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G. Ishikawa and T. Ozawa

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Goo ISHIKAWA[†] AND TETSUYA OZAWA[‡]

[†] Department of Mathematics, Hokkaido University, Sapporo 060, Japan.
[‡] Department of Mathematics, Nagoya University, Nagoya 464, Japan.

To the memory of Dmitriĭ Andrevich Gudkov

0. Introduction

The purpose of this paper is to give asymptotic estimates of the genus of a non-singular connected compact real algebraic surface in $\mathbf{R}^3$ by means of the degree of the surface. Let $g(2m)$ denote the maximal genus of such a surface of even degree $2m$. Then the result is formulated as follows:

THEOREM 1.

$$\frac{1}{4} \leqq \underline{\lim}_{m\to\infty} \frac{g(2m)}{(2m)^3} \leqq \overline{\lim}_{m\to\infty} \frac{g(2m)}{(2m)^3} \leqq \frac{1}{3},$$

In fact, we shall observe that, for $n = 2m$,

$$(*) \qquad \frac{1}{4}n^3 - \frac{1}{2}n^2 - \frac{1}{2}n + 1 \leqq g(n) \leqq \frac{1}{3}n^3 - n^2 + \frac{7}{6}n.$$

Furthermore we consider the problem to seek the exact value $g(4)$, that is, the maximal genus of a non-singular connected compact quartic surface in $\mathbf{R}^3$.

Let Σ_g denote, as usual, a connected compact orientable two-dimensional C^∞ manifold without boundary of genus g. Then our motivation lies in the following naive question:

What is the minimal degree $d(g)$ of real algebraic surfaces in $\mathbf{R}^3$ representing Σ_g?

Such a problem is understood more clearly in the projective space: In the three dimensional projective space $\mathbf{R}P^3$, we fix a projective plane $\mathbf{R}P^2$, the infinite projective plane, and identify $\mathbb{R}P^3 - \mathbb{R}P^2$ with $\mathbb{R}^3$. Then we consider the family $\mathcal{S}_n$ (resp. $\mathcal{S}^g$) of real algebraic surfaces S of degree n (resp. of topological genus g) in $\mathbb{R}P^3$ with the following properties: S is non-singular, connected and does not touch $\mathbb{R}P^2$. Remark that, if $\mathcal{S}_n \neq \emptyset$, then n is necessarily even; $n = 2m$. In this case we see that each surface $S \in \mathcal{S}_n$ is orientable. We denote by $g(S)$ the genus of $S \in \mathcal{S}_n$, and by $d(S)$ the degree of S. We set

$$d(g) = \inf\{d(S) \mid S \in \mathcal{S}^g\}, \quad \text{and} \quad g(2m) = \sup\{g(S) \mid S \in \mathcal{S}_{2m}\}.$$

Then it is easy to verify $d(g(2m)) = 2m, m \in \mathbb{N}$.

We show in this paper also the following asymptotic estimates of $d(g)$, which is closely related to Theorem 1:

THEOREM 2.

$$3 \leqq \varliminf_{g\to\infty} \frac{d(g)^3}{g} \leqq \varlimsup_{g\to\infty} \frac{d(g)^3}{g} \leqq 4.$$

The difficulty of the problem treated in this paper is caused from the two essential restrictions on surfaces: the connectivity and the compactness (in $\mathbb{R}^3$). The study of the topology of surfaces in $\mathbb{R}P^3$ gives however a nice information to the study of the topology of surfaces in $\mathbb{R}^3$.

The topology of curves in $\mathbb{R}P^2$ and surfaces in $\mathbb{R}P^3$ (the first half of Hilbert's 16th problem) is intimately studied by many authors: The isotopy classification of non-singular curves of degree 6 (resp. 7) in $\mathbb{R}P^2$ are given by Gudkov (resp. Viro); see [V4]. The isotopy classification of curves of degree 8 and 9 are now actively studied (cf. [K], [V5]).

The isotopy classification of non-singular surfaces of degree 4 in $\mathbb{R}P^3$ are established by Kharlamov [Kha2], [Kha3]. (See also [N], [Kha4], for the finer classification).

