HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 174: T. Ozawa, Wave propagation in even dimensional spaces, 15 pages. 1992.

♯ 175: S. Izumiya, Systems of Clairaut type, 7 pages. 1992.

♯ 176: A. Hoshiga, The initial value problems for quasi-linear wave equations in two space dimensions with small data, 25 pages. 1992.

♯ 177: K. Sugano, On bicommutators of modules over H-separable extension rings III, 9 pages. 1993.

♯ 178: T. Nakazi, Toeplitz operators and weighted norm inequalities, 17 pages. 1993.

♯ 179: O. Ogurisu, Existence and structure of infinitely degenerate zero-energy ground states of a Wess-Zumino type model in supersymmetric quantum mechanics, 26 pages. 1993.

♯ 180: O. Ogurisu, Ground state of a spin 1/2 charged particle in an even dimensional magnetic field, 9 pages. 1993.

♯ 181: K. Sugano, Note on H-separable Galois extension, 6 pages. 1993.

♯ 182: M. Yamada, Distance formulas of asymptotic Toeplitz and Hankel operators, 13 pages. 1993

♯ 183: G. Ishikawa, T. Ozawa, The genus of a connected compact real algebraic surface in the affine three space, 11 pages. 1993.

♯ 184: T. Hibi, Canonical modules and Cohen-Macaulay types of partially ordered sets, 6 pages. 1993.

♯ 185: Y. Giga, K. Yama-uchi, On a lower bound for the extinction time of surfaces moved by mean curvature, 16 pages. 1993.

♯ 186: Y. Kurokawa, On functional moduli for first order ordinary differential equations, 9 pages. 1993.

♯ 187: O. Ogurisu, Unitary equivalence between a spin 1/2 charged particle in a two-dimensional magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a two-dimensional electric field, 4 pages. 1993.

♯ 188: A. Jensen, T. Ozawa, Existence and non-existence results for wave operators for perturbations of the laplacian, 30 pages. 1993.

♯ 189: T. Nakazi, Multipliers of invariant subspaces in the bidisc, 12 pages. 1993.

♯ 190: S. Izumiya, G.T. Kossioris, Semi-local classification of geometric singularities for Hamilton-Jacobi equations, 24 pages. 1993.

♯ 191: N. Hayashi, T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, 21 pages. 1993.

♯ 192: J. Seade, T. Suwa, A residue formula for the index of a holomorphic flow, 22 pages. 1993.

♯ 193: H. Kubo, Blow-up of solutions to semilinear wave equations with initial data of slow decay in low space dimensions, 8 pages. 1993.

♯ 194: F. Hiroshima, Scaling limit of a model of quantum electrodynamics, 52 pages. 1993.

♯ 195: T. Ozawa, Y. Tsutsumi, Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions, 34 pages. 1993.

♯ 196: H. Kubo, Asymptotic behaviors of solutions to semilinear wave equations with initial data of slow decay, 25 pages. 1993.

♯ 197: Y. Giga, Motion of a graph by convexified energy, 32 pages. 1993.

♯ 198: T. Ozawa, Local decay estimates for Schrödinger operators with long range potentials, 17 pages. 1993.

♯ 199: A. Arai, N. Tominaga, Quantization of angle-variables, 31 pages. 1993.

HOLONOMIC SYSTEMS OF CLAIRAUT TYPE

Shyuichi IZUMIYA and Yasuhiro KUROKAWA

0. Introduction

The Clairaut equation (Alex Claude Clairaut [3], 1734) is one of the typical examples of first order differential equations which has a (classical) complete solution and a singular solution such that the singular solution is the envelope of a family of hyperplanes given by the complete solution. In this article we consider equations with the same geometric structure as the Clairaut equation. Here we give another example as follows :

$$\begin{cases} y - (\dfrac{\partial y}{\partial x_1})^2 = 0 \\ \quad \dfrac{\partial y}{\partial x_2} = 0. \end{cases}$$

We can exactly solve this equation and the complete solution is $y = \frac{1}{4}(x_1 + t)^2$, where t is a parameter. The complete solution of this equation does not consist of hyperplanes. However, the singular solution is the envelope of the family of graphs of the complete solution. We will refer such a system as a system of Clairaut type.

A system of first order differential equations with (classical) complete solution is called *a system of Clairaut type*. In [11] a characterization of systems of Clairaut type has been given. The next problem is to classify these systems by a natural equivalence relation. In this paper we give a generic classification of holonomic (i.e. maximally over determined) systems of Clairaut type under the equivalence relation given by the group of point transformations in the sense of Sophus Lie.

Since our concern is the local classification of differential equations, we can formulate as follows : Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of n-variables functions which may be considered as $\mathbb{R}^{2n+1}$ with natural coordinates given by $(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$. We have the canonical projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$, $\pi(x, y, p) = (x, y)$. Let θ be the canonical contact form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^{n} p_i dx_i$. Using this approach, a first order differential equation is most naturally interpreted as being a closed subset of

$J^1(\mathbb{R}^n, \mathbb{R})$. Unless the contrary is specifically stated, we adopt a submanifold of $J^1(\mathbb{R}^n, \mathbb{R})$ as an equation. We stick to holonomic systems in this article. *A holonomic system of first order differential equation germ (or, briefly, a holonomic system)* is defined to be an immersion germ $f : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$. We also say that f is *completely integrable* if there exists a submersion germ $\mu : (\mathbb{R}^{n+1}, 0) \to \mathbb{R}$ such that $\langle d\mu \rangle_{\mathcal{E}_u} \supset \langle f^* \theta \rangle_{\mathcal{E}_u}$, where $\mathcal{E}_u$ is the ring of smooth function germs on $(\mathbb{R}^{n+1}, 0)$ and its unique maximal ideal is denoted by $\mathfrak{M}_u$. Here, $u = (u_1, \ldots, u_{n+1})$ are canonical coordinates of $(\mathbb{R}^{n+1}, 0)$. We call μ *a complete integral of f* and the pair $(\mu, f) : (\mathbb{R}^{n+1}, 0) \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$ is called *a holonomic system with complete integral*. We observe that $\pi \circ f(\mu^{-1}(t))$ is the graph of the solution in $\mathbb{R}^n \times \mathbb{R}$. If $\pi \circ f(\mu^{-1}(t))$ are non-singular map germs for each $t \in (\mathbb{R}, \mu(0))$, then $\{\pi \circ f(\mu^{-1}(t))\}_{t \in \mathbb{R}}$ is the family of graphs of a (classical) complete solution (cf. [1,9]). We call such a system *a holonomic system of Clairaut type*. These situation lead us to the following definition. Let (μ, g) be a pair of a map germ $g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ and a submersion germ $\mu : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$. Then the diagram

