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ON BLOCH'S METACONJECTURE

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On Bloch's metaconjecture

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Introduction

In [6], Bloch raises the following conjecture:

For any smooth projective variety V, there exists a filtration on $CH_0(V)$, at least the beginning of which is given by

$$F^1CH_0(V) = \mathrm{Ker}(\deg : CH_0(V) \longrightarrow \mathbf{Z}),$$
$$F^2CH_0(V) = \mathrm{Ker}(F^1CH_0(V) \longrightarrow \mathrm{Alb}(V)).$$

(Presumably the filtration will in general have m steps where $m = \dim V$.) Let S be a surface and let z be a cycle on $V \times S$ with $\dim z = m = \dim V$. Then z induces a map

$$z : CH_0(V) \longrightarrow CH_0(S).$$

The above filtration being functorial for correspondences, we get also

$$[z]: gr^{\cdot} CH_0(V) \longrightarrow gr^{\cdot} CH_0(S).$$

CONJECTURE([6],1.8). *The map* $[z]$ *depends only upon the cohomology class* $\{z\} \in H^4(V \times S)$.

Moreover,

METACONJECTURE([6],1.10). *There is an equivalence of category between a suitable category of polarized Hodge structures of weight 2 and a category built up from* $gr^2CH_0(S)$.

The present article gives an affirmative answer to the conjecture and the metaconjecture above in a weak form. To give the reader a perspective, it will be worthwhile to describe an (ideal) picture the author has in mind, a very small portion of which is really proven.

To each smooth projective variety V over $\mathbb{C}$, there are filtrations $F^{\ell}CH^{p}(V)$, $'F^{\ell}CH^{p}(V)$ on $CH^{p}(V)$, which are functorial for correspondences:

$$\begin{array}{ccccccccc} CH^{p}(V) = F^{0}CH^{p}(V) & \supset & \cdots & \supset & F^{\ell}CH^{p}(V) & \supset & F^{\ell+1}CH^{p}(V) & \supset & \cdots \\ \| & & & & \cup & & \cup & & \\ 'F^{0}CH^{p}(V) & \supset & \cdots & \supset & 'F^{\ell}CH^{p}(V) & \supset & 'F^{\ell+1}CH^{p}(V) & \supset & \cdots \end{array} .$$

(i) $F = {}'F$ for 0-cycles: $F^{\ell}CH_{0}(V) = {}'F^{\ell}CH_{0}(V)$ for all ℓ, and $'F^{\ell}$ is generated by $F^{\ell}CH_{0}$.

(ii) Set $gr_{F}^{\ell}CH^{p}(V) = F^{\ell}CH^{p}(V)/F^{\ell+1}CH^{p}(V)$.

For a cycle $z \in CH^{p+q}(W \times V)$, we have

$$[z] : gr_{F}^{\ell}CH_{q}(W) \longrightarrow gr_{F}^{\ell}CH^{p}(V),$$

which depends only on the cohomology class $\{ z \} \in H^{2p+2q}(W \times V)$.

(iii) We set

$$Gr^{\ell}CH^{p}(V) = {}'F^{\ell}CH^{p}(V)/({}'F^{\ell}CH^{p}(V) \cap F^{\ell+1}CH^{p}(V)).$$

The standard conjecture holds, and we have the theory of Grothendieck's motives as in [12], i.e., the motives whose morphisms are given by algebraic cycles modulo numerical equivalence. Let $\mathcal{M}_{\ell}$ be the ($\mathbb{Q}$-abelian) category of motives of pure niveau ℓ. Then there exists an anti-equivalence of category between the pseudo-abelian envelope of the $\mathbb{Q}$-additive category built up from $Gr^{\ell}CH_{r}(V)$ and the category $\mathcal{M}_{\ell}$, given by

$$Gr^{\ell}CH_{r}(V) \longmapsto gr^{r}h^{2r+\ell}(V)(r),$$

(or equivalence of category given by $Gr^{\ell}CH^{p}(V) \longmapsto gr^{p-\ell}h^{2p-\ell}(V)(p-\ell)$.) where $gr^{r}h^{2r+\ell}(V)$ denotes the pure niveau ℓ part of the motive $h^{2r+\ell}(V)$, and (r) denotes the Tate twist.

(iv) $F^{1}CH(V)$ is the cycles homologically equivalent to zero, and

$Gr^1 CH^p(V)$ is naturally isomorphic to the algebraic part of the p-th intermediate jacobian $J^p_a(V)$ (or $gr^1_F CH^p(V)$ is isomorphic to the image of the Abel-Jacobi map).

(v) $F^{p+1} CH^p(V) = 0$.

The reason why one filtration is insufficient is this:

Note that $gr^{p-\ell} h^{2p-\ell}(V)(p-\ell)$ is generated by $gr^0 h^1$ of curves. Hence by (iii), $Gr^1 CH^p(V)$ is generated by jacobians (of curves), so that $Gr^1 CH^p(V)$ is generated by cycles algebraically equivalent to zero. But by (ii), $F^1 CH^p(V)$ cannot be the cycles algebraically equivalent to zero. More concretely, we have a 3-fold V such that $gr^1 h^3(V) = 0$, hence $J^2_a(V) = 0$, but the image of Abel-Jacobi map is non-zero, hence $gr^1_F CH^2(V) \neq 0$. In other words, $gr^{p-\ell} h^{2p-\ell}(V)$ controls not $gr^\ell_F CH^p(V)$, but another object $Gr^\ell CH^p(V)$, at least for $\ell = 1$. The nature of the part $F^\ell CH(V)/{}'F^\ell CH(V)$ remains mysterious. For example, it is, in general, not finitely generated ([7]).

For (v), we have no direct evidence for $p > 1$. Recall, however, another Bloch's conjecture that cubic equivalence for cycles of codimension 2 coincides with rational equivalence([4]). Also compare [17], n. 30.

We shall explain the organization and the relation with the above.

In §1, we introduce the notion of product of adequate equivalence relations. An adequate equivalence relation E consists of subgroups $ECH(V)$ of $CH(V)$ which are stable under the correspondences. For adequate equivalence relations E, E', the product denoted by $E*E'$

is the minimum adequate equivalence relation satisfying the condition

$x \in ECH(W)$, and $y \in E'CH(V) \Longrightarrow x \times y \in E*E'CH(W \times V)$.

As a consequence, the filtration given by powers of homological equivalence has the property of (ii) above. We also introduce the group $Gr^{\ell}CH^{p}(V)$.

§§2 and 3 are preliminaries:in §2, we define the fundamental class (or cohomology class) of families of subschemes, following [18], and prove that the subfunctor of product of Hilbert schemes corresponding to the pairs of subschemes having the same fundamental classes is representable. §3 is concerning the Chow schemes by [1], i.e., families of cycles on a scheme over a base scheme (of characteristic zero), and we show that the direct image morphism for a proper morphism is defined on the whole of the Chow scheme, when we add the cycle "zero" to the Chow scheme.

In §4, we show that on a smooth projective variety over an algebraically closed *uncountable* field of characteristic zero, for a family of cycles $\{Z(s)\}_{s \in S}$, if at each closed point s, Z(s) is equivalent to zero with respect to a power of homological equivalence, so is generically. This is an analogue of [14], 5.6.

From §5 on, the ground field is assumed to be the field of complex numbers.

In §5, we generalize the theorem 3.2 of [15], which, in particular, says, in Severi's terminology[17], that a family of 0-cycles on a surface in a class of cube of homological equivalence is *a circolazion algebrica*. Further, we introduce a category $\mathscr{C}(\ell)$ of $Gr^{\ell}CH_{r}(V)$ and define a functor

$$\eta : \mathscr{C}(\ell) \longrightarrow Hdg(\ell) , \quad Gr^{\ell}CH_{r}(V) \longmapsto gr^{r}H^{2r+\ell}(V,\mathbb{Q})(r),$$

as (iii) above, where $Hdg(\ell)$ denotes the category of effective polarizable $\mathbb{Q}$-Hodge structures of weight ℓ. We notice that $\mathscr{C}(\ell)$ is defined from *not* all of $Gr^{\ell}CH_r(V)$, but those with certain conditions with respect to V, ℓ, and r. For $\ell = 0$, $Gr^0CH_r(V)$ is the r-cycles modulo homological equivalence, and the condition is that the map $Gr^0CH^r(V)_{\mathbb{Q}} \longrightarrow Hom(Gr^0CH_r(V)_{\mathbb{Q}}, \mathbb{Q})$ induced by intersection pairing is injective, whose universal validity is equivalent to the standard conjecture in chracteristic zero case.

§6 is preliminary for faithfulness of η. We remark that a much stronger result could be obtained if we suppose the standard conjecture. The "trivial" case $\ell = 0$ is also mentioned: the restriction of η to the subcategory generated by $Gr^0CH_r(V)$ for which the above map $Gr^0CH^r(V)_{\mathbb{Q}} \longrightarrow Hom(Gr^0CH_r(V)_{\mathbb{Q}}, \mathbb{Q})$ is bijective is faithful.

In §7, the faithfulness of η for $\ell = 1$ is dealt with. Let $J^p_a(V)$ denote the algebraic part of intermediate jacobian. Then, the restriction of the functor η to the subcategory $\mathscr{C}'(1)$ of $\mathscr{C}(1)$ generated by $Gr^1CH^p(V)$ such that $J^p_a(V)$ and $J^a_{p-1}(V)$ are naturally dual is an equivalence of category between the $\mathbb{Q}$-additive categories $\mathscr{C}'(1)$ and $Hdg(1)$. We also show that $Gr^1CH^p(V)$ has a structure of abelian variety such that the natural map

$$ACH^p(V) \longrightarrow Gr^1CH^p(V)$$

is regular, where A denotes the algebraic equivalence. Further, the kernel of the natural map

$$Gr^1CH^p(V) \longrightarrow J^p_a(V)$$

is finite for arbitrary V and p. More precisely, the kernel of the map

$$ACH^p(V) \longrightarrow J^p_a(V)$$

is the product of algebraic equivalence and homological equivalence up to finite groups, and it is even bijective for $p = 1, 2$, or $\dim V$. This is an analogue of the converse to a theorem of Abel: cycles on a curve of degree zero which vanish in the jacobian are rationally equivalent to zero. Also cf. [19], p.534.

In §8, the faithfulness of η for $\ell = 2$ is treated. Here, we can prove very little. If we denote by $\mathscr{C}(2)_{surf}$ the subcategory of $\mathscr{C}(2)$ generated by Gr^2CH_0 of the surfaces, then the restriction

$$\mathscr{C}(2)_{surf} \subset \mathscr{C}(2) \xrightarrow{\eta} Hdg(2)$$

is faithful. We can define the motives h^2, gr^0h^2 of *surfaces* constructed from algebraic cycles, and let $\mathscr{M}_2$ be the category generated by gr^0h^2 of the surfaces. Then $\mathscr{M}_2$ is semi-simple $\mathbb{Q}$-abelian category and we have an anti-equivalence of categories

$$\mathscr{C}(2)_{surf} \xrightarrow{\sim} \mathscr{M}_2,$$

the composition of which with the Betti realization gives the above functor $\mathscr{C}(2)_{surf} \longrightarrow Hdg(2)$. The Bloch's metaconjecture is, in our context, equivalent to the fully faithfulness of the realization $\mathscr{M}_2 \longrightarrow Hdg(2)$, and it is equivalent to the Hodge conjecture on the products of surfaces.

A part of the work was done while the author was staying at the University of Chicago. He would like to express his sincere gratitude to the university and Prof. Spencer Bloch for the hospitality.

§ 1. Products of adequate equivalence relations

1.1. Let k be an algebraically closed field, and we work in the category of smooth projective varieties. First recall the definition of adequate equivalence relation.

DEFINITION 1.1.1([16]). *An adequate equivalence relation* E *is an equivalence relation on cycles such that*

i) *it is compatible with addition of cycles;*

ii) *Let* X *be a cycle on* V, and $W_1, \cdots, W_k$ *a finite number of subvarieties on* V. *Then there exists a cycle* X' *equivalent to* X *such that* X' *and* W *intersect properly;*

iii) *If* Z *is a cycle on* $V \times W$, *if* X *is a cycle on* V *equivalent to zero, and if* $Z(X) = pr_{W*}(Z.X\times W)$ *is defined, then the cycle* $Z(X)$ *on* W *is equivalent to zero.*

1.1.2. It is well-known that the rational equivalence relation, which we denote by O, is the finest adequate equivalence relation and the numerical equivalence relation is the non-trivial coarsest one. We denote the trivial adequate equivalence relation that all cycles are equivalent by I. The cycles on V modulo rational equivalence is called the Chow ring $CH(V)$ of V and it has a ring structure by intersection, and is graded by codimension. The codimension p part will be denoted by $CH^p(V)$.

1.2. Let E be an adequate equivalence relation and

$ECH(V) := \{$ cycles on V E-equivalent to zero$\}$/rational equivalence.

Then $ECH(V)$ has the following properties:

i) $ECH(V)$ is a graded submodule of $CH(V)$;

ii) IF $x \in ECH(V)$ and if $z \in CH(V\times W)$, then

$$z(x) := pr_{W*}(z.x\times 1) \in ECH(W).$$

PROPOSITION 1.2.1. *Giving an adequate equivalence relation* E *is equivalent to assigning* $ECH(V) \subset CH(V)$ *to each* V *which satisfies the condition* i) *and* ii) *of* 1.2.

Let E and E' be adequate equivalence relations. Then we define the adequate equivalence relations $E + E'$, $E \cap E'$ by

$$(E+E')CH(V) := ECH(V) + E'CH(V),$$

$$(E\cap E')CH(V) := ECH(V) \cap E'CH(V).$$

We shall denote $E \subset E'$ if $ECH(V) \subset E'CH(V)$ for all V.

1.3. For adequate equivalence relations E and E', we shall define their *product* denoted by $E*E'$ as follows:

$(E*E')CH(V)$ is a submodule of $CH(V)$ generated by the elements of the form $pr_{V*}(x.y)$, where $x \in ECH(T \times V)$, $y \in E'CH(T \times V)$, T is a (smooth projective) variety, $pr_V : T \times V \longrightarrow V$ is the projection.

LEMMA 1.3.1. $E*E'$ *satisfies the conditions of* 1.2. i),ii) *and hence defines an adequate equivalence relation. A cycle* Z *on* V *is* $E*E'$*-equivalent to zero if and only if* Z *is a sum of cycles of the forms* $pr_{V*}(X.Y)$, *where* X *is a cycle on* $T \times V$ E*-equivalent to zero and* Y *is a cycle on* $T \times V$ E'*-equivalent to zero and the cycles* X *and* Y *intersect properly, and where* T *is a variety and*

$pr_V : T \times V \longrightarrow V$ *is the projection.*

By linearity, it is sufficient to show that if $z \in CH(V \times W)$ and $x \in ECH(V \times T)$ and $y \in E'CH(V \times T)$, then

$$z(pr_V(x.y)) \in (E*E')CH(W).$$

$$z(pr_V(x.y)) = pr_W(z.pr_V(x.y)\times 1_W)$$
$$= pr_W(1_T\times z.x\times 1_W.y\times 1_W)$$

and $1_T\times z.x\times 1_W \in ECH(T \times V \times W)$, $y\times 1_W \in E'CH(T \times V \times W)$. The latter part results from the moving lemma.

LEMMA 1.4. *Let* E, E', E" *be adequate equivalence relations. Then the following are equivalent:*

i) *if* $x \in ECH(V)$ *and* $y \in E'CH(V)$, *then* $x.y \in E''CH(V)$ *for arbitrary* V.

ii) *if* $x \in ECH(V)$ *and* $y \in E'CH(W)$ *then* $x \times y \in E''CH(V \times W)$ *for arbitrary* V *and* W.

iii) $E*E' \subset E''$.

It is clear that i) implies iii) and ii) implies i). To see that iii) implies ii), let T := Spec k. Then $1_T\times x \times 1_W \in ECH(T \times V \times W)$, $1_T\times 1_V \times y \in E'CH(T \times V \times W)$, and

$$x \times y = pr_{V \times W}(1_T\times x \times 1_W.1_T\times 1_V \times y).$$

1.5. For adequate equivalence relations E, E', E", we have

$$(E + E') + E'' = E + (E' + E''),\ (E * E') * E'' = E * (E' * E''),$$
$$E + E' = E' + E, \qquad E * E' = E' * E\ ,$$
$$E + O = E, \qquad E * I = E$$

$$E * (E' + E'') = E * E' + E * E'',$$

$$E' \subset E'' \quad \text{implies} \quad E * E' \subset E * E''.$$

Let $E^{*\ell} := E * \cdots * E$ (ℓ times for $\ell > 0$, and $E^{*0} = I$), and set

$$gr_E^{\ell} CH(V) := E^{*\ell} CH(V)/E^{*(\ell+1)} CH(V).$$

By virtue of lemma 1.4, we have

LEMMA 1.5.1. *The ring structure of* $CH(V)$ *defines the bigraded ring structure on* $gr_E^{\cdot} CH^{\cdot}(V)$. *In particular,* $z \in CH^{p+q}(V \times W)$ *defines the map*

$$[z] : gr_E^{\ell} CH_q(V) \longrightarrow gr_E^{\ell} CH^p(W), \quad x \longmapsto z(x)$$

and it depends only on the class of z *in* $gr_E^0 CH(V \times W)$.

REMARKS 1.6.1. Let E, E' be adequate equivalence relations. Then $z \in CH(V)$ is in $(E*E')CH(V)$ if and only if there exits a finite number of $x_1, \cdots, x_k \in ECH(T \times V)$ and $y_1, \cdots, y_k \in E'CH(T \times V)$ such that $z = \sum pr_{V*}(x_i . y_i)$.

In fact, if $x \in ECH(T \times V)$ and $y \in E'CH(T \times V)$, then for any variety T', and a point t' of T', we have

$$pr_{V*}(x.y) = pr'_{V*}(t' \times x.1_{T'} \times y),$$

where $pr'_V : T' \times T \times V \longrightarrow V$ is the projection, and $t' \times x \in ECH(T' \times T \times V)$ and $1_{T'} \times y \in E'CH(T' \times T \times V)$.

1.6.2. More generally, let $E_1, \cdots, E_\ell$ be adequate equivalence relations and Z a cycle of codimension p on V. Then Z is $(E_1 * \cdots * E_\ell)$-equivalent to zero if and only if there exist a variety W, a (projective) morphism $f : W \longrightarrow V$, cycles X_{ij} of codimension p_{ij} on W, E_i-equivalent to zero ($1 \leq i \leq \ell$,

$1 \leq j \leq k$) such that $X_{i1}, \cdots, X_{ik}$ intersect properly on W,

$$\sum_j p_{ij} = p - \dim V + \dim W \quad \text{for all } i,$$

and that

$$Z = \sum_j f_*(X_{1j}. \cdots .X_{\ell j}).$$

By 1.4, it is clear that Z is $(E_1 * \cdots * E_\ell)$-equivalent to zero. To see the converse, by induction, it suffices to consider the case $\ell = 3$. Let $u \in (E_1 * E_2)CH(T \times V)$, $v \in E_3CH(T \times V)$. By linearity, we may assume that $u = pr_{T\times V*}(x.y)$, where $x \in E_1CH(T' \times T \times V)$ and $y \in E_2CH(T' \times T \times V)$. Then,

$$\begin{aligned} pr_{V*}(u.v) &= pr_{V*}(pr_{T\times V*}(x.y).v) \\ &= pr'_{V*}(x.y.1_{T'} \times v), \end{aligned}$$

where $pr'_V : T' \times T \times V \longrightarrow V$ is the projection and $1_{T'} \times v \in E_3CH(T' \times T \times V)$.

1.7. Let E be an adequate equivalence relation. We *define* the adequate equivalence relation $\langle E \rangle_0$ as the equivalence relation generated by 0-cycles E-equivalent to zero. More precisely,

$$\langle E \rangle_0 CH(V) = \sum z(CH_0(T)),$$

where T runs over all smooth projective varieties, and z runs over the cycles on $T \times V$. It is clear that $\langle E \rangle_0 CH(V)$ defines an adequate equivalence relation.

LEMMA 1.7.1. *Let* E, E' *be adequate equivalence relations.*

(i) $\langle E \rangle_0 \subset E$, *and* $\langle E' \rangle_0 \subset E$ *if and only if* $E'CH_0(V) \subset ECH_0(V)$ *for every variety* V.

(ii) $\langle E \rangle_0 * \langle E' \rangle_0 \subset \langle E*E' \rangle_0$.

(i) is trivial, and (ii) follows from the formula

$$z(x) \times z'(x') = (z \times z')(x \times x')$$

for $z \in CH(V \times T)$, $x \in ECH_0(T)$, $z' \in CH(V' \times T')$, and $x' \in E'CH_0(T')$.

Example 1.8. We work in the category of varieties over the complex numbers $\mathbb{C}$. We denote by $H_{\mathbb{Q}}$ the $\mathbb{Q}$-homological equivalence in $H^{\cdot}(V,\mathbb{Q})$ and $H = H_{\mathbb{Z}}$ the homological equivalence in $H^{\cdot}(V,\mathbb{Z})$, which are both adequate equivalence relations . We have a filtration of CH by powers of H:

(1.8.1) $I = H^{*0} \supset H = H^{*1} \supset H^{*2} \supset H^{*3} \supset \cdots \supset H^{*\ell} \supset H^{*(\ell+1)} \supset \cdots$.

