfolding of h_x is given by

$$h_{x,u} = -z^3 - 3(x_2 - u - x_1)z + 2(x_2 - u)(x_2 - u - x_1).$$

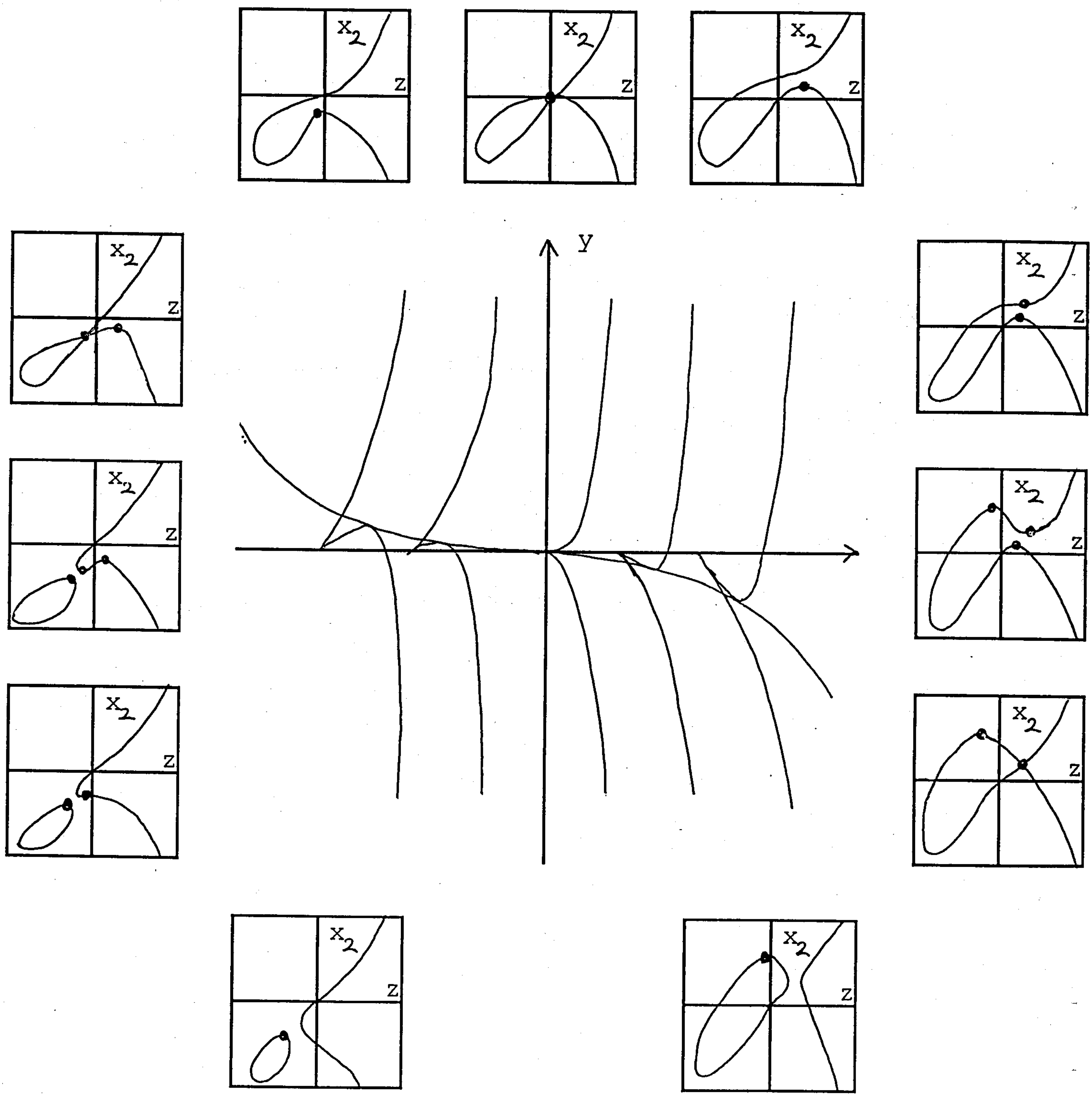
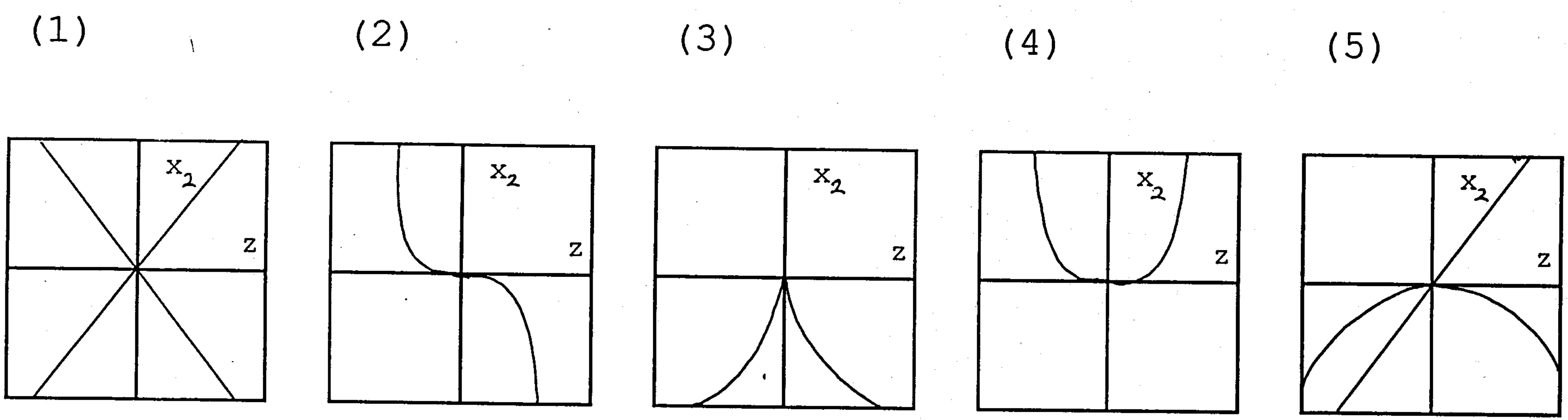


Fig. 5

The integral diagrams $\mathbb{R} \leftarrow \Sigma(x, h_x) \rightarrow \mathbb{R}^2$ for the above examples are respectively strictly equivalent to those in the following theorem.

Theorem 2.6 [19]. *The integral diagrams (*) $(f, \lambda) : \mathbb{R} \leftarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for generic differential equation in xy-plane are equivalent to one of the following five forms as divergent diagrams.*
(1) $f(z,x) = (z^2, x), \quad \lambda(z,x) = z + x,$
(2) $f(z,x) = (z^2, x), \quad \lambda(z,x) = z^3 + x,$
(3) $f(z,x) = (z^3 + xz, x), \quad \lambda(z,x) = z + \phi(f),$
(4) $f(z,x) = (z^3 + xz, x), \quad \lambda(z,x) = \frac{3}{4}z^4 + \frac{1}{2}xz^2 + x + \phi(f),$
(5) $f(z,x) = (z^3 + xz^2, x), \quad \lambda(z,x) = z^2 + x + \phi(f),$
where ϕ is an arbitrary smooth function defined on a neighbourhood of the origin in the xy-plane.

The above normal forms correspond respectively to the S-equivalence classes of the functions on varieties $\lambda_0 : f^{-1}(0,0) \rightarrow \mathbb{R}$ as follows. Here $f(x_1, x_2, z) = (x_1, h_x(z)), f^{-1}(0,0) \subset x_2z-$plane and x_2 stands for the restriction λ_0. The normal forms in the above theorems follow from Theorems 7.2 and 7.4 (with an easy formal argument for Case (5)). But we will not touch on the detailed analysis of these examples. Functions on varieties were classified by Goryunov [19] under a certain weak equivalence relation. An extended list of normal forms of integral diagrams for inperfect unfoldings of those functions in [19] is seen in the paper [23].



(1) $x_2^2 \pm z^2 = 0$, S-codim = 1

(2) $x_2 + z^3 = 0$, S-codim = 1

(3) $x_2^3 + z^2 = 0$, S-codim = 2

(4) $x_2 - z^4 = 0$, S-codim = 2

(5) $(x_2 - z)(x_2 + z^2) = 0$, S-codim = 2

Fig. 6

3. Equivalence relations and Infinitesimal stability for diagrams

Let $\lambda, \mu : \mathbb{R}^{n+k}, 0 \to \mathbb{R}, 0$ and $f, g : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^n, 0$. From now on the pair (f, λ) stands for the following diagram of the composition with the first projection π

$$\mathbb{R}^{n+k} \xrightarrow{(f,\lambda)} \mathbb{R}^{n+1} \xrightarrow{\pi} \mathbb{R}^n$$

as well as the divergent diagram

$$\mathbb{R} \xleftarrow{\lambda} \mathbb{R}^{n+k} \xrightarrow{f} \mathbb{R}^n.$$

The *solution web* $\mathcal{W}_{f,\lambda}$ for the divergent diagram (f, λ) is the family of the discriminant sets $D_t = D(f|\lambda^{-1}(t)) = D(f, \lambda) \cap \mathbb{R}^n \times t, t \in \mathbb{R}$, of the restrictions of f to the level surfaces of λ (see also §8). Two composite map germs $(f, \lambda), (g, \mu)$ are *equivalent* (resp. *topologically equivalent*) if there exist germs of diffeomorphisms (resp. homeomorphism) $\phi, \bar{\psi}, \psi$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{R}^{n+k} & \xrightarrow{(f,\lambda)} & \mathbb{R}^{n+1} & \xrightarrow{\pi} & \mathbb{R}^n \\ \phi \downarrow & & \bar{\psi} \downarrow & & \psi \downarrow \\ \mathbb{R}^{n+k} & \xrightarrow[(g,\mu)]{} & \mathbb{R}^{n+1} & \xrightarrow[\pi]{} & \mathbb{R}^n. \end{array}$$

We denote this diagram by $(\phi, \bar{\psi}, \psi) : (f, \lambda) \to (g, \mu)$ and call the triple an *equivalence.* We say $(f, \lambda), (g, \mu)$ are *equivalent as divergent diagrams* if $\bar{\psi} = (\psi \times \chi)$ with a germ of diffeomorphism $\chi : \mathbb{R}, 0 \to \mathbb{R}, 0$. By definition, $\psi(\mathcal{W}_{f,\lambda}) = \mathcal{W}_{g,\mu}$ holds: $\phi(D_t) = D'_{\chi(t)}$ for $t \in \mathbb{R}$, where D'_t denotes the solution of (g, μ). We say $(f, \lambda), (g, \mu)$ are *strictly equivalent as divergent diagrams* if χ is the identity.