$$(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^{n+1}, 0) \xrightarrow{g} (\mathbb{R}^n \times \mathbb{R}, 0)$$

or briefly (μ, g), is called *a (holonomic) integral diagram* if there exists a holonomic system $f : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ such that (μ, f) is an equation germ with complete integral and $\pi \circ f = g$, and we say that the integral diagram (μ, g) is *induced by f*. If f is a system of Clairaut type, then $(\mu, \pi \circ f)$ is called *of Clairaut type*. Furthermore we introduce an equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are *equivalent (respectively, strictly equivalent)* if the diagram

$$
\begin{array}{ccccc}
(\mathbb{R}, 0) & \xleftarrow{\mu} & (\mathbb{R}^{n+1}, 0) & \xrightarrow{g} & (\mathbb{R}^n \times \mathbb{R}, 0) \\
{\scriptstyle \kappa} \downarrow & & {\scriptstyle \psi} \downarrow & & \downarrow {\scriptstyle \phi} \\
(\mathbb{R}, 0) & \xleftarrow[\mu']{} & (\mathbb{R}^{n+1}, 0) & \xrightarrow[g']{} & (\mathbb{R}^n \times \mathbb{R}, 0)
\end{array}
$$

commutes for some diffeomorphism germs κ, ψ and ϕ (respectively, $\kappa = id_{\mathbb{R}}$).

In [10] it has been defined an equivalence relation among systems under the group of point transformations and shown that two completely integrable holonomic systems f and f' are equivalent if and only if induced integral diagrams $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent for generic (μ, f) and (μ', f') (see §1). Moreover, it has been given generic classifications of holonomic integral diagrams by the strict equivalence in the case when $1 \le n \le 3$. For general n, it is very hard to give classification by technical reasons. However, we give a generic classification of holonomic systems of Clairaut type as follows :

Theorem A. *For a generic holonomic system of Clairaut type*

$$(\mu, f) : (\mathbb{R}^{n+1}, 0) \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}),$$

the integral diagram $(\mu, \pi \circ f)$ is strictly equivalent to on of germs in the following list :
DA_1 ;

$$\mu = u_{n+1},$$
$$g = (u_1, \ldots, u_{n+1}).$$

DA_2 ;

$$\mu = u_{n+1} - \frac{1}{2}u_1,$$
$$g = (u_1, \ldots, u_n, u_{n+1}^2).$$

DA_ℓ $(3 \le \ell \le n+1)$;

$$\mu = u_{n+1},$$
$$g = \left(u_1, \ldots u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i\right).$$

$\widetilde{DA}_{n+2}$;

$$\mu_\alpha = u_{n+1} + \alpha \circ g \quad \text{for } \alpha \in \mathfrak{M}_{(x,y)},$$
$$g = \left(u_1, \ldots, u_n, u_{n+1}^{n+2} + \sum_{i=1}^{n} u_i u_{n+1}^i\right).$$

This theorem gives a generic classification of integral diagrams of Clairaut type under the strict equivalence. However, our purpose is to classify these subjects under the equivalence. We remark that each germs of types DA_ℓ $(1 \le \ell \le n+1)$ and $\widetilde{DA}_{n+2}$ are not equivalent. Thus the problem is reduced to classify germs which are contained in the family $\widetilde{DA}_{n+2}$ under the equivalence. This family is parametrized by function germs α which are called *functional moduli*. We shall characterize functional moduli relative to the equivalence. For the purpose, we now adopt coordinates $(x_1, \ldots, x_n, y)$ of $\mathbb{R}^n \times \mathbb{R}$ and define $\mathcal{D}^n = \{(x_1, \ldots, x_n, y) \in \mathbb{R}^n \times \mathbb{R} | F = \frac{\partial F}{\partial t} = \cdots = \frac{\partial^n F}{\partial t^n} = 0 \text{ for some } t\}$, where $F(t, x_1, \ldots, x_n, y) = t^{n+2} + x_1 t + \cdots + x_n t^n - y$. Then we have the following characterization theorem.

3

Theorem B. *Let (μ_α, g) be an integral diagram of $\widetilde{DA}_{n+2}$. Then for any α, there exists a function germ $\alpha' : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ such that*

(1) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$.

(2) $\alpha'|\mathcal{D}^n = 0$.

This theorem is a generalization of Dufour's result in [6]. He has also shown that the uniqueness of functional moduli relative to the equivalence, so that we now consider a generalization of his uniqueness result. Define

$$\Delta = \{(x_1, \ldots, x_n, y) \in \mathbb{R}^n \times \mathbb{R} | \text{The } (n+2)\text{-degree algebraic equation}$$

$$F_{(x_1, \ldots, x_n, y)}(t) = 0 \text{ has } (n+2)\text{-real roots with multiplicity}\},$$

where $F_{(x_1, \ldots, x_n, y)}(t) = F(t, x_1, \ldots, x_n, y)$. We say that α and α' are *equivalent as moduli* if there exists $a \in \mathbb{R} - \{0\}$ such that $a\alpha(x_1, \ldots, x_n, y) = \alpha'(a^{n+1}x_1, a^n x_2, \ldots, a^2 x_n, a^{n+2}y)$ for any $(x_1, \ldots, x_n, y) \in \Delta$. We remark that this definition of the equivalence among functional moduli is slightly different from Dufour's definition of it in [6]. If we adopt his definition, we cannot assert the necessity of the condition that functional moduli are equivalent. Then we correct the definition as the above.