We set

(1.8.2) $Gr^{\ell}CH(V) = \langle H^{*\ell}\rangle_0 CH(V)/(\langle H^{*\ell}\rangle_0 \cap H^{*(\ell+1)})CH(V)$

By 1.7.1, $Gr^{\cdot}CH^{\cdot}(V)$ has a bigraded ring structure, and for $z \in CH(T \times V)$, the induced map

$$[z] : Gr^{\ell}CH(T) \longrightarrow Gr^{\ell}CH(V)$$

depends only on the cohomology class of z. For 0-cycles, notice that $Gr^{\ell}CH_0(V)$ is the associated graded to the filtration 1.8.1.

Example 1.9. Let ACH(V) denote the classes of cycles which are algebraically equivalent to zero. Then ACH(V) defines an adequate equivalence relation, and $A^{*\ell}$ is nothing but the ℓ-cubic equivalence relation [16]. Note that $A = \langle H \rangle_0 = \langle H_{\mathbb{Q}} \rangle_0$.

LEMMA 1.10. *Let* E *and* E' *be adequate equivalence relations, and assume that* E'CH(V) *are divisible for all* V. *Then* E*E'CH(V) *are* *also divisible. In particular,* $A*ECH(V)$ *is divisible for each smooth projective variety* V.

Example 1.11. Let $T^p(V)$ denote the Griffiths intermediate jacobian; we have the Abel-Jacobi map

$$c^p : HCH^p(V) \longrightarrow T^p(V)$$

and the image of the restriction to $ACH^p(V)$ is, by definition, $J_a^p(V)$. For $z \in CH^{p+q}(W \times V)$, the diagram

$$\begin{array}{ccc} HCH_q(W) & \xrightarrow[z(?)]{} & HCH^p(V) \\ \downarrow & & \downarrow \\ T_q(W) & \longrightarrow & T^p(V) \end{array}$$

commutes, where the map below is induced by the fundamental class $\{z\} \in H^{2p+2q}(W \times V, \mathbb{Z})$. It follows that

$$\bar{J}CH^p(V) = \mathrm{Ker}(\ HCH^p(V) \longrightarrow T^p(V))$$

$$JCH^p(V) = \mathrm{Ker}(\ ACH^p(V) \longrightarrow J_a^p(V))$$

define adequate equivalence relations $\bar{J}$ and J. We have $J = \bar{J} \cap A$. It also follows from the diagram above that $c^p(H*H_{\mathbb{Q}}CH^p(V)) = 0$, which shows that $H^{*2} \subset H_{\mathbb{Q}}*H \subset \bar{J}$. In particular,

$$\langle H^{*2}\rangle_0 \subset \langle H_{\mathbb{Q}}*H\rangle_0 \subset \langle\ \bar{J}\ \rangle_0 \subset \bar{J} \cap \langle\ H\ \rangle_0 = \bar{J} \cap A = J.$$

Hence we have a surjective canonical map

$$\gamma^p : Gr^1CH^p(V) \longrightarrow J_a^p(V).$$

For $p = 1$, $JCH^1(V) = \bar{J}CH^1(V) = 0$, hence $H^{*2}CH^1(V) = 0$, and we have a bijection

$$\gamma^1 : Gr^1CH^1(V) \longrightarrow J_a^1(V) = Pic^0(V).$$

§ 2. Fundamental classes for Hilbert scheme

2.1. Let S be a locally noetherian scheme and $f : X \longrightarrow S$ be a compactifiable morphism, F an etale sheaf on S. For an integer n, we define

$$H_n(X/S,F) := H^0(S, R^{-n}f_*Rf^!F).$$

If $g : Y \longrightarrow S$ is a compactifiable morphism and $h : X \longrightarrow Y$ is a proper S-morphism, we have

$$h_* : H_n(X/S,F) \longrightarrow H_n(Y/S,F)$$

induced by adjunction $Rf_*Rf^!F = Rg_*Rh_*Rh^!Rg^!F \longrightarrow Rg_*Rg^!F$.

It is clear that $h \longmapsto h_*$ is functorial.

For a morphism $\varphi : S' \longrightarrow S$, we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\varphi]{} & S \end{array} .$$

Then we obtain $\varphi^* : H_n(X/S,F) \longrightarrow H_n(X'/S',\varphi^*F)$,

i.e., $H^0(S,R^{-n}f_*Rf^!F) \longrightarrow H^0(S,\varphi_*R^{-n}f'_*Rf'^!\varphi^*F)$ as follows:

By [3], 2.3.1, we have $\varphi'^*Rf^!F \longrightarrow Rf'^!\varphi^*F$, or

$Rf^!F \longrightarrow R\varphi'_*Rf'^!\varphi^*F$. Applying Rf_*, we get

$$Rf_*Rf^!F \longrightarrow Rf_*R\varphi'_*Rf'^!\varphi^*F = R\varphi_*Rf'_*Rf'^!\varphi^*F.$$

By Leray spectral sequence, we obtain

$$R^{-n}f_*Rf^!F \longrightarrow \varphi_*R^{-n}f'_*Rf'^!\varphi^*F.$$

The following diagram is commutative:

$$\begin{array}{ccc} H_n(X/S,F) & \xrightarrow{h_*} & H_n(Y/S,F) \\ {\scriptstyle \varphi^*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ H_n(X'/S',\varphi^*F) & \xrightarrow[h'_*]{} & H_n(Y'/S',\varphi^*F). \end{array}$$

2.2. Let $g : Z \longrightarrow S$ be a flat morphism of pure relative dimension r, and e be a prime integer invertible in S. By definition,

$$H_{2r}(Z/S,\mathbb{Z}_e(-r)) = H^0(S,R^{-2r}g_*Rg^!\mathbb{Z}_e(-r)).$$

On the other hand, we have

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_e,R^{-2r}g_*Rg^!\mathbb{Z}_e(-r)) &= \mathrm{Hom}(\mathbb{Z}_e,Rg_*Rg^!\mathbb{Z}_e(-r)[-2r])\\ &= \mathrm{Hom}(g^*\mathbb{Z}_e,Rg^!\mathbb{Z}_e(-r)[-2r])\\ &= \mathrm{Hom}(Rg_!g^*\mathbb{Z}_e(r)[2r],\mathbb{Z}_e)\\ &= \mathrm{Hom}(R^{2r}g_!g^*\mathbb{Z}_e(r),\mathbb{Z}_e).\end{aligned}$$

We have the trace map ([3], §2)

$$\mathrm{Tr}_g : R^{2r}g_!g^*\mathbb{Z}_e(r) \longrightarrow \mathbb{Z}_e,$$

hence the corresponding map $\mathrm{Tr}'_g : \mathbb{Z}_e \longrightarrow R^{-2r}g_*Rg^!\mathbb{Z}_e(-r)$.

Therefore we get $H^0(S,\mathbb{Z}_e) \longrightarrow H_{2r}(Z/S,\mathbb{Z}_e(-r))$.

Suppose Z is a closed subscheme of X over S : $j : Z \hookrightarrow X$.

The image of $1 \in H^0(S,\mathbb{Z}_e)$ by

$$H^0(S,\mathbb{Z}_e) \longrightarrow H_{2r}(Z/S,\mathbb{Z}_e(-r)) \xrightarrow{j_*} H_{2r}(X/S,\mathbb{Z}_e(-r))$$

will be called the *fundamental class of* Z/S and denoted by $\{Z/S\}$.

For $\varphi : S' \longrightarrow S$, and $Z' = Z \times_S S'$, the base-change of Z, we have $\varphi^*\{Z/S\} = \{Z'/S'\} \in H_{2r}(X'/S',\mathbb{Z}_e(-r))$.

If X is smooth of pure relative dimension m over S, denoting $p = m - r$, we have

$$\begin{aligned}\{Z/S\} \in H_{2r}(X/S,\mathbb{Z}_e(-r)) &= H^0(S,R^{-2r}f_*Rf^!\mathbb{Z}_e(-r))\\ &= H^0(S,R^{-2r}f_*\mathbb{Z}_e(p)[2m])\\ &= H^0(S,R^{2p}f_*\mathbb{Z}_e(p)).\end{aligned}$$

2.3. Suppose X is smooth projective over S of pure relative dimension m, and let $\mathrm{Hilb}_r(X/S)$ denote the set of subschemes flat of pure relative dimension r over S.

We have

$$\{\ /S\} : \mathrm{Hilb}_r(X/S) \longrightarrow H^0(S, R^{2p}f_*\mathbb{Z}_e(p)),\ Z \longmapsto \{Z/S\},$$

and set

$$\mathrm{Hilb}_r(X/S)^{\times 2,H} = \mathrm{Ker}(\mathrm{Hilb}_r(X/S) \times \mathrm{Hilb}_r(X/S) \underset{\{/S\}pr_2}{\overset{\{/S\}pr_1}{\rightrightarrows}} H^0(S, R^{2p}f_*\mathbb{Z}_e(p)).$$

It is clear that, for $\varphi : S' \longrightarrow S$,

$$\varphi^*: \mathrm{Hilb}_r(X/S)^{\times 2,H} \longrightarrow \mathrm{Hilb}_r(X'/S')^{\times 2,H}, (Z_1, Z_2) \longmapsto (Z_1', Z_2')$$

defines a functor $\mathrm{Hilb}^{\times 2,H}_{X/S,r}$ on locally noetherian schemes over S.

PROPOSITION 2.4. *With above hypotheses, the functor* $\mathrm{Hilb}^{\times 2,H}_{X/S,r}$ *is representable by an open subscheme of the product of Hilbert schemes* $\mathrm{Hilb}_{X/S,r} \underset{S}{\times} \mathrm{Hilb}_{X/S,r}$.

It is enough to show that if $(Z_1, Z_2) \in \mathrm{Hilb}_r(X/S) \times \mathrm{Hilb}_r(X/S)$ and if, for $s \in S$, $((Z_1)_s, (Z_2)_s) \in \mathrm{Hilb}_r(X_s/s)^{\times 2,H}$, then there exists an open neighbourhood U of s such that

$$((Z_1)_U, (Z_2)_U) \in \mathrm{Hilb}_r(X_U/U)^{\times 2,H}.$$

Let $\sigma = \{Z_1/S\} - \{Z_2/S\} \in H^0(S, R^{2p}f_*\mathbb{Z}_e(p))$. If $\bar{s}$ is a geometric point of s, the pull back of σ in $H^{2p}(X_{\bar{s}}, \mathbb{Z}_e(p))$ vanishes. It suffices to see that there exists an open neighbourhood U of s where $\sigma = 0$.

LEMMA 2.4.1. *Let* $f : X \longrightarrow S$ *be a smooth proper morphism and* s *a geometric point of* S, $\sigma \in H^0(S, R^nf_*\mathbb{Z}_e(k))$. *If the pull back of* σ *in* $H^n(X_s, \mathbb{Z}_e(k))$ *is zero, then* $\sigma = 0$ *on the connected component* *of* S *containing* s.

We have $\sigma = (\sigma_\nu) \in (H^0(S, R^n f_*(\mathbb{Z}/e^\nu(k))))_\nu$ and the hypothesis means that $(\sigma_\nu)_s = 0$ in $H^n(X_s, \mathbb{Z}/e^\nu(k)) = (R^n f_*(\mathbb{Z}/e^\nu(k)))_s$ for any ν, since f is proper. The morphism f is smooth proper, hence $R^n f_*(\mathbb{Z}/e^\nu(k))$ is a locally constant. Let U be an etale neighbourhood of s where $R^n f_*(\mathbb{Z}/e^\nu(k))$ is constant. Then

$$\sigma_\nu | U = 0 \iff (\sigma_\nu)_{\bar{t}} = 0 \text{ at some geometric point } \bar{t} \text{ of } U.$$

It follows that $\sigma_\nu = 0$ on the connected component of S containing s, and $\sigma = 0$ on it.

REMARKS 2.4.2. If S is the spectre of an algebraically closed field k, and Z is a closed subscheme of pure dimension r of a smooth projective variety V over k, then $\{Z/k\} \in H^{2p}(V, \mathbb{Z}_e(p))$ is the fundamental class of the cycle associated to the subscheme Z, cf. [18], 3.3.4.

2.4.3. The homological equivalence relation we have considered above is the $\mathbb{Z}_e$-homological equivalence. We can also consider the $\mathbb{Q}_e$-homological equivalence and in that case, the proposition remains true. In fact, with the notation of proof of the proposition, if the pull back of σ in $H^{2p}(X_{\bar{s}}, \mathbb{Q}_e(p))$ vanishes, then $k.\sigma = 0$ in $H^{2p}(X_{\bar{s}}, \mathbb{Z}_e(p))$ with $k \neq 0$, hence $k.\sigma$ vanishes in a neighbourhood of s with $\mathbb{Z}_e$-coefficient, hence σ vanishes there with $\mathbb{Q}_e$-coefficient.

2.4.4. Let E be a set consisting of some prime integer invertible in S. We could consider the intersection of $\mathbb{Z}_e$-homological equivalence, i.e.,

$\mathrm{Hilb}_r(X/S)^{\times 2, H, E}$

$= \{ (Z_1, Z_2);\ \{Z_1/S\} = \{Z_2/S\}$ in $H^0(S, R^{2p} f_* \mathbb{Z}_e(p))$ for $e \in E\}$.

In view of lemma 2.4.1, the functor $S \longmapsto Hilb_r(X/S)^{\times 2, H, E}$ is also representable by an open subscheme of $Hilb_{X/S,r} \underset{S}{\times} Hilb_{X/S,r}$. Moreover, we can replace the equivalent relation by the mixture of the type considered in 2.4.3.

§ 3. Direct image morphism of Chow schemes

3.1. Let S be a locally noetherian scheme. Recall that a morphism $h : X \longrightarrow S$ of finite type is called of pure relative dimension r if $X_s = h^{-1}(s)$ is of pure dimension r for every $s \in h(X)$. We set

$$X(r) = \{ x \in X \ ; \ \dim_x h^{-1}(h(x)) \geq r \ \}.$$

Then $X(r)$ is a closed subset of X.

Note that h is of pure relative dimension r if and only if $X(r) = X$, provided that all the fibres of h are of dimension $\leq r$.

PROPOSITION 3.2. *Let* X, Y *be* S-*schemes of finite type,* $f: X \longrightarrow Y$ *be a proper surjective* S-*morphism with* Y *irreducible and* $X \longrightarrow S$ *of pure relative dimension* r. *Suppose that there exists* $s \in S$ *such that* $\dim Y_s = r$. *Then* $Y \longrightarrow S$ *is of pure relative dimension* r.

The conclusion is equivalent to $Y = Y(r)$. If f is finite, then $f_s : X_s \longrightarrow Y_s$ is also finite, and it is clear that $Y = Y(r)$. In general case, let Y^o be the maximum open subscheme of Y such that $f^o = f_{(Y^o)}: X^o \longrightarrow Y^o$ is finite; then $Y^o \neq \phi$. In fact, consider $X_s \longrightarrow Y_s$. If $x \in X_s$ is the generic point of a component of X_s such that $\dim \overline{f(x)} = r$, then the restriction $\overline{x} \longrightarrow \overline{f(x)}$ of f_s is generically finite(the bars denote the closure in the fibres), and $f(x) \in Y^o$. Since $X_s \longrightarrow Y_s$ is surjective, such an x exists by hypothesis, hence $Y^o \neq \phi$. Let $g : Y \longrightarrow S$, and $g^o : Y^o \longrightarrow S$ be its restriction. Then $Y^{o}(r) := \{ y \in Y^{o} ; \dim_y g^{o-1}(g^{o}(y)) \geq r \}$ $= Y^{o} \cap Y(r)$. For $y \in Y^{o}$, $\dim_y g^{o-1}(g^{o}(y)) = \dim_y(Y^{o} \cap g^{-1}(g(y)))$ $= \dim_y g^{-1}(g(y))$. Since $f^{o}: X^{o} \longrightarrow Y^{o}$ is finite, $Y^{o}(r) = Y^{o}$, so that $Y^{o} = Y^{o}(r) = Y^{o} \cap Y(r) \subset Y(r) \subset Y$. Since the closure of Y^{o} is Y , $Y(r) = Y$.

LEMMA 3.3. *Let* S *be a locally noetherian scheme and* $f : X \longrightarrow Y$ *be a proper* S*-morphism, and suppose that* $X \longrightarrow S$ *is of pure relative dimension* r. *Then there exist closed subsets* Y_1, Y_2 *of* Y *such that* $f(X) = Y_1 \cup Y_2$ *and* $Y_1 \longrightarrow S$ *is of pure relative dimension* r , *and* $\dim (Y_2)_s < r$ *for any* $s \in S$.

We can suppose X reduced, and replacing f by $X \longrightarrow f(X)$, we may assume f is surjective. If Y is a union of closed subsets Y_λ, then for $y \in Y_s \subset Y$, since

$$\dim_y Y_s = \sup_\lambda \dim_y (Y_\lambda)_s \leq r,$$

$Y(r)$ is the union of $Y_\lambda(r)$. Let $X = \bigcup_\lambda X_\lambda$ is the decomposition into irreducible components. Then $Y = \bigcup_\lambda f(X_\lambda)$. Consider $X_\lambda \longrightarrow f(X_\lambda)$, and we have either $f(X_\lambda)(r) = f(X_\lambda)$ or $f(X_\lambda)(r) = \phi$ by 3.2. It will suffice to put $Y_1 = f(X)(r)$ and

$$Y_2 := \bigcup f(X_\lambda)$$

where the union is over those X_λ with $f(X_\lambda)(r) = \phi$.

3.4. Let S be an affine scheme of characteristic zero, and X be a smooth projective S-scheme of pure relative dimension m. Then for an integer p, $0 \leq p \leq m$, we have the Chow scheme $C^p_{X/S}$ of cycles of relative codimension p on X/S ([1]), while $C^p_{X/S}$ is, in fact, only an algebraic space in general. If X is a subscheme of $S \times P^N$ for some N, then, $C^p_{X/S}$ is embedded in $C^{N-m+p}_{S\times P^N/S} = C^{N-m+p}_{P^N} \times S$, and since $(C^{N-m+p}_{P^N})_{red}$ is the usual Chow variety of P^N, $C^p_{X/S}$ is an S-scheme, a countable union of proper S-schemes.

We set

$$\bar{C}^p_{X/S} = C^p_{X/S} \coprod o(S), \quad o(S) = S.$$

Intuitively, o(S) corresponds to the cycle "zero" of codimension p. We shall show that, for a proper S-morphism $f : X \longrightarrow Y$ of smooth projective S-schemes, we can define the direct image morphism

$$f_* : \bar{C}^p_{X/S} \longrightarrow \bar{C}^{p+n-m}_{Y/S}$$

of Chow schemes, where n is the relative dimension of Y/S. To do this, it suffices to define a morphism as functors.

Let S' be an S-scheme, and put

$$X' = X \underset{S}{\times} S', \quad Y' = Y \underset{S}{\times} S',$$
$$f' = f \times id_{S'} : X' \longrightarrow Y',$$
$$C^p(X'/S') = C^p_{X/S}(S'),$$
$$\bar{C}^p(X'/S') = \bar{C}^p_{X/S}(S') = C^p(X'/S') \coprod \{id_{S'}\}.$$

Recall that an element of $C^p(X'/S')$ is a pair (Z,c) of a closed subset $Z \subset X'$ of pure relative dimension $r = m - p$ over S', and an element $c \in H^p_Z(X',\Omega^p_{X'/S'})$ which satisfy some conditions (cf. [1], 4.1, 4.2). By Lemma 3.3, $Z' = f'(Z) = Z'_1 \cup Z'_2$, where Z'_1 is of pure relative dimension r over S, and Z'_2 is of relative dimension $< r$.

Note that

$$R\Gamma_{f'^{-1}(Z')}(X',?) = R\Gamma_{Z'}(Y',Rf'_*?).$$

Putting $d = m - n$, we have

$$
\begin{aligned}
\mathrm{Hom}(Rf'_{*}\Omega^{p}_{X'/S'},\Omega^{p-d}_{Y'/S'}[-d])
&= \mathrm{Hom}(\Omega^{p}_{X'/S'},Rf'^{!}\Omega^{p-d}_{Y'/S'}[-d]) \\
&= \mathrm{Hom}(\Omega^{p}_{X'/S'},Rf'^{!}\Omega^{n}_{Y'/S'}[n]\otimes(f'^{*}\Omega^{r}_{Y'/S'})^{\vee}[-m]) \\
&= \mathrm{Hom}(\Omega^{p}_{X'/S'},\Omega^{m}_{X'/S'}[m]\otimes(f'^{*}\Omega^{r}_{Y'/S'})^{\vee}[-m]) \\
&= \mathrm{Hom}(\Omega^{p}_{X'/S'}\otimes f'^{*}\Omega^{r}_{Y'/S'},\Omega^{m}_{X'/S'}),
\end{aligned}
$$

and the canonical map

$$\Omega^{p}_{X'/S'}\otimes f'^{*}\Omega^{r}_{Y'/S'} \longrightarrow \Omega^{p}_{X'/S'}\otimes \Omega^{r}_{X'/S'} \longrightarrow \Omega^{m}_{X'/S'},$$

hence, we get

$$Rf'_{*}\Omega^{p}_{X'/S'} \longrightarrow \Omega^{p-d}_{Y'/S'}[-d].$$

Therefore we obtain

$$
\begin{aligned}
H^{p}_{Z}(X',\Omega^{p}_{X'/S'}) \longrightarrow H^{p}_{f'^{-1}(Z')}(X',\Omega^{p}_{X'/S'}) &= H^{p}_{Z'}(Y',Rf'_{*}\Omega^{p}_{X'/S'}) \\
&\longrightarrow H^{p-d}_{Z'}(Y',\Omega^{p-d}_{Y'/S'}).
\end{aligned}
$$

LEMMA 3.5. *The canonical map*

$$H^{p-d}_{Z'_1}(Y',\Omega^{p-d}_{Y'/S'}) \longrightarrow H^{p-d}_{Z'}(Y',\Omega^{p-d}_{Y'/S'})$$

is an isomorphism.