An *unfolding of a composite map germ* (f, λ) *of dimension* s is a pair of a composite map germ (F, Λ) and a triple of imbeddings $i = (i_1, i_2, i_3)$ such that i_3 is transverse to $F = \pi \circ (F, \Lambda)$ and (f, λ) is given by the following commutative diagram of fiber product

$$\begin{array}{ccccc} \mathbb{R}^{n+s+k} & \xrightarrow{(F,\Lambda)} & \mathbb{R}^{n+s+1} & \xrightarrow{\pi} & \mathbb{R}^{n+s} \\ i_1 \uparrow & & i_2 \uparrow & & i_3 \uparrow \\ \mathbb{R}^{n+k} & \xrightarrow[(f,\lambda)]{} & \mathbb{R}^{n+1} & \xrightarrow[\pi]{} & \mathbb{R}^n. \end{array}$$

When $i_2 = i_3 \times \mathrm{id}_{\mathbb{R}}$, we call (F, Λ) an *unfolding of* (f, λ) *as a divergent diagram.* In both cases we denote as $i : (f, \lambda) \to (F, \Lambda)$ and call i a *morphism.*

Similarly to Thom-Mather theory for map germs, define the stability of the diagrams as follows. A diagram $(f, \lambda) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0 \to \mathbb{R}^n, 0$ is *stable as composite map germs* (resp. *strictly stable as divergent diagrams*) if for any representatives $\tilde{f}, \tilde{\lambda}$ of f, λ defined on a neighbourfood U of $0 \in \mathbb{R}^{n+k}$ there exists a neighbourhood V of $(\tilde{f}, \tilde{\lambda})$ in

$C^\infty(U,\mathbb{R}^n) \times C^\infty(U,\mathbb{R})$ in the Whitney topology with the following property: for any $(\tilde{g},\tilde{\mu}) \in V$ there is a $z \in U$ such that the germ of $(\tilde{g},\tilde{\mu})$ at z is equivalent as a composite map germ (resp. strictly equivalent as a divergent diagram) to (f,λ). We say (f,λ) is *homotopic stable* if any smooth deformation $(f_u,\lambda_u), u \in \mathbb{R}^s$ with $(f_0,\lambda_0) = (f,\lambda)$ is *trivial* i.e. there exist families of germs of diffeomorphisms $\phi_u : \mathbb{R}^{n+k},0 \to \mathbb{R}^{n+k},\phi_u(0),\bar{\psi}_u : \mathbb{R}^{n+1},0 \to \mathbb{R}^{n+1},\bar{\psi}_u(0)$ and $\psi_u : \mathbb{R}^n,0 \to \mathbb{R}^n,\psi_u(0)$ such that $\phi_0 = \mathrm{id}, \bar{\psi}_0 = \mathrm{id}, \psi_0 = \mathrm{id}$ and

$$(f_u,\lambda_u)\circ\phi_u = \bar{\psi}_u \circ (f,\lambda), \qquad f_u\circ\phi_u = \psi_u \circ f$$

holds for u in a neighbourhood of $0 \in \mathbb{R}^n$. And we say (f_u,λ_u) is *homotopic stable as a family of divergent diagrams* if $\bar{\psi}_u = \psi_u \times \chi_u$, $\lambda_u \circ \phi_u = \chi_u \circ \lambda$, and *strictly homotopic stable* if $\chi_u = \mathrm{id}$.

In the manner of Thom-Mather theory we say diagrams $(f,\lambda),(g,\mu)$ are S-*equivalent* or *contact equivalent* (*as divergent diagrams*)if their versal (stable) unfoldings (as divergent diagrams) are equivalent, $(f,\lambda),(g,\mu)$ are *strictly S-equivalent as divergent diagrams* if their versal unfoldings are strictly equivalent. We say $(f,\lambda),(g,\mu)$ are *equisingular* if they admit unfoldings (as divergent diagrams), which are topologically equivalent as composite map germs (see also §9).

Let $\epsilon(n)$ denote the local ring of smooth function germs on $\mathbb{R}^n$ at 0, $m(n)$ the maximal ideal consinting of function germs vanishing at 0, and $\theta(n)$ the $\epsilon(n)$-module of germs of smooth vector fields on $\mathbb{R}^n$ at 0. For a smooth map germ $f : \mathbb{R}^n,0 \to \mathbb{R}^p,0$, $\theta(f)$ denotes the $\epsilon(n)$-module of germs of smooth sections of the bundle $f^*T\mathbb{R}^p \to \mathbb{R}^n$, $tf : \theta(n) \to \theta(f)$ denotes the differential of f, and $\omega f = f^* : \theta(p) \to \theta(f)$ denotes the pull back. Note that tf is an $\epsilon(n)$-module homomorphism, while ωf is an $\epsilon(p)$-module homomorphism regarding $\theta(f)$ as an $\epsilon(p)$-module via the pull back $f^* : \epsilon(p) \to \epsilon(n)$. A map germ f is *infinitesimally stable* if the $\epsilon(p)$-module homomorphism $tf \; - \; \omega f : \theta(n) \oplus \theta(p) \to \theta(f)$ is surjective [25]. We say $(f,\lambda) : \mathbb{R}^{n+k},0 \to \mathbb{R}^{n+1},0 \to \mathbb{R}^n,0$ is *infinitesimally stable as a composite map germ* if the morphism $T_c(f,\lambda) : \theta(n+k) \oplus \theta(n+1) \oplus \theta(n) \to \theta(f,\lambda) \oplus \theta(\pi)$ is surjective, where $T_c(f,\lambda)$ is defined by

$$T_c(f,\lambda)(\alpha \oplus \beta \oplus \gamma) = (t(f,\lambda)(\alpha) - \omega(f,\lambda)(\beta), t\pi(\beta) - \omega\pi(\gamma))$$

and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the first projection [29,30,31]. We say (f,λ) is *infinitesimally stable as a divergent diagram* if the restriction of $T_c(f,\lambda)$ to $\theta(n+k) \oplus (\theta(n) \otimes \epsilon(n+1)) \oplus \theta(n)$ is surjective, regarding $\theta(n) \subset \theta(n+1)$ naturally by the inclusion $\pi^*T\mathbb{R}^n \subset T\mathbb{R}^{n+1}$. It is easy to see that (f,λ) is infinitesimally stable as a divergent diagram if and only if the morphism

$$T(f,\lambda) : \theta(n+k) \;\oplus\; \theta(n) \to \theta(\lambda) \oplus \theta(f)$$

defined by

$$T(f,\lambda)(\chi,\xi) = (t\lambda(\chi),\; tf(\chi) - \omega f(\xi))$$

is surjective.

4. Thom-Mather calculus

Let $g : \mathbb{R}^n,0 \to \mathbb{R}^p,0$ be a smooth map germ, A_1 a finitely generated $\epsilon(p)$-module, and A_2 and B finitely generated $\epsilon(n)$-modules. Let $\alpha : A_1 \to B$ be an $\epsilon(n)$-module homomorphism and $\beta : A_2 \to B$ an $\epsilon(p)$-module homomorphism via $g^* : \epsilon(p) \to \epsilon(n)$. The pull back g^* is said to be *an adequate homomorphism* and possesses the following property.

Lemma 4.1 [25]. *$\alpha(A_1) + \beta(A_2) = B$ if and only if*

$$\alpha(A_1) + \beta(A_2) + (g^*m(p) + m(n)^{k+1})B = B,$$

where $k = dim_{\mathbb{R}} A_1/m(p)A_1$, the number of generator of A_1.

This applies to the morphism

$$T(f,\lambda) : \theta(n+k) \oplus \theta(n) \to \theta(f) \oplus \theta(\lambda)$$

to give

Proposition 4.2. *The infinitesimal stability of a divergent diagram $(f,\lambda) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0 \to \mathbb{R}^n, 0$ is an algebraic condition on (n+1)-jets of f, λ.*

The infinitesimal stability of (f,λ) involves the various modules over the rings $\epsilon(n+k)$, $\epsilon(n)$ and the adequate homomorphism $f^* : \epsilon(n) \to \epsilon(n+k)$. By a generalisation of Thom-Mather theory due to Damon [12] we obtain the folloing theorems.

Theorem 4.3. *A divergent diagram (f,λ) is strictly stable if and only if infinitesimally stable.*

Theorem 4.4. *An infinitesimally stable divergent diagram (f,λ) is (n+1)-determined with respect to the strict equivalence relation : if (g,μ) has the same (n+1)-jet as (f,λ), then (g,μ) is strictly equivalent to (f,λ) as a divergent diagram.*

These results are known to fail for (weak) equivalence relation of divergent diagrams, as the infinitesimal stability involves the non-adequate diagram of homomorphisms [14]

$$\epsilon(1) \xrightarrow{\lambda^*} \epsilon(n+k) \xleftarrow{f^*} \epsilon(n).$$

On the other hand the morphism $T(f,\lambda)$ is a homomorphism over the composite of adequate homomorphisms

$$\epsilon(n+k) \xleftarrow{(f,\lambda)^*} \epsilon(n+1) \xleftarrow{\pi^*} \epsilon(n)$$

and Thom-Mather theory applies to the equivalence relation as composite map germs to give the following natural generalisation [14,16].

Theorem 4.5. *A diagram (f,λ) is stable as a composite map germ if and only if infinitesimally stable as a composite map germ.*

The clasification theorem of map germs by $\mathbb{R}$-algebra due to Mather [25] (see §6) is genaralised as follows.

Theorem 4.6. *Diagrams $(f,\lambda), (g,\mu)$ stable as composite map germs are C^∞-equivalent if and only if algebraically S-equivalent, i.e. roughly stating, the restrictions $\lambda_0 = \lambda|f^{-1}(0)$, $\mu_0 = \mu|g^{-1}(0)$ are C^∞ right-left equivalent (see §1and §5 for the definition).*

5. Criterion for infinitesimal stability for divergent diagrams

Let $\lambda : \mathbb{R}^{n+k}, 0 \to \mathbb{R}, 0$ be a smooth function germ and $f : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^n, 0$ a smooth map germ of the form

$$f = (f_u, u) : \mathbb{R}^{n+k-s} \times \mathbb{R}^s \to \mathbb{R}^{n-s} \times \mathbb{R}^s, u \in \mathbb{R}^s.$$

By definition, (f, λ) is infinitesimally stable as a divergent diagram if

$$T(f,\lambda) : \theta(n+k) \oplus \theta(n) \to \theta(\lambda) \oplus \theta(f)$$

is surjective, and by Lemma 3.1, if and only if

$$T(f,\lambda)(\theta(n+k) \oplus \theta(n)) + f^*m(n)(\theta(\lambda) \oplus \theta(f)) = \theta(\lambda) \oplus \theta(f).$$

Consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & & & \ker T(f,\lambda) & \longrightarrow & \mathbb{R}^s & & \\
 & & & & \downarrow & & \downarrow \Delta & & \\
0 \to & \frac{\theta(n+k-s)\oplus\theta(n-s)}{m(n-s)(\theta(n+k-s)\oplus\theta(n-s))} & \xrightarrow{i} & & \frac{\theta(n+k)\oplus\theta(n)}{m(n)(\theta(n+k)\oplus\theta(n))} & \xrightarrow{\pi} & \mathbb{R}^s \otimes \frac{\epsilon(n+k-s)}{m(n-s)} \oplus \mathbb{R}^s & \to 0 \\
 & \downarrow T(f_0,\lambda_0) & & & \downarrow T(f,\lambda) & & \downarrow & & \\
0 \longrightarrow & \frac{\theta(\lambda_0)\oplus\theta(f_0)}{m(n-s)(\theta(\lambda_0)\oplus\theta(f_0))} & \xrightarrow{i} & & \frac{\theta(\lambda)\oplus\theta(f)}{m(n)(\theta(\lambda)\oplus\theta(f))} & \xrightarrow{\pi} & \mathbb{R}^s \otimes \frac{\epsilon(n+k-s)}{m(n-s)} & \to 0 \\
 & \downarrow & & & \downarrow & & \downarrow & & \\
 & \operatorname{coker} T(f_0,\lambda_0) & \longrightarrow & & \operatorname{coker} T(f,\lambda) & \longrightarrow & 0. & &
\end{array}$$

