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Hochster's formula on Betti numbers and Buchsbaum complexes

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Abstract. The Betti numbers $\dim_k \underline{\mathrm{Tor}_i}^A(k[\Delta],k)$ with $i > v - d$ of the Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ of a Buchsbaum complex Δ of dimension $d - 1$ over a field k with v vertices are studied.

§1. Betti numbers of Stanley-Reisner rings

First, we recall some fundamental material for algebra, topology and combinatorics on simplicial complexes.

(1.1) Fix a finite set $V = \{x_1, x_2, \ldots, x_v\}$, called the vertex set, and let Δ be a simplicial complex on V. Thus Δ is a family of subsets of V such that (i) $\{x_i\} \in \Delta$ for each $1 \le i \le v$ and (ii) $\sigma \in \Delta$, $\tau \in \sigma$ imply $\tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. Let $d := \max\{\#(\sigma) ; \sigma \in \Delta\}$. Here $\#(\sigma)$ is the cardinality of σ as a finite set. Then the dimension of Δ is defined by $\dim \Delta = d - 1$. A simplicial complex Δ is called *pure* if every maximal face has the same cardinality.

When W is a subset of V, we write Δ_W for the simplicial complex $\{\sigma \in \Delta ; \sigma \subset W\}$ on the vertex set W. On the other hand, given a face σ of Δ, we define the subcomplex $\mathrm{link}_\Delta(\sigma)$

and $\mathrm{star}_\Delta(\sigma)$ of Δ by

$$\mathrm{link}_\Delta(\sigma) := \{ \tau \in \Delta ; \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in \Delta \}$$

$$\mathrm{star}_\Delta(\sigma) := \{ \tau \in \Delta ; \sigma \cup \tau \in \Delta \} .$$

Thus, in particular, $\mathrm{link}_\Delta(\emptyset) = \Delta$.

(1.2) Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring over a field k whose indeterminates are the elements of V with the standard grading, i.e., each $\deg x_i = 1$. Define I_Δ to be the ideal of A which is generated by those square-free monomials $x_{i1} x_{i2} \ldots x_{ir}$, $1 \le i_1 < i_2 < \ldots < i_r \le v$, such that $\{x_{i1}, x_{i2}, \ldots, x_{ir}\} \notin \Delta$, and set $k[\Delta] := A/I_\Delta$. The algebra $k[\Delta]$ over k is called the *Stanley-Reisner ring* of Δ over k ([5], [6]). From now on, we regard $k[\Delta]$ as a graded module over A with the "quotient grading." Then $\dim_A(k[\Delta]) = d$.

Let $\underline{H}_m{}^i(k[\Delta])$ be the i-th local cohomology module of $k[\Delta]$ over A with respect to the irrelevant maximal ideal $m = (x_1, x_2, \ldots, x_v)$ of A , i.e.,

$$\underline{H}_m{}^i(k[\Delta]) := \varinjlim_n \underline{\mathrm{Ext}}_A{}^i(A/m^n, k[\Delta]) ,$$

and $t := \mathrm{depth}_A(k[\Delta])$. Then (i) $\underline{H}_m{}^i(k[\Delta]) = 0$ unless $t \le i \le d$ and (ii) $\underline{H}_m{}^t(k[\Delta]) \neq 0$, $\underline{H}_m{}^d(k[\Delta]) \neq 0$. Consult, e.g., [8] for basic facts on local cohomology modules $\underline{H}_m{}^i(k[\Delta])$.

(1.3) We say that a simplicial complex Δ is *Cohen-Macaulay* (resp. *Buchsbaum*) over k if the module $k[\Delta]$ over A is Cohen-Macaulay (resp. Buchsbaum), i.e., $\underline{H}_m{}^i(k[\Delta]) = 0$ (resp. $\dim_k(\underline{H}_m{}^i(k[\Delta])) < \infty$) for every $0 \le i < d$. Let $\tilde{H}_i(\Delta;k)$ be the i-th reduced homology group of Δ with coefficients k . Then Δ is Cohen-Macaulay if and only if, for every face σ of Δ (possibly, $\sigma = \emptyset$) and for each $i \neq \dim(\mathrm{link}_\Delta(\sigma))$, we have $\tilde{H}_i(\mathrm{link}_\Delta(\sigma);k) = 0$. Every Cohen-Macaulay complex is pure. Moreover, a simplicial complex Δ is Buchsbaum if and only if Δ is pure and $\mathrm{link}_\Delta(\sigma)$ is Cohen-Macaulay for every non-empty face σ of Δ. We refer the reader to, e.g., [3], [7] and [8] for further information on Cohen-Macaulay and Buchsbaum complexes. See also [1].