Gudkov [G2] classifies also curves of degree 8 on the hyperboloid. Further Gudkov and Shustin [GS] give the classification of curves of degree 8 on the ellipsoid. (For this direction, see also [M], [Z1], [Z2]). It is remarkable that the topology of curves on a surface turns out to be closely related to the problem treated in this paper.

For further informations, for example, Rohlin's results, we should consult with, for instance, [GU], [G1], [AO], [W], [R], [GM], [V4], [V5].

For a curve in $\mathbb{R}P^2$, the isotopy type are decided by the number of connected components and their mutual positions. Turning to the affine case, we can pose the natural problem: Classify compact non-singular curves in $\mathbb{R}^2$ of degree $2m$.

Remark that two compact non-singular curves in $\mathbb{R}^2$ are isotopic if and only if they are isotopic in $\mathbb{R}P^2$. We further remark that any compact non-singular curve in $\mathbb{R}^2$ of (even) degree $\leqq 6$ are isotopic to a small perturbation of the multiple circle. Therefore the isotopy classification of non-singular compact curves in $\mathbb{R}^2$ does not give new results in these cases. In fact, this is clear for the case of degree 2 or 4. Viro [V3] remarks that any isotopy types of non-singular curves of degree 6 in $\mathbb{R}P^2$ can be obtained by a small perturbation of the union of three ellipses tangent one another in two points. Since such a singular curve is a small perturbation of the triple circle, we have the result.

For a surface in $\mathbb{R}P^3$, the isotopy type are decided by the number of connected components, the topology of each component and the mutual positions of the components.

We restrict ourselves to the connected case and concentrate to only the topology of surfaces in 3-space. Even under these restrictions, turning to the affine case still causes some non-trivial problem: An isotopy type of surface of even degree in $\mathbb{R}P^3$ is not necessarily realized by a compact surface in $\mathbb{R}^3$ because of the existence of non-contractible components.

Compared with $g(2m)$, we set $g'(2m) = \sup\{g(S)\}$ where S runs over the class of non-singular connected surfaces of degree $2m$ in $\mathbb{R}P^3$. Notice that estimating $g'(2m)$ is closely related to the following question appeared in [AO]:

What is the largest number of handles of a component of an algebraic surface of degree n in $\mathbb{R}P^3$?

Furthermore, to estimate $g(2m)$, it is also natural to consider a family S''_{2m} (of isotopy types) of non-singular connected surfaces S of degree $2m$ obtained from a small perturbation of the m-multiple sphere, and to set

$$g''(2m) = \sup\{g(S) \mid S \in \mathcal{S}''_{2m}\}.$$

Clearly we have $g''(2m) \leqq g(2m) \leqq g'(2m)$.

In this paper, in fact, we give a lower estimate of $g''(2m)$ and an upper estimate of $g'(2m)$ to get (*).

There are known two types of constructions of real surfaces with prescribed topological nature. These constructions are due to perturbations of singular surfaces: surfaces with transverse intersections (the classical method [V1], [I]) and doubled surfaces (Viro's method [V2]).

We utilize the former in §1 to give a lower estimate of $g''(2m)$: We construct a sequence of non-singular connected surfaces in $\mathbb{R}^3$ of degree n with genus of order $(1/4)n^3 + \cdots$, which are small deformations of multiples of a fixed ellipsoid. Also in §1, we give an upper estimate of $g'(2m)$ from the Petrovskii-Oleinik inequality. Remark that the sharpness of the Petrovskii-Oleinik inequality is unknown yet in general. Combined these estimates, we have (*), therefore Theorem 1. Theorem 2 is proved also in §1.

In the next section, we study the number $g(4)$.

The Viro's method is powerful to construct surfaces with many connected components. In §3, we examine to utilize this method to construct connected surfaces with many handles, which are small perturbations of the multiple sphere.

Though the results of this paper should be tentative, it would be useful to summarize here on the naive but interesting problem to stimulate further investigations.