Here $(f_0, \lambda_0$ denotes the restriction of (f,λ) to $\mathbb{R}^{n+k-s}$, the i's are the natural inclusions induced from the projections $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k-s}, \mathbb{R}^n \to \mathbb{R}^{n-s}$, $\mathbb{R}^s \otimes \frac{\epsilon(n+k-s)}{m(n-s)}$ stands for the $\epsilon(n+k-s)$-module of sections of the bundle $f^*T\mathbb{R}^s$ as well as $T\mathbb{R}^{n+k}/T\mathbb{R}^{n+k-s}$ restricted to $f^{-1}(0) \subset \mathbb{R}^{n+k-s}$, Δ is the diagonal imbedding and, $m(n-s), m(n)$ stand also for $f_o^*m(n-s), f^*m(n)$ respectively. By the snake lemma we obtain the long exact sequence

$$\mathbb{R}^s \xrightarrow{\delta((f_u,u),\lambda)} \operatorname{coker} T(f_0,\lambda_0) \longrightarrow \operatorname{coker} T(f,\lambda) \longrightarrow 0,$$

where $\delta((f_u,u),\lambda)$ is the connecting homomorphism. By chasing the big diagram we see

$$\delta((f_u,u),\lambda)(a) = \left(\sum_{i=1}^{s} a_i \, \partial\lambda/\partial u_i, \sum_{i=1}^{s} a_i \, \partial f_u/\partial u_i \right)$$

for $a \in \mathbb{R}^s$. Define the S-*codimension* of (f,λ) by

$$\text{S-codim}(f,\lambda) = \dim_{\mathbb{R}} \frac{\theta(\lambda) \oplus \theta(f)}{\operatorname{Im} T(f,\lambda) + f^*m(n)(\theta(\lambda) \oplus \theta(f))}.$$

We say (f,λ) is S-*finite* if it has finite S-codimension. By the above argument we obtain

Theorem 5.1. *Let $\{(f_i, \lambda_i), i = 1, \ldots, s\}$ be a generator over $\mathbb{R}$ of the module*

$$\frac{\theta(\lambda) \oplus \theta(f)}{Im\ T(f,\lambda) + f^*m(n)(\theta(\lambda) \oplus \theta(f))}.$$

Then (f, λ) admits the strictly stable unfolding $(F, \Lambda) : \mathbb{R}^{n+s+k}, 0 \longrightarrow \mathbb{R}^{n+s+1}, 0 \longrightarrow \mathbb{R}^{n+s}, 0$ as a divergent diagram defined by

$$F(z,u) = (f(z) + \sum_{i=1}^{s} u_i f_i,\ u), \quad \Lambda(z,u) = \lambda(z) + \sum_{i=1}^{s} u_i\lambda_i\ ,\ z \in \mathbb{R}^{n+k},\ u \in \mathbb{R}^s\ .$$

By Theorem 6.3, stable unfolding of (f, λ) is unique up to the strict equivalence of divergent diagrams.

Next consider the commutative diagram

$$\begin{array}{ccc}
0 \to \ker P_2\delta((f_u,u),\lambda) = \mathbb{R}^t & \to & \mathbb{R}^s \\
\delta'((f_u,u),\lambda)\downarrow & & \delta((f_u,u),\lambda)\downarrow \\
0 \longrightarrow \ker P_2 & \to & \text{coker } T(f_0,\lambda_0) \xrightarrow{P_2} \frac{\theta(f_0)}{tf_0(\theta(n+k-s)+f_0^*m(n-s)\theta(f_0)} \to 0,
\end{array}$$

where P_2 is induced from the second projection of $\theta(\lambda_0) \oplus \theta(f_0)$ and $\delta'((f_u,u),\lambda)$ is the restriction of the connecting homomorphism $\delta((f_u,u),\lambda)$. Clearly $\delta((f_u,u),\lambda)$ is surjective if and only if $P_2 \circ \delta((f_u,u),\lambda)$ is surjective (in other words, f is stable) and $\delta'((f_u,u),\lambda)$ is also surjective. Here $\theta_{f_0}(n+k-s)$ stands for the $\epsilon(n-s)$-module (via f_0^*) of germs of vector fields ξ on $\mathbb{R}^{n+k-s}$ such that $df(\xi)$ extend to smooth vector fields χ on $\mathbb{R}^{n-s}$ (in other words, χ is a lift of ξ). Assume f is stable as a map germ. Then

$$\begin{aligned}
\ker P_2 &= \theta(\lambda_0) \oplus 0/(\text{Im } T(f_0,\lambda_0)\ +\ f_0^*m(n-s)\theta(\lambda_0)) \cap (\theta(\lambda_0) \oplus 0) \\
&= \theta(\lambda_0)/tf_0(\theta_{f_0}(n+k-s))\ +\ f_0^*m(n-s)\theta(\lambda_0)
\end{aligned}$$

and $\delta'((f_u,u),\lambda)$ is given by

$$\delta'((f_u,u),\lambda)(b) = \sum_{i=1}^{t} b_i\ \partial\lambda/\partial u_i\ ,\ b \in \mathbb{R}^t = \ker P_2\delta((f_u,u),\lambda).$$

Let $\mathbb{R}^r \subset \mathbb{R}^s$ the complementary subspace of the $\mathbb{R}^t = \ker P_2\delta((f_u,u),\lambda)$ and regard $\mathbb{R}^s = \mathbb{R}^r \times \mathbb{R}^t$.

Proposition 5.2. *Assume (f,λ) is infinitesimally stable. Then f is equivalent to a trivial unfolding of a stable unfolding $f' = (f_u, u) : \mathbb{R}^{n+k-s} \times \mathbb{R}^r \to \mathbb{R}^{n-s} \times \mathbb{R}^r, u \in \mathbb{R}^r \subset \mathbb{R}^s, r < s$, and f' is minimal, i.e. f' is not equivalent to a trivial unfolding of a stable map germ.*

T. he sub-unfolding $f' = (f_u, u) : \mathbb{R}^{n+k-s} \times \mathbb{R}^r \to \mathbb{R}^{n-s} \times \mathbb{R}^r, u \in \mathbb{R}^r$ is stable by a result due to Mather[25] (see §4) and (by the versality of f') f is equivalent to the following trivial unfolding of f' with the parameter $\bar{u} \in \mathbb{R}^t$,

$$(f', \bar{u}) : \mathbb{R}^{n+k-t} \times \mathbb{R}^t \to \mathbb{R}^{n-t} \times \mathbb{R}^t.$$

Since f_0 has

$$\text{K-codimension} = \dim \frac{\theta(f_0)}{tf_0(\theta(n+k-s) + f_0^* m(n-s)\theta(f_0)} = s - t,$$

the unfolding f' of dimension $s-t$ is minimal. To show $r < s$, assume $r = s$. Then P_2 is an isomorphism, and $\delta((f_u, u), \lambda)$ is surjective if and only if $P_2 \circ \delta((f_u, u), \lambda)$ is surjective if and only if f is stable. Assume that f is minimal. Then the set S of those $x \in \mathbb{R}^{n+k}$ where the germ of f is K-equivalent to f consists of 0. On the other hand the set S is respected by strict equivalence of divergent diagrams while $\lambda(0)$ varies as λ is perturbed. Therefore (f, λ) can not be strictly stable and by Theorem 4.3 it is not infinitesimally stable.

We apply the above argument to the divergent diagram of the trivial unfolding $(f', \bar{u}), \bar{u} \in \mathbb{R}^t$ and λ. Then $\ker P_2\delta((f', \bar{u}), \lambda) = \mathbb{R}^t$ and

$$\text{Coker } P_2 = \frac{\theta(\lambda')}{tf'(\theta_{f'}(n+k-t) + f'^* m(n-s)\theta(\lambda')},$$

where λ' is the restriction of λ to $\mathbb{R}^{n+k-t}$ and Theorem 5.1 is interpreted as follows.

Theorem 5.3. *A divergent diagram (f, λ) is strictly stable if and only if $f = (f', \bar{u}), \bar{u} \in \mathbb{R}^t, f' : \mathbb{R}^{n+k-t}, 0 \to \mathbb{R}^{n-t}, 0$ being minimal and stable with suitable coordinates, and the connecting homomorphism*

$$\partial\lambda/\partial\bar{u} : \mathbb{R}^t \to \frac{\theta(\lambda')}{t\lambda'(\theta_{f'}(n+k-t)) + f' * m(n-t)\theta(\lambda')}$$

is surjective.

6. Classification by $\mathbb{R}$-algebra

We recall the classification theorem of stable map germs by $\mathbb{R}$-algebra due to Mather [22]. For a map germ $h : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ define the $\mathbb{R}$-algebra $Q(h) = \epsilon(n)/ < h >_{\epsilon(n)}$. Two map germs h, h' are *equivalent* if there exist germs of diffeomorphisms $\phi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\psi : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$ such that $\phi \circ h = h' \circ \psi$, and the map germs are *contact equivalent* if there exists an $\mathbb{R}$-algabra homomorphism $\phi^* : Q(h') \to Q(h)$, or roughly stating, if there exists a germ of diffeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\phi(h^{-1}(0)) = h'^{-1}(0)$. h is *infinitesimally stable* if $th(\theta(n)) - \omega h(\theta(p)) = \theta(h)$. Mather [25] proved

Theorem 6.1. *Let h, h' be infinitesimally stable and contact equivalent map germs and let $\phi^* : Q(h') \to Q(h)$ be an $\mathbb{R}$-algebra isomorphism. Then there exist germs of diffeomorphisms $\phi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\psi : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$ such that $\psi \circ h = h' \circ \phi$ and ϕ^* is induced by the pull back of ϕ.*

Let $h = (h_u, u), h' = (h'_u, u) : \mathbb{R}^{n+s}, 0 \to \mathbb{R}^{p+s}, 0, u \in \mathbb{R}^s$ be stable unfoldings of h_0, h'_0 respectively. By Theorem 6.1, h, h' are equivalent if and only if h_0, h'_0 are contact equivalent.

The K-codimension of h is defined by $\dim_{\mathbb{R}} \theta(h)/th(m(n)\theta(n)) + h^* m(p)\theta(h)$. We say h is K-finite if it has finite K-codimension. Tougeron and Mather [25,34] proved K-finiteness is a generic condition on the infinite jet space $J(n,p)$. A K-finite map germ h has a stable unfolding $H = (h_u, u) : \mathbb{R}^{n+s}, 0 \to \mathbb{R}^{p+s}, 0$, defined by $h_u = h + \sum_{i=1}^s u_i h_i$ with a generator $\{h_1, \ldots, h_s\}$ of $\theta(h)/th(\theta(n)) + h^* m(p)\theta(h)$. The above theorem implies

Corollary 6.2. *Two K-finite map germs are K-equivalent if and only if they admit unfoldings which are equivalent.*

Simalarly to the contact equivalence relation for map germs, we say two divergent diagrams $(f,\lambda),(g,\mu)$ are ***algebraically S-equivalent*** if there exist an $\mathbb{R}$-algebra isomorphism $\phi^*: Q(g) \to Q(f)$ and a germ of diffeomorphism $\chi: \mathbb{R},0 \to \mathbb{R},0$ such that $\phi^*(\mu) = \chi \circ \lambda$, and we say ***strictly algebraically S-equivalent*** if χ is the identity. The purpose of this section is to prove the following theorem.

