Star-shaped complexes and Ehrhart polynomials

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Version of May 28, 1993.

Star-shaped complexes and Ehrhart polynomials*)

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A polyhedral complex Γ in $\mathbb{R}^N$ is a finite set of convex polytopes in $\mathbb{R}^N$ such that (1.1) if $\mathcal{P} \in \Gamma$ and $\mathcal{F}$ is a face of $\mathcal{P}$ then $\mathcal{F} \in \Gamma$, and (1.2) if $\mathcal{P}, \mathcal{Q} \in \Gamma$ then $\mathcal{P} \cap \mathcal{Q}$ is a face of $\mathcal{P}$ and of $\mathcal{Q}$. We are concerned with a polyhedral complex Γ in $\mathbb{R}^N$ which satisfies the following conditions: (2.1) every vertex α of $\mathcal{P} \in \Gamma$ has integer coordinates, i.e., $\alpha \in \mathbb{Z}^N$, and (2.2) the underlying space $X := \cup_{\mathcal{P} \in \Gamma} \mathcal{P}$ ($\subset \mathbb{R}^N$) of Γ is homeomorphic to the d-ball. Let ∂X denote the boundary of X, thus ∂X is homeomorphic to the (d-1)-sphere. Given an integer $n > 0$, write nX for $\{ n\alpha ; \alpha \in X \}$ and define $i(X,n)$ to be $\#(nX \cap \mathbb{Z}^N)$, the cardinality of $nX \cap \mathbb{Z}^N$. In other words, $i(X,n)$ is equal to the number of rational points $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in X$ with each $n\alpha_i \in \mathbb{Z}$. It is known that (3.1) $i(X,n)$ is a polynomial in n of degree d, called the *Ehrhart polynomial* of X, (3.2) $i(X,0) = 1$, and (3.3) $(-1)^d i(X,-n) = \#[n(X - \partial X) \cap \mathbb{Z}^N]$ for every $1 \leq n \in \mathbb{Z}$. Define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

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*) This paper was presented in the meeting "Combinatorial Convexity and Algebraic Geometry" held at Mathematisches Forschungsinstitut Oberwolfach, March 28 - April 3, 1993.

$$(1-\lambda)^{d+1}\left[1+\sum_{n=1}^{\infty} i(X,n)\lambda^n\right]=\sum_{i=0}^{\infty}\delta_i\lambda^i.$$

Then (4.1) $\delta_0 = 1$ and $\delta_1 = \#(X \cap \mathbb{Z}^N) - (d+1)$, (4.2) $\delta_i = 0$ for each $i > d$, and (4.3) $\delta_d = \#[(X-\partial X) \cap \mathbb{Z}^N]$. We say that $\delta(X) = (\delta_0, \delta_1, \ldots, \delta_d)$ is the δ-vector of X. We refer the reader to, e.g., [4, Chap. IX] for geometric proofs of the above fundamental results due to Ehrhart. Note that, even though X is not necessarily convex, the proofs in [4] valid without modification since X is homeomorphic to the d-ball.