To end this section, we pose the following natural questions:

(i) Does there exist $\lim_{m\to\infty} g(2m)/(2m)^3$ (resp. $\lim_{g\to\infty} d(g)^3/g$)?

(ii) Can $\{g(2m+2) - g(2m)\}/(2m)^2$ be estimated asymptotically?

(iii) Is it right the statement that $d(g) = 2m+2$ if $g(2m) < g \leqq g(2m+2)$, for $m \in \mathbb{N}$?

(iv) Is $g(4)$ equal to 8 or 9?

The authors would like to thank Professor V.V. Nikulin for helpful comments.

1. Construction and Restriction

The following construction was given by the second author in 1987.

Let $f_i(x, y, z), 1 \leqq i \leqq m$, be polynomials of degree two with the following properties:

(0) $E_i = \{f_i(x, y, z) = 0\} \subset \mathbf{R}^3$ is an ellipsoid, $1 \leqq i \leqq m$.

(1) E_i and E_j intersect transversely and $E_i \cap E_j$ intersects to the xy-plane in four points, for $i \neq j, 1 \leqq i, j \leqq m$.

(2) $E_i \cap E_j \cap E_k = \emptyset$, for distinct $i, j, k, 1 \leqq i, j, k \leqq m$.

Remark that $E_i, 1 \leqq i \leqq m$, can be taken as a perturbation of a fixed ellipsoid in $\mathbf{R}^3$.

To see this, set

$$E_i : \quad (1 - \epsilon_i)^2 x^2 + (1 + \epsilon_i)^2 (y^2 + z^2) = 1,$$

$1 \leqq i \leqq m$, with $0 \leqq \epsilon_1 < \epsilon_2 < \cdots < \epsilon_m \ll 1$. (See Fig.1.) Then each E_i is a small perturbation of the sphere $x^2 + y^2 + z^2 = 1$, expanded along the x-axis and contracted along the yz-plane, and $\{E_i\}_{1 \leqq i \leqq m}$ has the required properties.

Choose a sufficiently small positive number $\epsilon > 0$ and real numbers $a_i, 1 \leqq i \leqq 2m$, with

$$-\epsilon < a_1 < a_2 < \cdots < a_{2m} < \epsilon.$$

Set $-h(z) = \prod_{i=1}^{2m} (z - a_i)$. Further set, for a sufficiently small positive number $\delta > 0$,

$$S : \quad \prod_{i=1}^{m} f_i(x, y, z) = \delta h(z).$$

Then we see S is a non-singular connected compact surface in $\mathbf{R}^3$; $S \in \mathcal{S}_{2m}$. Further we see

$$\chi(S) = 2m - 4 \cdot 2m \cdot \frac{m(m-1)}{2}.$$

In fact, along each intersection curve of E_i and E_j, the singular surface $\prod_{i=1}^{m} f_i(x,y,z)=0$ is perturbed as in Fig. 2.

Therefore

$$g(S)=2m^3-2m^2-m+1=\frac{1}{4}n^3-\frac{1}{2}n^2-\frac{1}{2}n+1.$$

This show the first inequality of (*).

REMARK 3: In the above construction, if we arrange the configuratoin of ellipsoids and h, then we can construct a surface in $\mathcal{S}_{2m}$ of genus g provided $g \leqq 2m^3-2m^2-m+1$.

Next we turn to the upper estimate:

LEMMA 4. *Let $S \subset \mathbf{R}P^3$ be a non-singular connected algebraic surface of degree $2m$. Then $g(S) \leqq (1/3)(2m)^3-(2m)^2+(7/6)(2m)$. Therefore $g'(2m) \leqq (1/3)(2m)^3-(2m)^2+(7/6)(2m)$.*

PROOF: By Petrovskii-Oleinik inequality [PO], (see also [Kha1], [Kho], [W], [A]), we see

$$|\chi(S)-1| \leqq \frac{2}{3}(2m)^3-2(2m)^2+\frac{7}{3}(2m)-1.$$

Since S is connected and orientable, we have $\chi(S)=2-2g(S)$. Thus we have Lemma 4.

