

**INDICES OF HOLOMORPHIC
VECTOR FIELDS RELATIVE
TO INVARIANT CURVES**

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INDICES OF HOLOMORPHIC VECTOR FIELDS RELATIVE TO INVARIANT CURVES

TATSUO SUWA

In [2], C. Camacho and P. Sad defined the index of a holomorphic vector field relative to an invariant non-singular curve and proved an index formula, which was used in showing the existence of a local integral curve for arbitrary holomorphic vector field with an isolated singularity at the origin in \mathbb{C}^2 . Then A. Lins Neto generalized, in [6], the definition of indices to the case where invariant curves may have singularities and proved an index formula for curves in $\mathbb{P}^2\mathbb{C}$, the complex projective plane. The formula was used in the study of the existence problem of algebraic solutions of dimension one singular foliations on $\mathbb{P}^2\mathbb{C}$. As one of the problems, he proposed to generalize his index formula to the case of dimension one singular foliations on an arbitrary complex surface. The purpose of this note is to remark that in fact his ideas apply to deal with the general case.

In section 1, we define the local index of the module $E = (v)$ generated by a holomorphic vector field v on \mathbb{C}^2 relative to an invariant curve C . As in [6], we make use of a meromorphic 1-form coming from a decomposition of a holomorphic 1-form ω which annihilates v . In our case, we take a decomposition of ω with respect to a defining function of C and restrict the meromorphic form to each irreducible component C_i of C , while in [6], the decomposition is taken from the beginning with respect to a defining function of each C_i . In either of these two definitions, the index is an analytic invariant and the difference is a topological invariant, which is given in terms of intersection numbers in (1.4) Proposition. It is also expressed in terms of some other invariants of curve singularities. An index formula for the indices defined in section 1 is proved in section 2 ((2.1) Theorem). The result is that, if C is a compact curve in a complex surface X invariant by a 1-dimensional singular foliation E on X , then the sum of the indices of E relative to C is equal to the self-intersection number C^2 of C . Thus in the presence of a foliation leaving C invariant, C^2 is localized to the singular points of the foliation. The proof is done by taking an embedded resolution $\pi : \tilde{X} \rightarrow X$ of C , comparing the indices of E relative to C and the indices of the foliation \tilde{E} induced on \tilde{X} from E relative to the proper transform \tilde{C} of C and applying the Camacho-Sad index formula for \tilde{E} and \tilde{C} . Thus the ingredients are essentially in [2] and [6]. Using the adjunction formula, the index formula is restated in (2.5) Theorem, which generalizes the index formula

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in [6]. In section 3, we give an interpretation of the index in terms of holonomy, which may be known to experts.

Elsewhere [11] a generalization in the higher dimensional case is studied along the line of [5] and that will include an alternative proof of the above index formula.

1. Indices of logarithmic vector fields

We denote by \mathcal{O} the ring of germs of holomorphic functions in $\mathbb{C}^2 = \{(x, y)\}$ at the origin 0. The \mathcal{O} -modules of germs of holomorphic vector fields and of holomorphic 1-forms are denoted, respectively, by Θ and Ω . Let C be the germ of a reduced curve in $\mathbb{C}^2 = \{(x, y)\}$ at 0 with defining function f in \mathcal{O} . We say that a germ v in Θ is logarithmic for C if $v(f)$ is in (f) , the ideal generated by f in \mathcal{O} ([1], [8]). Let $C = \cup_{i=1}^r C_i$ be the irreducible decomposition. It is not difficult to see that a germ v in Θ is logarithmic for C if and only if it is logarithmic for each C_i and that this is the case if and only if (a representative) of v is tangential to each C_i at each of its non-singular points sufficiently close to 0. Thus, if v is a logarithmic vector field for C , we also say that C is invariant by v . Sometimes C is called a separatrix of v . For a germ v in Θ , we let $S(v)$ denote (the germ at 0 of) the set $\{(x, y) \mid v(x, y) = 0\}$ and call it the singular set of v . If we write $v = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ with a and b in \mathcal{O} , $S(v)$ is the set of common zeros of a and b . In what follows we consider germs v with $\text{codim } S(v) \geq 2$, i.e., $S(v) \subset \{0\}$, which is equivalent to saying that a and b are relatively prime in \mathcal{O} . A germ v in Θ defines a dimension one (non-singular) foliation near 0 outside of $S(v)$. The foliation is equally defined by a 1-form ω which annihilates v . If $v = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$, we may write $\omega = b dx - a dy$.