SUBLEMMA 3.5.1. (cf.[2]) *Let* $g : Y \longrightarrow S$ *be a morphism of relative dimension* $< r$ *of locally noetherian schemes, then we have*

$$R^{i}g^{!}O_{S} = 0 \qquad \text{for} \quad i \leq -r.$$

The question is local on Y. For any $z \in Y$, We have a commutative diagram

$$
\begin{array}{ccc}
z \in U & \xrightarrow{\ j\ } & Y \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
A^{r-1}_{S} & \xrightarrow[\ a\ ]{} & S
\end{array}
$$

where j is an open immersion, and h is a quasi-finite morphism. By Zariski's Main theorem, there is a finite morphism $\bar{h} : V \longrightarrow A_S^{r-1}$ and an open immersion $k : U \longrightarrow V$ such that $h = \bar{h}\cdot k$. We have

$$R^i g^! O_S | U = R^i (g\cdot j)^! O_S$$
$$= R^i (a\cdot\bar{h}\cdot k)^! O_S$$
$$= R^{i+r-1}\bar{h}^! \Omega^{r-1}_{A_S^{r-1}/S} | U,$$
$$R\bar{h}^! \Omega^{r-1}_{A_S^{r-1}/S} = \tilde{h}^* R\underline{Hom}_{O_{A^{r-1}}}(\bar{h}_* O_V, \Omega^{r-1}_{A_S^{r-1}/S}),$$

where $\tilde{h} : (V, O_V) \longrightarrow (A_S^{r-1}, h_* O_V)$. Since $\tilde{h}$ is flat, we have

$$R^i\bar{h}^! \Omega^{r-1}_{A_S^{r-1}/S} = 0 \quad \text{for} \quad i < 0.$$

Therefore we have $R^i g^! O_S = 0$ for $i + r - 1 < 0$, i.e., for $i < -r + 1$.

LEMMA 3.5.2. *Let* $g : Y \longrightarrow S$ *be a smooth morphism of locally noetherian schemes of pure relative dimension* n, E *a locally free* O_Y*-Module of finite rank and* $Z \subset Y$ *a closed subscheme of relative dimension* $< r$ *over* S *and set* $p' = n - r$. *Then we have*

$$Ext^i(O_Z, E) = 0, \quad \textit{and} \quad H_Z^i(Y, E) = 0 \quad \textit{for} \quad i \le p'.$$

Let $j : Z \longrightarrow Y$ denote the closed immersion. We get

$$Ext^i(O_Z, E) = Hom(O_Z, E[i])$$
$$= Hom(O_Z, \Omega^n_{Y/S} \otimes \underline{Hom}(\Omega^n_{Y/S}, E)[i])$$
$$= Hom(O_Z \otimes \underline{Hom}(E, \Omega^n_{Y/S}), \Omega^n_{Y/S}[n][i-n])$$
$$= Hom(Rj_* j^* \underline{Hom}(E, \Omega^n_{Y/S}), Rg^! O_S[i-n])$$
$$= Hom(j^* \underline{Hom}(E, \Omega^n_{Y/S}), R(g\cdot j)^! O_S[i-n]),$$

and we have a spectral sequence

$$E_2^{a,i-a} = Ext^a(j^*\underline{Hom}(E,\Omega^n_{Y/S}),R^{i-a-n}(g\cdot j)^!O_S) \Longrightarrow Ext^i(O_Z,E).$$

By sublemma 3.5.1, $R^{i-a-n}(g\cdot j)^!O_S = 0$ for $i-a-n \leq -r$, i.e., for $i-a \leq p'$. Since $E_2^{a,i-a} = 0$ unless $a \geq 0$ and $i-a > p'$, $Ext^i(O_Z,E) = 0$ for $i \leq p'$. It follows that $H^i_Z(X,E) = 0$ for $i \leq p'$.

The proof of lemma 3.5 is now easy: we have an exact sequence

$$H^{p-d-1}_{Z'\setminus Z_1'}(Y'\setminus Z_1',\Omega^{p-d}_{Y'/S'}) \longrightarrow H^{p-d}_{Z_1'}(Y',\Omega^{p-d}_{Y'/S'}) \longrightarrow H^{p-d}_{Z'}(Y',\Omega^{p-d}_{Y'/S'}) \longrightarrow$$

$$\longrightarrow H^{p-d}_{Z'\setminus Z_1'}(Y'\setminus Z_1',\Omega^{p-d}_{Y'/S'})$$

and the both extremes vanish by virtue of sublemma 3.5.2, because $Z'\setminus Z_1' \subset Z_2'$.

3.6. We *define*

$$f_*' : \bar{C}^p(X'/S') \longrightarrow \bar{C}^p(Y'/S')$$

as follows: the image of $id_{S'}(= o(S)(S'))$ is $id_{S'} \in \bar{C}^p(Y'/S')$. For $(Z,c) \in C^p(X'/S')$, we have

$$\bar{f}'_* : H^p_Z(X',\Omega^p_{X'/S'}) \longrightarrow H^{p-d}_{Z'}(Y',\Omega^{p-d}_{Y'/S'}) \xleftarrow{\sim} H^{p-d}_{Z_1'}(Y',\Omega^{p-d}_{Y'/S'}).$$

If $Z_1' \neq \phi$, we put

$$f_*'((Z,c)) = (Z_1',\bar{f}'_*(c)),$$

and otherwise,

$$f_*'((Z,c)) = id_{S'}.$$

PROPOSITION 3.7. *Under the above hypothesis,* $f'_*((Z,c)) \in \bar{C}^p(Y'/S')$, *and we have a morphism of functors*

$$f_* : \bar{C}^p_{X/S} \longrightarrow \bar{C}^{p-d}_{Y/S}.$$

It suffices to see $f'_*((Z,c)) \in \bar{C}^p(Y'/S')$

Let $z' \in Z_1'$, and (U',B',φ') be a projection of Z_1' around z'. It is also a projection of Z_1' around z'' for any generization $z'' \in Z_1'$ of z', hence $(Z_1', \bar{f}'_*(c))$ is a Chow class at z' if it is a Chow class at z''. Let Z_1 be the pull-back of Z_1' by $Z \subset X' \xrightarrow{f} Y'$, and let Z_1'' be the closed set of Z_1' of points y such that the fiber of $Z_1 \longrightarrow Z_1'$ over y has positive dimension. Take an irreducible component of Z_1' ;it is not contained in Z_2', nor in Z_1'', i.e., z' has its generization $z'' \in Z_1' \setminus (Z_1'' \cup Z_2')$. To show $(Z_1', \bar{f}'_*(c))$ is a Chow class at z'', set $Y'' = Y' \setminus (Z_1'' \cup Z_2')$, $X'' = f'^{-1}(Y'')$, $f'' : X'' \longrightarrow Y''$ the base-change of f'. Then, we have

$$\bar{f}''_*(\text{restriction of } c \text{ to } H^p_{Z\cap X''}(X'', \Omega^p_{X'/S'}))$$
$$= \text{restriction of } \bar{f}'_*(c) \text{ in } H^{p-d}_{Z_1'\cap Y''}(Y'', \Omega^{p-d}_{Y''/S'}),$$

and $Z \cap X''$ is finite over Y''. In that case, the proof can be found in [1], 6.3.

3.8. With the notations and hypotheses in 3.2, let (Z,c) and (Z',c') be Chow classes. We have the sum $c + c'$ of c and c' by the natural maps

$$H^p_Z(X, \Omega^p_{X/S}) \longrightarrow H^p_{Z\cup Z'}(X, \Omega^p_{X/S}),$$
and
$$H^p_{Z'}(X, \Omega^p_{X/S}) \longrightarrow H^p_{Z\cup Z'}(X, \Omega^p_{X/S}),$$

respectively and $(Z \cup Z', c + c')$ is a Chow class, hence we get a morphism of functors $+ : C^p(X/S) \times C^p(X/S) \longrightarrow C^p(X/S)$. We extend it to the morphism of functors

$$+ : \bar{C}^p(X/S) \times \bar{C}^p(X/S) \longrightarrow \bar{C}^p(X/S)$$

as follows: it coincides with $+$ above on $C^p(X/S) \times C^p(X/S)$, and the first projection on $C^p(X/S) \times \{id_S\}$, the second projection on $\{id_S\} \times C^p(X/S)$, and the image of (id_S, id_S) is id_S. Therefore we obtain the morphism of algebraic spaces

$$+ : \bar{C}^p_{X/S} \underset{S}{\times} \bar{C}^p_{X/S} \longrightarrow \bar{C}^p_{X/S}.$$

§ 4. Genericity Theorem

4.1. In this section, the ground field k is supposed to be algebraically closed of characteristic zero and *uncountable*. Recall that we denote by $H_{\mathbb{Q}}$ the $\mathbb{Q}$-homological equivalence relation and we have the adequate equivalence relations $H_{\mathbb{Q}}^{*\ell}$ (See 1.5). The purpose of this section is to prove the following

THEOREM 4.2. *Let* V *be a smooth projective variety of dimension* m, S *a smooth variety,* ℓ *an integer and* Z *a cycle on* $S \times V$ *of codimension* p. *Assume that for an arbitrary closed point* $s \in S$, *the cycle* $Z(s)$ *is defined, and is* $H_{\mathbb{Q}}^{*\ell}$*-equivalent to zero. Then there exist a smooth variety* T, *a dominant morphism* $e : T \longrightarrow S$, *a smooth projective morphism* $\pi : \mathcal{F} \longrightarrow T$, *cycles* X_{ij} *of codimension* p_{ij} *on* $\mathcal{F} \times V$ ($1 \le i \le \ell$, $1 \le j \le k$) *such that*

i) $\sum_j p_{ij} = p + \dim \mathcal{F} - \dim T$;

ii) *For any* $t \in T$, $(j_t \times id_V)^*(X_{ij})$ *is* $\mathbb{Q}$*-homologous to zero on* $\mathcal{F}_t \times V$, *where* $j_t : \mathcal{F}_t \longrightarrow \mathcal{F}$ *is the inclusion.*

iii) $(e \times id_V)^*(Z) = \sum_j (\pi \times id_V)_*(X_{1j} . \cdots . X_{\ell j})$ *in* $CH(T \times V)$.

Let $\pi_\alpha : \mathcal{F}_\alpha \longrightarrow T_\alpha$ ($\alpha \in A$) be countable families of smooth projective morphisms such that T_α are affine algebraic schemes over k and that for any smooth projective variety W, there exist an $\alpha \in A$ and $t \in T_\alpha$ with $W \simeq (\mathcal{F}_\alpha)_t$.

For a smooth projective morphism $q : X \longrightarrow T$, integers $p_1, \cdots, p_\ell \ge 0$, let

$$\mathfrak{U} = \{(Z_1,\cdots,Z_\ell) \in C^{p_1}_{X/T} \times \cdots \times C^{p_\ell}_{X/T} \ ; \ (Z_1)_t,\cdots,(Z_\ell)_t \text{ intersect properly for } t \in T\}.$$

$\mathfrak{U}$ is an open subscheme of $\coprod_{i}{}_{/T}\, C^{p_i}_{X/T}$ and we have a morphism([1], 8.1).

$$\coprod_{i}{}_{/T}\, C^{p_i}_{X/T} \supset \mathfrak{U} \longrightarrow C^{\bar{p}}_{X/T}, \qquad \bar{p} = \sum_i p_i .$$
$$\psi \qquad\qquad \psi$$
$$(Z_1,\cdots,Z_\ell) \longmapsto Z_1 \dot{\underset{T}{}} \cdots \dot{\underset{T}{}} Z_\ell$$

By [1], 7.1.6, there is a morphism

$$\mathrm{Hilb}^{\times 2, H_{\mathbb{Q}}}_{X/T,r_i} \subset \mathrm{Hilb}^{p_i}_{X/T} \times \mathrm{Hilb}^{p_i}_{X/T} \longrightarrow C^{p_i}_{X/T} \times C^{p_i}_{X/T}$$

where $\mathrm{Hilb}^{\times 2, H_{\mathbb{Q}}}_{X/T,r_i}$ are defined in 2.4.3 (cf. also, 2.4; note that $\mathbb{Q}$-homological and $\mathbb{Q}_e$-homological equivalences coincide since we are in characteristic zero), r_i = rel.dim X/T - p_i, hence their product

$$\iota : \coprod_{i}{}_{/T}\, \mathrm{Hilb}^{\times 2, H_{\mathbb{Q}}}_{X/T,r_i} \longrightarrow \coprod_{i}{}_{/T}\, (C^{p_i}_{X/T} \times C^{p_i}_{X/T}).$$

Let $2^{[1,\ell]}$ be the set of maps from the interval $[1,\ell]$ of integers to the set $\{0,1\}$ and for $\sigma \in 2^{[1,\ell]}$, let

$$pr_\sigma : \coprod_{i}{}_{/T}\, (C^{p_i}_{X/T} \times C^{p_i}_{X/T}) \longrightarrow \coprod_{i}{}_{/T}\, C^{p_i}_{X/T}$$

be the product of projections $pr_{\sigma(i)}$ where $pr_{\sigma(i)}$ is the projection to the first factor if $\sigma(i) = 0$, and to the second factor if $\sigma(i) = 1$. Then we have

$$\bigcap_\sigma pr_\sigma^{-1}(\mathfrak{U}) \subset \coprod_{i}{}_{/T}\, (C^{p_i}_{X/T} \times C^{p_i}_{X/T}),$$

and,

$$\bigcap_\sigma \mathrm{pr}_\sigma^{-1}(\mathfrak{U}) \longrightarrow C^{\bar{p}}_{X/T} \times C^{\bar{p}}_{X/T}$$
$$\Psi \qquad\qquad\qquad \Psi$$
$$(Z^{(0)}_i, Z^{(1)}_i) \longmapsto \Big(\sum_{|\sigma|\equiv 0} \pi_i/_T\, Z^{(\sigma(i))}_i , \sum_{|\sigma|\equiv 1} \pi_i/_T\, Z^{(\sigma(i))}_i \Big),$$

where $|\sigma| = \sum_i \sigma(i)$, and $|\sigma| \equiv 0$ means that the summation is over all σ with even $|\sigma|$, and $|\sigma| \equiv 1$ means the summation over σ with odd $|\sigma|$.

Let $\mathscr{H}^{p_1,\cdots,p_\ell}_{X/T}$ be the pull-back of $\bigcap_\sigma \mathrm{pr}_\sigma^{-1}(\mathfrak{U})$ by the morphism ι.

Thus we get a morphism

$$\mathscr{H}^{p_1,\cdots,p_\ell}_{X/T} \longrightarrow C^{\bar{p}}_{X/T} \underset{T}{\times} C^{\bar{p}}_{X/T}.$$

Consider the morphisms

$$\mathscr{H}^{p}_{V\times\mathscr{F}_\alpha/S_\alpha} = \mathscr{H}^{p_1,\cdots,p_\ell}_{V\times\mathscr{F}_\alpha/S_\alpha} \longrightarrow C^{|p|}_{V\times\mathscr{F}_\alpha/S_\alpha} \underset{S_\alpha}{\times} C^{|p|}_{V\times\mathscr{F}_\alpha/S_\alpha} \longrightarrow \bar{C}^{p}_{V\times S_\alpha/S_\alpha} \times \bar{C}^{p}_{V\times S_\alpha/S_\alpha}$$
$$= \bar{C}^p_V \times \bar{C}^p_V \times S_\alpha \longrightarrow \bar{C}^p_V \times \bar{C}^p_V,$$

where $p = (p_1,\cdots,p_\ell)$, $|p| = \sum p_i$, $p = |p| - \mathrm{rel.dim}\, \mathscr{F}_\alpha/S_\alpha$, and $\bar{C}^p_{X/S} = C^p_{X/S} \coprod o(S)$ (cf. 3.4) and the second arrow is induced by the morphism $V \times \mathscr{F}_\alpha \longrightarrow V \times S_\alpha$.

For an integer $n \geq 1$ and a sequence of ℓ-tuples $p_1,\cdots,p_n$, putting

$$\mathscr{Q}^{p_1,\cdots,p_n}_\alpha = \coprod_j /S_\alpha\, \mathscr{H}^{p_j}_{V\times\mathscr{F}_\alpha/S_\alpha},$$

we get a morphism

$$\psi^{p_1,\cdots,p_n}_{V\times\mathscr{F}_\alpha/S_\alpha} = \psi^{p_1,\cdots,p_n}_\alpha : \mathscr{Q}^{p_1,\cdots,p_n}_\alpha \longrightarrow (\bar{C}^p_V \times \bar{C}^p_V)^{\times n} \longrightarrow \bar{C}^p_V \times \bar{C}^p_V,$$

the second arrow being the sum given by

$$((Y_1,Y'_1),\cdots,(Y_n,Y'_n)) \longmapsto \Big(\sum_i Y_i, \sum_i Y'_i\Big),$$

(cf. 3.8). For a k-rational point x of the left hand side, the image in $\bar{C}^p_V \times \bar{C}^p_V$ is given as follows:

Let s be the image of x in S_α. Then x consists of subschemes $(Z^{(0)}_{i,j}, Z^{(1)}_{i,j})$ of $V \times (\mathcal{F}_\alpha)_s$ of codimension $p_{i,j}$ (where $p_j = (p_{1,j}, \cdots, p_{\ell,j})$) such that the associated cycles to $Z^{(0)}_{i,j}$ and $Z^{(1)}_{i,j}$ are $\mathbb{Q}$-homologically equivalent on $V \times (\mathcal{F}_\alpha)_s$. The image of x in $\bar{C}^p_V \times \bar{C}^p_V$ corresponds to the pairs of cycles

$$\Big(\sum_j \sum_{|\sigma| \equiv 0} (\pi_\alpha)_*(Z^{(\sigma(1))}_{1,j} \cdot \ldots . Z^{(\sigma(\ell))}_{\ell,j}), \ \sum_j \sum_{|\sigma| \equiv 1} (\pi_\alpha)_*(Z^{(\sigma(1))}_{1,j} \cdot \ldots . Z^{(\sigma(\ell))}_{\ell,j}) \Big),$$

where for simplicity, we denote by $Z^{(\sigma(i))}_{i,j}$ the associated cycles on $V \times (\mathcal{F}_\alpha)_s$ to the subschemes $Z^{(\sigma(i))}_{i,j}$, and by π_α the morphism $\pi_\alpha : V \times (\mathcal{F}_\alpha)_s \longrightarrow V$. Since $\mathrm{Hilb}_{V \times \mathcal{F}_\alpha / S_\alpha, r} \longrightarrow C^p_{V \times \mathcal{F}_\alpha / S_\alpha}$ is surjective, any r-cycles on V which are $H^{*\ell}_{\mathbb{Q}}$-equivalent to zero can be written as the differences $z - z'$ of pairs (Z, Z') in this form for some α (cf. 1.6.2). We have a morphism defined by

$$C^p_V \times C^p_V \times \mathcal{Q}^{p_1, \cdots, p_n}_\alpha \xrightarrow{\ id \times \psi^{p_1, \cdots, p_n}_\alpha\ } C^p_V \times C^p_V \times \bar{C}^p_V \times \bar{C}^p \longrightarrow \bar{C}^p_V \times \bar{C}^p_V,$$
$$(Z_1, Z_2, Z_3, Z_4) \longmapsto (Z_2 + Z_3, Z_1 + Z_4)$$

denote by $\mathcal{R}^{p_1, \cdots, p_n}_\alpha$ the pull-back of the diagonal of $\bar{C}^p_V \times \bar{C}^p_V$, and consider the projection $\pi^{p_1, \cdots, p_n}_\alpha : \mathcal{R}^{p_1, \cdots, p_n}_\alpha \longrightarrow C^p_V \times C^p_V$. The union of the images for all $n, p_1 \cdots, p_n$ and α

$$\cup \operatorname{Im} \pi^{p_1, \cdots, p_n}_\alpha \subset C^p_V \times C^p_V$$

is the set of the pairs (Z, Z') of effective r-cycles which are $H^{*\ell}_{\mathbb{Q}}$-equivalent.

