

**SUBDIAGONAL ALGEBRAS FOR
SUBFACTORS**

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Series #201. May 1993

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1. Introduction

In this paper we study a relation between subdiagonal algebras and subfactors. After the systematic investigation on triangular algebras by Kadison and Singer [8], Arveson [1] introduced the notion of subdiagonal algebras to unify several aspects of non-selfadjoint operator algebras. Since then there has been many investigations on them. On the otherhand the work of Jones [7] on index for subfactors opened a rich theory of subfactors which has been developed by several people. The aim of the paper is an first attempt to understand existence (or non-existence) of analytic structure as subdiagonal algebras in the theory of subfactors.

Let N be a type II_1 factor, G a countable discrete group and $\alpha : G \rightarrow \text{Aut } N$ an outer action. Then the crossed product $M = N \rtimes G$ is also a type II_1 factor. Theorem 2 shows that there exists a bijective correspondence between the set of all maximal subdiagonal algebras for $N \subset M$ and the set of all positive cones of total orders on G . If we regard a subfactor as a quantization of a group, then the Theorem 2 means that a subdiagonal algebra is a quantization of (a positive cone of) a total order on a group. A totally ordered group G must be torsion free, in particular the order of G must be infinite, unless $G = \{1\}$. Therefore it is reasonable to conjecture that if N is a subfactor of M with $[M : N] < \infty$, then there exist no subdiagonal algebras with respect to the canonical conditional expectation $E : M \rightarrow N$ determined by the trace unless $M = N$. We shall confirm the conjecture in the case of subfactors N of a hyperfinite II_1 factor M with $[M : N] \leq 4$. Here the assumption of factorness is essential. In fact, if M is the $n \times n$ full matrix algebras and N is the diagonals (so N is abelian), then the set of all upper triangular matrices is a maximal subdiagonal algebra. In general, when N contains Cartan subalgebra of M , the complete study of subdiagonal algebras was done by Muhly, Saito and Solel [12].

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$

2. Crossed product case

Let M be a finite von Neumann algebras with a faithful normal normalized trace τ . We recall the definition of (σ -weakly closed) subdiagonal algebras by Arveson [1]. Let $\mathcal{A} \ni 1$ be a σ -weakly closed subalgebra of M and E a faithful normal conditional expectation from M onto $N = \mathcal{A} \cap \mathcal{A}^*$ such that $\tau(E(x)) = \tau(x)$ for $x \in M$. Then \mathcal{A} is called a maximal subdiagonal subalgebra of M with respect to E if the following conditions are satisfied:

- (1) $\mathcal{A} + \mathcal{A}^*$ is σ -weakly dense in M ,
- (2) $E(xy) = E(x)E(y)$ for $x, y \in \mathcal{A}$,
- (3) \mathcal{A} is maximal among those subalgebras of M satisfying (1) and (2).

We note that Exel showed that any σ -weakly closed finite subdiagonal algebra is automatically maximal in his excellent paper [4]. Let $\eta : M \rightarrow L^2(M, \tau)$ be the canonical injection. Let $H^2 = [\eta(\mathcal{A})]_2$ be the $\|\cdot\|_2$ -closure of $\eta(\mathcal{A})$ in $L^2(M)$. Put $\mathcal{I} = \{a \in \mathcal{A} | E(a) = 0\}$. Let $e_N : L^2(M) \rightarrow L^2(N)$ be a Jones projection. We also consider other projections $e_{\mathcal{A}} : L^2(M) \rightarrow H^2 = [\eta(\mathcal{A})]_2$ and $e_{\mathcal{I}} : L^2(M) \rightarrow [\eta(\mathcal{I})]_2$. We sometimes identify $\eta(M)$ with M .

Lemma 1. *Let M be a II_1 -factor and N a subfactor of M . Let \mathcal{A} be a maximal subdiagonal algebra with respect to the conditional expectation $E : M \rightarrow N$ determined by the trace. Then $e_N, e_{\mathcal{A}}, e_{\mathcal{I}}, J_M e_{\mathcal{A}} J_M$ and $J_M e_{\mathcal{I}} J_M$ are all in $N' \cap \langle M, e_N \rangle$ and we have that*