(1.1) Lemma. *Let v be a germ in Θ with $\text{codim } S(v) \geq 2$ and C a germ of reduced curve with defining function f . Then v is logarithmic for C if and only if there exist germs g and h in \mathcal{O} and η in Ω such that h and f are relatively prime and that*

$$(*) \quad g\omega = h df + f\eta.$$

Proof. The lemma is proved in [6] when f is irreducible. We may use the same argument if we note that, if f is reduced, after a suitable change of coordinates (x, y) of \mathbb{C}^2 , we may assume that $\frac{\partial f}{\partial y}$ and f are relatively prime. Thus suppose v is logarithmic for C . Then we have $a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = kf$ for some k in \mathcal{O} . We see that a and f are relatively prime, since by assumption, a and b are relatively prime and $\frac{\partial f}{\partial y}$ and f are relatively prime. Hence, by setting $g = \frac{\partial f}{\partial y}$, $h = -a$ and $\eta = k dx$, we have the identity (*). Conversely, the identity (*) implies that $h df \wedge \omega + f \eta \wedge \omega = 0$. This shows that $v(f) \in (f)$, since $df \wedge \omega = -v(f) dx \wedge dy$ and h and f are relatively prime. \square

Let v be a logarithmic vector field for a reduced curve C with defining function f . Also, let $C = \cup_{i=1}^r C_i$ be the irreducible decomposition of C and f

$= f_1 \cdots f_r$ the corresponding irreducible decomposition of f in \mathcal{O} . If we denote by $E = (v)$ the submodule of Θ generated by v , we may think of E as (the tangent sheaf of) a 1-dimensional singular foliation near 0 in \mathbb{C}^2 . Note that every germ in E is logarithmic for C . Thus for each i , $C_i - \{0\}$ is a leaf of the foliation. Now we define the index $\text{Ind}_0(E; C, C_i)$ of E relative to the pair (C, C_i) at 0 as follows. First let $\varphi_i : \Delta \rightarrow C_i$ be a local uniformization of C_i , where Δ is a disk about the origin in \mathbb{C} . Consider a decomposition $(*)$ of ω with respect to f . Since h and f are relatively prime, $\varphi_i^* \frac{\eta}{h}$ is a meromorphic 1-form on Δ . We denote by $\text{Res}_0 \varphi_i^* \frac{\eta}{h}$ its residue at 0; $\text{Res}_0 \varphi_i^* \frac{\eta}{h} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \varphi_i^* \frac{\eta}{h}$, where γ is a closed path in Δ going around 0 once counterclockwise, and we set

$$(1.2) \quad \text{Ind}_0(E; C, C_i) = -\text{Res}_0 \varphi_i^* \frac{\eta}{h}.$$

As in [6], it is shown that the right hand side in (1.2) depends only on E , C and C_i . We set

$$\text{Ind}_0(E; C) = \sum_{i=1}^r \text{Ind}_0(E; C, C_i).$$

Thus $\text{Ind}_0(E; C_i) = \text{Ind}_0(E; C_i, C_i)$, which is computed using a decomposition of ω with respect to f_i .

(1.3) *Remarks.* 1°. $\text{Ind}_0(E; C_i)$ is the index defined by Lins Neto in [6]. When C_i is non-singular, it also coincides with the index of Camacho and Sad [2].

2°. We denote by L the link of C at 0; $L = C \cap S_\epsilon^3$, where S_ϵ^3 is a 3-sphere in \mathbb{C}^2 with center at 0 and small radius ϵ , and by $B \rightarrow L$ the restriction of the closed disk bundle over $C' = C - \{0\}$ (closed tubular neighborhood of C' in \mathbb{C}^2). Note that L and B are naturally oriented. We have, by definition and the projection formula,

$$\text{Ind}_0(E; C) = \frac{\sqrt{-1}}{2\pi} \int_L \frac{\eta}{h} = \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int_\Gamma \frac{df}{f} \wedge \left(-\frac{\eta}{h} \right),$$

where Γ denotes the boundary of B with the reversed orientation. Later in section 2, we see that $\text{Ind}_0(E; C)$ represents a "localized self-intersection number" of C at 0. Noting that $-\frac{\eta}{h} = \frac{df}{f}$ along the integral curves, the above formula should be compared with the one for the intersection number at 0 of two curves C and D defined by functions f and g without common factors :

$$(C \cdot D)_0 = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{dg}{g} = \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int_\Gamma \frac{df}{f} \wedge \frac{dg}{g},$$

where L is the link of C and this time Γ is also taken as

$$\Gamma = \{ (x, y) \mid |f(x, y)| = |g(x, y)| = \epsilon \}$$

with the orientation such that $d(\arg f) \wedge d(\arg g)$ is positive ([3] Ch. 5). In [11], the above formula for the index is generalized to the higher dimensional case.