Since the set of possible $n, p_1, \cdots, p_n, \alpha$ is countable and the number of irreducible components of $\mathcal{R}^{p_1, \cdots, p_n}_{V \times \mathcal{F}_\alpha / S_\alpha}$ is countable, the above union is a countable union of irreducible subsets. Now, shrinking S if necessary, write Z as a difference of effective cycles which are non-degenerate on S: $Z = Z^{+} - Z^{-}$. It defines a morphism

$$\varphi : S \longrightarrow C_V^p \times C_V^p .$$

By hypotheses,

$$\operatorname{Im} \varphi \subset \bigcup \operatorname{Im} \pi_\alpha^{p_1,\cdots,p_n}$$

as k-rational point. Since the ground field k is uncountable, we can find n, $p_1,\cdots,p_n$, and α such that there exits a locally closed subvariety of $\mathcal{R}^{p_1,\cdots,p_n}_{V\times\mathcal{F}_\alpha/S_\alpha}$ such that the image of the restriction of $\pi_\alpha^{p_1,\cdots,p_n}$ to the subvariety contains the generic point of $\overline{\operatorname{Im}\varphi}$. Hence we have a diagram

$$\begin{array}{ccc}
S \underset{C_V^p \times C_V^p}{\times} \mathcal{R}^{p_1,\cdots,p_n}_{V\times\mathcal{F}_\alpha/S_\alpha} & \longrightarrow & \mathcal{R}^{p_1,\cdots,p_n}_{V\times\mathcal{F}_\alpha/S_\alpha} \\
\big\downarrow & & \big\downarrow \psi_\alpha^{p_1,\cdots,p_n} \\
S & \underset{\varphi}{\longrightarrow} & C_V^p \times C_V^p
\end{array}$$

and the left vertical arrow is dominant. There exist, therefore, a smooth affine variety T, and a dominant morphism $e : T \longrightarrow S$ which sit in the diagram

$$\begin{array}{ccc}
T & \longrightarrow & \mathcal{R}^{p_1,\cdots,p_n}_{V\times\mathcal{F}_\alpha/S_\alpha} \\
\big\downarrow e & & \big\downarrow \pi_\alpha^{p_1,\cdots,p_n} \\
S & \underset{\varphi}{\longrightarrow} & C_V^p \times C_V^p .
\end{array}$$

We have the morphism $T \longrightarrow S_\alpha$ and let

$$\mathcal{F} = \mathcal{F}_\alpha \underset{S_\alpha}{\times} T .$$

By base-change, we get an element ξ of $\mathcal{R}^{p_1,\cdots,p_n}_{V\times\mathcal{F}/T}(T)$ whose image by $\pi^{p_1,\cdots,p_n}_{V\times\mathcal{F}/T}$ is $\varphi\cdot e \in C_V^p \times C_V^p(T)$. Let the image of ξ under the morphism induced by ι

$$\mathcal{R}^{p_1,\cdots,p_n}_{\alpha} \longrightarrow \coprod_{i}{}_{/T} \coprod_{j}{}_{/T} \left(C^{p_{i,j}}_{V\times\mathcal{F}/T} \underset{T}{\times} C^{p_{i,j}}_{V\times\mathcal{F}/T}\right)$$

be $((Z^{(0)}_{i,j}, Z^{(1)}_{i,j}))$. If we denote the generic point of T by τ, the pull-backs $(Z^{(0)}_{i,j})_\tau$ and $(Z^{(1)}_{i,j})_\tau$ are the cycles on $V \times \mathcal{F}_\tau/\kappa(\tau)$. Let $\bar{Z}^{(0)}_{i,j}$ and $\bar{Z}^{(1)}_{i,j}$ be the closures of them in $V \times \mathcal{F}$, and put

$$X_{ij} = \bar{Z}^{(0)}_{i,j} - \bar{Z}^{(1)}_{i,j}.$$

Then X_{ij} and $e : T \longrightarrow S$ satisfy the conditions of the theorem.

§ 5. Definition of the functor

In the sequel, *the ground field is assumed to be the field of complex numbers.*

5.1. Recall the definition of coniveau filtration(cf. [15]): For a smooth variety, let

(5.1.1) $$N^pH^n(V,\mathbb{Q}) := \bigcup \operatorname{Im}(H^n_F(V,\mathbb{Q}) \longrightarrow H^n(V,\mathbb{Q}))$$
$$= \bigcup \operatorname{Ker}(\ H^n(V,\mathbb{Q}) \longrightarrow H^n(V\setminus F,\mathbb{Q})),$$

where F runs over the set of Zariski closed subset of V of codimension $\geq p$. $N^pH^n(V,\mathbb{Q})$ define a decreasing filtration of $H^n(V,\mathbb{Q})$ and we denote by $gr^pH^n(V,\mathbb{Q})$ the associated graded module:

$$gr^pH^n(V,\mathbb{Q}) = N^pH^n(V,\mathbb{Q})/N^{p+1}H^n(V,\mathbb{Q}).$$

We have $H^n(V,\mathbb{Q}) = N^0H^n(V,\mathbb{Q})$ and $N^pH^n(V,\mathbb{Q}) = 0$ if $n < 2p$. Note that $H^n(V,\mathbb{Q})$ has a mixed $\mathbb{Q}$-Hodge structure. In view of 5.1.1, $N^pH^n(V,\mathbb{Q})$ is a mixed Hodge sub-structure of $H^n(V,\mathbb{Q})$, and hence, $gr^pH^n(V,\mathbb{Q})$ has also a mixed $\mathbb{Q}$-Hodge structure. If V is projective, it is pure of weight n.

The coniveau filtration has the following functorial properties:

(i) for a morphism $f : V \longrightarrow W$, $N^pH^n(W,\mathbb{Q}) \subset H^n(W,\mathbb{Q})$ is mapped into $N^pH^n(V,\mathbb{Q})$ by the pull-back $f^* : N^n(W,\mathbb{Q}) \longrightarrow H^n(V,\mathbb{Q})$; hence f^* induces the map

$$f^* : gr^pH^n(W,\mathbb{Q}) \longrightarrow gr^pH^n(V,\mathbb{Q}).$$

(ii) for a proper morphism $f : V \longrightarrow W$, $N^pH^n(V,\mathbb{Q}) \subset H^n(V,\mathbb{Q})$ is mapped into $N^{p-d}H^{n-2d}(W,\mathbb{Q})(-d)$ by the push-forward $f^* : H^n(V,\mathbb{Q}) \longrightarrow H^{n-2d}(W,\mathbb{Q})(-d)$, where $d = \dim W - \dim V$. Hence f^* induces the map

$$f_* : gr^pH^N(V,\mathbb{Q}) \longrightarrow gr^{p-d}H^{n-2d}(W,\mathbb{Q})(-d).$$

(iii) The cup-product $\cup : H^n(V,\mathbb{Q}) \times H^{n'}(V,\mathbb{Q}) \longrightarrow H^{n+n'}(V,\mathbb{Q})$ maps $N^pH^n(V,\mathbb{Q}) \times N^{p'}H^{n'}(V,\mathbb{Q})$ into $N^{p+p'}H^{n+n'}(V,\mathbb{Q})$; hence we get

$$\cup : gr^pH^n(V,\mathbb{Q}) \times gr^{p'}H^{n'}(V,\mathbb{Q}) \longrightarrow gr^{p+p'}H^{n+n'}(V,\mathbb{Q}).$$

The fundamental class of an algebraic cycle z of codimension p on V will be denoted by $\{z\} \in gr^pH^{2p}(V,\mathbb{Q})(p) = N^pH^{2p}(V,\mathbb{Q})(p) \subset H^{2p}(V,\mathbb{Q})(p)$.

For smooth varieties T, V, with V projective, $\dim V = m$, and for $z \in CH^p(T \times V)$, ℓ an integer, $r = m - p$, we define a morphism of mixed Hodge structure

$$\{{}^tz\} : gr^rH^{2r+\ell}(V,\mathbb{Q})(r) \longrightarrow gr^0H^{\ell}(T,\mathbb{Q})$$

as the composite

$$gr^rH^{2r+\ell}(V,\mathbb{Q})(r) \xrightarrow{pr_V^*} gr^rH^{2r+\ell}(T \times V,\mathbb{Q})(r)$$
$$\longrightarrow gr^mH^{2m+\ell}(T \times V,\mathbb{Q})(m) \xrightarrow{pr_{T*}} gr^0H^{\ell}(T,\mathbb{Q}),$$

where the second map is defined by the cup-product with $\{{}^tz\} \in gr^pH^{2p}(T \times V,\mathbb{Q})(p)$.

THEOREM 5.2. *Let* V *be a smooth projective variety of dimension* m, S *a smooth variety,* $z \in CH^p(S \times V)$, $r = m - p$, *and* ℓ *an integer. If* $z(s) \in H_{\mathbb{Q}}^{*(\ell+1)}CH^p(V)$ *for all* $s \in S$, *then the map*

$$\{{}^tz\} : gr^rH^{2r+\ell}(V,\mathbb{Q})(r) \longrightarrow gr^0H^{\ell}(S,\mathbb{Q})$$

is zero.

Let Z be a cycle on $S \times V$ representing $z \in CH^P(S \times V)$. By shrinking S, if necessary, we may assume that $Z(s)$ are defined for all $s \in S$. Then $Z(s)$ are $H_{\mathbb{Q}}^{*(\ell+1)}$-equivalent to zero. By theorem

4.2, there exist a smooth variety T, a dominant morphism $e : T \longrightarrow S$, a smooth projective morphism $\pi : \mathcal{F} \longrightarrow T$, and cycles X_{ij} of codimension p_{ij} on $\mathcal{F} \times V$ ($0 \leq i \leq \ell$, $1 \leq j \leq n$) such that

$$\sum_j p_{ij} = \dim \mathcal{F} - \dim T + p;$$

For any $t \in T$, $X_{ij}|\mathcal{F}_t \times V$ are $\mathbb{Q}$-homologous to zero;

$$(e \times id_V)^*(Z) = \sum_j \pi_*(X_{0j}.\cdots.X_{\ell j}) \quad \text{in} \quad CH(T \times V).$$

We have a factorization

$$\{{}^t(e \times id_V)^*(Z)\} : gr^r H^{2r+\ell}(V,\mathbb{Q}(r)) \xrightarrow{\{{}^t z\}} gr^0 H^\ell(S,\mathbb{Q}) \xrightarrow{e^*} gr^0 H^\ell(T,\mathbb{Q}),$$

and e^* is injective(cf. [15], 1.7). The following lemma will complete the proof of the theorem:

LEMMA 5.2.1. *Let* T, X, V *be smooth varieties,* $g : X \longrightarrow V$ *be a morphism and* $f : X \longrightarrow T$ *a smooth proper morphism of relative dimension* m, Z_i ($0 \leq i \leq \ell$) *be cycles on* X *of codimension* p_i *such that the restriction of* Z_i *to a fiber* X_t *is* $\mathbb{Q}$-*homologically equivalent to zero. Put* $p = p_0 + \cdots + p_\ell$, $r = m - p$, *and*

$$z = \{Z_0\}\cup\cdots\cup\{Z_\ell\} \in H^{2p}(X,\mathbb{Q}(p)).$$

Then the map

$$H^{2r+\ell}(V,\mathbb{Q}(r)) \xrightarrow{g^*} H^{2r+\ell}(X,\mathbb{Q}(r)) \xrightarrow{\cup z} H^{2m+\ell}(X,\mathbb{Q}(m)) \xrightarrow{f_*} H^\ell(T,\mathbb{Q})$$

is zero.

We have the Leray spectral sequence

$$E_2^{p,n-p}(f) = H^p(T,R^{n-p}f_*\mathbb{Q}(k)) \Longrightarrow F^pH^n(X,\mathbb{Q}(k)).$$

(i) By intersection, we get a pairing of spectral sequence

$$\begin{array}{ccccc} H^p(T,R^{n-p}f_*\mathbb{Q}(k)) \times H^{p'}(T,R^{n'-p'}f_*\mathbb{Q}(k')) & \xrightarrow{\cup} & H^{p+p'}(T,R^{n+n'-p-p'}f_*\mathbb{Q}(k+k')) \\ \Downarrow & & \Downarrow \\ H^n(X,\mathbb{Q}(k)) \times H^{n'}(X,\mathbb{Q}(k')) & \xrightarrow{\cup} & H^{n+n'}(X,\mathbb{Q}(k+k')) \end{array}$$

in particular, we have

$F^pH^n(X,\mathbb{Q}(k)) \cup F^{p'}H^{n'}(X,\mathbb{Q}(k')) \subset F^{p+p'}H^{n+n'}(X,\mathbb{Q}(k+k'))$.

(ii) If $f' : X' \longrightarrow T$ is smooth of relative dimension m' and $h : X \longrightarrow X'$ is a proper T-morphism, and if $d = m - m'$, we have a morphism of spectral sequence

$$\begin{array}{ccc} E_2^{p,n-p}(f) = H^p(T,R^{n-p}f_*\mathbb{Q}(k)) & \Longrightarrow & F^pH^n(X,\mathbb{Q}(k)) \\ \downarrow h_* & & \downarrow h_* \\ E_2^{p,n-2d-p}(f') = H^p(T,R^{n-2d-p}f'\mathbb{Q}(k-d)) & \Longrightarrow & F^pH^{n-2d}(X',\mathbb{Q}(k-d)), \end{array}$$

in particular, $h_*\, F^pH^n(X',\mathbb{Q}(k)) \subset F^PH^{n-2d}(X',\mathbb{Q}(k-d))$.

By Lemma 2.4.1 (see also Remark 2.4.3.),

$$\{ Z_i \} \in F^1H^{2p_i}(X,\mathbb{Q}(p_i)).$$

For $\alpha \in H^{2r+\ell}(V,\mathbb{Q}(r))$, $g^*(\alpha) \in H^{2r+\ell}(X,\mathbb{Q}(r)) = F^0H^{2r+\ell}(X,\mathbb{Q}(r))$, and by (ii) and iterated use of (i), we obtain

$$f_*(z \cup g^*(\alpha)) = f_*(\{Z_0\}\cup\cdots\cup\{Z_\ell\}\cup g^*(\alpha)) \in F^{\ell+1}H^\ell(T,\mathbb{Q}) = 0,$$

hence, the lemma is proven.

5.3. For a smooth projective variety W and integers q, ℓ, we consider the condition:

$H(W,q,\ell)$: There exist smooth projective varieties T_j, and cycles u_j on $T_j \times W$ of codimension $\dim W - q$ such that

(i) the map

$$gr^qH^{2q+\ell}(W,\mathbb{Q}(q)) \longrightarrow \coprod_j gr^0H^\ell(T_j,\mathbb{Q})$$

induced by u_j is injective;

(ii) The following condition $H(T_j,\ell)$ holds for ℓ and all T_j.

$H(T,\ell)$: There exist smooth varieties S, $\mathcal{F}$, morphisms $\mathcal{F} \longrightarrow S$, and cycles $x_{1k},\cdots,x_{\ell k}$ $(1 \le k \le n,\ n \ge 1)$ on $\mathcal{F} \times T$ such that

i) $\mathcal{F} \longrightarrow S$ is smooth projective;

ii) $\sum_i \operatorname{codim} x_{ik} = \operatorname{rel.dim} \mathcal{F}/S + \dim T$, for all k;

iii) $x_{ik}|\mathcal{F}_s \times T$ are homologous to zero for all i, k and $s \in S$;

iv) The map

$$gr^0H^\ell(T,\mathbb{Q}) \longrightarrow gr^0H^\ell(S,\mathbb{Q})$$

induced by the cycle $\sum_k x_{1k}\cdot\cdots\cdot x_{\ell k}$ is injective.

(for $\ell = 0$, $\sum_k x_{1k}\cdot\cdots\cdot x_{\ell k} = n.1_{\mathcal{F}\times T} \neq 0$, and by ii), T must be a point; in that case, $H(T,0)$ always holds).

COROLLARY 5.4. *Let* V *and* W *be smooth projective varieties of dimension* m *and* n *respectively,* $z \in CH^{p+q}(W \times V)$, $r = m - p$, *and* ℓ *an integer, and suppose that the condition* $H(W,q,\ell)$ *holds. If the map*

$$[z] : Gr^\ell CH_q(W) \longrightarrow Gr^\ell CH_r(V)$$

(cf. 1.8) *is zero, then the map*

$$\{{}^tz\} : gr^rH^{2r+\ell}(V,\mathbb{Q}(r)) \longrightarrow gr^qH^{2q+\ell}(W,\mathbb{Q}(q))$$

is also zero.

With the notations of 5.3, we have

$$0 = [z\cdot u_j] : Gr^\ell CH_0(T_j) \xrightarrow{[u_j]} Gr^\ell CH_q(W) \xrightarrow{[z]} Gr^\ell CH_r(V)$$

and

$$\sum_j \{{}^t(z\cdot u_j)\} : gr^rH^{2r}(V,\mathbb{Q}(r)) \xrightarrow{\{{}^tz\}} gr^qH^{2q+\ell}(W,\mathbb{Q}(q))$$

$$\xrightarrow{\sum \{{}^t u_j\}} \coprod_j gr^0 H^{\ell}(T_j, \mathbb{Q}),$$

and since $\sum_j \{{}^t u\}$ is injective, we may assume that $H(W,\ell)$ holds and $q = 0$. The notation being as in the definition of $H(T,\ell)$ with $T = W$, let $x = \sum_k x_{1k} \cdot \cdots \cdot x_{\ell k}$. If $\pi : \mathcal{F} \longrightarrow S$ is the morphism, then we set $y = (\pi \times id_T)_*(x) \in CH(S \times T)$. For $s \in S$,

$$y(s) \in \langle H^{*\ell}\rangle_0 CH_0(T),$$

and $z \circ y(s) \in H^{*(\ell+1)} CH_r(V)$. By the theorem, we have that

$$\{{}^t(z\circ y)\} : gr^r H^{2r+\ell}(V,\mathbb{Q}(r)) \xrightarrow{\{{}^t z\}} gr^0 H^{\ell}(W,\mathbb{Q}(q)) \xrightarrow{\{{}^t y\}} gr^0 H^{\ell}(S,\mathbb{Q})$$

is zero. But by 5.3, iv), $\{{}^t y\}$ is injective, so that $\{{}^t z\}$ is zero.

5.5. We shall reformulate the corollary 5.4. To do so, we introduce a pseudo-abelian category $\mathcal{C}(\ell)$. First, we define an additive category $\mathcal{C}^*(\ell)_{\mathbb{Z}}$ as follows:

Objects: formal sum $\coprod_i Gr^{\ell} CH_{r_i}(V_i)$, where the condition $H(V_i, r_i, \ell)$ holds for each smooth projective variety V_i.

Morphisms : $Hom(\ Gr^{\ell}CH_q(W),\ Gr^{\ell}CH_r(V)) =$

$= \{\ [z] : Gr^{\ell}CH_q(W) \longrightarrow Gr^{\ell}CH_r(V)\ ;\ z \in CH^{p+q}(W \times V),\ p + r = \dim V\ \}$

and for general objects, we define

$$Hom(\coprod_j Gr^{\ell}CH_{q_j}(W_j), \coprod_i Gr^{\ell}CH_{r_i}(V_i))$$

$$= \coprod_{i,j} Hom(Gr^{\ell}CH_{q_j}(W_j), Gr^{\ell}CH_{r_i}(V_i)).$$

It is clear that $\mathcal{C}^*(\ell)_{\mathbb{Z}}$ is an additive category, and we define a $\mathbb{Q}$-additive category $\mathcal{C}^*(\ell)$ having the same objects as $\mathcal{C}^*(\ell)_{\mathbb{Z}}$ and

$$Hom_{\mathcal{C}^*(\ell)}(M,N) = Hom_{\mathcal{C}^*(\ell)_{\mathbb{Z}}}(M,N) \otimes \mathbb{Q}.$$

Then the pseudo-abelian category $\mathcal{C}(\ell)$ is obtained as the pseudo-abelian envelope of $\mathcal{C}^*(\ell)$.

Let Hdg be the category of polarizable $\mathbb{Q}$-Hodge structures and $\mathrm{Hdg}(\ell)$ be the full subcategory of Hdg whose objects are effective of weight ℓ. As noted above, $\mathrm{gr}^r H^{2r+\ell}(V,\mathbb{Q}(r)) \in \mathrm{Hdg}(\ell)$. By Corollary 5.4, we have

COROLLARY 5.6. *We have an additive contravariant functor*

$$\eta : \mathcal{C}(\ell) \longrightarrow \mathrm{Hdg}(\ell), \quad \mathrm{Gr}^{\ell}\mathrm{CH}_r(V) \longmapsto \mathrm{gr}^r H^{2r+\ell}(V,\mathbb{Q}(r)).$$

LEMMA 5.7. (i) *If* z *is a cycle of codimension* $q + \dim W' - q'$ *on the product* $W \times W'$ *of smooth projective varieties such that the induced map*

$$({}^t z) : \mathrm{gr}^{q'} H^{2q'+\ell}(W',\mathbb{Q}(q)) \longrightarrow \mathrm{gr}^q H^{2q+\ell}(W,\mathbb{Q}(q))$$

is injective and if the condition $H(W,q,\ell)$ *holds, then the condition* $H(W',q',\ell)$ *also holds.*

(ii) *If* $H(T,\ell)$ *holds, then* $H(T,0,\ell)$ *also holds.*

For (i), let T_j's and u_j's be as in 5.3 (for $H(W,q,\ell)$). Then, the map

$$\mathrm{gr}^{q'} H^{2q'+\ell}(W',\mathbb{Q}(q')) \xrightarrow{({}^t z)} \mathrm{gr}^q H^{2q+\ell}(W,\mathbb{Q}(q)) \longrightarrow \coprod_j \mathrm{gr}^0 H^{\ell}(T_j,\mathbb{Q})$$

induced by $z \circ u_j$ is injective, and $H(T_j,\ell)$ hold for all T_j.