$$e_N + e_{\mathcal{I}} + J_M e_{\mathcal{I}} J_M = 1$$

Proof. Since $N\mathcal{A} \subset \mathcal{A}$, $e_{\mathcal{A}}$ is in N' . Similarly $\mathcal{A}N \subset \mathcal{A}$ implies that $e_{\mathcal{A}} \in (J_M N J_M)' = \langle M, e_N \rangle$. We have $e_{\mathcal{A}} = e_N + e_{\mathcal{I}}$ because $H^2 = L^2(N) \oplus [\eta(\mathcal{I})]_2$. Thus $e_N, e_{\mathcal{A}}$ and $e_{\mathcal{I}}$ are in $N' \cap \langle M, e_N \rangle$. Since $\eta(\mathcal{A}^*) = J_M \eta(\mathcal{A})$, we have that $e_{\mathcal{A}^*} = J_M e_{\mathcal{A}} J_M$, and similarly we have $e_{\mathcal{I}^*} = J_M e_{\mathcal{I}} J_M$. Since $\langle M, e_N \rangle = J_M N' J_M$, these operators also in $N' \cap \langle M, e_N \rangle$. Furthermore $L^2(M) = [\eta(\mathcal{A}^*)]_2 \oplus L^2(N) \oplus [\eta(\mathcal{A})]_2$ shows that $J_M e_{\mathcal{I}} J_M + e_N + e_{\mathcal{I}} = 1$.

We recall a construction of subdiagonal algebras in crossed products due to Arveson[1]. Let G be a countable discrete group. Consider a subsemigroup S of G satisfying $S \cap S^{-1} = \{1\}$ and $S \cup S^{-1} = G$. Define the relation $x \leq y$ in G by $x^{-1}y \in S$. Then \leq is a total order on G and is invariant under left multiplication. Clearly $S = \{x \in G | x \geq 1\}$. Any (left-invariant) total order can be realized as above by such a subsemigroup S and we call that S is a positive cone of a total order on G .

Let N be a type II_1 -factor and $\alpha : G \rightarrow \text{Aut } N$ an outer action. Consider a crossed product $M = N \rtimes G$. Let $a = \sum_g a_g \lambda_g$ be a "Fourier" expansion of $a \in M$. Then there exists a conditional expectation $E : M \rightarrow N$ such that $E(\sum_g a_g \lambda_g) = a_1$ and $\tau \circ E = \tau$. We also note that since $N' \cap M = \mathbb{C}$, the above E is the unique conditional expectation of M onto N , see [2] and [3;1.5.5]. Let S be a positive cone of a total order on G . Let \mathcal{A} be the σ -weak closure of the set of all finite sums $\sum_g a_g \lambda_g$, where $a_g \in N$ and $a_g = 0$ except for finitely many $g \in S$. Then by a result of Exel [4], \mathcal{A} is a maximal subdiagonal algebra

of M with respect to $E : M \rightarrow N$. By Arveson [1;Corollary 2.2.4], We have that

$$\begin{aligned} \mathcal{A} &= \{x \in M \mid E(x\lambda_g) = 0 \text{ for all } g \in S \setminus \{1\}\} \\ &= \{x = \sum_g a_g \lambda_g \in M \mid a_g = 0 \text{ for all } g \not\leq 1\} \\ &= H^2 \cap M \end{aligned}$$

The following Theorem 2 shows that any maximal subdiagonal algebra of M has this form:

Theorem 2. *Let N be a type II_1 -factor, G a countable discrete group and $\alpha : G \rightarrow \text{Aut } M$ an outer action. Consider a II_1 -factor $M = N \rtimes_{\alpha} G$ and the unique conditional expectation $E : M \rightarrow N$. Then there exists a bijective correspondence between the set of all maximal subdiagonal algebras \mathcal{A} of M with respect to E and the set of all positive cones S of (left-invariant) total orders on G .*

Proof. Let S be a positive cone of a total order on G . Then the corresponding maximal subdiagonal algebra \mathcal{A} is given by the σ -weak closure of the set of all finite sums $\sum_g a_g \lambda_g$, where $a_g \in N$ and $a_g = 0$ except for finitely many $g \in S$ as discussed above.

Conversely let \mathcal{A} be a maximal subdiagonal algebra of M with respect to E . Then the projection $e_{\mathcal{A}} : L^2(M) \rightarrow H^2$ is in $N' \cap \langle M, e_N \rangle$ by Lemma 1. If we identify $\langle M, e_N \rangle$ with $N \otimes B(\ell^2(G))$, then N is described by diagonal operators

$$\{\oplus_g \alpha_g^{-1}(n) \in N \otimes B(\ell^2(G)) \mid n \in N\} .$$

Since α is an outer action, we have that

$$N' \cap \langle M, e_N \rangle \cong \ell^\infty(G) .$$

Therefore we can identify the projection $e_{\mathcal{A}}$ with a characteristic function $\chi_S \in \ell^\infty(G)$ for some subset $S \subset G$. This implies that