(1.4) Proposition. *In the above situation, we have*

$$\text{Ind}_0(E; C, C_i) = \text{Ind}_0(E; C_i) + \sum_{j \neq i} (C_i \cdot C_j)_0,$$

where $(C_i \cdot C_j)_0$ denotes the intersection number of C_i and C_j at 0.

Proof. Let $g_j \omega = h df_j + f_j \eta_j$ be a decomposition of ω with respect to f_j . Note that we may choose h independent of j (see the proof of (1.1) Lemma). If we set $F_j = f_1 \cdots \hat{f}_j \cdots f_r$ (omit f_j), we have $F_j g_j \omega = h F_j df_j + f_j \eta_j$. Hence we get

$$g\omega = h df + f\eta \quad \text{with} \quad g = \sum_{j=1}^r F_j g_j \quad \text{and} \quad \eta = \sum_{j=1}^r \eta_j.$$

Therefore, we have

$$\text{Ind}_0(E; C, C_i) = -\text{Res}_0 \varphi_i^* \frac{\eta}{h} = -\text{Res}_0 \varphi_i^* \frac{\eta_i}{h} - \sum_{j \neq i} \text{Res}_0 \varphi_i^* \frac{\eta_j}{h}.$$

Since $\varphi_i^* \omega = 0$, we have

$$\text{Res}_0 \varphi_i^* \frac{\eta_j}{h} = -\text{Res}_0 \varphi_i^* \frac{df_j}{f_j} = -(C_i \cdot C_j)_0.$$

□

(1.5) Example. Let C be a reduced curve with defining function f . The vector field $v = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$ is logarithmic for C and its annihilator is given by $\omega = df$. We may think of this as a decomposition of ω with respect to f . Thus if we set $E = (v)$, we have $\text{Ind}_0(E; C, C_i) = 0$ for each irreducible component C_i . Hence we get $\text{Ind}_0(E; C_i) = -\sum_{j \neq i} (C_i \cdot C_j)_0$ (Cf. [6] p.201).

(1.6) Example. Let $E = (v)$ be generated by $v = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$ with λ and μ non-zero complex numbers and set $C_1 = \{y = 0\}$ and $C_2 = \{x = 0\}$. The vector field v is logarithmic for $C = C_1 \cup C_2$. Since $\omega = \mu y dx - \lambda x dy$ is a 1-form which annihilates v , we have $\text{Ind}_0(E; C_1) = \frac{\mu}{\lambda}$ and $\text{Ind}_0(E; C_2) = \frac{\lambda}{\mu}$. Hence $\text{Ind}_0(E; C, C_1) = \frac{\mu}{\lambda} + 1$, $\text{Ind}_0(E; C, C_2) = \frac{\lambda}{\mu} + 1$ and $\text{Ind}_0(E; C) = \frac{(\lambda + \mu)^2}{\lambda \mu}$. We may also compute these invariants directly using the decomposition $(y - x)\omega = (\lambda x + \mu y)d(xy) - (\lambda + \mu)xy(dx + dy)$ of ω with respect to xy .

(1.7) Example. Let $E = (v)$ be generated by $v = 2x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y}$ and set $C_i = \{f_i = 0\}$ with $f_1 = y$, $f_2 = x$ and $f_3 = x^3 - y^2$. We see easily that v is logarithmic for $C = C_1 \cup C_2 \cup C_3$. We have $\text{Ind}_0(E; C_1) = \frac{3}{2}$ and $\text{Ind}_0(E; C_2) = \frac{2}{3}$. To compute