Theorem C. *Let (μ_α, g) and $(\mu_{\alpha'}, g)$ be integral diagrams of $\widetilde{DA}_{n+2}$ such that $\alpha|\mathcal{D}^n = \alpha'|\mathcal{D}^n = 0$. Then (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent if and only if α and α' are equivalent as moduli.*

We emphasize that this theorem asserts that the equivalence classes of functional moduli α with $\alpha|\mathcal{D}^n = 0$ are the complete invariant for generic classifications of holonomic systems of Clairaut type under the equivalence relation given by the group of point transformations.

We define $\mathfrak{M}(\mathcal{D}^n) = \{\alpha \in \mathfrak{M}_{(x,y)} | \alpha|\mathcal{D}^n = 0\}$ and $\mathcal{M}(\widetilde{DA}_{n+2}) = \mathfrak{M}(\mathcal{D}^n)/\sim$, where $\sim$ denotes the equivalence relation as moduli. The above theorem asserts that the moduli space for generic holonomic systems of Clairaut type is $\mathcal{M}(\widetilde{DA}_{n+2})$.

In §1 we shall prepare basic tools to prove theorems including quick reviews of ([9,10]). We shall give a proof of Theorem A in §2. Theorems B and C will be proved in §3 by using an analogous method of Dufour[6].

All map germs considered here are differentiable of class C^∞, unless stated otherwise.

1. Preparations (Including quick reviews of [9,10])

In this section we review some results on completely integrable systems and the theory of Legendrian unfoldings [9,10] which will be used in later sections.

Firstly we introduce an equivalence relation among systems under the group of point transformations of $\mathbb{R}^n \times \mathbb{R}$. A *point transformation* ϕ on $\mathbb{R}^n \times \mathbb{R}$ is, by definition, a diffeomorphism of $\mathbb{R}^n \times \mathbb{R}$ onto itself.

To define a lift of ϕ, we give a contact manifold which is a fiberwise compactification of $J^1(\mathbb{R}^n, \mathbb{R})$. Let $\tilde{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ be a projective cotangent bundle over $\mathbb{R}^n \times \mathbb{R}$ which contains $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ as an affine part. Then we have a canonical contact lift $\hat{\phi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ of ϕ. Let $f,\ g : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ be equation germs. Following Lie, we say that f and g are *equivalent as equations* if there exist a diffeomorphism germ $\psi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ and a point transformation $\phi : (\mathbb{R}^n \times \mathbb{R}, \pi(z_0)) \to (\mathbb{R}^n \times \mathbb{R}, \pi(z_1))$ such that the lift $\hat{\phi}$ of ϕ satisfies that $\hat{\phi}(z_0) = z_1$ and $\hat{\phi} \circ f = g \circ \psi$, where $z_0 = f(0)$ and $z_1 = g(0)$. In [10] it has been shown the following theorem.

Theorem 1.1. *Let (μ, f) and $(\mu', f') : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}), 0 \times v)$ be holonomic system with complete integral such that the set of critical points of $\pi \circ f$ and $\pi \circ f'$ are closed sets without interior points. Then the followings are equivalent:*

(1) f and f' are equivalent as equations.

(2) $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent as integral diagrams.

Secondly, we briefly review the theory of one-parameter Legendrian unfoldings in [,]. We now consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on the space. Let $(t, x_1, \ldots, x_n)$ be the canonical coordinate on $\mathbb{R} \times \mathbb{R}^n$ and $(t, x_1, \ldots, x_n, y, q, p_1, \ldots, p_n)$ be the corresponding coordinate on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then the canonical 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i dx_i - q dt = \theta - q dt$. We also have the natural projection

$$\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$$

defined by $\Pi(t, x, y, q, p) = (t, x, y)$. We call the above 1-jet bundle *an unfolded 1-jet bundle*. Let (μ, f) be a holonomic system with complete integral. Then there exists a unique element $h \in \mathcal{E}_u$ such that $\ell^*\theta = h \cdot d\mu$. Define a map germ

$$\ell_{(\mu, f)} : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by
$$\ell_{(\mu,f)}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

Then we can easily show that $\ell_{(\mu,f)}$ is a Legendrian immersion germ. We call $\ell_{(\mu,f)}$ a *complete Legendrian unfolding associated with* (μ, f). By the aid of the notion of Legendrian unfoldings, holonomic systems of Clairaut type are characterized as follows :

Proposition 1.2. [9]. *Let* $(\mu, f) : (\mathbb{R}^{n+1}, 0) \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$ *be a holonomic system with complete integral. Then* (μ, f) *is a holonomic system of Clairaut type if and only if* $\ell_{(\mu,f)}$ *is Legendrian non-singular.*

A complete Legendrian unfolding $\ell_{(\mu,f)}$ associated with (μ, f) is called *a Legendrian unfolding of Clairaut type* if $\ell_{(\mu,f)}$ is a holonomic system of Clairaut type.

Returning to the study of equations with complete integral, we now establish the notion of the genericity.

Let $U \subset \mathbb{R}^{n+1}$ be an open set. We denote by $\mathrm{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$ the set of systems with complete integral $(\mu, f) : U \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$. We also define $L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ to be the set of complete Legendrian unfoldings $\ell_{(\mu,f)} : U \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of either spaces is said to be *generic* if it is an open dense subset in the space.

The genericity of a property of germs are defined as follows. Let P be a property of equation germs with complete integral $(\mu, f) : (\mathbb{R}^{n+1}, 0) \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$ (respectively, Legendrian unfoldings $\ell_{(\mu,f)} : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$. For an open set $U \subset \mathbb{R}^r$, we define $\mathcal{P}(U)$ to be the set of $(\mu, f) \in \mathrm{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$ (respectively, $\ell_{(\mu,f)} \in L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})))$ such that the germ at x whose representative is given by (μ, f) (respectively, $\ell_{(\mu,f)}$) has property P for any $x \in U$.