$$H^2 = \oplus_{g \in S} [N\lambda_g]_2$$

By [16;Theorem 1], we have $\mathcal{A} = H^2 \cap M$. Therefore

$$S = \{g \in G \mid \lambda_g \in \mathcal{A}\}$$

Since \mathcal{A} is a subdiagonal algebra, we have that $\mathcal{A}\mathcal{A} \subset \mathcal{A}$, $\mathcal{A} + \mathcal{A}^*$ is σ -weakly dense in M and $\mathcal{A} \cap \mathcal{A}^* = N$. These properties implies that $SS \subset S$, $S \cup S^{-1} = G$ and $S \cap S^{-1} = \{1\}$. Thus S is a positive cone of a total order on G .

From the construction the above correspondence between positive cones S and subdiagonals \mathcal{A} is bijective by noticing the fact that $\mathcal{A} = H^2 \cap M$.

Remark. In the above Theorem 2, the factorness of N is essential. In fact let $N = \mathbb{C}^n$ and $G = \mathbb{Z}/n\mathbb{Z}$. The action α is given by cyclic permutations of cordinales of \mathbb{C}^n . Then the crossed product $M = N \rtimes G$ is isomorphic to the $n \times n$ full matrix algebra. There exist no positive cones S of total orders on G but the set \mathcal{A} of all upper triangular matrices is a maximal subdiagonal algebra of M with respect to $E : M \rightarrow N$.

Corollary 3. *Let N be a type II_1 -factor and $\alpha : G \rightarrow \text{Aut } N$ an outer action of a finite group G . Consider a crossed product $M = N \rtimes_{\alpha} G$ and the conditional expectation $E : M \rightarrow N$. Then there exists no subdiagonal algebras of M with respect to E unless $G = \{1\}$.*

Proof. Since the order of G is finite, there exists no positive cones of total orders on G unless $G = \{1\}$. Therefore by Theorem 2 there exists no subdiagonal algebras of M with respect to E .

If the order of G is infinite, there occur several cases on the number of maximal subdiagonal algebras of $M = N \rtimes G$.

Examples. We use the notation as in Theorem 2.

(1) If $G = \mathbb{Z}$, then there exist exactly two maximal subdiagonal algebras of $M = N \rtimes G$ with respect to E corresponding to \mathbb{Z}_+ and $-\mathbb{Z}_+$ by Theorem 2. The subdiagonal algebra $\mathcal{A} = N \rtimes \mathbb{Z}_+$ is called an analytic crossed product and has been studied deeply, for example see [9],[10] and [11].

(2) If $G = \mathbb{Z}^2$, then there exist infinitely many maximal subdiagonal algebra \mathcal{A} of $M = N \rtimes \mathbb{Z}^2$ with respect to E . In fact consider maximal subdiagonal algebras $\mathcal{A} = \mathcal{A}_{\theta}$ for positive irrational numbers θ corresponding to positive cones

$$S_{\theta} = \{(m, n) \in \mathbb{Z}^2 \mid \theta m + n \geq 0\} .$$

(3) Let $G = \prod_{n=1}^{\infty} G_n$, where $G_n \cong \mathbb{Z}/2\mathbb{Z}$. Then there exists no subdiagonal algebras of $M = N \rtimes G$ with respect to E , because a totally ordered group must be torsion free.

(4) Let $G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then there exists no subdiagonal algebras of $M = N \rtimes G$ with respect to E as in (3).

3. Finite index case

A totally ordered group must be torsion free, In particular the order of G must be infinite unless $G = \{1\}$. If we regard a subfactor as a quantization of a group, then Theorem 2 means that a subdiagonal algebra is a quantization of a positive cone of a total order on a group. Therefore it is reasonable to conjecture that if N is a subfactor of M with $[M : N] < \infty$, then there exist no subdiagonal algebras with respect to the canonical conditional expectation $E : M \rightarrow N$ determined by the trace, unless $M = N$. With a help of the classification result of subfactors of a hyperfinite II_1 -factor (see [5], [6], [13], [14] and [15]), we shall confirm the conjecture in the case of subfactors N of a hyperfinite II_1 factor M with $[M : N] \leq 4$.

Theorem 4. *Let M be a hyperfinite II_1 -factor and N a subfactor of M . If $[M : N] \leq 4$, then there exist no maximal subdiagonal algebras of M with respect to the conditional expectation $E : M \rightarrow N$ determined by the trace, unless $M = N$.*

Proof. We assume that $M \neq N$. Hence we have that

$$\dim N' \cap \langle M, e_N \rangle \geq 3$$

by Lemma 1. Therefore by a result of the classification of subfactors mentioned above, the principal graph of the subfactor $N \subset M$ is one of $D_4, A_n^{(1)}, A_{\infty, \infty}, D_n^{(1)}$ and D_{∞} .