$\text{Ind}_0(E; C_3)$, note that $y\omega = xdf_3 - 3f_3dx$, $\omega = 3ydx - 2xdy$. Since a uniformization for C_3 is given by $(x, y) = (t^2, t^3)$, We have $\text{Ind}_0(E; C_3) = -\text{Res}_0 \frac{-3dt^2}{t^2} = 6$. Using $(C_1, C_2)_0 = 1$, $(C_2, C_3)_0 = 2$ and $(C_3, C_1)_0 = 3$, we have $\text{Ind}_0(E; C, C_1) = \frac{3}{2} + 3 + 1 = \frac{11}{2}$, $\text{Ind}_0(E; C, C_2) = \frac{2}{3} + 1 + 2 = \frac{11}{3}$, $\text{Ind}_0(E; C, C_3) = 6 + 2 + 3 = 11$ and $\text{Ind}_0(E; C) = \frac{11^2}{6}$. We may also compute these invariants directly using the decomposition $(-x^4 + 4x^3y - y^3 + 3xy^2)\omega = (2x + 3y)df - 11f(dx + dy)$ of ω with respect to $f = f_1f_2f_3$.

Let C be a germ of reduced curve in \mathbb{C}^2 at 0 with defining function f , $C = \cup_{i=1}^r C_i$ the irreducible decomposition of C and $f = f_1 \cdots f_r$ the corresponding irreducible decomposition of f . For the pair (C, C_i) , we define a non-negative integer $c_0(C, C_i)$ as in [4] 1. Thus let $\varphi_i: \Delta \rightarrow C_i$ be a uniformization of C_i as before. We have $\varphi_i^*df = 0$. Hence $\varphi_i^*(\frac{dx}{\partial f}) = -\varphi_i^*(\frac{dy}{\partial f})$ and since either $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ is not divisible by f_i , it represents a meromorphic 1-form on Δ . We let $c_0(C, C_i)$ be the order of the pole at 0 of this meromorphic 1-form. Note that it is zero if and only if 0 is a non-singular point of C . Since $\varphi_i^*f_i = 0$, we have $\varphi_i^*\frac{\partial f}{\partial x} = \varphi_i^*(f_1 \cdots \frac{\partial f_i}{\partial x} \cdots f_r)$ and $\varphi_i^*\frac{\partial f}{\partial y} = \varphi_i^*(f_1 \cdots \frac{\partial f_i}{\partial y} \cdots f_r)$. Hence we see that

$$(1.8) \quad c_0(C, C_i) = c_0(C_i, C_i) + \sum_{j \neq i} (C_i \cdot C_j)_0.$$

We set

$$c_0(C) = \sum_{i=1}^r c_0(C, C_i).$$

This number is related to the Milnor number of C as follows. Suppose $\frac{\partial f}{\partial y}$ and f are relatively prime. Then denoting by D and C_y , respectively, the divisor of $\frac{\partial f}{\partial y}$ and the y -axis, we have

$$c_0(C, C_i) = (D \cdot C_i)_0 - (C_y \cdot C_i)_0 + 1.$$

By a similar computation of intersection numbers as in [6] p.218, it is shown that

$$(1.9) \quad c_0(C) = \mu_0(C) + r - 1,$$

where $\mu_0(C)$ denotes the Milnor number of C at 0. Especially we have $c_0(C_i, C_i) = \mu_0(C_i)$, the Milnor number of C_i at 0. We remark that the invariant $c_0(C)$ coincides with 2δ ([4] Appendix I), where δ is the invariant defined in [9] Ch. IV, §1, and that the identity (1.9) coincides with the one proved in [7] Theorem 10.5. From (1.4) Proposition and the identities (1.8) and (1.9), we may write

$$(1.10) \quad \text{Ind}_0(E; C, C_i) - c_0(C, C_i) = \text{Ind}_0(E; C_i) - \mu_0(C_i).$$

and

$$(1.11) \quad \text{Ind}_0(E; C) - \mu_0(C) + 1 = \sum_{i=1}^r (\text{Ind}_0(E; C_i) - \mu_0(C_i) + 1).$$