Some algebraic technique[1] is indispensable for the study of combinatorics on δ-vectors. Fix a field k and let $\xi_1, \xi_2, \ldots, \xi_N, t$ be (commutative) indeterminates over k. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in nX \cap \mathbb{Z}^N$, then we set $\xi^\alpha t^n = \xi_1^{\alpha 1} \xi_2^{\alpha 2} \ldots \xi_N^{\alpha N} t^n$. We write $[A_k(\Gamma)]_n$ for the vector space spanned by all monomials $\xi^\alpha t^n$ with $\alpha \in nX \cap \mathbb{Z}^N$. Thus, in particular, $\dim_k [A_k(\Gamma)]_n = i(X,n)$. Let $A_k(\Gamma)$ denote $\oplus_{n \ge 0} [A_k(\Gamma)]_n$ with $[A_k(\Gamma)]_0 = k$ and define multiplication $(\xi^\alpha t^n)(\xi^\beta t^m)$ of monomials $\xi^\alpha t^n$ and $\xi^\beta t^m$ in $A_k(\Gamma)$ as follows: $(\xi^\alpha t^n)(\xi^\beta t^m) = \xi^{\alpha+\beta} t^{n+m}$ if there exists $\mathcal{P} \in \Gamma$ with $\alpha \in n\mathcal{P}$ and $\beta \in m\mathcal{P}$; $(\xi^\alpha t^n)(\xi^\beta t^m) = 0$ otherwise. Then $A_k(\Gamma)$ is a noetherian (i.e., finitely generated graded) algebra over k and the Hilbert series $F(A_k(\Gamma),\lambda) := \sum_{n=0}^{\infty} \dim_k [A_k(\Gamma)]_n \lambda^n$ is $(\delta_0 + \delta_1\lambda + \delta_2\lambda^2 + \ldots + \delta_d\lambda^d)/(1-\lambda)^{d+1}$. Let $\Omega(A_k(\Gamma)) = \oplus_{n \ge 1} [\Omega(A_k(\Gamma))]_n$ be the graded ideal of $A_k(\Gamma)$ which is generated by those monomials $\xi^\alpha t^n$ such that $0 < n \in \mathbb{Z}$ and $\alpha \in n(X-\partial X) \cap \mathbb{Z}^N$. Since X is homeomorphic to the d-ball, $A_k(\Gamma)$ is Cohen-Macaulay [8, Lemma 4.6]. Thus, a well-known technique of commutative algebra enables us to obtain $\delta(X) \ge 0$, i.e., each $\delta_i \ge 0$ (cf. Stanley [6]). On the other hand, $\Omega(A_k(\Gamma))$ is the canonical module of $A_k(\Gamma)$, see [7, p.81].

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1) We refer to, e.g., [4, Chap. IV] for "Commutative Algebra for Combinatorialists."

We say that X is "star-shaped" with respect to a point $\alpha \in X - \partial X$ if $t\alpha + (1-t)\beta \in X - \partial X$ for every point $\beta \in X$ and for each real number $0 < t < 1$.

THEOREM. We employ the same notation as above. Suppose that the set $(X-\partial X)\cap\mathbb{Z}^N$ is non-empty and that the underlying space X is star-shaped with respect to some $v_1 \in (X-\partial X)\cap\mathbb{Z}^N$. Then the δ-vector $\delta(X) = (\delta_0,\delta_1,\ldots,\delta_d)$ of X satisfies the linear inequalities as follows:

(5.1) $\delta_0 + \delta_1 + \ldots + \delta_i \le \delta_d + \delta_{d-1} + \ldots + \delta_{d-i}$, $0 \le i \le [d/2]$;

(5.2) $\delta_1 \le \delta_i$, $2 \le i < d$.

Sketch of proof. First, recall that a simplicial complex in $\mathbb{R}^N$ is a polyhedral complex Δ in $\mathbb{R}^N$ such that every convex polytope belonging to Δ is a simplex in $\mathbb{R}^N$. Fix an arbitrary simplicial complex $\Delta(0)$ in $\mathbb{R}^N$ with the vertex set $\partial X\cap\mathbb{Z}^N$ whose underlying space is the boundary ∂X of X. Since X is star-shaped with respect to $v_1 \in (X-\partial X)\cap\mathbb{Z}^N$, we can define the cone $\Delta(1)$ over $\Delta(0)$ with apex v_1, i.e., $\Delta(1)$ is the simplicial complex in $\mathbb{R}^N$ which consists of those simplices σ such that either $\sigma \in \Delta(0)$ or σ is the convex hull of $\tau \cup \{v_1\}$ in $\mathbb{R}^N$ for some $\tau \in \Delta(0)$. The vertex set of $\Delta(1)$ is $(\partial X\cap\mathbb{Z}^N)\cup\{v_1\}$ and the underlying space of $\Delta(1)$ is X. Let $(X-\partial X)\cap\mathbb{Z}^N = \{v_1,v_2,\ldots,v_\ell\}$ and, for each $2 \le j \le \ell$, construct a simplicial complex $\Delta(j)$ with the vertex set $(\partial X\cap\mathbb{Z}^N)\cup\{v_1,v_2,\ldots,v_j\}$ and with the underlying space X by the same way as in [5]. We write Δ for $\Delta(\ell)$. Then the element $\theta = \xi^{v1}t + \xi^{v2}t + \ldots + \xi^{v\ell}t$ of $[\Omega(A_k(\Delta))]_1$ is a non-zero divisor on $A_k(\Delta)$. Hence, it follows from a standard technique of commutative algebra [9] (see also [2]) that $\sum_{0\le j\le i} \delta_j \le \sum_{0\le j\le i} \delta_{d-j}$ for every $0 \le i \le [d/2]$. On the other hand, let $h(\Delta) = (h_0,h_1,\ldots,h_d,0)$ be the h-vector (e.g., [7]) of the simplicial complex Δ. Then $h_1 \le h_i$ for each $2 \le i < d$ (cf. [5]). Also, $h_1 = \delta_1$. Since $h_i \le \delta_i$, $0 \le i \le d$, by [1], we have $\delta_1 \le \delta_i$ for each $2 \le i < d$ as desired. Q. E. D.