(ii) is trivial by taking the diagonal as $u = u_1$ in the definition of $H(T,0,\ell)$.

PROPOSITION 5.8. *For a smooth projective variety* V, *the condition* $H(V,0,1)$ *holds.*

By virtue of 5.7, the weak Lefschetz theorem, and the fact that $gr^0H^1 = H^1$, we are reduced to proving $H(T,1)$ where T is a curve (cf. proof of 5.9). Let $S = T$, $\mathcal{F} = T \times T$, $pr_2 : \mathcal{F} \longrightarrow S$ the second projection, $t \in T$ a point, and let $x = t \times \Delta_T - t \times T \times t$, a cycle on $\mathcal{F} \times T$ of codimension 2 . Then the conditions i) - iv) of 5.3 are satisfied.

PROPOSITION 5.9. *For a smooth projective variety* V, *the condition* $H(V,0,2)$ *holds.*

Let $i : V' \hookrightarrow V$ be a smooth hyperplane section. Then

$$i^* : H^2(V,\mathbb{Q}) \longrightarrow H^2(V',\mathbb{Q})$$

is injective if $\dim V' \geq 2$. Note that

$$N^1H^2(V,\mathbb{Q}) = H^2(V,\mathbb{Q}) \cap H^{1,1}(V) = \mathbb{Q}(-1)^{\oplus \rho}.$$

Since $Hom_{Hdg}(gr^0H^2(V,\mathbb{Q}),\mathbb{Q}(-1)) = 0 = Hom_{Hdg}(\mathbb{Q}(-1),gr^0H^2(V,\mathbb{Q}))$, we have the canonical decomposition

$$H^2(V,\mathbb{Q}) = gr^0H^2(V,\mathbb{Q}) \oplus gr^1H^2(V,\mathbb{Q})$$

and $i^* : gr^0H^2(V,\mathbb{Q}) \longrightarrow gr^0H^2(V',\mathbb{Q})$ is also injective. Therefore, there exist a surface S, and $j : S \longrightarrow V$ such that the map

$$j^* : gr^0H^2(V,\mathbb{Q}) \longrightarrow gr^0H^2(S,\mathbb{Q})$$

is injective. Then, by lemma 5.7, it suffices to show $H(S,0,2)$.

If $b : S' \longrightarrow S$ is surjective, $b^* : gr^0H^2(S,\mathbb{Q}) \longrightarrow gr^0H^2(S',\mathbb{Q})$ is injective, hence by 5.7, we can suppose S has a fibration $\pi : S \longrightarrow C$ over a curve C with smooth generic fibre, and a section $\sigma : C \longrightarrow S$.

LEMMA 5.9.1 *Let* S *be a smooth projective surface. Then there exists a 2-cycle* Z *on* $S \times S$ *with* $\mathbb{Q}$*-coefficients inducing the projector* $H^{\cdot}(S,\mathbb{Q}) \longrightarrow gr^0 H^2(S,\mathbb{Q})$, i.e., *the induced map* $H^n(S,\mathbb{Q}) \longrightarrow H^n(S,\mathbb{Q})$ *are zero for* $n \neq 2$, $N^1H^2(S,\mathbb{Q}) \longrightarrow N^1H^2(S,\mathbb{Q})$ *is also zero, and the map* $gr^0H^2(S,\mathbb{Q}) \longrightarrow gr^0H^2(S,\mathbb{Q})$ *is the identity.*

Sketch of proof. Let $\mathcal{V}$ be the full subcategory of smooth projective schemes consisting of schemes whose components V satisfy the condition $B(V)$ of [9](See 7.9.1). Note that the condition B is stable under product, that the Künneth components of the class of the diagonal of V are algebraic (loc. cit., 2.5, 2.9), that the condition $I(V,L)$ (loc. cit.) holds for those schemes by the Hodge theory, and that all the curves and all the surfaces (and all the abelian varieties) belong to $\mathcal{V}$. Starting from $\mathcal{V}$, employing algebraic cycles modulo numerical equivalence as morphisms, we can construct the category $\mathcal{M}$ of true motives as in [12]. The category $\mathcal{M}$ is semi-simple, and we have a faithful (tensor) functor

$$H : \mathcal{M} \longrightarrow \mathrm{Hdg}$$

with $H(h^n(V)) = H^n(V,\mathbb{Q})$, the Betti realization. By [8], there exist a finite number of curves $C_1, \cdots, C_k$ and morphisms $\varphi_i : C_i \longrightarrow S$ such that the image of

$$\coprod_i H^0(C_i,\mathbb{Q}) \xrightarrow{\sum \varphi_{i*}} H^2(S,\mathbb{Q})(1)$$

is $N^1H^2(S,\mathbb{Q})(1)$. Since C_i, $S \in \mathrm{Ob}\,\mathcal{V}$, we have as well

$$\coprod_i h^0(C_i) \xrightarrow{\sum \varphi_{i*}} h^2(S)(1).$$

in $\mathcal{M}$. Denote the image by I. Since the category $\mathcal{M}$ is semi-simple, we have the projector $p : h^2(S)(1) \longrightarrow I \subset h^2(S)(1)$.

The composite of the morphism $h(S)(1) \longrightarrow h^2(S)(1)$ with $id - p$ is represented by a 2-cycle with $\mathbb{Q}$-coefficient on $S \times S$ which has the required properties, by considering the Betti realization.

LEMMA 5.9.2. *For a surface* S *which has a fibration* $\pi : S \longrightarrow T$ *over a curve with smooth generic fiber and a section* $\sigma : C \longrightarrow S$, *the condition* $H(S,2)$ *holds.*

Let C_0 be an open subset of C such that $\pi_0 : S_0 := \pi^{-1}(C_0) \longrightarrow C_0$ is smooth, and set $\mathcal{F} = S_0 \underset{C}{\times} S$. We have the projections $\pi_1 : \mathcal{F} \longrightarrow S_0$ and $\pi_2 : \mathcal{F} \longrightarrow S$, and put

$$\bar{\pi}_1 = \pi_1 \times id : \mathcal{F} \times S \longrightarrow S_0 \times S, \text{ and}$$
$$\bar{\pi}_2 = \pi_2 \times id : \mathcal{F} \times S \longrightarrow S \times S.$$

Note that π_1 is smooth projective, so that $\mathcal{F}$ is smooth.
Let Z be the cycle with $\mathbb{Q}$-coefficients as in 5.7.1, and let N be a sufficiently large integer > 0 such that $Z_1 = N.{}^tZ$ has $\mathbb{Z}$-coefficients, and put $X_1 = \bar{\pi}_2{}^*(Z_1)$, a 3-cycle on $\mathcal{F} \times S$. To the C-morphisms $\psi_1 : S_0 \longrightarrow S$, the inclusion, and $\psi_2 = \sigma \cdot \pi : S_0 \longrightarrow S$, there correspond the morphisms $\tau_1, \tau_2 : S_0 \longrightarrow \mathcal{F}$, and $\bar{\tau}_1, \bar{\tau}_2 : S_0 \times S \longrightarrow \mathcal{F} \times S$, the base-changes. Finally, we set

$$X_2 = \bar{\tau}_{1*}(1_{S_0 \times S}) - \bar{\tau}_{2*}(1_{S_0 \times S}).$$

For $s \in S_0$, putting $c = \pi(s)$, $\mathcal{F}_s = s \times S_c = S_c$, and we have

$$j^*(X_2) = \psi_1(s) \times S - \psi_2(s) \times S,$$

where $j : S_c \times S = \mathcal{F}_s \times S \longrightarrow \mathcal{F} \times S$, and $j^*(X_2)$ is homologous to zero on $\mathcal{F}_s \times S$. Denoting the natural inclusion $S_c \times S \longrightarrow S \times S$ by j', we have $j^*(X_1) = (\bar{\pi}_2 \cdot j)^*(Z_1) = j'^*(Z_1) = Z_1 | S_c \times S$.

In the Künneth decomposition

$$H^4(S \times S, \mathbb{Q}(2)) = \coprod_{0 \le i \le 4} H^{4-i}(S,\mathbb{Q}) \otimes H^i(S,\mathbb{Q})(2),$$

Z_1 has no other than $H^2(S) \otimes H^2(S)$-component, and further, in the decomposition

$$\begin{aligned} H^2(S,\mathbb{Q}) \otimes H^2(S,\mathbb{Q})(2) &= gr^0H^2(S,\mathbb{Q}) \otimes gr^0H^2(S,\mathbb{Q})(2) \oplus gr^0H^2(S,\mathbb{Q}) \otimes gr^1H^2(S,\mathbb{Q})(2) \\ &\oplus gr^1H^2(S,\mathbb{Q}) \otimes gr^0H^2(S,\mathbb{Q})(2) \oplus gr^1H^2(S,\mathbb{Q}) \otimes gr^1H^2(S,\mathbb{Q})(2), \end{aligned}$$

Z_1 has only $gr^0H^2 \otimes gr^0H^2$-component. Hence $Z_1 | S_c \times S$ is $\mathbb{Q}$-homologous to zero, by $gr^0H^2(S_c,\mathbb{Q}) = 0$. Taking N larger if necessary, we may assume that it is $\mathbb{Z}$-homologous to zero.

We claim that

$$\{{}^t(\bar{\pi}_{1*}(X_1.X_2))\} = N.\psi_1{}^* : gr^0H^2(S,\mathbb{Q}) \longrightarrow gr^0H^2(S_0,\mathbb{Q}),$$

hence, injective. We have $X_1.X_2 = X_1.\bar{\tau}_{1*}(1_{S_0 \times S}) - X_1.\bar{\tau}_{2*}(1_{S_0 \times S})$, and

$$\bar{\pi}_{1*}(X_1.\bar{\tau}_{i*}(1_{S_0 \times S})) = \bar{\pi}_{1*}(\bar{\tau}_{i*}(\bar{\tau}_i{}^* \bar{\pi}_2{}^*(Z_1))) = (\psi_i \times id_S)^*(Z_1),$$

for $i = 1, 2$. Therefore,

$$\{{}^t(\pi_{1*}(X_1.X_2))\} : gr^0H^2(S,\mathbb{Q}) \xrightarrow{\{{}^tZ_1\}} gr^0H^2(S,\mathbb{Q}) \xrightarrow{\psi_1{}^* - \psi_2{}^*} gr^0H^2(S_0,\mathbb{Q}).$$

On $gr^0H^2(S,\mathbb{Q})$, $\{{}^tZ_1\} = N.id$, $\psi_2{}^* = 0$ because

$$\psi_2{}^* : gr^0H^2(S,\mathbb{Q}) \xrightarrow{\sigma^*} gr^0H^2(C,\mathbb{Q}) = 0 \xrightarrow{\pi^*} gr^0H^2(S_0,\mathbb{Q}).$$

This completes the proof of 5.9.2 and hence that of 5.9.

§ 6. Faithfulness : Preliminary

LEMMA 6.1. *Let* T *and* V *be smooth projective varieties and*

$$H^{2p}(T\times V,\mathbb{Q})(p) = \bigoplus_{\ell} H^{\ell}(T,\mathbb{Q})\otimes H^{2p-\ell}(V,\mathbb{Q})(p)$$

be the Künneth decomposition, and let $z \in N^pH^{2p}(T\times V,\mathbb{Q})(p)$ *be decomposed into*

$$z = \sum_{\ell} z_{\ell},\qquad z_{\ell}\in H^{\ell}(T,\mathbb{Q})\otimes H^{2p-\ell}(V,\mathbb{Q})(p).$$

Then $\quad z_{\ell}\in H^{\ell}(T,\mathbb{Q})\otimes N^{p-\ell}H^{2p-\ell}(V,\mathbb{Q})(p)$ *for all* ℓ.

Let $e_1,\cdots,e_b \in H^{\ell}(T)$ be a basis, and $e_1^*,\cdots, e_b^*\in H^{2t-\ell}(T)$ be its dual basis: $\langle e_i, e_j^*\rangle = \delta_{ij}$ ($t = \dim T$. We omit the coefficients and twists). If $z_{\ell} = \sum_j e_j\otimes x_j$, $x_j \in H^{2p-\ell}(V)$, then

$$x_j = pr_{V*}(\, e_j^*\otimes 1 \cup z_{\ell}),$$

and $e_j^*\otimes 1\cup z_k \in H^{2t-\ell+k}(T)\otimes H^{2p-\ell}(V)$, so that $pr_{V*}(\, e_j^*\otimes 1\cup z_k) = 0$ for $k\neq \ell$. Hence, we have

$$x_j = pr_{V*}(\, e_j^*\otimes 1\cup z).$$

If $t\leq \ell$, then,

$$e_j^*\otimes 1\cup z\in N^pH^{2t-\ell+2p}(T\times V),$$

and $x_j = pr_{V*}(\, e_j^*\otimes 1\cup z)\in N^{p-t}H^{2p-\ell}(V)\in N^{p-\ell}H^{2p-\ell}(V)$.

If $t > \ell$, then $H^{2t-\ell}(T) = N^{t-\ell}H^{2t-\ell}(T)$, and

$$e_j^*\otimes 1\cup z\in N^{p+t-\ell}H^{2t-\ell+2p}(T\times V),$$

hence, $\quad x_j\in N^{p-\ell}H^{2p-\ell}(V)$.

LEMMA 6.2. *Under the hypotheses of lemma* 6.1, *we fix an* ℓ *and we assume further that*

(i) $D(V,r,\ell)$: *the intersection pairing*

$$N^rH^{2r+\ell}(V,\mathbb{Q})(r) \otimes N^{p-\ell}H^{2p-\ell}(V,\mathbb{Q})(p) \longrightarrow$$
$$\longrightarrow H^{2r+\ell}(V,\mathbb{Q}) \otimes H^{2p-\ell}(V,\mathbb{Q})(p) \longrightarrow H^{2m}(V,\mathbb{Q})(m) = \mathbf{Q}$$

is perfect ($p + r = m = \dim V$);

(ii) *the map*

$$\{{}^tz\} : gr^rH^{2r+\ell}(V,\mathbb{Q})(r) \longrightarrow gr^0H^\ell(T,\mathbb{Q})$$

is zero.

If $\ell \leq 2$, *then* $N.z_\ell$ *is represented by an algebraic cycle degenerate on* T *for some* $N \neq 0$.

The assumption (ii) implies $\{{}^tz\}(N^rH^{2r+\ell}(V)) \subset N^1H^\ell(T)$. By (i) and lemma 6.1, $z_\ell \in N^1H^\ell(T) \otimes N^{p-\ell}H^{2p-\ell}(V)$, so that if $\ell < 2$, then $z_\ell = 0$, trivially. Let $\ell = 2$, and $e_1,\cdots,e_\rho \in N^1H^2(T)$ and $e_1^*,\cdots,e_\rho^* \in N^{t-1}H^{2t-2}(T)$ be dual bases(By Lefschetz, $N^1H^2(T)$ and $N^{t-1}H^{2t-2}(T)$ are dual via intersection). Let $z_2 = \sum_j e_j \otimes x_j$, $x_j \in H^{2p-2}(V)$. Then as in the proof of lemma 6.1, we have

$$x_j = pr_{V*}(\, e_j^* \otimes 1 \cup z\,) \in N^{p-1}H^{2p-2}(V),$$

and $z_2 \in N^1H^2(T) \otimes N^{p-1}H^{2p-2}(V)$.

REMARK 6.2.1. For $\ell = 1$, *without* the condition $D(V,r,\ell)$, we have: if the map

$$\{\, z\,\} : gr^0H^1(T,\mathbb{Q})(1) \longrightarrow gr^{p-1}H^{2p-1}(V,\mathbb{Q})(p)$$

is zero, then $N.z_1$ is represented by an algebraic cycle degenerate on T for some $N \neq 0$.

The proof is similar to that of 6.2.

REMARK 6.3. If we assume the condition (i) of lemma 6.2 for $\ell = 0$

universally, i.e., for any V, r, then we can get rid of the restriction $\ell \leq 2$. In fact, the universal validity of the condition (i) means that the condition D of [9] holds universally. Since we are in the field of complex numbers, the condition I(X,L) holds universally, hence we can assume the standard conjecture.

LEMMA 6.3.1. *We assume the standard conjecture for all varieties over the field of complex numbers. Let* $F \subset V$ *be a Zariski closed subset of a smooth projective variety,* Z *an algebraic cycle of codimension* p *on* V *such that* $Z \mid V \setminus F$ *is homologous to zero on* $V \setminus F$. *Then* Z *is* $\mathbb{Q}$*-homologically equivalent to a cycle supported by* F.

Let $F_1, \cdots, F_s$ be the irreducible components of F. Then denoting the resolutions of F_i by $\varphi_i : \tilde{F}_i \longrightarrow F_i \subset V$ and the codimension of F_i in V by p_i, we have an exact sequence

$$\oplus H^{2p-2p_i}(\tilde{F}_i,\mathbb{Q})(p-p_i) \xrightarrow{\oplus \varphi_i} H^{2p}(V,\mathbb{Q})(p) \longrightarrow H^{2p}(V \setminus F,\mathbb{Q})(p),$$

[8],8.2.8. Since we suppose the standard conjecture, we have also the theory of motives, and a morphism of motives

$$\oplus h^{2p-2p_i}(\tilde{F}_i)(p-p_i) \xrightarrow{\oplus \varphi_i} h^{2p}(V)(p).$$

The above map is the Betti realization of this morphism. Let I be its image. By semi-simplicity of the category of motives, I is a direct summand of $h^{2p}(V)(p)$, and there is a morphism $\sigma : I \longrightarrow \oplus h^{2p-2p_i}(\tilde{F}_i)(p-p_i)$ with $(\oplus \varphi_i)\cdot\sigma = id_I$. σ is induced by an algebraic cycles on $V \times \tilde{F}_i$. Considering the Betti-realization, we have a map

$$\sigma' : H^{2p}(V,\mathbb{Q})(p) \longrightarrow \oplus H^{2p-2p_i}(\tilde{F}_i\mathbb{Q})(p-p_i)$$

induced by algebraic correspondence with $(\oplus \varphi_i)\cdot\sigma' = \mathrm{id}$ on $H_B(I)$. If $\{ z \} \in H^{2p}(V,\mathbb{Q})(p)$ is the fundamental class of Z, then, $\{ z \} \in H_B(I)$ by hypotheses, and $\sigma'(\{ z \}) \in \oplus H^{2p-2p_i}(\tilde{F}_i,\mathbb{Q})(p-p_i)$ is then represented by algebraic cycles Y_i. The algebraic cycle $\sum \varphi_{i*}(Y_i)$ supported by F is $\mathbb{Q}$-homologically equivalent to Z.

Returning to the proof of the remark, z_ℓ is algebraic and $z_\ell \in N^1H^\ell(T) \otimes N^{p-\ell}H^{2p-\ell}(V)$ as in the proof of 6.2, i.e., there is a closed subset $F \neq T$ such that z_ℓ is homologous to zero on $(T \setminus F) \times V$, and z_ℓ is $\mathbb{Q}$-homologically equivalent to a cycle on $T \times V$ supported by $F \times V$.

LEMMA 6.4. *Let* T *be a smooth projective variety of dimension* t, *and assume the condition* B(T) *of* [9] *holds. Let* Δ_T *be the diagonal and*

$$\{ \Delta_T \} = \sum_i \Delta_i \quad in \quad H^{2t}(T \times T,\mathbb{Q})(t)$$

be the Künneth decomposition, $\Delta_i \in H^i(T,\mathbb{Q}) \otimes H^{2t-i}(T,\mathbb{Q})(t)$, *and* $H \in CH^1(T)$ *be the hyperplane section. Then there exits an integer* $N > 0$ *such that* $N.\Delta_i$ *is represented by the algebraic cycle* $(H^{i-t}\times 1_T).\Delta'_i$ *for* $i \geq t$ *and is represented by* $(1_T \times H^{t-i}).\Delta''_i$ *for* $t > i$, *where* Δ'_i *and* Δ''_i *are algebraic cycles on* $T \times T$.

Let $h \in H^2(T,\mathbb{Q})(1)$ be the class of H. Then we have isomorphisms

$$L^{t-i} : H^i(T,\mathbb{Q}) \longrightarrow H^{2t-i}(T,\mathbb{Q})(t-i), \ x \longmapsto h^{t-i} \cup x$$

$(i \leq t)$, and the inverses are algebraic by B(T). If $i \geq t$, then $\Delta_i = (L^{i-t}\otimes \mathrm{id})(((L^{i-t})^{-1}\otimes \mathrm{id})(\Delta_i))$, and

$((L^{i-t})^{-1}\otimes \mathrm{id})(\Delta_i) \in H^{2t-i}(T,\mathbb{Q}) \otimes H^{2t-i}(T,\mathbb{Q})(2t-i)$ is algebraic, hence there is an algebraic cycle Δ'_i on $T \times T$ which represents $((L^{i-t})^{-1}\otimes \mathrm{id})(N.\Delta_i)$. The case of $i < t$ is similar.