The property P is said to be *generic* if for some neighbourhood U of 0 in $\mathbb{R}^r$, the set $\mathcal{P}(U)$ is a generic subset in $\mathrm{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$ (respectively, $L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_* : L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \to \mathrm{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

defined by $(\Pi_1)_*(\ell_{(\mu,f)}) = \Pi_1 \circ \ell_{(\mu,f)} = (\mu, f)$, where $\Pi_1 : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$ is the canonical projection. Then it has been shown that the following fundamental theorem.

Theorem 1.3. [9,10]. *The continuous map*

$$(\Pi_1)_* : L(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \to \mathrm{Int}(U, \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

is a homeomorphism.

On the other hand, since $\ell_{(\mu,f)}$ is a Legendrian immersion germ, then there exists a generating family of $\mathcal{L}$ by the Arnol'd-Zakalyukin's theory ([1,12]). In this case the generating family is naturally constructed by an one-parameter family of generating families associated with (μ,ℓ). Let $F : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ be a function germ such that $d_2F|0 \times \mathbb{R}^n \times \mathbb{R}^k$ is non-singular, where $d_2F(t,x,q) = (\frac{\partial F}{\partial q_1}(t,x,q), \ldots, \frac{\partial F}{\partial q_k}(t,x,q))$. We call F a *generalized phase family*. Then $C(F) = d_2F^{-1}(0)$ is a smooth $(n+1)$-manifold germ and $\pi_F : (C(F), 0) \to \mathbb{R}$ is a submersion germ, where $\pi_F(t,x,q) = t$. We call the submanifold $C(F)$ a *catastrophe set of* F. Define

$$\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\Phi}_F(t,x,q) = (x, F(t,x,q), \frac{\partial F}{\partial x}(t,x,q))$$

and

$$\Phi_F : (C(F), 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t,x,q) = (t, x, F(t,x,q), \frac{\partial F}{\partial t}(t,x,q), \frac{\partial F}{\partial x}(t,x,q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that $(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|C(F) \cdot dt|C(F) = 0$. By the definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin ([1,12]), we can show the following proposition.

Proposition 1.4. *All Legendrian unfolding germs are constructed by the above method.*

Let (μ, f) be a holonomic system of Clairaut type. By Proposition 1.2, $\ell_{(\mu,f)}$ is Legendrian non-singular. Then we can choose a family of function germ

$$F : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$$

such that $\mathrm{Image}\, j^1 F_t = f(\mu^{-1}(t))$ for any $t \in (\mathbb{R})$ and

$$j_1^1 F : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

is an immersion germ, where $F_t(x) = F(t, x)$ and $j_1^1 F(t, x) = j^1 F_t(x)$. The fact that $j_1^1 F$ is an immersion leads us to the following equality :

$$\operatorname{rank}\left(\frac{\partial F}{\partial t} \frac{\partial^2 F}{\partial t \partial x_j}\right) = 1.$$

In this case, we have $(C(F), 0) = (\mathbb{R} \times \mathbb{R}^n, 0)$ and

$$\Phi_F = j^1 F : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}),$$

so that it is a complete Legendrian unfolding associated with $(\pi_{r-n}, j_1^1 F)$. Thus the generating family of a Legendrian unfolding of Clairaut type is given by the above germ.

2. Proof of Theorem A

The main idea of the proof is to define an equivalence relation which can ignore functional moduli and to do everything in terms of generating families for Legendrian unfoldings like as those of in [10]. Let (μ, g) and (μ', g') be holonomic integral diagrams. Then (μ, g) and (μ', g') are $\mathcal{R}^+$-*equivalent* if there exist a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Psi(t, x, y) = (t + \alpha(x, y), \psi(x, y))$ and a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ such that $\Psi \circ (\mu, g) = (\mu', g') \circ \Phi$. We remark that if (μ, g) and (μ', g') are $\mathcal{R}^+$-equivalent by the above diffeomorphisms, then we have $\mu(u) + \alpha \circ g(u) = \mu' \circ \Phi(u)$ and $\psi \circ g(u) = g' \circ \Phi(u)$ for any $u \in (\mathbb{R}^{n+1}, 0)$. Thus the diagram $(\mu + \alpha \circ g, g)$ is strictly equivalent to (μ', g').

We now define the corresponding equivalence relation among Legendrian unfoldings. Let $\ell_{(\mu, f)}$, $\ell_{(\mu', f')} : (\mathbb{R}^{n+1}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0)$ be Legendrian unfoldings. We say that $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are $S.P^+$-*Legendrian equivalent (respectively, S.P-Legendrian equivalent)* if there exist a contact diffeomorphism germ $K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0')$, a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ and a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0)) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0'))$ of the form $\Psi(t, x, y) = (t + \alpha(x, y), \psi(x, y))$ (respectively, $\Psi(t, x, y) = (t, \psi(x, y))$) such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \mathcal{L} = \mathcal{L}' \circ \Phi$. It is clear that if $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are $S.P^+$-Legendrian equivalent (respectively, $S.P$-Legendrian equivalent), then $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are $\mathcal{R}^+$-equivalent (respectively, strictly equivalent). By Theorem 1.1 in [12], the converse is also true for generic (μ, f) and (μ', f'). The notion of the stability of Legendrian unfoldings with respect to $S.P^+$-Legendrian equivalence (respectively, $S.P$-Legendrian equivalence) is analogous to the usual notion of the

8

stability of Legendrian immersion germs with respect to Legendrian equivalence (cf. Part III in [1]).