2. The index formula

Let X be a complex analytic surface, i.e., a complex manifold of complex dimension two. The sheaf of germs of holomorphic functions on X is denoted by \mathcal{O}_X . A dimension one singular foliation is defined by a rank one locally free subsheaf E of the tangent sheaf Θ_X of X . The singular set $S(E)$ of E is defined to be the set of points in X where the quotient sheaf Θ_X/E fails to be \mathcal{O}_X -free. Thus a dimension one singular foliation is represented by the following data: (1) an open covering $\{U_\alpha\}_{\alpha \in A}$ of X , (2) (not identically zero) holomorphic vector field v_α on each U_α and (3) non-vanishing holomorphic function $e_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ with $v_\alpha = e_{\alpha\beta}v_\beta$ on $U_\alpha \cap U_\beta$ for each pair (α, β) . The sheaf E above is the sheaf of germs of holomorphic sections of the line bundle dual to the one associated to the cocycle $\{e_{\alpha\beta}\}$. If we set $S(v_\alpha) = \{x \in U_\alpha \mid v_\alpha(x) = 0\}$, these sets patch together to obtain the analytic set $S(E)$. From now on we assume that $\text{codim } S(E) \geq 2$. Thus $S(E)$ consists of isolated points in X . Also, the integral curves of v_α in $U_\alpha - S(v_\alpha)$ patch together to obtain a decomposition of $X - S(E)$ into disjoint union of 1-dimensional submanifolds (integral curves of E). Note that a dimension one singular foliation is also defined as a rank one locally free subsheaf of the cotangent sheaf Ω_X of X (the annihilator of E). For more details about singular foliations, we refer to [10]. Let C be a reduced and (globally) irreducible curve in X . We say that C is invariant by E , if a generator of E_p (and hence every germ in E_p) is logarithmic for C at each point p of C . Suppose C is invariant by E and set $S(E; C) = S(E) \cap C$. Note that $S(E; C)$ contains the set $\text{Sing}(C)$ of singular points of C . At each point p in $S(E; C)$, we have the index $\text{Ind}_p(E; C)$ defined in section 1.

(2.1) Theorem. *Let E be a dimension one singular foliation on a complex surface X and C a reduced and irreducible curve in X invariant by E . If C is compact, we have*

$$\sum_{p \in S(E; C)} \text{Ind}_p(E; C) = C^2.$$

First we consider the following situation. Let C be a germ of reduced curve in \mathbb{C}^2 at 0 with defining function f and $C = \cup_{i=1}^r C_i$ its irreducible decomposition. Also, let v be a vector field logarithmic for C and set $E = (v)$. Denoting by X a small neighborhood of 0 in \mathbb{C}^2 , we take an embedded resolution $\pi : \tilde{X} \rightarrow X$ of the singularity of C . Thus \tilde{X} is a complex surface and π a proper holomorphic map onto X . The divisor of π^*f can be written as

$$\sum_{i=1}^r \tilde{C}_i + \sum_{j=1}^s m_j D_j,$$

where (1) the \tilde{C}_i 's are mutually disjoint non-singular curves and for each i , $\pi|_{\tilde{C}_i} : \tilde{C}_i \rightarrow C_i$ is a resolution of the singularity of C_i , (2) each D_j is a projective line and m_j a positive integer, (3) if we set $D = \cup_{j=1}^s D_j$, $\pi|_{\tilde{X}-D}$ is biholomorphic

onto $X - \{0\}$ and (4) each \tilde{C}_i intersects with D at one point q_i which is a non-singular point of D and the intersection is transverse. The foliation E canonically induces a singular foliation \tilde{E} on \tilde{X} ([2], [6]).

(2.2) Lemma. In the above situation, we have

$$\text{Ind}_0(E; C, C_i) = \text{Ind}_{q_i}(\tilde{E}; \tilde{C}_i) + \left(\sum_{j=1}^s m_j D_j \cdot \tilde{C}_i \right)_{q_i}.$$

Proof. A similar assertion for $\text{Ind}_0(E; C_i)$ is proved in [6] p.210. We may prove the lemma along the same line. Thus let

$$g\omega = h df + f\eta$$

be a decomposition for ω with respect to f . We have

$$(2.3) \quad \pi^* g \cdot \pi^* \omega = \pi^* h \cdot d(\pi^* f) + \pi^* f \cdot \pi^* \eta.$$