If the principal graph is D_4 , then $M = N \rtimes \mathbb{Z}/3\mathbb{Z}$. Thus there exist no maximal subdiagonal algebras of M in the case by Corollary 3.

If the principal graph is $D_4^{(1)}$, then $M = N \rtimes \mathbb{Z}/4\mathbb{Z}$ or $M = N \rtimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$. Thus the theorem holds in the case by Corollary 3 similarly.

Consider the case that the principal graph is $D_n^{(1)}$ ($n \geq 5$) or D_{∞} . Then there exist two outer automorphisms α and β with period 2 on a hyperfinite II_1 -factor R such that

$$N = R^{\beta} \subset M = R \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z} .$$

Since $N' \cap \langle M, e_N \rangle \cong \mathbb{C}^3$, $\{1, e_N^M, e_R^M\}$ forms a linear basis of it. Therefore the minimal projections of $N' \cap \langle M, e_N \rangle$ are $e_N^M, e_R^M - e_N^M$ and $1 - e_R^M$. We can also write

$$R = N \rtimes \mathbb{Z}/2\mathbb{Z} = N + Nv$$

for some implementing unitary v with $v^2 = 1$. Then

$$e_R^M = e_N^M + ve_N^M v^* .$$

Therefore we have that

$$e_N^M + ve_N^M v^* + (1 - e_R^M) = 1 .$$

On the contrary we suppose that there were subdiagonal algebra \mathcal{A} . Then Lemma 1 shows that

$$e_N^M + e_{\mathcal{I}} + J_M e_{\mathcal{I}} J_M = 1 .$$

We may assume and do assume that $e_{\mathcal{I}} = ve_N^M v^*$ without loss of generality. We shall identify M with $\eta(M)$. Then by [16; Theorem 1] we have

$$\mathcal{I} = [\mathcal{I}]_2 \cap M = e_{\mathcal{I}} L^2(M) \cap M = (ve_N^M v^*) L^2(M) \cap M = [vN]_2 \cap M$$

Therefore v is in \mathcal{I} . Since v is a unitary with $v^2 = 1$, $v = v^* \in \mathcal{I} \cap \mathcal{I}^* = N$. This contradicts to that $E(v) = 0$. Thus there exist no subdiagonal algebras in the case of $D_n^{(1)}$ or D_∞ .

Finally consider the case that the principal graph is one of $A_n^{(1)}$ or $A_{\infty, \infty}$. Then we see that $N' \cap \langle M, e_N \rangle$ is isomorphic to $M_4(\mathbb{C})$, $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ or $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$. In any case there exists a faithful representation $\pi : N' \cap \langle M, e_N \rangle \rightarrow B(\mathbb{C}^4)$ such that $\text{rank } \pi(e_N^M) = 1$ and $\text{rank } \pi(I) = 4$. Let θ be the anti-automorphism on $N' \cap \langle M, e_N \rangle$ defined by $\theta(x) = J_M x^* J_M$. By Lemma 1 we have $e_N + e_{\mathcal{I}} + J_M e_{\mathcal{I}} J_M = I$. Since $\theta(e_{\mathcal{I}}) = J_M e_{\mathcal{I}} J_M$, we have

$$\text{rank } \pi(e_{\mathcal{I}}) = \frac{4 - \text{rank } \pi(e_N)}{2} = \frac{3}{2}.$$

This is a contradiction. Thus the assertion is verified for all cases.

Remark. There is another way to show the non-existence of subdiagonal algebra in general cases including non-hyperfinite factors. Consider the anti-automorphism $\theta(x) = J_M x^* J_M$ for $x \in N' \cap \langle M, e_N \rangle$. Suppose that the anti-automorphism θ preserves the trace, (for example, say, N is an extremal subfactor of II_1 -factor M with a finite depth). Then Lemma 1 implies that

$$\tau(e_{\mathcal{I}}) = \frac{1 - [M : N]^{-1}}{2}.$$

We can also calculate the possible values of traces of projections in $N' \cap \langle M, e_N \rangle$ using the Perron-Frobenius eigenvector for the principal graph. In certain cases $\tau(e_{\mathcal{I}})$ does not belong to the possible values of trace of projections. In this way we can also show the non-existence of subdiagonal algebras easily in certain cases.

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