Theorem 6.3. *Let $(f,\lambda),(g,\mu): \mathbb{R}^{n+k},0 \to \mathbb{R}^{n+1},0 \to \mathbb{R}^n,0$ be strictly stable divergent diagrams.*
(A) The following conditions are equivalent.
(1) $(f,\lambda),(g,\mu)$ are strictly equivalent as divergent diagrams.
(2) $(f,\lambda),(g,\mu)$ are strictly algebraically S-equivalent.
(B) The following conditions are equivalent.
(3) $(f,\lambda),(g,\mu)$ are equivalent as divergent diagrams,
(4) $(f,\lambda),(g,\mu)$ are equivalent as composite map germs,
(5) $(f,\lambda),(g,\mu)$ are algebraically S-equivalent.

From which we obtain

Corollary 6.4. *Assume $(f,\lambda),(g,\mu)$ have finite S-codimension.*
(A) The following conditions are equivalent.
(1) $(f,\lambda),(g,\mu)$ are strictly S-equivalent.
(2) $(f,\lambda),(g,\mu)$ are strictly algebraically S-equivalent.
(B) The following conditions are equivalent.
(3) $(f,\lambda),(g,\mu)$ are S-equivalent.
(4) $(f,\lambda),(g,\mu)$ are algebraically S-equivalent.

Proof of Theorem 6.3. The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear. If $(f,\lambda),(g,\mu)$ are algebraically S-equivalent, there exists a difeomorphism χ of $\mathbb{R},0$ such that (f,λ), $(g,\chi\circ\mu)$ are strictly algebraically S-equivalent. So it suffices to prove the implication (2) $\Rightarrow$ (1). Clearly the infinitesimal stability of $(f,\lambda),(g,\mu)$ implies those of f,g. By the definition of strict algebraic S-equivalence relation there exists an $\mathbb{R}$-algebra isomorphism $\phi^*: Q(g) \to Q(f)$ such that $\phi^*(\mu) = \lambda$. By Theorem 6.1 there exist germs of diffeomorphisms $\phi: \mathbb{R}^{n+k},0 \to \mathbb{R}^{n+k},0$ and $\psi: \mathbb{R}^n,0 \to \mathbb{R}^n,0$ such that $g\circ\phi = \psi\circ f$ and the isomorphism ϕ^* is induced from ϕ. Define $\mu' = \mu\circ\phi$. Then (f,μ') is strictly equivalent to (g,μ) and strictly stable as a divergent diagram, $\mu' \equiv \lambda \bmod \; < f >_{\epsilon(n)}$, and $(f,\lambda),(f,\mu')$ are strictly algebraically S-equivalent: $\lambda \equiv \mu' \bmod \; < f >_{\epsilon(n+k)}$.

Claim. *Assume that there exists a family of strictly stable divergent diagrams $(f,\lambda_v), v \in \mathbb{R}$ with $\lambda_0 = \lambda, \lambda_1 = \mu'$ and $\lambda_v \equiv \lambda \bmod \; < f >_{\epsilon(n+k)}$. Then $(f,\lambda),(f,\mu')$ are strictly equivalent.*

Proof. Define the unfolding $(F,\Lambda): \mathbb{R}^{n+1+k} \to \mathbb{R}^{n+1+1} \to \mathbb{R}^{n+1}$ by $F(z,v) = (f(z),v)$, $\Lambda(z,v) = \lambda_v(z), z \in \mathbb{R}^{n+k}, v \in \mathbb{R}$. Regard $\theta(n+k) \subset \theta(n+1+k), \theta(n) \subset \theta(n+1)$ naturally and let $\theta_\pi(F_{0,v_0}) \subset \theta(F_{0,v_0})$ denote the sub-module of germs of sections of the sub-bundle $F^*T\mathbb{R}^n \subset F^*T\mathbb{R}^{n+1}$ at $(0, v_0) \in \mathbb{R}^{n+1} \times \mathbb{R}^k = \mathbb{R}^{n+1+k}$. By the definition of infinitesimal stability

$$T(f, \lambda_{v_0})(\theta(n+k) \oplus \theta(n)) = \theta(\lambda_{v_0}) \oplus \theta(f),$$

from which

$$T(F,\Lambda)_{0,v_0}(\theta(n+k) \oplus \theta(n)) \equiv \theta(\Lambda_{0,v_0}) \oplus \theta_\pi(F_{0,v_0}) \mod F^*_{0,v_0}m(1)(\theta(\Lambda_{0,v_0}) \oplus \theta_\pi(F_{0,v_0})),$$

where $m(1)$ denotes the maximal ideal of the ring of smooth functions of v at $0 \in \mathbb{R}$ and $(F,\Lambda)_{0,v_0}$ denotes the germ of (F,Λ) at $(0,v_0) \in \mathbb{R}^{n+1+k}$ and $\theta(n+k) \subset \theta(n+1+k), \theta(n) \subset \theta(n+1)$. From this we obtain

$$\begin{aligned} &T(F,\Lambda)_{0,v_0}((\theta(n+k) \otimes \epsilon(n+1+k)) \oplus (\theta(n) \otimes \epsilon(n+1))) \\ &\qquad \equiv \theta(\Lambda_{0,v_0}) \oplus \theta_\pi(F_{0,v_0}) \mod F^*_{0,v_0}m(1)(\theta(\Lambda_{0,v_0}) \oplus \theta_\pi(F_{0,v_0})) \end{aligned}$$

and by Lemma 4.1

$$T(F,\Lambda)_{0,v_0}((\theta(n+k) \otimes \epsilon(n+1+k)) \oplus (\theta(n) \otimes \epsilon(n+1))) = \theta(\Lambda_{0,v_0}) \oplus \theta_\pi(F_{0,v_0}),$$

from which

$$\begin{aligned} &T(F,\Lambda)_{0,v_0}\{F^*_{0,v_0}m(n)\{(\theta(n+k) \otimes \epsilon(n+1+k)) \oplus (\theta(n) \otimes \epsilon(n+1))\}\} \\ &\qquad = F^*_{0,v_0}m(n)(\theta(\Lambda_{0,v_0}) \oplus \theta_\pi(F_{0,v_0})). \end{aligned}$$

By the assumption of the claim

$$\partial\lambda_v/\partial v|_{v=v_0} \in F^*_{0,v_0}m(n)\theta(\Lambda_{o,v_0})$$

hence

$$T(F,\Lambda)_{0,v_0}(\partial/\partial v,\ \partial/\partial v) = \partial\lambda_v/\partial v|_{v=v_0} \oplus 0 = T(F,\Lambda)_{0,v_0}(\chi, \xi)$$

with a $\chi \in F^*_{0,v_0}m(n)(\theta(n+k) \otimes \epsilon(n+k+1))$ and a $\xi \in F^*_{0,v_0}m(n)(\theta(n) \otimes \epsilon(n+1))$, from which we obtain

$$T(F,\Lambda)_{0,v_0}(\partial/\partial v - \chi, \partial/\partial v - \xi) = 0,$$

in other words

$$t\Lambda_{0,v_0}(\partial/\partial v - \chi) = 0 \quad \text{and} \quad tF_{0,v_0}(\partial/\partial v - \chi) = \partial/\partial v - \xi\ .$$

Integrating these vector fields $\partial/\partial v - \chi, \partial/\partial v - \xi$ along the v-axis, we obtain a local C^∞-trivialisation of the family (f, λ_v) at $v = v_0$. By the compactness of the unit interval the local triviality glues to give a strict equivalence of (f,λ) and (f,μ'). This completes the proof of the claim.

To complete the proof of Theorem 6.3 We show that there exist germs of diffeomorphisms ϕ of $\mathbb{R}^{n+k}, 0$, ψ of $\mathbb{R}^n$ with $f \circ \phi = \psi \circ f$ and one parameter family of strictly stable divergent diagrams (f, λ_v) such that $\lambda_0 = \lambda, \lambda_1 = \mu' \circ \phi$ and $\lambda_v \cong \lambda \mathrm{mod} < f >_{\epsilon(n+k)}$. Then by the above claim (f,λ) is strictly equivalent to $(f, \mu' \circ \phi)$ which is strictly equivalent to $(\psi \circ f \circ \psi^{-1}, \mu') = (f, \mu')$.

Construction of ϕ, ψ and the stable homotopy λ_v.

We use the notations in §4. Here the trivial parameter $\bar{u} \in \mathbb{R}^t$ is denoted by u for simplicity. Assume $f = (f', u), f' : \mathbb{R}^{n+k-t}, 0 \to \mathbb{R}^{n-t}, 0,\ u \in \mathbb{R}^t$, and f' is minimal. By the definition of the connecting homomorphism $\delta(f, \lambda)$, $\mathbb{R}^t = \ker P_2\delta(f,\lambda)$. Let λ'_v denote the restriction of λ_v to $\mathbb{R}_{n+k-t}$. By the criterion for the infinitesimal stability (Theorem 5.3), (f, λ_v) is strictly stable as a divergent diagram if and only if the connecting homomorphism

$$\delta(f,\lambda) : \mathbb{R}^t \longrightarrow \frac{\theta(\lambda'_v)}{t\lambda'_v(\theta_{f'}(n+k-s)) + f'^{*}_v m(n-t)\theta(\lambda'_v)}$$

is surjective. By the assumption that $\lambda_v \equiv \lambda \bmod < f >_{\epsilon(n+k)}$, it follows $\lambda'_v \equiv \lambda' \bmod < f' >_{\epsilon(n+k-t)}$. By the minimality of f', all lifts χ of f' are tangent to the variety of f' and $f'_*\chi \in \theta(n-t)$ vanishes at the origin (if $f'_*\chi(0) \neq 0, f'$ admits a trivialisation integrating χ and $f'_*\chi$). Therefore the module

$$K = t\lambda'_v(\theta_{f'}(n+k-t)) + f'^{*}m(n-t)\epsilon(n+k-t)$$

is independent of $v \in \mathbb{R}$.

By assumption $(f, \lambda_0), (f, \lambda_1)$ are infinitesimally stable hence the connecting homomorphism $\delta(f, \lambda_v) : \mathbb{R}^t \to \theta(n+k-t)/K$ is surjective for $v = 0, 1$. If the connecting homomorphisms for $v = 0, 1$ are not isomorphisms with different orientation with each other, the homomorphisms are joined by a smooth family of linear surjections $L_v : \mathbb{R}^t \to \theta(n+k-t)/K$. If the connecting homomorphisms are isomorphisms with different orientation, define $\phi(x, u) = (x, u_1, \ldots, u_{t-1}, -u_t)$ and $\psi(y, u) = (y, u_1, \ldots, u_{t-1}, -u_t)$. Then $f \circ \phi = \psi \circ f$ and $\delta(f, \lambda_0), \delta(f, \lambda \circ \phi)$ have the same orientation and admits the smooth surjective family L_v. It remains an easy exercise to the reader to realise the homotopy L_v by a connecting homomorphism $\delta(f, \lambda_v)$ for a smooth homotopy λ_v. This completes the proof of the theorem.