COROLLARY 6.5. *Let* T *and* V *be smooth projective varieties of dimension* $\ell \leq 2$ *and* m, *respectively,* $z \in CH^P(T \times V)$, *and assume that the condition* $D(V,r,\ell)$ *of* 6.2 *holds for* V, *and that*

$$0 = ({}^t z) : gr^r H^{2r+\ell}(V,\mathbb{Q})(r) \longrightarrow gr^0 H^{\ell}(T,\mathbb{Q}).$$

Then for some $N \neq 0$, *we have*

$$0 = N.[\ z\ ] : Gr^{\ell}CH_0(T) \longrightarrow Gr^{\ell}CH_r(V).$$

Since $\ell \leq 2$, the condition $B(T)$ holds, hence, we have cycles Δ'_i, Δ''_i on $T \times T$, an integer N as in 6.4, and $z_i = z \circ \Delta_i$ is the Künneth components of z. If $i > \ell$, $N.z_i = z\circ((H^{i-\ell}\otimes 1_T).\Delta_i')$ induces the map

$$Gr^{\ell}CH_0(T) \xrightarrow{H^{i-\ell}} Gr^{\ell}CH_{-(i-\ell)}(T) \xrightarrow{\Delta'_i} Gr^{\ell}CH_0(T) \xrightarrow{[\ z\ ]} Gr^{\ell}CH_r(V),$$

which vanishes by dimension reason. If $i < \ell$, then $N.z_i = z \circ ((1_T \times H^{\ell-i}).\Delta''_i)$ gives the map

$$Gr^{\ell}CH_0(T) \xrightarrow{\Delta''_i} Gr^{\ell}\ CH^i(T) \xrightarrow{H^{\ell-i}} Gr^{\ell}CH_0(T) \xrightarrow{[\ z\ ]} Gr^{\ell}CH_r(V).$$

Since $0 \leq i < \ell \leq 2$, $Gr^{\ell}CH^i(T) = 0$ (cf. 1.11), $[N.z_i] = 0$. Taking N large enough, we may assume that $N.z_{\ell}$ is represented by an algebraic cycle on $T \times V$ degenerate on T. Hence we have

$$N.[\ z\ ] = [\ N.z_{\ell}\ ] : Gr^{\ell}CH_0(T) \longrightarrow Gr^{\ell}CH_r(V).$$

REMARK 6.5.1. For $\ell = 1$, in correspondence with 6.2.1, by similar proof, we have:

if

$$0 = \{ z \} : gr^0 H^1(T,\mathbb{Q})(1) \longrightarrow gr^{p-1} H^{2p-1}(V,\mathbb{Q})(p),$$

then, for some $N \neq 0$,

$$0 = N.[z] : Gr^1 CH_0(T) \longrightarrow Gr^1 CH^p(V)$$

LEMMA 6.6. *Let* $\mathcal{A}$ *be an additive category, and* $\mathcal{A}'$ *its full subcategory,* $\mathcal{B}$, *and* $\mathcal{B}'$ *be their respective pseudo-abelian envelopes. Then, the canonical functor* $\mathcal{B}' \longrightarrow \mathcal{B}$ *is fully faithful. We identify* $\mathcal{B}'$ *with its image and we shall say that the pseudo-abelian subcategory* $\mathcal{B}'$ *is generated by the objects of* $\mathcal{A}'$.

The proof is straightforward, and omitted.

6.7. Let $\mathscr{C}'(0)$ be the pseudo-abelian subcategory of $\mathscr{C}(0)$ generated by $Gr^0 CH_r(V)$ with the condition D(V,r,0), and we have the restriction $\mathscr{C}'(0) \subset \mathscr{C}(0) \xrightarrow{\eta} Hdg(0)$, which we denote also by η.

PROPOSITION 6.7.1. *The functor* $\eta : \mathscr{C}'(0) \longrightarrow Hdg(0)$ *is faithful.*

Note that $gr^r H^{2r}(V,\mathbb{Q}) = Gr^0 CH^r(V) \otimes \mathbb{Q}$, and the proposition results from duality of vector spaces by the condition D(V,r,0).

REMARK 6.8. The condition H(V,r,0) means that the map

$$gr^r H^{2r}(V,\mathbb{Q})(r) \longrightarrow (gr^p H^{2p}(V,\mathbb{Q}))^*$$

induced by intersection is injective. The condition D(V,r,0) is that this is bijective, hence D(V,r,0) ===> H(V,r,0). Incidentally, the fully-faithfulness of η in 6.7 is equivalent to the Hodge conjecture for cycles of codimension r of varieties V with $D(V,r,0)$.

§ 7. Faithfulness : case of niveau 1

Theorem 7.1. *Let* V *and* T *be a smooth projective varieties,* $z \in CH^{p+q}(T \times V)$. *If the map*

$$\{ z \} : gr^{q-1}H^{2q-1}(T,\mathbb{Q}) \longrightarrow gr^{p-1}H^{2p-1}(V,\mathbb{Q})$$

is zero, then the map

$$[z] : Gr^1CH_q(T) \longrightarrow Gr^1CH^p(V)$$

is also zero.

Let $t = \dim T$ and $q' = t - q$. To see $[z] = 0$, it suffices to show that for any curve C and for any $u \in CH^{q'}(C \times T)$, the composite

$$[z \circ u] : Gr^1CH_0(C) \xrightarrow{[u]} Gr^1CH_q(T) \xrightarrow{[z]} Gr^1CH_r(V)$$

vanishes because $\langle H\rangle_0CH_q(T) = ACH_q(T)$ is generated by $u(ACH_0(C))$ for all C and u. But we have

$$gr^0H^1(C,\mathbb{Q}) \xrightarrow{[u]} gr^{q-1}H^{2q-1}(T,\mathbb{Q}) \xrightarrow{[z]} gr^{p-1}H^{2p-1}(V,\mathbb{Q})$$

is zero by hypothesis. Since $Gr^1CH_0(C)$ is divisible, $[z \circ u] = 0$ by 6.2.1.

COROLLARY 7.2. *Let* T *and* V *be smooth projective varieties,* $z \in CH^{p+q}(T \times V)$, $m = \dim V$, *and* $r = m - p$. *Assume the condition* D(V,r,1) *of* 6.2 *holds. If the map*

$$\{ {}^tz \} : gr^rH^{2r+1}(V,\mathbb{Q})(r) \longrightarrow gr^qH^{2q+1}(T,\mathbb{Q})(q)$$

is zero, then the map

$$[z] : Gr^1CH_q(T) \longrightarrow Gr^1CH_r(V)$$

is also zero.

As in the proof of 7.1, we may assume that $q = 0$, and $\dim T = 1$. Note that $D(T,1,1)$ always holds, and we have $N^{p-1}H^{2p-1}(V,\mathbb{Q}) \simeq gr^{p-1}H^{2p-1}(V,\mathbb{Q})$. The condition $D(V,r,1)$ means that the map $\{{}^t z\}$ above is dual to the map

$$\{ z \} : gr^0H^1(T,\mathbb{Q})(1) \longrightarrow gr^{p-1}H^{2p-1}(V,\mathbb{Q})(p),$$

and we conclude by 7.1.

7.3. Let Pic^pV denote the higher Picard variety in the sense of [13]. Then we have the canonical map([13], 5.1)

$$\pi_V^{(p)} : J_a^p(V) \longrightarrow Pic^p(V).$$

LEMMA 7.3.1. *For a variety* V, *integers* p, r *with* $p + r = m = \dim V$, consider the following conditions:

(i) *The condition* $H(V,r,1)$ *holds, i.e.,* $Gr^1CH_r(V) \in ob\ \mathcal{C}(1)$.

(ii) *the condition* $D(V,r,1)$ *holds, i.e., the pairing*

$$N^rH^{2r+1}(V,\mathbb{Q})(r) \times N^{p-1}H^{2p-1}(V,\mathbb{Q})(p) \subset H^{2r+1}(V,\mathbb{Q})(r) \times H^{2p-1}(V,\mathbb{Q})(p) \xrightarrow{\ \cup\ } H^{2m}(V,\mathbb{Q})(m) = \mathbb{Q}$$

is perfect.

(iii) *the canonical map* $\pi_V^{(p)} : J_a^p(V) \longrightarrow Pic^p(V)$ *is an isogeny.*

(iv) *the canonical map* $\pi_V^{(r+1)} : J_a^{r+1}(V) \longrightarrow Pic^{r+1}(V)$ *is an isogeny.*

Then we have the implications (ii) <===> ((iii) and (iv)), *and* (i) <===> (iv).

By [13], 1.2, since the Abel-Jacobi map $\alpha : ACH^p(V) \longrightarrow J_a^p(V)$ is regular, we have a cycle $\mathfrak{P}$ on $J_a^p(V) \times V$ of codimension p such that the induced map

$$h_{\mathfrak{P}} : J_a^p(V) \xrightarrow{\mathfrak{P}((?)-(0))} ACH^p(V) \xrightarrow{\ \alpha\ } J_a^p(V)$$

is multiplication by an integer $k > 0$. We can choose $\mathfrak{P}$ so that the resulting k is minimum. Similarly, we choose a cycle $\mathfrak{P}'$ on $J_a^{r+1}(V) \times V$ of codimension $r+1$ and the integer k'. If we denote the p-th intermediate Jacobian by $T^p(V)$, we have $J_a^p(V) \subset T^p(V)$ and $H_1(T^p(V),\mathbb{Q}) = H^{2p-1}(V,\mathbb{Q})(p)$. We identify $H_1(J_a^p(V),\mathbb{Q})$ with $N^{p-1}H^{2p-1}(V,\mathbb{Q})(p)$ by

$$\begin{array}{ccc} H_1(J_a^p(V),\mathbb{Q}) & \hookrightarrow & H_1(T^p(V),\mathbb{Q}) \\ \| & & \| \\ N^{p-1}H^{2p-1}(V,\mathbb{Q})(p) & \subset & H^{2p-1}(V,\mathbb{Q})(p). \end{array}$$

Then $H_1(h_{\mathfrak{P}},\mathbb{Q})$ is identified with the multiplication by k. Now consider the cycle ${}^t\mathfrak{P}'\circ\mathfrak{P} \in CH^1(J_a^p(V) \times J_a^{r+1}(V))$, which induces the map

$$h_{{}^t\mathfrak{P}'\circ\mathfrak{P}} = [{}^t\mathfrak{P}'\circ\mathfrak{P}]' : J_a^p(V) \longrightarrow ACH^1(J_a^{r+1}(V)) = (J_a^{r+1}(V))^\vee.$$

Denoting the inclusions $N^{p-1}H^{2p-1}(V,\mathbb{Q})(p) \subset H^{2p-1}(V,\mathbb{Q})(p)$ and $N^rH^{2r+1}(V,\mathbb{Q})(r) \subset H^{2r+1}(V,\mathbb{Q})(r)$ by j and j', respectively, we have the commutative diagram

$$\begin{array}{ccccc} H_1(J_a^p(V),\mathbb{Q}) & & \xrightarrow{H_1([{}^t\mathfrak{P}'\circ\mathfrak{P}],\mathbb{Q})} & & H_1((J_a^{r+1}(V))^\vee,\mathbb{Q}) \\ \| & & & & \| \\ N^{p-1}H^{2p-1}(V,\mathbb{Q})(p) & \xrightarrow{k.j} & H^{2p-1}(V,\mathbb{Q})(p) & \xrightarrow{k'.{}^tj'} & (N^rH^{2r+1}(V,\mathbb{Q})(r))^*, \end{array}$$

where $H^{2r+1}(V,\mathbb{Q})(r)$ is regarded as the dual of $H^{2p-1}(V,\mathbb{Q})(p)$ via intersection product. Therefore the map

$${}^tj'\cdot j : N^{p-1}H^{2p-1}(V,\mathbb{Q})(p) \longrightarrow (N^rH^{2r+1}(V,\mathbb{Q})(r))^*$$

is associated to the pairing in (ii). Hence,

$$\text{(ii)} \Longleftrightarrow [{}^t\mathfrak{P}'\circ\mathfrak{P}] \text{ is an isogeny .}$$

Now, we have the diagram

$$\begin{array}{ccccc} J_a^p(V) & \xrightarrow{\mathfrak{P}((?)-(0))} & ACH^p(V) & \xrightarrow{{}^t\mathfrak{P}'} & ACH^1(J_a^{r+1}(V)) \\ & & \alpha \downarrow & & \vdots \uparrow \\ & & J_a^p(V) & \xrightarrow{\pi_V^{(p)}} & Pic^p(V) \end{array},$$

and the dotted morphism is obtained by the universality of $Pic^p(V)$, cf.[13], 3.5. Since $h_{\mathfrak{P}}$ is an isogeny, and $\pi_V^{(p)}$ is surjective, $[{}^t\mathfrak{P}'\circ\mathfrak{P}]$ is an isogeny $\Longrightarrow$ $\pi_V^{(p)}$ is an isogeny, i.e., (iii). By symmetry, we have also (ii) ===> (iv).

Since $J_a^p(V) \longrightarrow Pic^pV$ is surjective, there exists a morphism $s : Pic^p(V) \longrightarrow J_a^p(V)$ such that $\pi_V^{(p)}\circ s = m$, the multiplication by $m \in \mathbb{Z}$. The pull-back $p = (s \times id_V)^*(\mathfrak{P}) \in CH^p(Pic^p(V) \times V)$ induces the mapping

$$m.k : Pic^p(V) \xrightarrow{s} J_a^p(V) \xrightarrow{\mathfrak{P}(?-0)} ACH^p(V) \longrightarrow J_a^p(V) \xrightarrow{\pi_V^{(p)}} Pic^p(V).$$

Similarly, we have a morphism $s' : Pic^{r+1}(V) \longrightarrow J_a^{r+1}(V)$ with $\pi_V^{(r+1)}\circ s' = m'$, $m' \in \mathbb{Z}$, and the pull-back $p' = (s' \times id_V)^*(\mathfrak{P}')$ induces $m'.k' : Pic^{r+1}(V) \longrightarrow Pic^{r+1}(V)$. Then, ${}^tp'\circ p = (s' \times s)^*({}^t\mathfrak{P}'\circ\mathfrak{P}) \in CH^1(Pic^p(V) \times Pic^{r+1}(V))$ induces an isogeny $Pic^p(V) \longrightarrow (Pic^{r+1}(V))^\vee$, (cf. [13], 4,4.) which is factorized as

$$Pic^p(V) \xrightarrow{s} J_a^p(V) \xrightarrow{k} J_a^p(V) \xrightarrow{[{}^t\mathfrak{P}'\circ\mathfrak{P}]} (J_a^{r+1}(V))^\vee \xrightarrow{s'^\vee} (Pic^{r+1}(V))^\vee.$$

The conditions (iii) and (iv) mean that s and s' are isogenies, hence $[{}^t\mathfrak{P}'\circ\mathfrak{P}]$ is an isogeny, and we have the condition (ii).

We shall show that (iv) ===> (i). With notations as above, the cycle ${}^tp \in CH^p(V \times Pic^p(V))$ induces a homomorphism $J_a^{r+1}(V) \longrightarrow (Pic^p(V))^\vee$, which is factorized as

$$J_a^{r+1}(V) \xrightarrow{\pi_V^{r+1}} Pic^{r+1}(V) \xrightarrow{\lambda} (Pic^p(V))^\vee.$$

We know λ is an isogeny(loc. cit.). If π_V^{r+1} is an isogeny, the H_1 of the above mapping is an isomorphism, which is identified with

$$\{{}^t p\} : gr^r H^{2r}(V,\mathbf{Q})(r) = N^r H^{2r+1}(V,\mathbf{Q})(r) \longrightarrow gr^0 H^1(Pic^p(V),\mathbf{Q}),$$

and, the condition $H(Pic^p(V),0,1)$ holds by 5.8.

Assume (i). By definition, there exist varieties T_j, cycles u_j of codimension $p = \dim V - r$ such that

$$(\{{}^t u_j\}) : gr^r H^{2r+1}(V,\mathbb{Q})(r) \longrightarrow \coprod_j gr^0 H^1(T_j,\mathbb{Q}) = \coprod_j H^1(T_j,\mathbb{Q})$$

is injective. Denoting the graph of the projection $T = \prod_k T_k \longrightarrow T_j$ by Γ_j, and putting $u = \sum_j u_j \circ \Gamma_j$, a cycle on $T \times V$ of codimension p, we get an injection

$$\{{}^t u\} : gr^r H^{2r+1}(V,\mathbb{Q})(r) \longrightarrow H^1(T,\mathbb{Q}) = \coprod_j H^1(T_j,\mathbb{Q}).$$

This is H_1 of

$$[{}^t u] : J_a^{r+1}(V) \longrightarrow J_a^1(T),$$

and the kernel of $[{}^t u]$ is finite. By the universality of $Pic^{(r+1)}(V)$, $[{}^t u]$ is factorized as

$$J_a^{r+1}(V) \xrightarrow{\pi_V^{(r+1)}} Pic^{(r+1)}(V) \longrightarrow J_a^1(T),$$

hence $\pi_V^{(r+1)}$ is an isogeny, i.e., (iv).

COROLLARY 7.4. *Let* $\mathscr{C}'(1)$ *be the subcategory of* $\mathscr{C}(1)$ *generated by* $Gr^1 CH_r(V)$ *for* V *with* D(V,r,1). *The restriction of the functor* $\eta : \mathscr{C}(1) \longrightarrow Hdg(1)$ *to* $\mathscr{C}'(1)$ *gives an anti-equivalence of categories*

$$\eta : \mathscr{C}'(1) \xrightarrow{\sim} Hdg(1).$$

By definition, the condition D(V,r,1) is satisfied if

$Gr^1CH(V) \in \mathscr{C}'(1)$. It is shown that η is faithful (7.2), and is fully-faithful by virtue of 7.3.1, and [13], 4.6.

Theorem 7.5. *Let* V *be a smooth projective variety,* p *be an integer. Then,*

(i) $Gr^1CH^p(V)$ *has a structure of abelian variety, and the canonical map* $ACH^p(V) \longrightarrow Gr^1CH^p(V)$ *is regular: i.e., for an arbitrary smooth projective variety* T, *a cycle* $z \in CH^p(T \times V)$, *and* $t_0 \in T$, *the map*

$$T \longrightarrow Gr^1CH^p(V),\ t \longmapsto z((t)-(t_0))$$

is a morphism of varieties.

(ii) *The canonical mapping (cf.* 1.11)

$$\gamma^p : Gr^1CH^p(V) \longrightarrow J_a^p(V)$$

is surjective and the kernel is finite.

(iii) If $\langle H\rangle_0CH^p(V)_{tors} \longrightarrow J_a^p(V)$ *is injective, then* γ^p *is bijective.*

LEMMA 7.5.1. *There exist an abelian variety* A *of dimension* a, $u \in CH^p(A \times V)$ *such that the induced mapping*

$$[\ u\ ] : gr^{a-1}H^{2a-1}(A,\mathbb{Q}) \longrightarrow gr^{p-1}H^{2p-1}(V,\mathbb{Q})$$

is bijective. Moreover, putting $'Gr^1CH^p(V) = ACH^p(V)/A{*}HCH^p(V)$, *the mapping*

$$\{\ u\ \} : 'Gr^1CH_0(A) \longrightarrow 'Gr^1CH^p(V)$$

is surjective.

As in the proof of 7.3.1, (ii) ===> (i), we have a surjective map $H_1(P) \longrightarrow N^{p-1}H^{2p-1}(V)$ induced by an algebraic cycle. Since the kernel of $H_1(P) \longrightarrow N^{p-1}H^{2p-1}(V)$ is a sub-Hodge structure of weight -1 of $H_1(P)$, there exits an abelian variety K of P such that $0 \longrightarrow H_1(K) \longrightarrow H_1(P) \longrightarrow N^{p-1}H^{2p-1}(V)$ is exact. Let A be an abelian subvariety of P such that $A + K = P$ and $A \cap K$ is finite. Then the map $H_1(A) \longrightarrow N^{p-1}H^{2p-1}(V)$ induced by an algebraic cycle $u \in CH^p(A \times V)$ is an isomorphism. Replacing u by $u - 0 \times u(0)$, we may assume $u(0) = 0$.

We shall show that $[\, u \,] : {}'Gr^1CH_0(A) \longrightarrow {}'Gr^1CH^p(V)$ is surjective. It suffices to show that $u : A \longrightarrow {}'Gr^1CH^p(V)$, $x \longmapsto u(x)$ is surjective.