By the above construction and by Lemma 4, we have (*) and therefore Theorem 1.

PROOF OF THEOREM 2: By Lemma 4, we have

$$g \leqq \frac{1}{3}d(g)^3-d(g)^2+\frac{7}{6}d(g),$$

and therefore we see $d(g)\to\infty, (g\to\infty)$. Then, by Remark 3, we have

$$\frac{1}{4}d(g)^3-\frac{1}{2}d(g)^2-\frac{1}{2}d(g)+1 \leqq g,$$

for sufficiently large g. Hence we have Theorem 2.

2. Quartic surface

Let us consider the case of quartic surfaces ($n = 4$).

First we remark that $g'(4) = 10$ ([GU], [Kha2]).

By Kharlamov [Kha3], if a surface S of degree 4 in $\mathbb{R}P^3$ is homotopic to zero, then $\dim H_*(S;\mathbb{Z}/2) \leqq 20$. Thus $g(4) \leqq 9$. In fact, there exists a surface of degree 4 in $\mathbb{R}P^3$ which is homeomorphic to Σ_9 and is homotopic to zero.

On the other hand we have

LEMMA 5. $g''(4) = 8$.

PROOF: By Gudkov-Shustin [GS], there exists a curve on the ellipsoid which consists of nine ovals outside each other. Then by Viro's method [V2] (see also §3), we have $g''(4) \geqq 8$. Conversely we will show $g''(4) \leqq 8$.

Let fix the ellipsoid $E : F_0 = 0$. For any rigid isotopy type ([G]) nearby the doubled ellipsoid, we can take a representative $S : F = F_0^2 + G = 0$, such that G is sufficiently small and the surface $\{G = 0\}$ is transverse to E. Then we see S is homeomorphic to the double of the domain $E_- = \{G \leqq 0\}$ in E (cf. [V2]). Thus $\chi(S) = 2\chi(E_-)$. If S is connected, then E_- must be connected. In this case, $C = \{G = 0\} \cap E$ is a curve of degree 8 with ovals lying outside each other. Then the number of ovals of C is less than or equal to 9 (cf. [GS]). Therefore $\chi(E_-) \geqq -7$ and hence $g(S) \leqq 8$.

As an conclusion we see $g(4)$ is equal to 8 or 9.

REMARK 6: Viro [V2] constructs an M-curve on the hyperboloid of degree 8 of type $\frac{9}{1}$, using the result of Polotovskii [P]. Then the problem is that a curve of this type can be contained in $\mathbb{R}^3$ or not. This is equivalent to the existence of a curve of type $\frac{9}{1}$ on $\mathbb{R}P^1 \times \mathbb{R}P^1$ of bidegree $(4,4)$ which does not intersect to $\mathbb{R}P^1 \times \{*\}$. If it is affirmative, then we have $g(4) = 9$. (This is pointed out by V.V. Nikulin to the first author.)

REMARK 7: The inequalities of (*) in Introduction have the form $7 \leqq g(4) \leqq 10$, for

$m = 2$, and each inequality of (*) therefore never gives the exact values $g(2m)$ in the non-asymptotic sense.

3. Doubling construction

In this section we give an asymptotic estimate of genuses of surfaces obtained by the iteration of perturbations from doubled surfaces [V2]: We remark that, by the iterating applications of doubling procedures using only a sequence of curves on the ellipsoid, the maximal genuses of resulting surfaces increase at most of order (1/4)(degree)^3.

First set $F_1 = x^2+y^2+z^2-1$ and $A(1) = \{F_1 = 0\}(= S^2 \subset \mathbf{R}^3)$. Let $B(k) = \{G_k = 0\}$ be a surface of degree 2^k transversal to $A(1)$ and non-singular along $B(k) \cap A(1), k = 1, 2, \ldots$. Set $C(k) = B(k) \cap A(1)$ and assume that $C(k) \cup C(k')$ consists of ovals lying outside to each other in $A(1)$, for $k \neq k'$. Suppose G_k is positive on the interiors of ovals of $C(k)$. (Otherwise it suffices to replace G_k by $-G_k$).