On the other hand, in [2], we study the integers $\alpha^*(\Delta) = \alpha^*(\Delta;k)$ and $\gamma^*(\Delta) = \gamma^*(\Delta;k)$ defined as follows:

$$\alpha^*(\Delta) := \max \{ j ; \underline{H}_m{}^i(k[\Delta]) = 0 \text{ for each } 0 \le i < j \ (\le d) \}$$

$$\gamma^*(\Delta) := \max \{ j ; \dim_k(\underline{H}_m{}^i(k[\Delta])) < \infty \text{ for each } 0 \le i < j \ (\le d) \}.$$

Thus $1 \le \alpha^*(\Delta) \le \gamma^*(\Delta) \le d$ and $\alpha^*(\Delta) = \mathrm{depth}_A(k[\Delta])$. Moreover, the simplicial complex Δ is Cohen-Macaulay (resp. Buchsbaum) if and only if $\alpha^*(\Delta) = d$ (resp. $\gamma^*(\Delta) = d$). Note that the integer $\alpha^*(\Delta)$ (resp. $\gamma^*(\Delta)$) is equal to the topological invariant $\alpha(\Delta) + 1$ (resp. $\gamma(\Delta) + 1$) in Munkres [4].

(1.4) The i-th *Betti number* $\beta_i{}^A(k[\Delta])$ of the module $k[\Delta]$ over A is defined to be

$$\beta_i{}^A(k[\Delta]) := \dim_k \underline{\mathrm{Tor}}_i{}^A(k[\Delta],k).$$

Let $\rho := v - \alpha^*(\Delta)$. Then $\beta_i{}^A(k[\Delta]) = 0$ unless $0 \le i \le \rho$. The following formula on Betti numbers $\beta_i{}^A(k[\Delta])$ is given by Hochster [3, Theorem (5.1)] :

$$\beta_i{}^A(k[\Delta]) = \sum_{W \subset V} \dim_k(\widetilde{H}_{v-\#(W)-i-1}(\Delta_{V-W};k)). \qquad (1)$$

We are now in the position to state our main result in this paper.

(1.5) THEOREM. Let Δ be a simplicial complex on the vertex set $V = \{x_1,x_2, \ldots ,x_v\}$ of dimension $d - 1$, $A = k[x_1, \ldots , x_v]$ the polynomial ring over a field k, and $k[\Delta] = A/I_\Delta$. Suppose that $(1 \le)\ \alpha^*(\Delta) < \gamma^*(\Delta)\ (\le d)$. Then, for each integer i with $v - \gamma^*(\Delta) < i \le v - \alpha^*(\Delta)$, we have

$$\beta_i{}^A(k[\Delta]) = \sum_{j=0}^{v-\alpha^*(\Delta)-i} \binom{v}{j} \dim_k(\tilde{H}_{v-i-1-j}(\Delta;k)) .$$

(1.6) COROLLARY. Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_v\}$ of dimension $d-1$, $A = k[x_1, \ldots, x_v]$ the polynomial ring over a field k, and $k[\Delta] = A/I_\Delta$. Suppose that Δ is Buchsbaum, but not Cohen-Macaulay. Then, for each integer i with $v - d < i \le v - \mathrm{depth}_A(k[\Delta])$, we have

$$\beta_i{}^A(k[\Delta]) = \sum_{j=0}^{v-\mathrm{depth}_A(k[\Delta])-i} \binom{v}{j} \dim_k(\tilde{H}_{v-i-1-j}(\Delta;k)) .$$

§2. Proof of Theorem (1.5)

We inherit the notation in the preceding section.

(2.1) LEMMA. $\alpha^*(\mathrm{star}_\Delta(\sigma)) \ge \gamma^*(\Delta)$ for every non-empty face σ of Δ.

Proof. See [2, Lemma (2.7)] for an algebraic proof based on [7, Theorem 4.1, p.70]. Also, consult [4, Lemma (6.1)] for a topological proof. Q.E.D.

(2.2) LEMMA. $\underline{H}_m{}^j(k[\Delta]) \cong \underline{H}_m{}^j(k[\Delta_{V-\{x\}}])$ for every $x \in V$ and for each $j < \gamma^*(\Delta) - 1$.

Proof. We have an exact sequence

$$0 \to k[\mathrm{star}_\Delta(\{x\})] \to k[\Delta] \to k[\Delta_{V-\{x\}}] \to 0$$

as graded modules over A. See, e.g., [2, Theorem (1.7)]. Hence, there exists a long exact sequence

$$0 \to \underline{H}_m{}^0(k[\mathrm{star}_\Delta(\{x\})]) \to \underline{H}_m{}^0(k[\Delta]) \to \underline{H}_m{}^0(k[\Delta_{V-\{x\}}])$$

$$\to \underline{H}_m{}^1(k[\mathrm{star}_\Delta(\{x\})]) \to \underline{H}_m{}^1(k[\Delta]) \to \underline{H}_m{}^1(k[\Delta_{V-\{x\}}])$$