7. Versality and fuction moduli

Proposition 7.1. *S-codimension is invariant under S-equivalence relation.*

Proof. It is easy to see the invariance under the transformation $(f, \lambda) \to (f, \chi \circ \lambda)$ with a diffeomorphism $\chi : \mathbb{R}, 0 \to \mathbb{R}, 0$. So we prove the invariance under the strict S-equivalence relation. Assume $(f, \lambda), (g, \mu)$ are strictly S-equivalent divergent diagrams, S-codim $(f, \lambda) = s$, and let $\{(f_i, \lambda_i), i = 1, \ldots, s\}$ be a generator of

$$\frac{\theta(\lambda) \oplus \theta(f)}{T(f,\lambda)(\theta(n+k)) + f^{*}m(n)(\theta(\lambda) \oplus \theta(f))}.$$

By the definition of strict S-equivalence, $(f, \lambda), (g, \mu)$ admit unfoldings

$$(F, \Lambda), (G, \Gamma) : \mathbb{R}^{n+t+k}, 0 \to \mathbb{R}^{n+t+1}, 0 \to \mathbb{R}^{n+t}, 0,$$

which are strictly equivalent. Let $(\phi, \psi \times \mathrm{id}, \psi) : (F, \Lambda) \to (G, \Gamma)$ be the equivalence (see §1 for the definition). Define $(\bar{F}, \bar{\Lambda}) : \mathbb{R}^{n+s+t+k} \to \mathbb{R}^{n+s+t+1} \to \mathbb{R}^{n+s+t}$ by $\bar{\Lambda} = \Lambda + \sum_{i=1}^{s} v_i \lambda_i, \bar{F} = (F + \sum_{i=1}^{s} v_i f_i,\ v)$ and define $(\bar{G}, \bar{\Gamma}), \bar{G} = (G + \sum_{i=1}^{s} v_i f'_i,\ v), \bar{\Gamma} = \Gamma + \sum_{i=1}^{s} v_i \lambda'_i$ so that $((\phi, v), (\psi, v) \times \mathrm{id}, (\psi, v)) : (\bar{F}, \bar{\Lambda}) \to (\bar{G}, \bar{\Gamma})$ is a strict equivalence. By the versality of the restriction $(F', \Lambda') : \mathbb{R}^{n+s+k} \to \mathbb{R}^{n+s+1} \to \mathbb{R}^{n+s}$ of $(\bar{F}, \bar{\Lambda})$, $(\bar{F}, \bar{\Lambda})$ is strictly equivalent to the trivial unfolding of (F', Λ') of dimension t, hence $(\bar{G}, \bar{\Gamma})$ is also strictly equivalent to a trivial unfolding (G'', Γ'') of a (G', Γ') of dimension t. Let $e = (\phi'', \bar{\psi}'', \psi'') : (\bar{G}, \bar{\Gamma}) \to (G'', \Gamma'')$ be a strict equivalence, $\pi : (G'', \Gamma'') \to (G', \Gamma')$ the projection and $j : (g, \gamma) \to (\bar{G}, \bar{\Gamma})$ the natural morphism. Then $\pi \circ e \circ j : (g, \mu) \to (G', \Gamma')$ is a morphism to the stable unfolding. Therefore S-codim $(g, \mu) \leq$ S-codim $(f, \lambda) = s$.

Theorem 7.2. *Let $i : (f, \lambda) \to (F, \Lambda), (F, \Lambda) : \mathbb{R}^{n+s+k} \to \mathbb{R}^{n+s+1} \to \mathbb{R}^{n+s}$ a morphism into a strictly stable unfolding of divergent diagram of dimension s. Then*

(1) For a divergent diagram (g, μ) strictly algebraically S-equivalent to (f, λ), there exists a morphism $j : (g, \mu) \to (F, \Lambda)$,

(2) For a smooth family of divergent diagrams $(f_v, \lambda_v), v \in \mathbb{R}^r$ with $(f_0, \lambda_0) = (f, \lambda)$, there exists a smooth family of morphisms $i_v : (f_v, \lambda_v) \to (F, \Lambda)$: there exist smooth families of smooth imbeddings i_{1v}, i_{3v} with $i_{10} = id, i_{30} = id$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\lambda_v} & \mathbb{R}^{n+k}, 0 & \xrightarrow{f_v} & \mathbb{R}^n, 0 \\ \| & & \downarrow{\scriptstyle i_{1v}} & & \downarrow{\scriptstyle i_{3v}} \\ \mathbb{R} & \xleftarrow[\Lambda]{} & \mathbb{R}^{n+s+k}, * & \xrightarrow[F]{} & \mathbb{R}^{n+s}, * \end{array}$$

for $v \in \mathbb{R}^r$ nearby 0 ($i_v = (i_{1v}, i_{2v}, i_{3v}) : (f_v, \lambda_v) \to (F, \Lambda)$ and $i_{2v} = i_{3v} \times id$).

Proof. By Proposition 7.1 the S-codimension is invariant under the algebraic S-equivalence, so (g, μ) admits a stable unfolding $k : (g, \mu) \to (G, \Gamma)$ of the same dimension. By Theorem 6.3, there exists a strict equivalence $h : (G, \Gamma) \to (F, \Lambda)$. The composition $h \circ k$ is a morphism of the divergent diagrams (g, μ) to (F, Λ).

To prove (2) we repeat a similar argument. Define $(F', \Lambda') : \mathbb{R}^{n+r+k} \to \mathbb{R}^{n+r+1} \to \mathbb{R}^{n+r}$ by $\Lambda'(x, v) = \lambda_v(x), F'(x, v) = (f_v(x), v), x \in \mathbb{R}^{n+k}, v \in \mathbb{R}^r$, and $(F'', \Lambda'') : \mathbb{R}^{n+r+s+k} \to \mathbb{R}^{n+r+s+1} \to \mathbb{R}^{n+r+s}$ by its trivial unfolding of dimension s. Let $(F''', \Lambda''') : \mathbb{R}^{n+r+s+k} \to \mathbb{R}^{n+r+s+1} \to \mathbb{R}^{n+r+s}$ be the trivial unfolding of (F, Λ) of dimension r. Clearly these $(F'', \Lambda''), (F''', \Lambda''')$ are algebraically strictly S-equivalent, and by Theorem 6.3, there exists a strict equivalence $j : (F'', \Lambda'') \to (F''', \Lambda''')$. Let $P : (F''', \Lambda''') \to (F, \Lambda)$ be the natural projection: the triple of the natural projections of $\mathbb{R}^{n+r+s+k}, \mathbb{R}^{n+r+s+1}, \mathbb{R}^{n+r+s}$ onto $\mathbb{R}^{n+k}, \mathbb{R}^{n+1}, \mathbb{R}^n$. Then the composition $P \circ j$ restricts to a morphism of (f_v, λ_v) to (F, Λ). This completes the proof of Theorem 7.2.

Let $(f, \lambda) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0 \to \mathbb{R}^n, 0$ be a divergent diagram of S-codimension s, and assume f is stable. Let $\{\lambda_i, i = 1, \ldots, s\}$ be a generator of $\theta(\lambda)/t\lambda(\theta_f(n+k))$ over $\epsilon(n)$ via f^*. By Theorem 5.1, (f, λ) admits the stable unfolding $(F, \Lambda) : \mathbb{R}^{n+s+k} \to \mathbb{R}^{n+s+1} \to \mathbb{R}^{n+s}$

of dimeision s

$$\Lambda(z,u)=\lambda(z)+\sum_{i=1}^{s}u_i\lambda_i\ ,\quad F(z,u)=(f(z),u),\quad z\in\mathbb{R}^{n+k},u\in\mathbb{R}^s.$$

Write those $i_v:(f_v,\lambda_v)\to(F,\Lambda)$ in Theorem 7.2 (2) as

$$i_{1v}(z)=(\phi_v(z),\alpha_v(z)),\quad j_{3v}(q)=(\psi_v(q),\beta_v(q))$$

with smooth map germs $\alpha_v:\mathbb{R}^{n+k}\to\mathbb{R}^s,\beta_v:\mathbb{R}^n\to\mathbb{R}^s$. By the commutative diagram in the theorem, $\alpha_v(z)=\beta_v(f_v(z))$ holds. Then ϕ_v,ψ_v are germs of diffeomorphisms of $(\mathbb{R}^{n+k},0),(\mathbb{R}^n)$ for small v and the following diagram commutes

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\lambda_v} & \mathbb{R}^{n+k},0 & \xrightarrow{f_v} & \mathbb{R}^n,0\\
\Vert & & \phi_v\downarrow & & \downarrow\psi_v\\
\mathbb{R} & \xleftarrow[\Lambda\circ i_{1v}]{} & \mathbb{R}^{n+k},* & \xrightarrow[f]{} & \mathbb{R}^n,*\ .
\end{array}$$

Therefore we obtain

Theorem 7.3. *Let $(f_v,\lambda_v),v\in\mathbb{R}^r$ be a smooth family of divergent diagrams, (F,Λ) the above strictly stable unfolding and assume f_0 is stable. Then (f_v,λ_v) is strictly equivalent to a germ of (f,λ'_v) at an $z_v\in\mathbb{R}^{n+k}$ nearby 0, where λ'_v is of the form*

$$\begin{aligned}\lambda'_v(z)=\Lambda\circ i_{1v}(z)&=\lambda_0(z)+\sum_{j=1}^{s}\alpha_{v,j}(z)\cdot\lambda_j(z)\\&=\lambda_0(z)+\sum_{j=1}^{s}\beta_{v,j}(f(z))\cdot\lambda_j(z).\end{aligned}$$

The second term in the theorem is called *function moduli.* Let M denote the $\epsilon(n)$-module generated by $\lambda_i,i=1,\ldots,s$. Then

The number of generator of $M=\dim{}_{\mathbb{R}}M/m(n)M=$ S-codim (f,λ).

Furthermore we can prove

Proposition 7.4. *For any smooth family of divergent diagrams $(f,\lambda_u),\lambda_u\in M,u\in\mathbb{R}^s$, $(f,\lambda_u),(f,\lambda'_u)$ are not strictly equivalent for distinct u,u' sufficiently close each other.*

By a similar argument we prove

Theorem 7.5 (Versality theorem). *Let (g,μ) be strictly algebraically S-equivalent to an (f,λ) and assume f,g are stable and f is minimal: f is not equivalent to a trivial unfolding of an f'. Then (g,μ) is strictly equivalent to a diagram $(f,\lambda'), \lambda' \in M$.*

Proof. Let i_1, i_3 be the imbeddings in Theorem 7.1 and $\pi_1 : \mathbb{R}^{n+s+k} \to \mathbb{R}^{n+k}, \pi_3 : \mathbb{R}^{n+s} \to \mathbb{R}^n$ be the natural projections. Then g is given by the fibre product of f and $\pi_3 \circ i_3$

$$\begin{array}{ccc} \mathbb{R}^{n+k},0 & \xrightarrow{f} & \mathbb{R}^n,0 \\ {\scriptstyle \pi_1 \circ i_1}\uparrow & & {\scriptstyle \pi_3 \circ i_3}\uparrow \\ \mathbb{R}^{n+k},0 & \xrightarrow[g]{} & \mathbb{R}^n,0\,. \end{array}$$

By the minimality of f, $\pi_3 \circ i_3$ is necessarily transversal to $0 \in \mathbb{R}^n$ hence it is a germ of diffeomorphism. This argument is verified as follows. The germ f admits a representative $\tilde{f}$ defined on a neighbourhood U of $0 \in \mathbb{R}^{n+k}$ with the following property [32]. The set S of those $x \in U$ where the germ of $\tilde{f}$ is contact equivalent to f is a submanifold, and $\tilde{f}$ restricts to an imbedding of S to $\mathbb{R}^n$. The induced map germ g is stable if and only if $\pi_3 \circ i_3$ is transversal to the image $\tilde{f}(S)$ at the origin. If $S \neq 0$, then there exists an imbedding $i : \mathbb{R}^{n-s},0 \to \mathbb{R}^n,0$ transverse to $f(S)$ and the fibre product of f and i defines a stable map germ f_i and f is equivalent to the trivial unfolding of f_i of dimension $s \neq 0$. So the minimality of f is equivalent to $S = 0$. The rest of the proof follows the above argument in the proof of Theorems 7.2 and 7.3.