If we denote $(\sum_{j=1}^s m_j D_j \cdot \tilde{C}_i)_{q_i}$ by m , we may choose local coordinates (s, t) on \tilde{X} about q_i so that \tilde{C}_i is defined by $s = 0$ and $\sum_{j=1}^s m_j D_j$ by $t^m = 0$ in a neighborhood of q_i and that t is a local coordinate on \tilde{C}_i about q_i . Note that the left hand side of (2.3) is a multiple of a 1-form defining the foliation \tilde{E} in a neighborhood of q_i . The right hand side of (2.3) can be written as

$$\tilde{h} ds + s\tilde{\eta}$$

with $\tilde{h} = \pi^* h \cdot ut^m$ and $\tilde{\eta} = ut^m \pi^* \eta + \pi^* h \cdot d(ut^m)$. Since h and f are relatively prime, \tilde{h} and s are also relatively prime. Hence, denoting the embedding $\tilde{C}_i \hookrightarrow \tilde{X}$ by ι and setting $\varphi = \pi \circ \iota$, we have

$$\text{Ind}_{q_i}(\tilde{E}; \tilde{C}_i) = -\text{Res}_0 \iota^* \frac{\tilde{\eta}}{\tilde{h}} = -\text{Res}_0 \left(\varphi^* \frac{\eta}{h} + \frac{d(\iota^* u \cdot t^m)}{\iota^* u \cdot t^m} \right) = \text{Ind}_0(E; C, C_i) - m,$$

since $\iota^* u$ is a unit. \square

Now, to prove (2.1) Theorem, we take a (global) embedded resolution $\pi : \tilde{X} \rightarrow X$ of the singularities of C . Thus \tilde{X} is a complex surface and π a proper holomorphic map onto X . The total transform of C can be written as

$$\tilde{C} + \sum_{j=1}^s m_j D_j,$$

where this time (1) \tilde{C} is a compact connected non-singular curve and $\pi|_{\tilde{C}} : \tilde{C} \rightarrow C$ is a resolution of the singularities of C , (2) each D_j is a projective line and m_j a positive integer, (3) if we set $D = \cup_{j=1}^s D_j$, $\pi|_{\tilde{X}-D}$ is biholomorphic onto $X - \text{Sing}(C)$ and (4) \tilde{C} intersects with D at a finite number of points which are non-singular points of D and each intersection is transverse. The foliation E canonically induces a singular foliation \tilde{E} on \tilde{X} . Then by (2.2) Lemma, the sum $\sum \text{Ind}(E; C)$ of the indices of E relative to C is equal to the sum $\sum \text{Ind}(\tilde{E}; \tilde{C})$ of the indices of \tilde{E} relative to \tilde{C} plus $\sum m_j D_j \cdot \tilde{C}$, the total intersection number of $\sum m_j D_j$ and \tilde{C} . By the Camacho-Sad index theorem in [2], we have $\sum \text{Ind}(\tilde{E}; \tilde{C}) = \tilde{C}^2$. The theorem follows if we apply the projection formula $C^2 = \tilde{C}^2 + \sum m_j D_j \cdot \tilde{C}$.

(2.4) *Remark.* Thus in the presence of a foliation E leaving C invariant, the self-intersection number is localized to the set $S(E; C)$, which includes the singular set $\text{Sing}(C)$ of C .

(2.1) Theorem is restated as

(2.5) **Theorem.** Let X , E and C be as in (2.1) Theorem. For each p in $S(E; C)$, let $\cup_{i=1}^{r_p} C_{p,i}$ be the irreducible decomposition of C at p . Then we have

$$\sum_{p \in S(E; C)} \sum_{i=1}^{r_p} \text{Ind}_p(E; C_{p,i}) = -K_X \cdot C - \chi(\tilde{C}) + \sum_{p \in S(E; C)} \sum_{i=1}^{r_p} \mu_p(C_{p,i})$$

and

$$\sum_{p \in S(E; C)} \text{Ind}_p(E; C) = -K_X \cdot C - \chi(\tilde{C}) + \sum_{p \in S(E; C)} c_p(C),$$

where K_X denotes the canonical bundle of X and $\chi(\tilde{C})$ the Euler-Poincaré characteristic of a non-singular model \tilde{C} of C .

Proof. These follow from (2.1) Theorem, (1.10) and the adjunction formula ([4] 2.)

$$-\chi(\tilde{C}) = C^2 + K_X \cdot C - \sum_{p \in \text{Sing}(C)} c_p(C).$$

□

(2.6) *Remarks.* 1°. In the right hand side of each of the above identities, the set $S(E; C)$ may be replaced by $\text{Sing}(C)$.

2°. The first formula above is proved in [6] Theorem A when $X = \mathbb{P}^2\mathbb{C}$.