On the other hand, we can interpret the above equivalence relation in terms of generating families. For the purpose, we use some notations and results in [1,4,8,10,12]. Let $\tilde{F}, \tilde{G} :$ $(\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type. We say that $\tilde{F}$ and $\tilde{G}$ are P-$\mathcal{C}^+$-equivalent (respectively, P-$\mathcal{C}$-equivalent) if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (t + \alpha(x, y), \phi_1(x, y), \phi_2(x, y))$ (respectively, $\Phi(t, x, y) = (t, \phi_1(x, y), \phi_2(x, y))$) such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(t,x,y)}} = \langle G \rangle_{\mathcal{E}_{(t,x,y)}}$ where $\langle G \rangle_{\mathcal{E}_{(t,x,y)}}$ is the ideal generated by G in $\mathcal{E}_{(t,x,y)}$. We also say that $\tilde{F}(t, x, y)$ is $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|\mathbb{R} \times 0$ if

$$\mathcal{E}_t = \langle \frac{df}{dt} \rangle_{\mathbb{R}} + \langle f \rangle_{\mathcal{E}_t} + \langle \frac{\partial F}{\partial x_1}|\mathbb{R} \times 0, \ldots, \frac{\partial F}{\partial x_n}|\mathbb{R} \times 0, 1 \rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_t = \langle f \rangle_{\mathcal{E}_t} + \langle \frac{\partial F}{\partial x_1}|\mathbb{R} \times 0, \ldots, \frac{\partial F}{\partial x_n}|\mathbb{R} \times 0, 1 \rangle_{\mathbb{R}}).$$

By the similar arguments like as those of Theorems 20.8 and 21.4 in [1], we can show the following :

Theorem 2.1. Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type Φ_F, Φ_G respectively. Then

(1) Φ_F and Φ_G are S.P+ (respectively, S.P)-Legendrian equivalent if and only if $\tilde{F}$ and $\tilde{G}$ are P-$\mathcal{C}^+$ (respectively, $\mathcal{C}$)-equivalent.

(2) Φ_F is S.P+ (respectively, S.P)-Legendrian stable if and only if $\tilde{F}$ is a P-$\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|\mathbb{R} \times 0$.

The following theorem is a corollary of Damon's general versality theorem in [4].

Theorem 2.2. Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type such that Φ_F, Φ_G are S.P+ (respectively, S.P)-Legendrian stable. Then Φ_F, Φ_G are S.P+ (respectively, S.P)-Legendrian equivalent if and only if $f = F|\mathbb{R} \times 0$, $g = G|\mathbb{R} \times 0$ are $\mathcal{C}$-equivalent (i.e. $\langle f \rangle_{\mathcal{E}_t} = \langle g \rangle_{\mathcal{E}_t}$).

Then the classification theory of function germs by the $\mathcal{C}$-equivalence is quite useful for

our purpose. For each function germ $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, we set

$$\mathcal{C}\text{-cod}\,(f) = \dim_{\mathbb{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t},$$

$$\mathcal{C}^+\text{-cod}\,(f) = \dim_{\mathbb{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t} + \langle \frac{df}{dt} \rangle_{\mathbb{R}},$$

$$\mathcal{K}\text{-cod}\,(f) = \dim_{\mathbb{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t} + \langle \frac{df}{dt} \rangle_{\mathcal{E}_t}.$$

Then we have the following well-known classification (cf. [8]).

Lemma 2.3. *Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ with $\mathcal{K}\text{-cod}(f) < \infty$. Then f is $\mathcal{C}$-equivalent to the map germ $t^{\ell+1}$ for some $\ell \in \mathbb{N}$.*

By the direct calculation, we have

$$\mathcal{C}\text{-cod}\,(t^{\ell+1}) = \ell + 1,$$

$$\mathcal{C}^+\text{-cod}\,(t^{\ell+1}) = \ell.$$

Thus we can easily determine $\mathcal{C}$ (respectively, $\mathcal{C}^+$)-versal deformations of the above germs by the usual method as follows :
The $\mathcal{C}$-versal deformation :

$$t^{\ell+1} + \sum_{i=0}^{\ell} u_{i+1} t^i.$$

The $\mathcal{C}^+$-versal deformation :

$$t^{\ell+1} + \sum_{i=0}^{\ell-1} u_{i+1} t^i.$$

We now ready to prove Theorem A.

Proof of Theorem A. Let (μ, f) be a holonomic system of Clairaut type such that the corresponding Legendrian unfolding $\ell_{(\mu,f)}$ is S.P$^+$-Legendrian stable. By the assumption, the generating family $\tilde{F}(t, x, y)$ of $\ell_{(\mu,f)}$ is $\mathcal{C}^+$-versal deformation of $f = F|\mathbb{R} \times 0$. By Theorem 2.2 and Lemma 2.3, $\tilde{F}(t, x, y)$ is P-$\mathcal{C}^+$-equivalent to one of germs in the following list :

DA_ℓ $(1 \le \ell \le n+1)$; $t^\ell + \sum_{i=1}^{\ell-1} x_i t^i + \sum_{j=\ell}^{n} x_j - y,$

$\widetilde{DA}_{n+2}$; $t^{n+2} + \sum_{i=1}^{n} x_i t^i - y,$

We now detect the corresponding normal forms of integral diagrams as follows :

DA_ℓ $(1 \le \ell \le n+1)$; We can choose

$$F(t, x,) = t^\ell + \sum_{i=1}^{\ell-1} x_i t^i + \sum_{j=\ell}^{n} x_j$$

10

as a generalized phase family, so that

$$\Phi_F = (t, x_1, \ldots, x_n, \pm t^\ell + \sum_{i=1}^{\ell-1} x_i t^i + \sum_{j=\ell}^{n} x_j,$$

$$\pm \ell t^{\ell-1} + \sum_{i=1}^{\ell-1} i x_i t^{i-1}, t, \ldots, t^{\ell-1}, 1, \ldots, 1).$$

Then we can easily calculate that the corresponding integral diagram is strictly equivalent to

$$\mu = u_{n+1},$$

$$g = (u_1, \ldots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i).$$

This is the normal form in the case of $3 \le \ell \le n+1$. If $\ell = 2$, we have

$$\mu = u_{n+1},$$

$$g = (u_1, \ldots, u_n, u_{n+1}^2 + u_1 u_{n+1}).$$

We now define a transformation by

$$U_i = u_i \ (i = 1, \ldots, n), \ U_{n+1} = u_{n+1} + \frac{1}{2} u_1,$$

then (μ, g) is strictly equivalent to

$$\mu = u_{n+1} - \frac{1}{2} u_1,$$

$$g = (u_1, \ldots, u_n, (u_{n+1}^2 - \frac{1}{4} u_1^2)).$$

We also apply a transformation which is defined by

$$X_i = x_i \ (i = 1, \ldots, n), \ Y = y + \frac{1}{4} x_1^2,$$

then we have the normal form.