8. Genericity of S-finiteness

A *pro-algebraic set* $\Sigma \subset J(n+k) \times J(n+k,n)$ is defined by a projective system $\pi_{rs} : \Sigma^r \to \Sigma^s, r > s$ of algebraic sets $\Sigma^r \subset J^r(n+k) \times J^r(n+k,n)$, where π_{rs} is the natural projections of of r-jet space to s-jet space for $r > s$. The codimension of Σ is *infinite* if the codimension of Σ^r tends to infinity as r tends to infinity. In this paper we say a property of infinite jets is *generic* on $J(n+k) \times J(n+k,n)$ if it holds on a complement of a pro-algebraic set of infinite codimension.

Let $(f,\lambda),(g,\mu) : \mathbb{R}^{n+k},0 \to \mathbb{R}^{n+1},0 \to \mathbb{R}^n,0$ be divergent diagrams and $(F,\Lambda) : \mathbb{R}^{n+s+k},0 \to \mathbb{R}^{n+s+1},0 \to \mathbb{R}^{n+s},0$ a strictly stable unfolding of (f,λ). By the (n+s+1)-determinacy of (F,Λ) (Theorem 4.4), if g,μ are (n+s+1)-flat, then $(F+g,\Lambda+\mu)$ is strictly equivalent to (F,Λ) hence $(f+g,\lambda+\mu)$ is strictly S-equivalent to (f,λ). Therefore

Proposition 8.1. *Assume a divergent diagram (f,λ) has S-codimension s. Then (f,λ) is $(n+s+1)$-S-determined: the S-equivalence class is determined by the $(n+s+1)$-jet of (f,λ).*

This argument tells also

Proposition 8.2. *The condition S-codim $(f,\lambda) > s$ is an algebraic condition on $(n+s+1)$-jet of (f,λ), and S-finiteness is a condition on intinite jets.*

Proof. By Theorem 5.1, the property of having S-codimension $\leq$ s is equivalent to having strictly stable unfolding of dimension $\leq$ s. By Theorem 4.4, this is a property of (n+s+1)-jet and, by Lemma 4.1, equivalent to the inequality

$$\dim_{\mathbb{R}} \frac{\theta(\lambda) \oplus \theta(f)}{\mathrm{Im}\, T(f,\lambda) + m(n+k)^{n+s+1} + f^* m(n)(\theta(\lambda) \oplus \theta(f))} \leq s \,,$$

which is the intersection of varieties of $s \times s$-minors of a certain matrix with functions of $(n+s+1)$-jet of (f, λ) at 0 as entries.

The purpose of this section is to prove

Theorem 8.3. *S-finiteness holds in general in the infinite jet space.*

Proof. Let $\Sigma^s \subset J^{n+s+1}(n+k) \times J^{n+s+1}(n+k,n)$ be the algebraic set of $(n+s+1)$-jets of those (f, λ) with S-codimension > s. We prove

$$\mathrm{codim}\, \Sigma^s \to \infty \quad \text{as} \quad s \to \infty.$$

Consider the complexification $\Sigma^s_{\mathbb{C}}$ of Σ^s in the jet space of complex alanytic diagrams $J^{n+s+1}(n+k)_{\mathbb{C}} \times J^{n+s+1}(n+k,n)_{\mathbb{C}}$. We claim that codim $\Sigma^s_{\mathbb{C}} \to \infty$ as $s \to \infty$. To show this it suffices to find an S-finite jet in the preimage $\pi^{-1}_{rs}(\Sigma^s_{\mathbb{C}})$ with an $r > s$ for any given s (this argument is attributed to duPlessis [32]).

From now let $(f, \lambda) : \mathbb{C}^{n+k}, 0 \to \mathbb{C}^{n+1}, 0 \to \mathbb{C}^n, 0$ be a divergent diagram of complex analytic map germs and let $\tilde{f} : U \to V, \tilde{\lambda} : U \to W$ be representative of f, λ defined on open neighbourhoods of 0. Let $O(U)$ denote the sheaf of germs of holomorphic functions on U, $\theta(U)$ the sheaf of $O(U)$-modules of germs of holomorphic vector fields on U, and let $\theta(\tilde{\lambda}), \theta(\tilde{f})$ be respectively the sheaf of $O(U)$-module of germs of holomorphic sections of the bundles $\tilde{\lambda}^*TW, \tilde{f}^*TV$ on U. Define the morphisms $t\tilde{\lambda} : \theta(U) \to \theta(\tilde{\lambda}), t\tilde{f} : \theta(U) \to \theta(\tilde{f})$ and $\omega\tilde{f} : \theta(V) \to \theta(\tilde{f})$ similarly to those in §4 by the differentials of $\tilde{\lambda}, \tilde{f}$ and the pull back of $\tilde{f}$ and define

$$T(\tilde{f}, \tilde{\lambda}) : \theta(U) \oplus \theta(V) \to \theta(\tilde{\lambda}) \oplus \theta(\tilde{f})$$

similarly to the smooth case in §4 by

$$T(\tilde{f}, \tilde{\lambda})(\chi, \xi) = (t\tilde{\lambda}(\chi),\ t\tilde{f}(\chi) - \omega\tilde{f}(\xi)).$$

Consider the exact sequence

$$\theta(U) \xrightarrow{t\tilde{\lambda} \oplus t\tilde{f}} \theta(\tilde{\lambda}) \oplus \theta(\tilde{f}) \longrightarrow C \longrightarrow 0.$$

The cokernel C is coherent sheaf of $O(U)$-modules by a result due to Oka, of which the support is the set of those $z \in U$ where one of the following conditions holds:

(1) $\tilde{f}$ is singular

(2) $\tilde{f}$ is non singular and the restriction of $\tilde{\lambda}$ to $\tilde{f}^{-1}(\tilde{f}(z))$ is singular.

And the stalk of C on z is

$$C_z = \frac{\theta(\tilde{\lambda})_z \oplus \theta(\tilde{f})_z}{T(\tilde{f},\tilde{\lambda})_z(\theta(U)_z \oplus 0)}.$$

In [30] it is shown that the following condition is a generic condition in the complex jet space: f, g aadmit representatives $\tilde{f}, \tilde{\lambda}$ with the following conditions (i) - (v),

(i) $\tilde{f}^{-1}(0)$ has an isolated singularity at 0,
(ii) the restriction $\tilde{\lambda} : \tilde{f}^{-1}(0) \to \mathbb{C}$ is non singular on a punctured neighbourhood of 0,
(iii) codim $\Sigma(\tilde{f})$ at $0 = k+1$, codim $\Sigma(\tilde{\lambda}) = n+k$,

By shrinking U, V

(iv) $\tilde{f}|\Sigma(\tilde{f})$ is a proper and finite to one map to the image and $\tilde{f}^{-1}(0) \cap \Sigma(\tilde{f}) = 0$,
(v) $\Sigma(\tilde{\lambda}) = 0$ or empty.

The direct image $\tilde{f}_* C$ is a coherent sheaf of $O(V)$-modules by the finite coherence theorem. Next consider the exact sequence

$$O(V) \xrightarrow{\bar{\omega}\tilde{f}_*} \tilde{f}_* C \longrightarrow C' \longrightarrow 0,$$

where $\bar{\omega}\tilde{f}$ is induced from $\omega\tilde{f} : \theta(V) \to \theta(\tilde{f})$. The cokernel C' is again a coherant sheaf of $O(V)$-modules and the stalk of C' on 0 is

$$C'_0 = \frac{\theta(\lambda) \oplus \theta(f)}{T(\lambda, f)(\theta(n+k) \oplus \theta(n))}.$$

By the coherence the stalk C'_0 is finitely generated over $O(V)_0$ and

$$\begin{aligned}\text{S-codim}\,(f,\lambda) &= \dim_{\mathbb{C}} \frac{\theta(\lambda) \oplus \theta(f)}{T(\lambda, f)(\theta(n+k) \oplus \theta(n)) + f^* m(n)(\theta(\lambda) \oplus \theta(f))} \\ &= \dim_{\mathbb{C}} C'_0 / m(n) C'_0 < \infty.\end{aligned}$$

The genericity for Conditions (i)-(v) shows the claim.

9. Web structure of divergent diagrams

Let us recall some results from the theory of composite map germs [30,31]. Regard a diagram $(f,\lambda) : \mathbb{R}^{n+k} \to \mathbb{R}^{n+1} \to \mathbb{R}^n$ as a composition of (f,λ) with the first projection π of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$. Assume this admits a stable unfolding as a composite map germs. Then (f,λ) admits a representative $\tilde{f} : U \to V, \tilde{\lambda} : U \to W$ defined on open neighbourhoods U, V, W of the origin of the source and the target such that

(1) The restriction of $(\tilde{f},\tilde{\lambda})$ to the singular point set $\Sigma(\tilde{f},\tilde{\lambda})$ of $(\tilde{f},\tilde{\lambda})$ is proper and finite to one, $\Sigma(\tilde{f},\tilde{\lambda}) \cap (\tilde{f},\tilde{\lambda})^{-1}(0) = 0$ and the image $D(\tilde{f},\tilde{\lambda}) = (\tilde{f},\tilde{\lambda})(\Sigma(\tilde{f},\tilde{\lambda})) \subset V \times W$ is closed,

(2) The second projection π restricts on $D(\tilde{f},\tilde{\lambda})$ to a proper and finite to one mapping, $D(\tilde{f},\tilde{\lambda}) \cap \pi^{-1}(0) = 0$ and the image $\pi(D(\tilde{f},\tilde{\lambda})) \subset V$ is closed, If Λ is nonsingular, we may also assume, by shrinking U,

(3) $\tilde{\lambda}$ is non singular if λ is nonsingular.

We call a representative with these properties a *good representative*. The germs of $D(\tilde{f},\tilde{\lambda}), \pi(D(\tilde{f},\tilde{\lambda}))$ at 0 are determined by (f,λ). When (f,λ) has finite S-codimension, it admits a stable unfolding as a divergent diagram, which is stable as a composite map germ by Theorem 5.1 and Theorem 6.3. So the above result applies to the diagram (f,λ). On the discriminant set $D(\tilde{f},\tilde{\lambda})$, the n+1 st coordinate function t of $\mathbb{R}^{n+1}$ cuts a singular foliation, of which the leaves project to the solutions D_t of the PDE associated to the diagram. We denote the "foliation" by those D_t by $\mathcal{W}_{\tilde{f},\tilde{\lambda}}$ and call the *solution web of* $(\tilde{f},\tilde{\lambda})$. The germ of $\mathcal{W}_{\tilde{f},\tilde{\lambda}}$ at 0 is determined by the germ (f,λ) and denoted $\mathcal{W}_{f,\lambda}$.