In the above Sketch of proof, let $A_k(\Delta)^*$ denote the graded subalgebra of $A_k(\Delta)$ generated by $[A_k(\Delta)]_1$ over k. Then $A_k(\Delta)^*$ coincides with the Stanley-Reisner ring [7] of the simplicial complex Δ. Thus $A_k(\Delta)^*$ is Cohen-Macaulay with the Hilbert series $F(A_k(\Delta)^*,\lambda) = (h_0+h_1\lambda+h_2\lambda^2+\ldots+h_d\lambda^d)/(1-\lambda)^{d+1}$. Moreover, $A_k(\Delta)$ is finitely generated as a module over $A_k(\Delta)^*$.

EXAMPLE. Let $N = d = 3$ and $X = \mathcal{P} \cup \mathcal{Q}$, where $\mathcal{P} \subset \mathbb{R}^3$ (resp. $\mathcal{Q} \subset \mathbb{R}^3$) is the tetrahedron with the vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,-1,-1)$ (resp. $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$). Then $(X-\partial X)\cap\mathbb{Z}^3 = \{(0,0,0)\}$ and X is not star-shaped with respect to $(0,0,0)$. However, X is star-shaped with respect to, e.g., $(1/3,1/3,1/3)$. We have $\delta(X) = (1,2,1,1)$ which fails to satisfy (5.1) for $i = 1$ and (5.2) for $i = 2$.

COROLLARY ([5], [9]). Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and suppose that $(\mathcal{P}-\partial\mathcal{P})\cap\mathbb{Z}^N$ is non-empty. Then the δ-vector $\delta(\mathcal{P}) = (\delta_0,\delta_1,\ldots,\delta_d)$ of $\mathcal{P}$ satisfies the following linear inequalities:

(6.1) $\delta_0 + \delta_1 + \ldots + \delta_i \le \delta_d + \delta_{d-1} + \ldots + \delta_{d-i}$, $0 \le i \le [d/2]$;

(6.2) $\delta_1 \le \delta_i$, $2 \le i < d$.

We conclude the paper with a remark about the question when $A_k(\Gamma)$ is Gorenstein. For a while, we assume that $N = d$ and the origin of $\mathbb{R}^d$ is contained in the interior of X. We say that $\delta(X) = (\delta_0,\delta_1,\ldots,\delta_d)$ is symmetric if $\delta_i = \delta_{d-i}$ for every $0 \le i \le d$. It follows from, e.g., [3] that X is star-shaped with respect to the origin if $\delta(X)$ is symmetric. On the other hand, $\delta(X)$ is symmetric if and only if there exists a polyhedral complex Γ in $\mathbb{R}^d$ with the underlying space X such that $A_k(\Gamma)$ is Gorenstein, i.e., the canonical module $\Omega(A_k(\Gamma))$ of $A_k(\Gamma)$ is generated by a single element of $A_k(\Gamma)$.

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