Let B be an abelian variety and $z \in CH^p(B \times V)$, $z(0) = 0$. Put $w = 1_B \times u + 1_A \times z \in CH^p(B \times A \times V)$. We have

$$u : A \simeq 0 \times A \hookrightarrow B \times A \xrightarrow{w} {}'Gr^1CH^p(V),$$

$$z : B \simeq B \times 0 \hookrightarrow B \times A \xrightarrow{w} {}'Gr^1CH^p(V).$$

Let $K_1 \subset B \times A$ be an abelian subvariety such that

$$H_1(K) = \mathrm{Ker}(H_1(B \times A) \xrightarrow{w} H^{2p-1}(V)).$$

By 7.1, $K \subset B \times A \xrightarrow{w} {}'Gr^1CH^p(V)$ vanishes. Therefore, we obtain

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & B \times A & \longrightarrow & (B \times A)/K & \longrightarrow & 0, \\ & & & & & \overset{w}{\searrow} & \vdots & & \\ & & & & & & {}'Gr^1CH^p(V) & & \end{array}$$

and

$$\begin{array}{ccccccccc} & & & & H_1(A) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & H_1(K) & \longrightarrow & H_1(B \times A) & \longrightarrow & H_1((B \times A)/K) & \longrightarrow & 0, \\ & & & & & \overset{\{w\}}{\searrow} & \vdots & & \\ & & & & & & N^{p-1}H^{2p-1}(V) & & \end{array}$$

where the dotted maps are not necessarily algebraic. Since

$\{u\} : H_1(A) \longrightarrow H_1((B \times A)/K) \hookrightarrow N^{p-1}H^{2p-1}(V)$ is bijective, so is the map $H_1(A) \longrightarrow H_1((B \times A)/K)$. Hence $A \longrightarrow (B \times A)/K$ is an isogeny, in particular, is surjective. It therefore follows that

$$\mathrm{Im}(B \longrightarrow Gr^1CH^p(V)) \subset \mathrm{Im}(B \times A \xrightarrow{w} {'Gr}^1CH^p(V))$$
$$\subset \mathrm{Im}((B \times A)/K \longrightarrow {'Gr}^1CH^p(V)) = \mathrm{Im}(A \xrightarrow{u} {'Gr}^1CH^p(V))$$

Since, for any element of ${'Gr}^1CH^p(V)$, we can find an abelian variety B and $z \in CH^p(B \times V)$ as above such that the element is contained in the image of $B \xrightarrow{z} {'Gr}^1CH^p(V)$, we see that $A \xrightarrow{u} {'Gr}^1CH^p(V)$ is surjective.

We shall prove the theorem 7.5. Note that we have

$${'\gamma}^p : {'Gr}^1CH^p(V) \longrightarrow Gr^1CH^p(V) \xrightarrow{\gamma^p} J^p_a(V).$$

Let $'N$ be the kernel of $A \xrightarrow{u} {'Gr}^1CH^p(V)$. The map $A \xrightarrow{u} {'Gr}^1CH^p(V) \xrightarrow{'\gamma^p} J^p_a(V)$ is an isogeny, since its H_1 is identified with the bijection $H_1(A) \xrightarrow{u} N^{p-1}H^{2p-1}(V)$. Hence $'N$ is contained in the kernel, and finite. By the surjectivity of $A \xrightarrow{u} {'Gr}^1CH^p(V)$, and of the maps in the factorization of ${'\gamma}^p$, the kernel of each of these maps is finite.

Suppose $ACH^p(V)_{tors} \longrightarrow J^p_a(V)_{tors}$ is injective, and put

$$\overline{K} = \mathrm{Ker}(ACH^p(V) \longrightarrow J^p_a(V)).$$

For $k \in \mathbb{Z}$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{K} & \longrightarrow & ACH^p(V) & \longrightarrow & J^p_a(V) & \longrightarrow & 0 \\ & & \downarrow \times k & & \downarrow \times k & & \downarrow \times k & & \\ 0 & \longrightarrow & \overline{K} & \longrightarrow & ACH^p(V) & \longrightarrow & J^p_a(V) & \longrightarrow & 0 \end{array}$$

and we see that $\overline{K}$ is torsion-free. From $A{*}HCH^p(V) \subset \overline{K}$ follows that $A{*}HCH^p(V)$ is torsion-free and divisible (cf. 1.10). Hence $\mathrm{Ker}({'\gamma}^p) = \overline{K}/A{*}HCH^p(V)$ is torsion-free. Since it is finite,

$\mathrm{Ker}('\gamma^p) = 0$. As its quotient, $\mathrm{Ker}(\gamma^p) = 0$, too. In particular, for $p = \dim V$, the maps $'\gamma^p$ and γ^p are bijective by [12].

We shall prove (i). Putting $N = \mathrm{Ker}(\ A \xrightarrow{u} \mathrm{Gr}^1 CH^p(V))$, a finite group, we have

$$A/N \simeq \mathrm{Gr}^1 CH^p(V).$$

The right hand side has a structure of abelian variety, and we endow the left hand side with the structure of abelian variety via the isomorphism above. We shall show that the natural homomorphism

$$ACH^p(V) \longrightarrow \mathrm{Gr}^1 CH^p(V)$$

is regular. Let T and z be as in (i), and B be the Albanese variety of T : $\beta : T \longrightarrow B$, with $\beta(t_0) = 0$. Assume $z = (\beta \times id_V)^*(z')$, $z' \in CH^p(B \times V)$. Then,

$$z : T \xrightarrow{\beta} B \xrightarrow{z'} \mathrm{Gr}^1 CH^p(V).$$

With the notations of the proof of lemma 7.5.1, z replaced by z', we have

$$\begin{array}{ccccc} & & B & & \\ & & \downarrow & & \\ A & \longrightarrow & B \times A & \longrightarrow & (B \times A)/K. \\ & \searrow^{u} & & & \downarrow \\ & & & & \mathrm{Gr}^1 CH^p(V) \end{array}$$

Let $N' = \mathrm{Ker}(A \longrightarrow (B \times A)/K)$. We get $(B \times A)/K = A/N'$ and $N' \subset N$. Then,

$$A/N = (A/N')/(N/N') = ((B \times A)/K)/(N/N'),$$

and the map

$$B \longrightarrow (B \times A)/K \longrightarrow ((B \times A)/K)/(N/N') = A/N$$

is a morphism.(Notice we are in characteristic 0.) Therefore, $z : T \xrightarrow{\beta} B \xrightarrow{z'} \mathrm{Gr}^1 CH^p(V)$ is also a morphism. Next we shall assume $\dim T = 1$. Let J be the jacobian of T and $\mathfrak{P}$ be the Poincaré divisor on $J \times T$. The map

$$\{\mathfrak{B}\} : H_1(J) \longrightarrow H^1(T) = H_1(T)$$

is the inverse of $\beta : H_1(T) \longrightarrow H_1(J)$. For $z \in CH^p(T \times V)$, let $z' = z \circ \mathfrak{B} \in CH^p(J \times V)$. We have

$$\{z\} = \{(\beta \times id_V)^*(z')\} : H_1(T) \xrightarrow{\beta_*} H_1(J) \xrightarrow{\{\mathfrak{B}\}} H_1(T) \xrightarrow{\{z\}} H^{2p-1}(V),$$

hence $z = (\beta \times id_V)^*(z') : T \longrightarrow Gr^1CH^p(V)$ and $J \simeq Gr^1CH_0(T) \longrightarrow Gr^1CH^p(V)$ is a morphism. Consider the general case. Let C be a general curve of $T : i : C \hookrightarrow T$. Then $i_* : Alb(C) \longrightarrow Alb(T)$ is surjective, and we have

$$\begin{array}{ccccc} C & \xrightarrow{i} & T & \xrightarrow{z} & ACH^p(V) \\ \downarrow & & \downarrow & & \downarrow \\ Alb(C) & \xrightarrow{i_*} & Alb(T) & \xrightarrow{z} & Gr^1CH^p(V). \end{array}$$

As shown above, $z \circ i_* : Alb(C) \longrightarrow Gr^1CH^p(V)$ is a morphism, and so is the map $Alb(T) \longrightarrow Gr^1CH^p(V)$ by the surjectivity of i_*. It follows that $z : T \longrightarrow Gr^1CH^p(V)$ is a morphism.

Corollary 7.6. *For* $p = 0, 1, 2, \dim V$, *the canonical map*

$$\gamma^p : Gr^1CH^p(V) \longrightarrow J^p_a(V)$$

is bijective.

We may assume $p = 2$. By virtue of [10], for any prime e, we have an isomorphism $CH^2(V)(e) \simeq N^1H^3(V,\mathbb{Q}_e/\mathbb{Z}_e(2))$, where $CH^p(V)(e)$ is the e-torsion subgroup of $CH^p(V)$, and the map is induced by Bloch's map [5]. Summing up over all primes, we get

$$ACH^2(V)_{tors} \subset CH^2(V)_{tors} \simeq N^1H^3(V,\mathbb{Q}/\mathbb{Z}) \subset H^3(V,\mathbb{Q}/\mathbb{Z}) \simeq T^2(V)_{tors},$$

which is induced from the Abel-Jacobi map

$$ACH^2(V) \longrightarrow J^2_a(V) \subset T^2(V).$$

REMARK. 7.7. In the course of the proof of 7.5, we have proven that the subgroups

$$A*HCH^p(V) \subset \langle H\rangle_0 CH^p(V) \cap H^{*2}CH^p(V) \subset \mathrm{Ker}(\langle H\rangle_0 CH^p(V) \longrightarrow J^p_a(V))$$

coincide up to finite groups, and if the assumption 7.5, (iii) is satisfied, then they coincide precisely.

REMARKS 7.8.1. The condition $B(V)$ implies the condition $D(V,r,\ell)$ for arbitrary r, and ℓ: let $h \in N^1H^2(V,\mathbb{Q})(1)$ be the class of a hyperplane section. We set

$$H^{m-j}_{pr}(V,\mathbb{Q}) := \mathrm{Ker}(\ h^{j+1}\cup : H^{m-j}(V,\mathbb{Q}) \longrightarrow H^{m+j+2}(V,\mathbb{Q})(j)).$$

Then, by Hard Lefschetz theorem, $x \in H^n(V,\mathbb{Q})$ is uniquely decomposed into

$$x = \sum_{i\geq i_0} h^i \cup x_i,$$

where $i_0 = \max(n-m,0)$, $x_i \in H^{n-2i}_{pr}(V,\mathbb{Q})(-i)$. We put

$$\Lambda\, x = \sum_{i\geq i_1} h^{i-1} \cup x_i, \quad i_1 = \max(n-m,1).$$

We have $\Lambda \in \bigoplus_n \mathrm{Hom}(H^n(V),H^{n-2}(V)(-1)) = H^{2m-2}(V\times V)(m-1)$.

The condition $B(V)$ is that $\Lambda \in N^{m-1}H^{2m-2}(V,\mathbb{Q})(m-1)$.

Note that $h \cup N^pH^n(V,\mathbb{Q}) \subset N^{p+1}H^{n+2}(V,\mathbb{Q})(1)$, and

$$\Lambda\ N^pH^n(V,\mathbb{Q}) \subset N^{p-1}H^{n-2}(V,\mathbb{Q})(-1) \quad \text{if } \Lambda \text{ is algebraic.}$$

In particular,

$$N^pH^n(V,\mathbb{Q}) = \coprod_{i\geq i_0} h^i \cup N^{p-i}H^{n-2i}_{pr}(V,\mathbb{Q})(-i),$$

where $N^pH^n_{pr}(V,\mathbb{Q}) = N^pH^n(V,\mathbb{Q}) \cap H^n_{pr}(V,\mathbb{Q})$.

Now assume $2r+\ell \leq m$. Put $j = m-(2r+\ell)$, then $m+j = 2p-\ell$, and we have

$$h^j \cup : N^rH^{2r+\ell}(V,\mathbb{Q})(r) \xrightarrow{\ \sim\ } N^{p-\ell}H^{2p-\ell}(V,\mathbb{Q})(p-\ell).$$

If $x = \sum h^i \cup x_i$, $x_i \in N^{r-i}H^{2r-2i+\ell}_{pr}(V,\mathbb{Q})(r-i)$, then, $h^{k+j} \cup x_k \in N^{p-\ell}H^{2p-\ell}(V,\mathbb{Q})(p-\ell)$, and

$$x \cup (h^{k+j} \cup x_k) = (h^{2k} \cup x_k \cup x_k) \neq 0 \quad \text{unless} \quad x_k = 0,$$

by the Hodge's positivity, which implies $D(V,r,\ell)$. If $2r + \ell > m$, exchanging the roles of $N^rH^{2r+\ell}(V,\mathbb{Q})(r)$ and $N^{p-\ell}H^{2p-\ell}(V,\mathbb{Q})(p-\ell)$, we conclude that $B(V)$ implies $D(V,r,\ell)$ for any r, and ℓ.

Incidentally, put

$$Gr^pH^{m+i}(V,\mathbb{Q})$$
$$= \{ x \in N^pH^{m+i}(V,\mathbb{Q}) \ ; \ x \text{ is orthogonal to } N^{p-i+1}H^{m-i}(V,\mathbb{Q})(m) \}$$

for $i \in \mathbb{Z}$. If the condition $B(V)$ holds, then we have

$$Gr^pH^n(V,\mathbb{Q}) \simeq gr^pH^n(V,\mathbb{Q}),$$

and a perfect pairing

$$Gr^rH^{2r+\ell}(V,\mathbb{Q})(r) \times Gr^{p-\ell}H^{2p-\ell}(V,\mathbb{Q})(p-\ell) \longrightarrow H^{2m}(V,\mathbb{Q})(m) = \mathbb{Q}$$

via cup-product, where $r + p = m$, since $D(V,r,\ell)$ and $D(V,r+1,\ell-2)$ hold.

7.8.2. If the condition $B(V)$ holds, then

$$A*HCH^p(V) \otimes \mathbb{Q} = \mathrm{Ker}(ACH^p(V) \longrightarrow J^p_a(V)) \otimes \mathbb{Q},$$

for arbitrary p. In particular, the equations hold for an abelian variety [9], or a variety V with

$$H^n(V,\mathbb{Q}) = 0 \qquad \text{for odd } n \neq \dim V, \text{ and}$$
$$H^n(V,\mathbb{Q}) = \mathbb{Q}(-n/2) \quad \text{for even } n \neq \dim V.$$

Note that a complete intersection in $\mathbb{P}^N$ satisfies these conditions.

§ 8. Faithfulness : case of niveau 2

THEOREM 8.1. *Let* T, V *be smooth projective varieties* m = dim V, $z \in CH^{p+q}(T \times V)$ *and assume the condition* D(V,r,2):

$$N^r H^{2r+2}(V,\mathbb{Q})(r) \otimes N^{p-2} H^{2p-2}(V,\mathbb{Q})(p-1) \longrightarrow H^{2m}(V,\mathbb{Q})(m) = \mathbb{Q}$$

is a perfect pairing, where p = m - r. *If*

$$0 = \{{}^t z\} : gr^r H^{2r+2}(V,\mathbb{Q})(r) \longrightarrow gr^q H^{2q+2}(T,\mathbb{Q}),$$

then, we have

$$0 = [\ z\ ] : Gr^2 CH_q(T) \longrightarrow Gr^2 CH_r(V).$$

LEMMA 8.1.1. *The adequate equivalence relation* $\langle H^{*2}\rangle_0$ *is generated by* $\langle H^{*2}\rangle_0 CH_0$ *of surfaces. More precisely, for an arbitrary smooth projective variety* V, *we have*

$$\langle H^{*2}\rangle_0 CH(V) = \sum z(\langle H^{*2}\rangle_0 CH_0(S)),$$

where S *ranges over all surfaces, and* z *ranges* over *all of elements of* $CH(S \times V)$.

We denote the right hand size by ECH(V). It is clear that E gives an adequate equivalence relation, and that $\langle H^{*2}\rangle_0 \supset E$. Note that $\langle H^{*2}\rangle_0$ is generated by $H^{*2}CH_0$, and, by 7.6, and 7.7, we have $\langle H^{*2}\rangle_0 CH_0 = A{*}HCH_0$, hence that

$$ECH(V) = \sum z(A{*}HCH_0(S)),$$

where S runs over all surfaces and $z \in CH(S \times V)$. We may assume dim V > 2 and it is sufficient to show that $A{*}HCH_0(V) = ECH_0(V)$, i.e., that for a smooth projective variety T, $x \in HCH^p(T \times V)$, $y \in ACH^q(T \times V)$, with p + q = dim T + dim V, we have

$$pr_{V*}(x,y) \in ECH_0(V).$$

By definition, there exist a curve C, $u \in CH^q(C \times T \times V)$, and points a and b of C such that $y = u(\gamma)$, where $\gamma = (a) - (b)$. Since

$$pr_{V*}(x.y) = pr_{V*}(1_C \times x.u.\gamma \times 1_T \times 1V),$$

it suffices to show that $1_C \times x.u.\gamma \times 1_T \times 1_V \in ECH_0(C \times T \times V)$. We are thus reduced to show the following assertion:

Let V *be a smooth projective variety of dimension* > 2 *with a morphism* $\pi : V \longrightarrow C$ *to a curve,* $x \in HCH_1(V)$, $\gamma \in ACH_0(C)$, *Then* $x.\pi^*(\gamma) \in ECH(V)$.

Let X be a 1-cycle representing x, and let $Supp(X)$ denote the support of X with reduced scheme structure. Blowing-up V at singular points of $Supp(X)$, we get $b : \tilde{V} \longrightarrow V$ such that the proper transform of $Supp(X)$ is smooth. Then the proper transform $\tilde{X}$ of X is a 1-cycle whose support is smooth and $b_*(\tilde{X}) = X$. By the following sublemma, we can find a smooth hyperplane section $V' \subset V$, with respect to some embedding into a projective space, containing the support of $\tilde{X}$, if $\dim V > 2$.

SUBLEMMA 8.1.2. *Let* X *be an* r*-dimensional smooth subscheme of a smooth projective variety* V, I_X *the ideal sheaf of* X *in* V, L *an ample line bundle. If* $2r < \dim V$, *a general member of* $|I_X \otimes L^{\otimes n}|$ *is a smooth variety containing the scheme* X *for sufficiently large* n.

For sufficiently large n, the map

$$H^0(V, I_X \otimes L^{\otimes n}) \otimes O_V \longrightarrow I_X \otimes L^{\otimes n}$$

is surjective. Then $(I_X/I_X^{\ 2}) \otimes L^{\otimes n}$ is generated by the global sections of $H^0(V, I_X \otimes L^{\otimes n})$. Since the rank of the vector bundle $I_X/I_X^{\ 2}$ on X is $\dim V - r > r$, the image of a general member s of $|I_X \otimes L^{\otimes n}|$ by the canonical map $I_X \otimes L^{\otimes n} \longrightarrow (I_X/I_X^{\ 2}) \otimes L^{\otimes n}$ vanishes nowhere. Then, $V' = (s) \subset V$ is smooth at the points of X. By Bertini's theorem, it is smooth off X, whence the sublemma 8.1.2.

We return to the proof of 8.1.1. Taking hyperplane sections repeatedly, we obtain a smooth surface $S \overset{i}{\hookrightarrow} \tilde{V}$ containing the support of $\tilde{X}$. Let $b' = b \circ i$. Denoting by X' the 1-cycle $\tilde{X}$ regarded as a cycle on S, we have $b'_*(X') = X$. In the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0 V = Gr^1 CH^1(V) & \xrightarrow{\quad b'^* \quad} & Gr^1 CH^1(S) \\ .X \downarrow & & \downarrow .X' \\ \mathrm{Alb}\ V = Gr^1 CH_0(V) & \xleftarrow[\quad b'_* \quad]{} & Gr^1 CH_0(S), \end{array}$$

the horizontal map below is an isogeny, since

$$b'^* : H^1(V) \xrightarrow{\ b^*\ } H^1(\tilde{V}) \xrightarrow{\ i^*\ } H^1(S)$$

is an isomorphism. The cycle X is homologous to zero, hence the left vertical arrow vanishes, which means that $b'^*(\alpha).X' = 0$ in $Gr^1 CH_0(S)$, for any $\alpha \in Gr^1 CH^1(V) = ACH^1(V)$. In other words, $b'^*(\alpha).X' \in A{*}HCH_0(S)$. It follows that

$$\alpha.X = b'_*(b'^*(\alpha).X') \in ECH_0(V).$$

It is now enough to take $\alpha \in \pi^*(\gamma)$.

We shall prove the theorem 8.1. By means of 8.1.1, we are reduced to the case where $q = 0$ and T is a surface, as in the proof of theorem 7.1. By virtue of 6.5, there exists an integer $N \neq 0$ with

$$0 = N.[z] : Gr^2CH_0(T) \longrightarrow Gr^2CH_r(V).$$

Since $A*HCH_0(T) = \langle H^{*2}\rangle_0 CH_0(T)$ as noted above, $Gr^2CH_0(T)$ is divisible by 1.10, so that $[\,z\,] = 0$.

8.2. We shall define the pseudo-abelian category $\mathscr{C}'(2)$, as in 5.5, starting from $Gr^{\ell}CH_r(V)$ with $H(V,r,2)$ and $D(V,r,\ell)$. Then, $\mathscr{C}'(2)$ is a full subcategory of $\mathscr{C}(2)$ (6.6) and we have the composite

$$\mathscr{C}'(2) \subset \mathscr{C}(2) \xrightarrow{\eta} Hdg(2),$$

which we shall also denote by η. Note that $Gr^2CH_0(V)$ are objects of $\mathscr{C}'(2)$ for all smooth projective varieties V, since the condition $D(V,0,2)$ holds trivially for $r = 0$, and the condition $H(V,0,2)$ holds by 5.9.

COROLLARY 8.3. *The contravariant functor*

$$\eta : \mathscr{C}'(2) \longrightarrow Hdg(2)$$

is faithful.