$\widetilde{DA}_{n+2}$; In this case the generalized phase family is given by

$$F(t, x, q) = t^{n+2} + \sum_{i=1}^{n} x_i t^i.$$

11

By the same calculations as those of the case when $\ell \leq n + 1$, we can show that the corresponding integral diagram is $\mathcal{R}^+$-equivalent to

$$\mu = u_{n+1},$$

$$g = (u_1, \ldots, u_n, \pm u_{n+1}^{n+2} + \sum_{i=1}^{n} u_i u_{n+1}^i).$$

Since the generalized phase family is $\mathcal{C}^+$-versal and not $\mathcal{C}$-versal, then the integral diagram is strictly equivalent to the normal form. This completes the proof of Theorem A.

3. Proof of Theorems B and C

For the proof of Theorem B, we now introduce another equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be holonomic integral diagrams. Then (μ, g) and (μ', g') are *weak equivalent* if there exist a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Psi(t, x, y) = (\psi_1(t, x, y), \psi_2(x, y))$ and a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ such that $\Psi \circ (\mu, g) = (\mu', g') \circ \Phi$. By the definition, if either (μ, g) and (μ, g') are equivalent or $\mathcal{R}^+$-equivalent, then these are weak equivalent.

There also exists the corresponding equivalence relation among Legendrian unfoldings. Let $\ell_{(\mu, f)}$, $\ell_{(\mu', f')} : (\mathbb{R}^{n+1}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0)$ be Legendrian unfoldings. We say that $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are *P-Legendrian equivalent* if there exist a contact diffeomorphism germ $K : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0')$, a diffeomorphism germ $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ and a diffeomorphism germ $\Psi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0)) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), \Pi(z_0'))$ of the form $\Psi(t, x, y) = (\psi(t, x, y), \psi(x, y))$ such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \mathcal{L} = \mathcal{L}' \circ \Phi$. It is clear that if $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are P-Legendrian equivalent, then $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are weak equivalent. By Theorem 1.1 in [12], the converse is also true for generic (μ, f) and (μ', f'). We can also define the notion of the stability of Legendrian unfoldings with respect to P-Legendrian equivalence exactly the same way as in the previous section. The corresponding equivalence relation among generating families are also given as follows : Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type. We say that $\tilde{F}$ and $\tilde{G}$ are P-$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (\phi_1(t, x, y), \phi_2(x, y), \phi_3(x, y))$ such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(t,x,y)}} = \langle G \rangle_{\mathcal{E}_{(t,x,y)}}$. We also say that $\tilde{F}(t, x, y)$ is $\mathcal{K}$-versal deformation of $f = F | \mathbb{R} \times 0$ if

$$\mathcal{E}_t = \langle \frac{df}{dt}, f \rangle_{\mathcal{E}_t} + \langle \frac{\partial F}{\partial x_1} | \mathbb{R} \times 0, \ldots, \frac{\partial F}{\partial x_n} | \mathbb{R} \times 0, 1 \rangle_{\mathbb{R}}.$$

By the similar arguments like as those of Theorems 2.1 and 2.2, we can show the following :

Theorem 3.1. *Let $\tilde{F}, \tilde{G} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type Φ_F, Φ_G respectively. Then*

(1) Φ_F and Φ_G are P-Legendrian equivalent if and only if $\tilde{F}$ and $\tilde{G}$ are P-$\mathcal{K}$-equivalent.

(2) Φ_F is P-Legendrian stable if and only if $\tilde{F}$ is a P-$\mathcal{K}$-versal deformation of $f = F|\mathbb{R} \times 0$.

(3) Suppose that $\tilde{F}, \tilde{G}$ be generating families of P-Legendrian stable Legendrian unfoldings Φ_F, Φ_G of Clairaut type. Then Φ_F, Φ_G are P-Legendrian equivalent if and only if $f = F|\mathbb{R} \times 0$, $g = G|\mathbb{R} \times 0$ are $\mathcal{K}$-equivalent.

Since $f = F|\mathbb{R} \times 0$ is a function germ of one-variable, we have $\langle \frac{df}{dt} \rangle_{\mathbb{R}} + \langle f \rangle_{\mathcal{E}_t} = \langle \frac{df}{dt}, f \rangle_{\mathcal{E}_t}$, so that $\tilde{F}$ is a C^+-versal deformation of f if and only if it is a $\mathcal{K}$-versal deformation. By Theorems 2.1 and 3.1, a Legendrian unfolding of Clairaut type is $S.P^+$-Legendrian stable if and only if it is P-Legendrian stable. Then the following lemma is a corollary of Theorem A.

Lemma 3.2. *Let (μ', g') be an integral diagram of Clairaut type which is equivalent to (μ_α, g) for some $\alpha \in \mathfrak{M}_{(x,y)}$. Then there exists $\alpha' \in \mathfrak{M}_{(x,y)}$ such that (μ', g') and $(\mu_{\alpha'}, g)$ are strictly equivalent.*

The main idea of the proof is a generalization of methods which have been developed in ([6]). Let (μ, g) be an integral diagram which is equivalent to (μ_α, g) for some $\alpha \in \mathfrak{M}_{(x,y)}$. Define

$$S_\mu = \{(c_1, \ldots, c_{n+2}) | \text{All of } c_1, \ldots, c_{n+2} \text{ are different and } \bigcap_{j=1}^{n+2} g(\mu^{-1}(c_j)) \neq \emptyset\}.$$

We call it *the Carneiro hypersurface* of (μ, g). We can detect a defining equation of the closure of the Carneiro hypersurface of (μ_α, g).

Proposition 3.3. *The closure $\overline{S_{\mu_\alpha}}$ of the Carneiro surface of (μ_α, g) in $\mathbb{R}^{n+2}$ is defined by*

$$s_1 = (n+2)\alpha(x_1, \ldots, x_n, y),$$

where

$$s_j = \sum_{\{i_1, \ldots, i_j\} \subset \{1, \ldots, n+2\}} c_{i_1} \cdots c_{i_j}$$

13

for $j = 1, \ldots, n+2$ and

$$x_{n+2-k} = \sum_{i=0}^{k-1} b_i s_{k-i} s_1^i \quad (k = 2, \ldots, n+1)$$

$$y = \sum_{i=0}^{n+1} b_i s_{n+2-i} s_1^i$$

for some real numbers b_i which depend on n, k.