Theorem 9.1. *Let $(f,\lambda) : \mathbb{R}^{n+k} \to \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a strictly stable divergent diagram. Then*

(1) λ is non singular,

(2) f is stable,

(3) $(f,\lambda) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0$ is stable,

(4) the rectriction of f, $f_0 : \lambda^{-1}(0), 0 \to \mathbb{R}^n, 0$ is stable.

Proof. (1),(2) and (3) follow from the surjectivity of the morhpism $T(f,\lambda)$. By definition, (f,λ) is infinitesimally stable if $T(f,\lambda)$ is surjective if and only if

$$tf(\ker t\lambda) + \omega f(\theta(n)) = \theta(f).$$

Restricting the equality to the nonsingular variety $\lambda^{-1}(0)$, we obtain

$$tf_0(\theta(\lambda^{-1}(0))_0) + \omega f_0(\theta(n)) = \theta(f_0),$$

where $\theta(\lambda^{-1}(0))$ denotes the module of germs of smooth vector fields on $\lambda^{-1}(0)$ at 0. Therefore f_0 is stable by definition. In particular Theorem 9.1(4) assirts

Corollary 9.2. *Solution web of a strictly stable divergent diagram is a one parameter family of discriminant sets of stable map germs.*

The germ of the solution web $\mathcal{W}_{\tilde{f},\tilde{\lambda}}$ at q is determined by the multi germ of $(\tilde{f},\tilde{\lambda})$ at $S(q) = \{z_1,\dots,z_m\} = \Sigma(\tilde{f},\tilde{\lambda}) \cap \tilde{f}^{-1}(q)$, which is called the *multi germ of* $(\tilde{f},\tilde{\lambda})$ *over* q, denoted $(\tilde{f},\tilde{\lambda})_{S(q)}$ and displayed as

$$\begin{array}{ccccc}
\mathbb{R}^{n+k}, z_1 & \xrightarrow{(\tilde{f},\tilde{\lambda})_{z_1}} & \mathbb{R}^{n+1}, ((\tilde{f}(z_1), \tilde{\lambda}(z_1)) & \longrightarrow & \mathbb{R}^n, q \\
\vdots & & \vdots & & \| \\
\mathbb{R}^{n+k}, z_m & \xrightarrow{(\tilde{f},\tilde{\lambda})_{z_m}} & \mathbb{R}^{n+1}, ((\tilde{f}(z_m), \tilde{\lambda}(z_m)) & \longrightarrow & \mathbb{R}^n, q.
\end{array}$$

All results in the previous sections are naturally generalised to the above multi germ. For example a multi germ of a representative $(\tilde{g},\tilde{\mu})$ at $S(q') = \{z'_1,\dots,z'_m\}$ is *equivalent* (resp. *topologically equivalent*) *to* $(\tilde{f},\tilde{\lambda})_{S(q)}$ *as composite map germs* if there exists one to one correspondence $\sigma : S(q) \to S(q')$ and equivalence (resp. topological equivalence) $(\phi_i, \bar{\psi}_i, \psi_i)$ of the germ of $(\tilde{f},\tilde{\lambda})$ at z_i to the germ of $(\tilde{g},\tilde{\mu})$ at $z_{\sigma(i)}$ for $i = 1,\dots,m$ such that $\psi = \psi_i : \mathbb{R}^n, q \to \mathbb{R}^n, q'$ is independent of i. We say these germs are *strictly equivalent as divergent diagrams* if $\bar{\psi}_i = \psi \times \mathrm{id}$ for $i = 1,\dots,m$. An *unfolding* of the multi germ of $(\tilde{f},\tilde{\lambda})$ at $S(q)$ of dimension s is an m-tuple of unfoldings $j_i = (j_{i1}, j_{i2}, j_{i3}) : (\tilde{f},\tilde{\lambda})_{z_i} \to (F_i, \Lambda_i)_{z_i \times 0}$ such that $F_i(z_i \times 0) = q \times 0$ for $i = 1,\dots,m$ and $j_{i3} : \mathbb{R}^n, q \to \mathbb{R}^{n+s}, q \times 0$ is independent of i. The multi germ $(\tilde{f},\tilde{\lambda})_{S(q)}$ is *infinitesimally strictly stable* if the restriction of

$$\bigoplus_{i=1}^{m} T(\tilde{f},\tilde{\lambda})_{z_i} : \bigoplus_{i=1}^{m} \theta(n+k) \oplus \theta(n) \to \bigoplus_{i=1}^{m} \theta(\tilde{f}_{z_i}) \oplus \theta(\tilde{\lambda}_{z_i})$$

to the diagonal set of those $(\chi_i, \xi)_{i=1,\dots,m} \in \bigoplus_{i=1}^{m} \theta(n+k) \oplus \theta(n)$ is surjective. Two multi germs of divergent diagrams are *strictly S-equivalent* if their infinitesimally stable (hence strictly stable) unfoldings are strictly equivalent.

Proposition 9.3. *An infinitesimally strictly stable divergent diagram (f,λ) admits a good representative $(\tilde{f},\tilde{\lambda})$ such that the multi germs $(\tilde{f},\tilde{\lambda})_{S(q)}, q \in V$ are infinitesimally strictly stable.*

Proof. By Theorem 4.4, (f,λ) is strictly equivalent to a divergent diagram of polynomial map germs (f',λ'). Let $(\tilde{f}'_{\mathbb{C}},\tilde{\lambda}'_{\mathbb{C}})$ be a good representative of the complexification of (f',λ') defined on open neighbourhoods $U \subset \mathbb{C}^{n+k}, V \subset \mathbb{C}^n, W \subset \mathbb{C}$ of the origin. Since $S(q) \subset S(q)_{\mathbb{C}} = \Sigma(\tilde{f}'_{\mathbb{C}},\tilde{\lambda}'_{\mathbb{C}}) \cap \tilde{f}'^{-1}_{\mathbb{C}}(q)$ for $q \in V \cap \mathbb{R}^n$, it suffices to prove the statement for the complexification. The set Σ_{uns} of those $q \in V$, for which the multi germ of the complexification at $S(q)_{\mathbb{C}}$ is not infinitesimally stable is a support of a certain coherent sheaf hance an analytic subset by the argument in the proof of Theorem 8.3. Therefore the infinitesimal stability of the multi germs holds on V by shrinking V.

Theorem 9.1 is generalised for multi germs as follows.

Proposition 9.4. *Let $(\tilde{f}',\tilde{\lambda}')$ be a representative with the property in Proposition 9.3. Then for any distinct $z_i \in \tilde{\lambda}^{-1}(t_i), i = 1,\dots,m$ the multi germ $(\tilde{f}_{t_i z_i})_{i=1,\dots,m}$ is stable, and in particular $\tilde{f}_{t_i}$ are in general position for any distinct $t_1,\dots,t_m$. Here $\tilde{f}_{t_i}$ denotes the restriction of $\tilde{f}$ to $\tilde{\lambda}^{-1}(t_i)$.*

Let $(\tilde{f},\tilde{\lambda})$ be a good representative of (f,λ). The solution web $\mathcal{W}_{\tilde{f},\tilde{\lambda}}$ is a *non singular m-web* at q if all solutions $D_{t_i}, i = 1,\dots,m$ passing through q are smooth, in general position at q, and D_t forms non singular foliation at q as t varies nearby t_i for $i = 1,\dots,m$. The solution web $\mathcal{W}_{\tilde{f},\tilde{\lambda}}$ is a *regular m-web* at q if its germ at q is a non singular web: a configuration of non singular m-foliations in general position (D_{t_i} are possibly singular at q).

Proposition 9.5. *Let $(f,\lambda),(\tilde{f},\tilde{\lambda})$ be as above. Then the following conditions are equivalent.*
(1) The web $\mathcal{W}_{\tilde{f},\tilde{\lambda}}$ is a non singularm-web at a q.

(2) $\pi : D(\tilde{f}, \tilde{\lambda}) \to W$ *is a non singular m-sheet covering over q.*
(3) $\tilde{\lambda}_q : \tilde{f}^{-1}(q) \to \mathbb{R}$ *is a Morse function with m distinct critical values.*

Proof. The (1) $\Rightarrow$ (2) is clear. To prove (2) $\Rightarrow$ (1), assume Codition (2), and that $(\tilde{f}, \tilde{\lambda})$ has a hold singularity at a $z \in U$ and $\tilde{f}(z) = q$. The level surfaces of $\tilde{\lambda}$ cut a non singular foliation on the discriminalt set $D(\tilde{f}, \tilde{\lambda})$ at a smooth point $(\tilde{f}(z), \tilde{\lambda}(z))$ since the family $\tilde{f}_t$ is trivial by Theorem 9.1 and Corollary 9.2, and the foliation projects to a germ of non singular foliation at q. By Proposition 9.4, the projects of the germs of foliations on $D(f, \lambda)$ at $D(\tilde{f}, \tilde{\lambda}) \cap \pi^{-1}(q)$ are in general position and form a non singular m-web at q. The part (3) $\Rightarrow$ (2) is clear as a Morse function is stable under deformation. Conversely, Condition (2) implies that the singularities of λ_q are of Morse type and the critical values are all distinct.

The web number (the maximum of m) is less than or equal to the sheet number of the first projection of $D(\tilde{f}, \tilde{\lambda})$ by definition in the introduction. By Proposition 9.5 if (f, λ) is strictly stable, the web is nonsingular at a generic point $q \in \mathbb{R}^n$ and the web number is equal to the sheet number of the projection and less than or equal to n. Therefore we obtain

Corollary 9.6. *Let* $(f, \lambda) : \mathbb{R}^{n+k} \to \mathbb{R}^{n+1} \to \mathbb{R}^n$ *be a divergent diagram with S-codimension s. Then* (f, λ) *has web number less than or equal to n+s.*

Proof. It suffices to prove for strictly stable diagrams since the solution web of (f, λ) is obtained by cutting that of a stable unfolding. A strictly stable (f, λ) admits a good representative $(\tilde{f}, \tilde{\lambda})$ with the property in Proposition 9.4. By the transversality of $\tilde{f}_{t_i}$, there exist at most n solutions D_{t_i} passing through q for any $q \in V$.

In the complex analytic case a result due to Goryunov [20] can be stated as follows.