3°. In the theorems (2.1) and (2.5), X need not be compact.

3. A remark on the indices

As is known the Camacho-Sad index can be interpreted in terms of holonomy. In this section, we remark that a similar interpretation is possible for the singular invariant curve case as well.

For simplicity, we consider only (locally) irreducible curves. Thus let C be the germ of a reduced and irreducible curve in $\mathbb{C}^2 = \{(x, y)\}$ at 0 with defining function f . By changing the coordinate system if necessary, we may assume that f is a Weierstrass polynomial in y . Thus C has a uniformization $\varphi : \Delta \rightarrow C$ of the form $\varphi(t) = (t^m, y(t))$ and f and $\frac{\partial f}{\partial y}$ are relatively prime. On $C' = C - \{0\}$, $\frac{\partial f}{\partial y}$ does not vanish and in a neighborhood of C' , (x, f) form a coordinate system and x is a coordinate on C' . Let v be a logarithmic vector field for C and E the foliation generated by v . In the (x, f) coordinate system, take a point $(x_0, 0)$ on C for sufficiently small $|x_0|$ and consider the complex line \mathbb{C}_{x_0} given by $x = x_0$. Also for small $|f_0|$, let $P = (x_0, f_0)$ be a point on \mathbb{C}_{x_0} and let the integral curve through P be C_P , which is a non-singular (complex) curve in $\mathbb{C}^2 - \{0\}$. Let t_0 be a point in Δ such that $x_0 = t_0^m$. A closed path in Δ based at t_0 and going around the origin once counterclockwise corresponds, by φ , to a closed path γ_0 in C based at $(x_0, 0)$ and going around 0 once along the link L of C . We denote by γ the lift of γ_0 in C_P with the initial point at P . Its end point is in \mathbb{C}_{x_0} and can be written as $(x_0, H(f_0))$, where $H(f_0)$ is a holomorphic function of f_0 (the holonomy of the foliation at P along γ). Since $H(0) \neq 0$ we may write $H(f_0) = \sum_{i \geq 1} c_i (f_0)^i$. Note that c_1 is non-zero and its argument is determined by the path γ .

(3.1) Proposition. *In the above situation, we have*

$$\text{Ind}_0(E; C) = \frac{1}{2\pi\sqrt{-1}} \log c_1.$$

Proof. Consider the decomposition $g\omega = hdf + f\eta$ for an annihilator ω of v as in (1.1) Lemma. Since g and f are also relatively prime, g is non-vanishing in a neighborhood of C' . Thus the integral curves of v near C' are obtained by solving the differential equation $hdf + f\eta = 0$ or

$$(3.2) \quad -\frac{\eta}{h} = \frac{df}{f}.$$

Suppose the integral curve C_P is given, in a neighborhood of P , by

$$f = \psi(x, f_0).$$

Thus it satisfies (3.2) and we have $\psi(x_0, f_0) = f_0$. Now outside of 0, we may write $-\frac{\eta}{h} = a(x, f)dx + b(x, f)df$ with $a(x, f)$ and $b(x, f)$ holomorphic in (x, f) . From (3.2), we have

$$\left(a(x, \psi(x, f_0)) + b(x, \psi(x, f_0)) \frac{d\psi}{dx}(x, f_0) \right) dx = d \log \psi(x, f_0),$$

for x near x_0 . If we denote by $A_0(x)$ a primitive function of the left hand side above, fixing the argument of $\psi(x, f_0)$ suitably, we have $A_0(x) = \log \psi(x, f_0)$. We consider the analytic continuation of $A_0(x)$ along γ to get a holomorphic function $A_1(x)$ in a neighborhood of x_0 . The analytic continuation of $\psi(x, f_0)$ along γ is $\psi(x, H(f_0))$ in a neighborhood of x_0 and its argument is determined from that of $\psi(x, f_0)$. We have

$$A_1(x) - A_0(x) = \log \frac{\psi(x, H(f_0))}{\psi(x, f_0)}.$$

If we set $x = x_0$, the left hand side becomes $-\int_{\gamma} \frac{\eta}{h}$ and the right hand side $\log \frac{H(f_0)}{f_0}$. Finally if we make f_0 approach 0, we get $-\int_{\gamma_0} \frac{\eta}{h} = \log c_1$. Thus we obtain the formula. \square

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