Proof. Let $t_1, \ldots, t_{n+2}$ with $t_1 \le \cdots \le t_{n+2}$ be the (n+2)-real roots of $F_{(x,y)}(t) = 0$ for each $(x, y) \in \Delta$. We define map germs $c_j \colon \Delta \to (\mathbb{R}, 0)$ by $c_j(x, y) = t_j + \alpha(x, y)$, where $j = 1, \ldots, n+2$. Then we set $C = (c_1, \ldots, c_{n+2}) \colon \Delta \to (\mathbb{R}^{n+2}, 0)$. Moreover let $s = (s_1, \ldots, s_{n+2}) \colon (\mathbb{R}^{n+2}, 0) \to (\mathbb{R}^{n+2}, 0)$ be a map germ defined by $s_j =$ elementary symmetric polynomial of degree j. By the relation between roots and coefficients we have

$$t_1 + \cdots + t_{n+2} = 0$$

$$\sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, n+2\}} t_{i_1} \cdots t_{i_k} = (-1)^k x_{n+2-k} \quad (k = 2, \ldots, n+1)$$

$$t_1 \cdots t_{n+2} = (-1)^{n+2}(-y).$$

Putting $s \circ C(x, y) = (s_1, \ldots, s_{n+2})$, we have

$$s_1 = (n+2)\alpha(x, y)$$

$$s_k = \sum_{i=2}^{k} \binom{n+2-i}{k-i} (-1)^i x_{n+2-i} \alpha(x, y)^{k-i} + \binom{n+2}{k} \alpha(x, y)^k \quad (k = 2, \ldots, n+1)$$

$$s_{n+2} = (-1)^{n+2}(-y) + \sum_{i=2}^{n+1} (-1)^i x_{n+2-i} \alpha(x, y)^{n+2-i} + \alpha(x, y)^{n+2},$$

where $\binom{m}{r}$ denotes the binomial coefficient and we presumably use $\binom{m}{0} = 1$.

From these formulae and by direct calculations we can inductively show that x_{n+2-k}, y are uniquely expressed as in the statement of this proposition. Furthermore we can show that $\overline{S_{\mu_\alpha}} = \bigcup_{\sigma \in \mathfrak{S}_{n+2}} \mathrm{Image}\,(c_{\sigma(1)}, \ldots, c_{\sigma(n+2)})$, where $\mathfrak{S}_{n+2}$ is the symmetric group of degree $n+2$. It follows that $\overline{S_{\mu_\alpha}}$ is included by the hypersurface which is defined by the equation $s_1 = (n+2)\alpha(x, y)$, where (x, y) is the preimage of $(s_1, \ldots, s_{n+2})$ by the map ger $s \circ C$ and s_j is the elementary symmetric polynomial of degree j with respect to $c_1, \ldots, c_{n+2}$. Therefore we obtain a defining equation of $\overline{S_{\mu_\alpha}}$. This completes the proof.

By the same reasons as those of in [6], we have the following lemma.

Lemma 3.4. *Under the same hypothesis as in Lemma 3.3, $\alpha|\mathcal{D}^n = 0$ if and only if*

$$\left.\begin{array}{l}(c_1,\ldots,c_{n+2}) \in \overline{S_{\mu_\alpha}} \\ c_2 = \cdots = c_{n+2}\end{array}\right\} \Rightarrow c_1 = -(n+1)c_{n+2}.$$

We now ready to prove Theorem B.

Proof of Theorem B. By Proposition 3.3 and the implicit function theorem, there exists a function germ $h: (\mathbb{R},0) \to (\mathbb{R},0)$ whose form is given by $h(c) = -(n+1)c + o(|c|)$ such that

$$\overline{S_{\mu_\alpha}} \cap \{(c_1,c_2,\ldots,c_{n+2})|c_2 = \cdots = c_{n+2}\} = \{(h(c_{n+2}),c_2,\ldots,c_{n+2})|c_2 = \cdots = c_{n+2}\}.$$

Applying a linearization theorem of Sternberg, there exists a diffeomorphism germ $\lambda :$ $(\mathbb{R},0) \to (\mathbb{R},0)$ such that $\lambda \circ h \circ \lambda^{-1}(c) = -(n+1)c$. Thus we have $\lambda(c_1)+\lambda(c)+\cdots+\lambda(c) = 0$ for any $(c_1,c,\ldots,c) \in \overline{S_{\mu_\alpha}}$. By this relation and the fact that the $(n+2)$-fold product of λ sends $\overline{S_{\mu_\alpha}}$ onto $\overline{S_{\lambda\circ\mu_\alpha}}$, we can show that if $(c_1,\ldots,c_{n+2}) \in \overline{S_{\lambda\circ\mu_\alpha}}$ with $c_2 = \cdots = c_{n+2}$ then $c_1 = -(n+1)c_{n+2}$. Furthermore, by Lemma 3.2 there exists $\alpha' \in \mathfrak{M}_{(x,y)}$ such that the following diagram commutes

$$\begin{array}{ccccc}
(\mathbb{R},0) & \overset{\lambda\circ\mu_\alpha}{\longleftarrow} & (\mathbb{R}^{n+1},0) & \overset{g}{\longrightarrow} & (\mathbb{R}^n \times \mathbb{R},0) \\
\| & & \downarrow{\psi} & & \downarrow{\phi} \\
(\mathbb{R},0) & \underset{\mu_{\alpha'}}{\longleftarrow} & (\mathbb{R}^{n+1},0) & \underset{g}{\longrightarrow} & (\mathbb{R}^n \times \mathbb{R},0).
\end{array}$$

The commutativity of the diagram implies that $\overline{S_{\lambda\circ\mu_\alpha}} = \overline{S_{\mu_{\alpha'}}}$. Thus by Lemma 3.4 we have $\alpha'|\mathcal{D}^n = 0$. This completes the proof of Theorem B.