Proposition 9.7. *Let* $(f, \lambda) : \mathbb{C}^{n+k}, 0 \to \mathbb{C}^{n+1}, 0 \to \mathbb{C}^n$ *be strictly stable and minimal, i.e.* (f, λ) *is not strictly euivalent to a trivial unfolding of a strictly stable diagram. Then* (f, λ) *has the web number n.*

Theorem 9.8. *Let* $(f, \lambda) : \mathbb{C}^{n+k}, 0 \to \mathbb{C}^{n+1}, 0 \to \mathbb{C}^n$ *be a strictly stable divergent diagram. Then the singular locus* $\mathrm{Sing}(\mathcal{W}_{f,\lambda})$ *of the solution web of the PDE associated to the diagram is a germ of a hypersurface. If the diagram is simple, i.e.* λ_0 *is simple in the sense of the singularity of functions, the complement of* $\mathrm{Sing}(\mathcal{W}_{f,\lambda})$ *is a* $K(\pi, 1)$ *space. Here* π *is a finite index subgroup of the braid group* $S(d)$*, d being the web number, and the index of* π *is the intersection number of* $D_{t_1} \cdots D_{t_d}$ *for generic distinct* $t_1, \ldots, t_d$ *close to 0.*

Proof. We may assume that (f, λ) is minimal. Then there exist n distinct solutions $D_{t_1}, \ldots, D_{t_n}$ passing through each $q \in \mathbb{C}^n - \mathrm{Sing}(\mathcal{W}_{f,\lambda})$. Here $t_1, \ldots, t_n$ are the critical values of λ_q. Let $\mathbb{C}^n/\sim$ be the quotient space by the permutation of the entries, and define $\Phi : \mathbb{C}^n - \mathrm{Sing}(\mathcal{W}_{\mathcal{I}}) \to \mathbb{C}^n/\sim$ by $\Phi(q) = (t_1, \ldots, t_n)$. Since $D_{t_1}, \ldots, D_{t_n}$ are in general position (Proposition 9.4), Φ is locally diffeomorphic. Since the first projection of $D(\tilde{f}, \tilde{\lambda})$ is proper for a good representative $(\tilde{f}, \tilde{\lambda})$, $t_1, \ldots, t_n$ are bounded and Φ extends analytically to $\mathrm{Sing}(\mathcal{W}_{f,\lambda})$. The hypothesis that λ_0 is simple is equivalent to $\Phi^{-1}(0) = 0$ by definition. Therefore Φ is a covering with the branch locus Sing($\mathcal{W}_{f,\lambda}$). The complement of the discriminat set (the image of the generalised diagonal set) in $\mathbb{C}^n/\sim$ is a $K(B(n),1)$ space. Therefore the complement of Sing($\mathcal{W}_{f,\lambda}$) is a $K(\pi,1)$ space. The index is the sheet number of Φ, which is the intersection number of $D_{t_1}\cdots D_{t_n}$.

10. Equisingularity of PDE

A *non singular m-web of* $\mathbb{R}^n$ *of codimension* 1 is an m-tuple of non singular foliations $(\mathcal{F}_1,\ldots,\mathcal{F}_m)$ in general position. Dufour [15] proved

Theorem 10.1. *Let $\mathcal{W}_1=(\mathcal{F}_1,\ldots,\mathcal{F}_m), \mathcal{W}_2=(\mathcal{F}'_1,\ldots,\mathcal{F}'_m), n<m$, be germs of C^∞-smooth non singular m-webs of codimension 1 at $0\in\mathbb{R}^n$. Let $f:\mathbb{R}^n,0\to\mathbb{R}^n,0$ be a germ of homeomorphism such that $h(\mathcal{W}_1)=\mathcal{W}_2: h(\mathcal{F}_i)=\mathcal{F}'_i$ for $i=1,\ldots,m$. Then h is a germ of C^∞-diffeomorphism.*

It is easy to see that the equivalence classes of (n+1)-tuple of germs of functions $(f_1,\ldots,f_{n+1})$ of n+1-web have infinite codimension in the infinite jet space $J(n,1)\times\cdots\times J(n,1)$. The above theorem tells that topological equivalence classes of (n+1)-functions are all of infinite codimension. This tells that topological equivalence of the solution webs is still too strong to give a "reasonable" view point to classify PDE's with complete integrals. The author suggests the following equivalence relation regarding Theorem 6.3 B. Two S-finite divergent diagrams $(f,\lambda),(g,\mu)$ are *equisingular* if their stable unfoldings $(F,\Lambda),(G,\Gamma)$ of dimension s are topologically equivalent as composite map germs: there exist germs of homeomorphisms $\phi,\bar{\psi}$ and ψ such that the following diagram commutes

$$\begin{array}{ccccc}
\mathbb{R}^{n+k+s},0 & \xrightarrow{(F,\Lambda)} & \mathbb{R}^{n+s+1},0 & \xrightarrow{\pi} & \mathbb{R}^{n+s},0 \\
\phi\downarrow & & \downarrow\bar{\psi} & & \downarrow\psi \\
\mathbb{R}^{n+k+s},0 & \xrightarrow{(G,\Gamma)} & \mathbb{R}^{n+s+1},0 & \xrightarrow{\pi} & \mathbb{R}^{n+s},0\,.
\end{array}$$

Define the equisingularity for multi germs of divergent diagrams in the manner of §8. Apparently this seems to be too weak, while we obtain the following theorems.

Theorem 10.2. *Let $(f,\lambda),(g,\mu)$ be strictly stable as divergent diagrams and topologically equivalent as composite map germs, and $(\phi,\bar{\psi},\psi)$ a topological eqiuvalence. Then the number of the solutions D_t of (f,λ) passing through $q\in\mathbb{R}^n$ is the same as that of (g,μ) passing through $\psi(q)$.*

Proof. Damon [11] proved that the singular point set is topologically invariant for stable map germs. So we obtain $\bar{\psi}(D(f,\lambda))=D(g,\mu)$. The number of solutions of (f,λ) passing through q is the munber of the element of the fiber of the projection π of $D(f,\lambda)$ to $\mathbb{R}^n$. The projection π of $D(f,\lambda)$ is topologically conjugate to that of $D(g,\mu)$ by the homeomorphisms $\bar{\psi},\psi$, from which the statement follows.

Proposition 10.3. *Let $(f,\lambda),(g,\mu)$ be as above. Then the germs of their solutions D_0-0,D'_0-0 passing through the origin are S-equivalent: the germs of $(D_0-0)\times\mathbb{R},(D'_0-0)\times\mathbb{R}$ at the origin are homeomorphic. In particular D_0-0,D'_0-0 are homotopic.*

Proof. By Proposition 9.1, the restriction $f_0 = f|\lambda^{-1}(0)$ is stable hence the family of the restrictions $f_t = f|\lambda^{-1}(t)$ is C^∞-trivial and the discriminant $D(f,\lambda)$ of (f,λ) is diffeomorphic to the cylinder $D(f_0) \times \mathbb{R}$. By the topological invariance of the discriminant set [11], $\bar{\psi}(D(f,\lambda)) = D(g,\mu)$. Now recall the definition of homotopy type of a germ of a subset $X \subset \mathbb{R}^n$ at $x \in X$. Let $U_0 \supset U_1 \supset \cdots$ be an open neighbourhood system at an $x \in X$ and assume the inclusions $U_i \subset U_{i+1}$ are homotopy equivalences. The homotopy type of the germ of X at x is defined by those of U_i (This is independent of the system U_i). Stable map germs are equivalent to polynomial map germs by the finite determinacy [25], and their discriminant sets are cone like and admit Whitney regular semialgebraic stratifications. By the first isotopy lemma [26], the homotopy types of the discriminant sets are well defined (the homotopy type of a germ of a cone V in $\mathbb{R}^n$ with vertex 0 is determined by the S-equivalence class of the link $V \cap S^\epsilon$ with a small sphere S^ϵ centered at 0, and invariant under germs of homeomorphisms of the cone at the vertex). By Proposition 10.3 the germs of $(D_0 - 0) \times \mathbb{R}, (D_0' - 0) \times \mathbb{R}$ at 0×0 are homeomorphic. Clearly these germs are homotopic to $D_0 - 0, D_0' - 0$ respectively.

11. Stability of the contour of PDE with complete integral

Two S-finite divergent diagrams $(f,\lambda), (g,\mu)$ have the same *contour* if the munti germs of good representatives $(\tilde{f},\tilde{\lambda}), (\tilde{g},\tilde{\mu})$ over $q \in \mathbb{R}^n$ have the same equisingular type for q in a neighbourhood of 0, and the contours of the diagrams are *homeomorphic* if there exists a germ of hmeomorphism $\psi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ such that the multi germ of $(\tilde{f},\tilde{\lambda})$ over q has the same equisingular type as that of $(\tilde{g},\tilde{\mu})$ over $\psi(q)$ for q on a neighbourhood of the origin (for the definition of the multi germs over q, see after Corollary 9.2). A contour of an S-finite divergent diagram (f,λ) is *stable* if for any representative $\tilde{f},\tilde{\lambda}$ defined on a neighbourhood U of $0 \in \mathbb{R}^{n+k}$ there exists a neighbourhood V of $(\tilde{f},\tilde{\lambda})$ in $C^\infty(U,\mathbb{R}^n) \times C^\infty(U,\mathbb{R})$ in Whitney topology with the following property: For any $(\tilde{g},\tilde{\mu}) \in V$ there exists an $z \in U$ such that the contour of the germ of $(\tilde{g},\tilde{\mu})$ at z is homeomorphic to that of $(\tilde{f},\tilde{\lambda})$. The purpose of this section is to prove

Theorem 11.1. *The stability of the contour of (f,λ) is a condition on its infinite jet, and holds on a complement of a subset of $J(n+k,n) \times J(n+k,1)$ of codimension $\geq n+k+1$.*

First recall a result for composite map germs

Theorem 11.2 [30]. *A stable composite map germ $(f,\lambda) : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^{n+1}, 0 \to \mathbb{R}^n, 0$ admits a "canonical" stratification*

$$\mathbb{R}^{n+k}, \mathcal{S}_1 \xrightarrow{(f,\lambda)} \mathbb{R}^{n+1}, \mathcal{S}_2 \xrightarrow{\pi} \mathbb{R}^n, \mathcal{S}_3$$

which is Whitney regular and Thom regular (for the definitions, see [18]).

Thom's isotopy theorem [18,26] applies to the morphism of unfolding to give

Theorem 11.3. *Let $i = (i_1, i_2, i_3) : (f,\lambda) \to (F,\Lambda)$ be a morphism of a versal unfolding of dimension s. If the imbedding i_k is transverse to the canonical stratification $\mathcal{S}_k$ for $k = 1,2,3$, then the contour of (f,λ) is stable.*

Proof of Theorem 11.1. In the paper [30] the author proved that the transversality of morphisms $i : (f,\lambda) \to (F,\Lambda)$ is a condition on the infinite jet of (f,λ) (and the first projection

$\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$). And the set Σ of non transverse jets is a subset of codimension greater than or equal to n+k+1, which is semialgebraic, invariant under the C^∞-equivalence of composite map germs, and locally defined in terms of finte jets at each point with respect to the projective topology induced from finite jet spaces. The set $\mathcal{D}$ of infinite jets of composite map germs

$$\mathbb{R}^{n+k}, 0 \xrightarrow{H} \mathbb{R}^{n+1}, 0 \xrightarrow{G} \mathbb{R}^n, 0$$

satisfying the conditions rank $d(G)(0) = n$, rank $dH(0) \geq n$ is open in $J(n+k, n+1) \times J(n+1, n)$, on which the Σ cuts a subset of codimension $\geq n+k+1$. Infinite jets of composite map germs in $\mathcal{D}$ are equivalent to the form

$$(f, \lambda) : \mathbb{R}^{n+k}, 0 \longrightarrow \mathbb{R}^{n+1}, 0 \xrightarrow{\pi} \mathbb{R}^n, 0$$

such that rank $d(f)(0) \geq n-1$, and π is the first projection. The equivalence is canonically constructed to give a semialgebraic submersion of $\mathcal{D}$ to the jet space $J(n+k, n+1)$ of (f, λ). Under this projection the set of non-transverse jets $\Sigma \cap \mathcal{D}$ projects to a semialgebraic set of codimension $\geq n+k+1$. The transversality holds on the complement of the image of the projection. The stability of the contour of those divergent diagrams with transverse jets follow from Theorem 11.3.

We say a contour of an (f, λ) is *topologically finitely determined* if there is a positive integer l such that if (g, μ) has the same l-jet as (f, λ), then the contour of (g, μ) is homeomorphic to the contour of (f, λ). The following statement seems to be true.

The topological finite determinacy of the contour holds in general in the jet space $J(n+k, n+1)$.

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