Set $F_2 = F_1^2 + \delta_1 G_2, (0 < \delta_1 \ll 1)$, and inductively, $F_{k+1} = F_k^2 + \delta_k G_{k+1}, (0 < \delta_k \ll \delta_{k-1})$. Then the surface $A(k)$ is non-singular connected of degree 2^k, and is a small perturbation of the 2^{k-1}-multiple sphere.

PROPOSITION 8. *For the genus $g(A(k))$ of $A(k)$, we have*

$$\overline{\lim_{k\to\infty}} \frac{g(A(k))}{(2^k)^3} \leqq \frac{1}{4}.$$

PROOF: Denote by $\ell(k)$ the number of ovals of $C(k)$. Then, by Oleinik inequality ([O], [Kha1]), we have $\ell(k) \leqq (3/4)(2^k)^2 - 2^k + 2$. Then the number of ovals $\tilde{\ell}(k+1)$ of the curve $B(k+1) \cap A(k)$ on $A(k)$ is equal to $2^{k-1}\ell(k+1)$. Therefore

$$\tilde{\ell}(k+1) \leqq \frac{3}{16}(2^{k+1})^3 - \frac{1}{4}(2^{k+1})^2 + \frac{1}{2}2^{k+1}.$$

Besides, we see $A(k+1)$ is homeomorphic to the double of the complement of interiors of ovals of $B(k+1) \cap A(k)$ to $A(k)$, ([V2]). Thus

$$g(A(k+1)) = 2g(A(k)) + \tilde{\ell}(k+1) - 1.$$

Therefore we have

$$\frac{g(A(k+1))}{(2^{k+1})^3} \leqq \frac{1}{4}\frac{g(A(k))}{(2^k)^3} + \frac{3}{16} + b_k,$$

with $b_k \to 0, (k \to \infty)$. Now we set $a = \overline{\lim}_{k\to\infty} g(A(k))/(2^k)^3$. Then, since a is finite by Theorem 1, we have

$$a \leqq \frac{1}{4}a + \frac{3}{16}.$$

Thus we have $a \leqq 1/4$.

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Fig. 1 (m = 4)

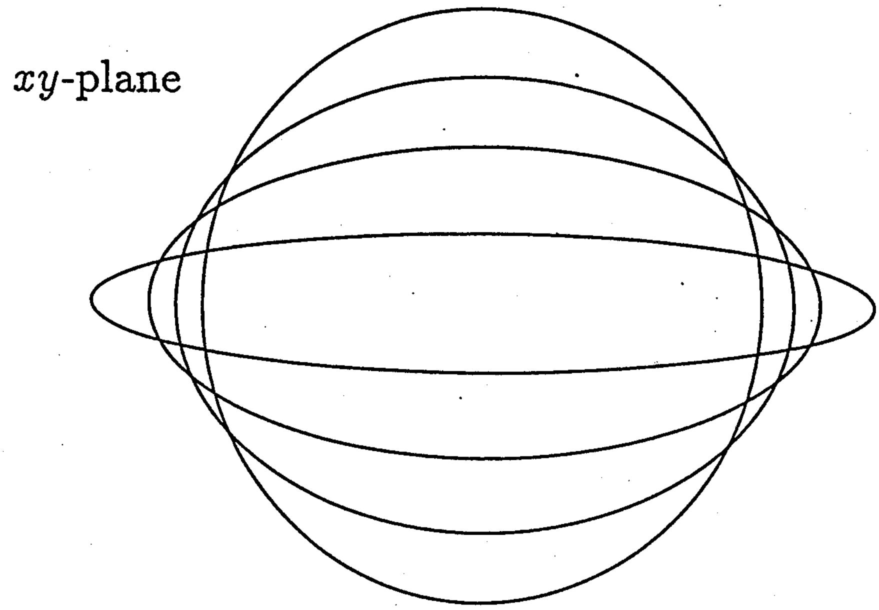

Fig. 2