Proof of Theorem C. Firstly we prove the necessity of the condition that functional moduli are equivalent. Assume that (μ_α,g) and $(\mu_{\alpha'},g)$ are equivalent, where $\alpha|\mathcal{D}^n = \alpha'|\mathcal{D}^n = 0$. Let us assume that $\lambda \circ \mu_\alpha = \mu_{\alpha'} \circ \psi$ and $\phi \circ g = g \circ \psi$ for some diffeomorphism germs $\lambda: (\mathbb{R},0) \to (\mathbb{R},0), \psi: (\mathbb{R}^{n+1},0) \to (\mathbb{R}^{n+1},0)$ and $\phi: (\mathbb{R}^n \times \mathbb{R},0) \to (\mathbb{R}^n \times \mathbb{R},0)$. We determine the form of λ. Since $(\lambda \times \cdots \times \lambda)(\overline{S_{\mu_\alpha}}) = \overline{S_{\mu_{\alpha'}}}$ and by Lemma 3.4 we have $\lambda(-(n+1)c) = -(n+1)\lambda(c)$, so that we inductively have

$$\lambda(c) = (-(n+1))^p \lambda\left(\frac{c}{(-(n+1))^p}\right)$$

for all integers p. By the same arguments as those of Dufour ([5] p.466,[6] p.231,[7] p.274) we can show that the form of λ is given by $\lambda(c) = ac \ (a \neq 0)$. Also we determine the

form of $\phi|\Delta$. Let $\mathcal{C}$ be the map germ as in Proposition 3.3. By the assumption we have $(\lambda \times \cdots \times \lambda) \circ \mathcal{C} = (c_{\sigma(1)}, \ldots, c_{\sigma(n+2)}) \circ \phi|\Delta$ for some $\sigma \in \mathfrak{S}_{n+2}$. We recall the fact that $s \circ \mathcal{C}$ is invertible and the inverse map germ is given as in Proposition 3.3. Thus we have $\phi(x, y) = (a^{n+1}x_1, a^n x_2, \ldots, a^2 x_n, a^{n+2}y)$ on Δ, where $a \in \mathbb{R} - \{0\}$ such that $\lambda(c) = ac$. Now for each $(x, y) \in \Delta$ write $\mathcal{C}(x, y) = (c_1, \ldots, c_{n+2})$. Then by Proposition 3.3 we have $c_1 + \cdots + c_{n+2} = (n+2)\alpha(x, y)$ and $ac_1 + \cdots + ac_{n+2} = (n+2)\alpha'(\phi(x, y))$. Hence $a\alpha(x, y) = \alpha'(\phi(x, y))$ on Δ. Therefore we obtain the necessity.

Conversely suppose that $a\alpha(x_1, \ldots, x_n, y) = \alpha'(a^{n+1}x_1, a^n x_2, \ldots, a^2 x_n, a^{n+2}y)$ on Δ for some $a \in \mathbb{R} - \{0\}$. Then we have the following commutative diagram

$$
\begin{array}{ccccc}
(\mathbb{R}, 0) & \xleftarrow{\mu_\alpha} & g^{-1}(\Delta) & \xrightarrow{g} & \Delta \\
\lambda \downarrow & & \downarrow \psi & & \downarrow \phi \\
(\mathbb{R}, 0) & \xleftarrow{\mu_{\alpha'}} & g^{-1}(\Delta) & \xrightarrow{g} & \Delta
\end{array}
$$

where $\lambda(c) = ac, \psi(u_1, \ldots, u_{n+1}) = (a^{n+1}u_1, a^n u_2, \ldots, a^2 u_n, au_{n+1}), \phi(x_1, \ldots, x_n, y) = (a^{n+1}x_1, a^n x_2, \ldots, a^2 x_n, a^{n+2}y)$.

The proof is based on the following result of Carneiro [2].

Theorem 3.5. *Let (μ_α, g) and $(\mu_{\alpha'}, g)$ be integral diagrams of $\widetilde{DA}_{n+2}$. Then (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent if and only if there exists a diffeomorphism germ $\lambda \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $(\lambda \times \cdots \times \lambda)(\overline{S_{\mu_\alpha}}) = \overline{S_{\mu_{\alpha'}}}$.*

We remark that the commutativity of the above diagram implies that the $(n+2)$-fold product of λ sends $\overline{S_{\mu_\alpha}}$ onto $\overline{S_{\mu_{\alpha'}}}$. Thus it follows from Theorem 3.5 that (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent. This completes the proof of Theorem C.

References

1. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I*, Birkhäuser, 1986.

2. M. J. Dias Carneiro, *Singularities of envelopes of families of submanifolds in $\mathbb{R}^N$*, Ann. scient. Ec. Norm. Sup (1983), 173–192.

3. A. C. Clairaut, *Solution de plusieurs problems*, Histore de l'Academie royale de Sciences, Paris (1734), 196–215.

4. J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Memoirs of AMS. **50** (1984).

5. J. -P. Dufour, *Families de courbes planes différentiables*, Topology **22** (1983), 449–474.

6. J. -P. Dufour, *Modules pour les familles de courbes planes*, Annal. Inst. Fourier **39** (1989), 225–238.

7. J. -P. Dufour and P. Jean, *Rigidity of webs and families of hypersurfaces*, Singularities and Dynamical Systems (ed S. N. Pnevmatikos), North-Holland (1985), 271–283.

8. Gibson, C.G., *Singular Points of Smooth Mappings*, Pitman, London, 1979.

9. S. Izumiya, *The theory of Legendrian unfoldings and first order differential equaitions*, to appear in Proceedings of the Royal Society of Edingburh.
10. S. Izumiya, *Completely integrable holonomic systems of first order differential equations*, preprint.
11. S. Izumiya, *Systems of Clairaut type*, preprint.
12. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, Journal of Soviet Math. **27** (1983), 2713–2735.

Department of Mathematics, Faculty of Science, Hokkaido University , Sapporo 060 JAPAN
E-mail address: izumiya@math.hokudai.ac.jp and y-kurokw@math.hokudai.ac.jp