$$\to \ . \ . \ .$$

$$\to \underline{H}_m{}^j(k[\mathrm{star}_\Delta(\{x\})]) \to \underline{H}_m{}^j(k[\Delta]) \to \underline{H}_m{}^j(k[\Delta_{V-\{x\}}])$$

$$\to \ . \ . \ .$$

of local cohomology modules. Since $\underline{H}_m{}^j(k[\mathrm{star}_\Delta(\{x\})]) = (0)$ for every $i < \alpha^*(\mathrm{star}_\Delta(\{x\}))$, Lemma (2.1) guarantees that $\underline{H}_m{}^j(k[\mathrm{star}_\Delta(\{x\})]) = (0)$ for every $i < \gamma^*(\Delta)$. Thus $\underline{H}_m{}^j(k[\Delta]) \cong \underline{H}_m{}^j(k[\Delta_{V-\{x\}}])$ for each $j < \gamma^*(\Delta) - 1$ as reqired. **Q.E.D.**

(2.3) LEMMA. $\gamma^*(\Delta_{V-W}) \geq \gamma^*(\Delta) - \#(W)$ **for every** $W \subset V$.

Proof. By Lemma (2.2), $\dim_k(\underline{H}_m{}^i(k[\Delta_{V-\{x\}}])) < \infty$ for each $0 \leq i < \gamma^*(\Delta) - 1$. Hence $\gamma^*(\Delta_{V-\{x\}}) \geq \gamma^*(\Delta) - 1$. Thus $\gamma^*(\Delta_{V-W}) \geq \gamma^*(\Delta_{V-(W-\{x\})}) - 1$ for every $x \in W$. Hence $\gamma^*(\Delta_{V-W}) \geq \gamma^*(\Delta) - \#(W-\{x\}) - 1 = \gamma^*(\Delta) - \#(W)$ as desired. **Q.E.D.**

(2.4) PROPOSITION. $\underline{H}_m{}^j(k[\Delta]) \cong \underline{H}_m{}^j(k[\Delta_{V-W}])$ **for every** $W \subset V$ **and for each** $j < \gamma^*(\Delta) - \#(W)$.

Proof. Let $W = \{x_{i1}, x_{i2}, \ldots, x_{is}\}$ and, for each $0 \leq \ell \leq s$, $W(\ell) = \{x_{i1}, x_{i2}, \ldots, x_{i\ell}\}$. Lemma (2.2) enables us to see $\underline{H}_m{}^j(k[\Delta_{V-W(\ell)}]) \cong \underline{H}_m{}^j(k[\Delta_{V-W(\ell+1)}])$ for each $0 \leq \ell < s$ and for every $j < \gamma^*(\Delta_{V-W(\ell)}) - 1$. On the other hand, by Lemma (2.3), $\gamma^*(\Delta_{V-W(\ell)}) \geq \gamma^*(\Delta) - \#(W(\ell))$ ($> \gamma^*(\Delta) - \#(W)$). Thus $\underline{H}_m{}^j(k[\Delta]) \cong \underline{H}_m{}^j(k[\Delta_{V-W}])$ for each $j < \gamma^*(\Delta) - \#(W)$. **Q.E.D.**

(2.5) COROLLARY. For every subset W of the vertex set V and for each $j < \gamma^*(\Delta) - \#(W)$, $\dim_k(\tilde{H}_{j-1}(\Delta;k))$ is equal to $\dim_k(\tilde{H}_{j-1}(\Delta_{V-W};k))$.

Proof. It follows from, e.g., [7, Theorem 4.1, p.70] that $\dim_k(\underline{H}_m{}^i(k[\Delta])) = \dim_k \tilde{H}_{i-1}(\Delta;k)$ if $\dim_k(\underline{H}_m{}^i(k[\Delta])) < \infty$.

Q.E.D.

We are now in the position to give our proof of Theorem (1.5). Suppose that $\alpha^*(\Delta) < \gamma^*(\Delta)$. Let i be an integer with $v - \gamma^*(\Delta) < i \leq v - \alpha^*(\Delta)$ and W a subset of V. By Corollary (2.5), we have the equality

$$\dim_k(\tilde{H}_{v-\#(W)-i-1}(\Delta;k)) = \dim_k(\tilde{H}_{v-\#(W)-i-1}(\Delta_{V-W};k))$$

since $v-\#(W)-i < \gamma^*(\Delta) - \#(W)$. Hence, by virtue of Eq. (1),

$$\beta_i{}^A(k[\Delta]) = \sum_{W \subset V} \dim_k(\tilde{H}_{v-\#(W)-i-1}(\Delta_{V-W};k))$$

$$= \sum_{W \subset V} \dim_k(\tilde{H}_{v-\#(W)-i-1}(\Delta;k))$$

$$= \sum_{j=0}^{v-\alpha^*(\Delta)-i} \binom{v}{j} \dim_k(\tilde{H}_{v-i-1-j}(\Delta;k))$$

as required. Q.E.D.

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