8.4. Let $\mathscr{C}(2)_{surf}$ be the full pseudo-abelian subcategory of $\mathscr{C}(2)$ obtained from $Gr^2CH_0(S)$ with surfaces S, and let $\mathscr{M}_2$ be the full subcategory of motives defined in 5.9.1, consisting of the subobjects of sums of $gr^0h^2(S)$, where S is a surface, $gr^0h^2(S) = h^2(S)/N^1h^2(S)$, and $N^1h^2(S)$ is the submotive of $h^2(S)$ whose Betti realization is $N^1H^2(S,\mathbb{Q})$ (cf. 5.7.1). Then $\mathscr{M}_2$ is a semi-simple abelian subcategory of $\mathscr{M}$. Note that by Betti realization, we have a faithful functor

$$H : \mathscr{K}_2 \longrightarrow \mathrm{Hdg}(2).$$

$\mathscr{C}(2)_{\mathrm{surf}}$ is a full subcategory of $\mathscr{C}'(2)$ and, we have the restriction $\eta : \mathscr{C}(2)_{\mathrm{surf}} \longrightarrow \mathrm{Hdg}(2)$, which is factorized as

$$\eta : \mathscr{C}(2)_{\mathrm{surf}} \xrightarrow{\eta'} \mathscr{K}_2 \xrightarrow{H} \mathrm{Hdg}(2),$$

where $\eta' : \mathscr{C}(2)_{\mathrm{surf}} \longrightarrow \mathscr{K}_2$ is given by $\mathrm{Gr}^2CH_0(S) \longmapsto gr^0h^2(S)$.

COROLLARY 8.5. *The functor* η' *gives an anti-equivalence of categories*:

$$\eta' : \mathscr{C}(2)_{\mathrm{surf}} \xrightarrow{\sim} \mathscr{K}_2.$$

In particular, the category $\mathscr{C}(2)_{\mathrm{surf}}$ *is a semi-simple* $\mathbb{Q}$*-abelian category.*

We have shown that η' is faithful. By definition, the morphisms from $gr^0h^2(S)$ to $gr^0h^2(S')$ are induced by algebraic cycles of codimension 2 on $S' \times S$. Hence it is clear that η' is fully-faithful, and its essential image is $\mathscr{K}_2$, because $\eta'(\mathrm{Gr}^2CH_0(S)) = gr^0h^2(S)$.

REMARKS 8.6.1. Since $gr^0H^2(S,\mathbb{Q})$ and $gr^0H^2(S,\mathbb{Q})(2)$ are dual via intersection, we could formulate the corollary 8.5 as

$$\mathscr{C}(2)_{\mathrm{surf}} \longrightarrow \mathscr{K}_2, \quad \mathrm{Gr}^2CH_0(S) \longmapsto gr^0h^2(S), \quad [z] \longmapsto \{z\}$$

is an equivalence of categories.

8.6.2. Bloch's original metaconjecture is equivalence with a subcategory of Hdg(2). It is, now, equivalent to the fully-faithfulness of η. It is easily seen that the fully-faithfulness is equivalent to the Hodge conjecture on the products of surfaces.

8.6.3. By 8.1.1, for any smooth projective variety V, $Gr^2CH_0(V)$ is generated by those of surfaces as abelian group. We do not know, however, whether the inclusion from $\mathscr{C}(2)_{surf}$ into the category generated by all $Gr^2CH_0(V)$ is an equivalence of categories, or more generally, whether $\mathscr{C}'(2)_{surf} \hookrightarrow \mathscr{C}(2)$ is an equivalence of categories. Notice that if the standard conjecture B(V) holds universally, then the conditions H(V,r,2) and D(V,r,2) are true, $Gr^2CH_r(V)$ is an object of $\mathscr{C}'(2)$ for arbitrary V and r, hence, $\mathscr{C}'(2) = \mathscr{C}(2)$, and $\mathscr{C}(2)_{surf} \hookrightarrow \mathscr{C}(2)$ is an equivalence of categories and, moreover, they are equivalent to the category $\mathscr{M}_2$ via the functor η.

REMARK 8.7. So far, we have assumed that the ground field k is the complex numbers. Some statements remain true even if k is algebraically closed of characteristic zero. For example, theorems 7.1 and 8.1 are those ones when the Betti cohomology is replaced by etale cohomology or De Rham cohomology, the proof being reduced to the case of complex numbers by the comparison theorem. However, theorem 4.1 (hence 5.1) makes essential use of the hypothesis that the ground field is uncountable, and it is plausible that it is false if k is the algebraic closure of the field of rational numbers. Hence it might be a right formulation to define first a functor of the form

$$\mathscr{M}_\ell \longrightarrow \mathscr{C}(\ell)$$

and to show it is (fully) faithful when k is, for example, the field of complex numbers.

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A remark on truncated Krichever maps

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Introduction

In this note, I report on some remarks about the truncation of the so-called Krichever map, which appears in the algebro-geometric formulation of the conformal field theory (of abelian type or with $\hat{U}(1)$-symmetry), cf.[KNTY].

The dressed moduli spaces appear as a projective limit of their truncated analogues. However a naive way to truncate the Krichever map fails (cf.Remark(3.2)). Then a way is just to truncate the Fock spaces and Grassmannians. This view-point is carried out in §2,3 (but not fully yet) and is motivated by the truncated KP hierarchy, due to Harada, Noumi et al. [H],[N]. Another way is to reinterpret the Krichever map in terms of "curves" in the sense of Cartier. This view-point is roughly sketched in §4 and is a simple application of ideas in [KSU2]. The author hopes to proceed further in this direction.

Through these observations, we can verify that the usage of infinite degree of freedom in not superficial contrary to the author's common sense with respect to simple structure of the dressed moduli spaces.

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staying in the Japan-U.S. Mathematics Institute (JAMI) at the Johns Hopkins University. He would like to express sincere gratitude to its staffs for hospitality. Thanks are also due to Kenji Ueno, Toshiyuki Katsura, Jack Morava and Masatoshi Noumi for useful conversation.

§1 Review of a geometric realization of abelian conformal field theory after [KNTY]

1.1 Conformal field theory deals with representations of the Virasoro Lie algebra. Krichever's construction connects the geometry of moduli spaces of algebraic curves with extra data and a field theory of free fermions (i.e. Clifford algebra) through an infinite dimensional Grassmannian manifold. Basic ingredients are the bosonization, τ-functions of the KP hierarchy in soliton theory. Representations in this context are Fock representations of central charge 1 (or $-2(6j^2-6j+1)$ for $j\in\frac{1}{2}\mathbb{Z}$).

1.2 According to Mumford's formulation, one can associate a commutative subalgebra of the ring of formal differential operators in one variable to a datum $\mathfrak{X} = (C,Q,z)$ consisting of a projective smooth curve C over $\mathbb{C}$, a point $Q \in C$ and a formal local coordinate at Q (cf.[S,Appendix0]). Here we have chosen a theta characteristic $\mathcal{L}$ ($\mathcal{L}^{\otimes 2} = \Omega^1_C$) which supplies the torsion-free sheaf of rank 1 required in addition to the above datum.

The subspace $U(\mathfrak{X})$ of the ring of ordinary differential operators is actually realized as $H^0(C-Q,\mathcal{L}) = H^0(C,\mathcal{L}(*Q))$ in

$\mathbb{C}((z))\,dz = \mathbb{C}((z)) =: \hat{K}$. So $U(\mathfrak{X})$ defines a point of the Grassmannian $\mathrm{Grass}(\hat{K})$:

$$\mathrm{Grass}(\hat{K}) = \{\ U \subset \hat{K}\ ;\ U \text{ is a closed subspace of finite index }\}.$$

$\hat{K}$ is a linear topological space, defined by the filtration $F^p\hat{K} := z^p\mathbb{C}[[z]]$. Put $F_p = F^{-p}$ as usual. Closedness of U is defined by this topology. Being of finite index means that both of Ker and Coker of the natural map $U \longrightarrow \hat{K} \longrightarrow \hat{K}/F^0\hat{K}$ are finite dimensional. The index of U is by definition dim Ker - dim Coker of the above map.

The Grassmanian is a scheme and is a disjoint union of its components $\mathrm{Grass}^d(\hat{K})$ of those U of index d :

$$\mathrm{Grass}(\hat{K}) = \coprod_{d\in\mathbb{Z}} \mathrm{Grass}^d(\hat{K}) .$$

Let $\hat{\mathcal{M}}_g$ denote the dressed moduli space, the collection of the above data $\mathfrak{X} = (C,Q,z)$ together with a level ℓ-structure, $4|\ell$. Then $\hat{\mathcal{M}}_g$ has a structure of scheme and C is naturally equipped with a theta characteristic. Then the above correspondece $\mathfrak{X} \longmapsto U(\mathfrak{X})$ is a morphism $U : \hat{\mathcal{M}}_g \longrightarrow \mathrm{Grass}^0(\hat{K})$, which is called the Krichever map.

1.3 Our Grassmannian has a kind of Plücker embedding :

$$\mathrm{Pl} : \mathrm{Grass}(\hat{K}) \longrightarrow \mathbb{P}(\hat{\mathcal{F}}) = \coprod_{p\in\mathbb{Z}} \mathbb{P}(\hat{\mathcal{F}}_p) .$$

Here $\hat{\mathcal{F}}$ is the module of semi-infinite exterior products, i.e., the Fock space. $\hat{\mathcal{F}}$ has a topological basis $\{|M\rangle\}$ indexed by a pair $M = (p,\lambda)$, $p\in\mathbb{Z}$, $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ a Young diagram (i.e. $\lambda_1 \geq \ldots \geq \lambda_\ell > 0$): $\hat{\mathcal{F}} = \prod \mathbb{C}|M\rangle = \bigoplus_{p\in\mathbb{Z}} \prod_{\lambda} \mathbb{C}|(p,\lambda)\rangle$. $\hat{\mathcal{F}}_p = \prod_{\lambda} \mathbb{C}|(p,\lambda)\rangle$ is the charge (or index) p part of $\hat{\mathcal{F}}$ and Pl preserves the index.

Consider formally the exterior product

$$z^{p+\lambda_1-1} \wedge z^{p+\lambda_2-2} \wedge \cdots \wedge z^{p+\lambda_\ell-\ell} \wedge z^{p-(\ell+1)} \wedge z^{p-(\ell+2)} \wedge \cdots$$

which can be thought as $|(p,\lambda)\rangle$. Then to a point $U \in \mathrm{Grass}(\hat{K})$, one can take an admissible basis of U so that its semi-infinite exterior product det U is well-defined up to a constant multiple, just as in the finite dimensional case.

1.4 It is the so-called bosonization that makes the composition $Pl \circ U$ calculable in some sense. The standard way to introduce bosonization is to define a Hamiltonian time-evolution using current operators. We look it differently in view of bosonization over $\mathbb{Z}$. Consider $\mathcal{F} = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{\lambda} \mathbb{C}|(p,\lambda)\rangle \subset \hat{\mathcal{F}}$ and $\mathcal{H} = \mathbb{C}[t_1, t_2, \ldots] \otimes \mathbb{C}[u, u^{-1}]$, $\hat{\mathcal{H}} = \mathbb{C}[[t_1, t_2, \ldots]] \otimes \mathbb{C}[u, u^{-1}]$ (deg $t_i = i$). Denote the Schur polynomial associated to λ by $\chi_\lambda(t)$. Then the bosonization means the isomorphism of $\mathbb{C}$-vector spaces :

$$B : \mathcal{F} \longrightarrow \mathcal{H} \quad ; \ |(p,\lambda)\rangle \mapsto \chi_\lambda(t)u^p.$$

1.5 Summing up, we have the following maps :

$$\hat{\mathcal{M}}_g \xrightarrow{U} \mathrm{Grass}(\hat{K}) \xhookrightarrow{Pl} \mathbb{P}(\hat{\mathcal{F}}) \xrightarrow[\sim]{B} \mathbb{P}(\hat{\mathcal{H}}) .$$

Note that $\mathrm{Grass}(\hat{K})$, $\mathbb{P}(\hat{\mathcal{F}})$ and $\mathbb{P}(\hat{\mathcal{H}})$ have a natural $\mathbb{C}^*$-bundle on each of them. The main result of abelian conformal field theory is that we can lift $B \circ Pl \circ U$ to $\hat{\mathcal{H}}^\times$ using the Riemann theta function and find a system of differential equations characterizing the lift.

In this note, we consider only the preparatory part of this formulation in the truncated case.

§2 Various Grassmannians and truncated Fock spaces

Grassmannians

2.1 We recall some facts about the structure of Grassmannians. cf. [SS],[N],[K].

Denote the Grassmannian of closed subspaces of dimension (resp. codimension) m of a (linearly topologized) vector space V by $\mathrm{Grass}(m,V)$ (resp. $\mathrm{Grass}(V,m)$). $\mathrm{Grass}(m,V)$ has a structure of smooth $\mathbb{C}$-scheme, naturally embedded into $\mathbb{P}(\Lambda^m V)$ by the Plücker coordinates. Then $\mathrm{Grass}(m,V)$ is the quotient by $GL(1)$ of $\widetilde{\mathrm{Grass}}(m,V)$ which consists of those vectors of $\Lambda^m V-\{0\}$ whose (inhomogenious) Plücker coordinates satisfy the Plücker relations :

$$\begin{array}{ccc} \widetilde{\mathrm{Grass}}(m,V) & \longrightarrow & \Lambda^m V-\{0\} \\ \downarrow & & \downarrow \\ \mathrm{Grass}(m,V) & \longrightarrow & \mathbb{P}(\Lambda^m V) \end{array}$$

We will consider only those Grassmannians which are related to $\hat{K}=\mathbb{C}((z))$ and its topological basis $\{z^n\}_{n\in\mathbb{Z}}$.

Take $V=F^{-m}/F^n$. Then $\mathrm{Grass}(m,V)$ is the usual finite dimensional Grassmannian which is denoted as $GM(m,n)$ in [SS] and is a scheme of finite type over $\mathbb{C}$, embedded into $\mathbb{P}(\Lambda^m(F^{-m}/F^n))$ by the Plücker coordinates.

Take $V=F^{p-m}\hat{K}$. Then $\mathrm{Grass}(m,F^{p-m}\hat{K})$ appears in [H],[N] and is a $\mathbb{C}$-scheme of countable type in the sense of Kashiwara [K].

Truncated Fock spaces

2.2 Now we turn to the (truncated) Fock spaces. The bosonization suggests how to truncate the Fock space $\mathcal{F}_p$ or $\hat{\mathcal{F}}_p$ $(p\in\mathbb{Z})$.

The boson Fock space $\mathcal{H}$ can be considered as the character ring tensored with $\mathbb{C}$ over $\mathbb{Z}$ of the general linear group $GL(\infty)$: $\mathcal{H} = \varinjlim_m R(GL(m))\otimes_{\mathbb{Z}}\mathbb{C}$ cf.[DJKM],[KSU1].

To trace the inverse image of $R(GL(m))$ under the bosonization B, recall that $B(|(p,\lambda)\rangle) = \chi_\lambda(t)u^p$ and $|(p,\lambda)\rangle = z^{p+\lambda_1-1}\wedge\cdots\wedge z^{p+\lambda_\ell-\ell}\wedge z^{p-(\ell+1)}\wedge\cdots$ where $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ is a partition. In conformity with the notation in [KNTY],[KSU1] we also use $e^\mu = z^{\mu-1/2}$ $(\mu\in\mathbb{Z}+\frac{1}{2})$. Denote $e^{p-\ell+1/2}\wedge e^{p-(\ell+1)+1/2}\wedge\cdots$ by $e(p-\ell)$.

The Young diagrams which correspond to finite dimensional irreducible representations of $GL(m)$ are contained in the zone of depth m. Thus by the above remark, $B^{-1}(R(GL(m))\otimes\mathbb{C})$ is $\Lambda^m(F^{p-m}\hat{K})\wedge e(p-m)$. Therefore we are led to the

<u>Definition</u> (truncated Fock spaces)

$$\mathcal{F}_p^{(m)} := \Lambda^m(F^{p-m}K) \subset \hat{\mathcal{F}}_p^{(m)} := \Lambda^m(F^{p-m}\hat{K})$$
$$\mathcal{H}_p^{(m)} := R(GL(m))\otimes\mathbb{C}u^p$$

2.3 We have the following natural maps :

(a) restriction :

$$\begin{array}{ccc} \mathcal{F}_p^{(m+1)} & \xrightarrow{\pi} & \mathcal{F}_p^{(m)} \\ \downarrow\wr\, B & & \downarrow\wr\, B \\ \mathcal{H}_p^{(m+1)} & \xrightarrow{\pi} & \mathcal{H}_p^{(m)} \end{array} \quad ; \quad \chi_\lambda \mapsto \begin{cases} \chi_\lambda & \text{if } \lambda \text{ has depth} \le m \\ 0 & \text{otherwise} \end{cases}$$

(b) inclusion : $\iota : \mathcal{F}_p^{(m)} \longrightarrow \mathcal{F}_p^{(m+1)}$

$$x\wedge e(p-m) \mapsto x\wedge e^{p-m+1/2}\wedge e(p-(m+1))$$

We have also π and ι for $\hat{\mathcal{F}}_p^{(m)}$.

$\pi : \Lambda^{m+1}(F^{p-(m+1)}K) \to \Lambda^m(F^{p-m}K)$ and $\iota : \Lambda^m(F^{p-m}K) \to \Lambda^{m+1}(F^{p-(m+1)}K)$ can be obtained from the short exact sequence :

$$0 \to F^{p-m}K \to F^{p-(m+1)}K \to \mathbb{C}z^{p-(m+1)} \to 0 .$$

It is clear that $\mathcal{F}_p$ is the inductive limit of $(\mathcal{F}_p^{(m)}; \iota)$ and $\hat{\mathcal{F}}_p$ is tha projective limit of $\{\hat{\mathcal{F}}_p^{(m)}, \pi\}$.

Relation among various Grassmannians

2.4 For $m \le \ell$, $n \le k$, we have the maps

$$\Lambda^{\ell}(F^{-\ell}/F^{k}) \underset{\iota}{\overset{\pi}{\rightleftarrows}} \Lambda^{\ell-m}(F^{-\ell}/F^{-m}) \otimes \Lambda^{m}(F^{-m}/F^{n})$$
$$\qquad\qquad\qquad\qquad \underset{e^{-\ell} \wedge \dots \wedge e^{-m-1}}{\Psi}$$

induced from the decomposition $F^{-\ell}/F^{k} = (F^{-\ell}/F^{-m}) \oplus (F^{-m}/F^{n}) \oplus (F^{n}/F^{k})$.

We can similarly consider $\Lambda^{m}(F^{-m}/F^{n}) \rightleftarrows \Lambda^{m}((F^{-m}/F^{n}) \oplus F^{n}) = \Lambda^{m}(F^{-m}\hat{K})$.

Thus we obtain the commutative diagram :

$$\begin{array}{ccc} \Lambda^{m}(F^{-m}/F^{n}) & \underset{\iota}{\overset{\pi}{\leftrightarrows}} & \Lambda^{\ell}(F^{-\ell}/F^{k}) \\ \downarrow & & \downarrow \\ \hat{\mathcal{F}}^{(m)} & \underset{\iota}{\overset{\pi}{\leftrightarrows}} & \hat{\mathcal{F}}^{(\ell)} \end{array}.$$

This induces

$$\begin{array}{ccc} \widetilde{\mathrm{Grass}}(m, F^{-m}/F^{n}) & \underset{\iota}{\overset{\pi}{\leftrightarrows}} & \widetilde{\mathrm{Grass}}(\ell, F^{-\ell}/F^{k}) \\ \downarrow & & \downarrow \\ \mathrm{Grass}(m, F^{-m}/F^{n}) & \underset{\iota}{\overset{\pi}{\leftrightarrows}} & \mathrm{Grass}(\ell, F^{-\ell}/F^{k}) \end{array}.$$

Proposition [SS]

(1) $\mathrm{Grass}(\hat{K})$ (resp. $\widetilde{\mathrm{Grass}}(\hat{K})$) is the projective limit of $\{\mathrm{Grass}(m, F^{-m}/F^{n}), \pi\}$ (resp. $\{\widetilde{\mathrm{Grass}}(m, F^{-m}/F^{n}), \pi\}$).

(2) $\mathrm{Grass}(\hat{K})_{fin} = \varinjlim \{\mathrm{Grass}(m, F^{-m}/F^{n}), \iota\}$ (as sets) is dense in $\mathrm{Grass}(\hat{K})$.

2.5 In a similar way, using the restriction and the inclusion $\mathcal{F}_p^{(m)} \rightleftarrows \mathcal{F}_p^{(m+1)}$, we have maps for Grassmannians :

$$\mathrm{Grass}(m, F^{p-m}\hat{K}) \underset{\iota}{\overset{\pi}{\leftrightarrows}} \mathrm{Grass}(m+1, F^{p-(m+1)}\hat{K}) \quad .$$

ι sends a subspace U to $U \oplus \mathbb{C}z^{p-(m+1)}$.

Proposition (1) $\mathrm{Grass}^{p}(\hat{K}) = \varprojlim_{m} \mathrm{Grass}(m, F^{p-m}\hat{K})$ as schemes and

$\varinjlim_{m} \mathrm{Grass}(m, F^{p-m}\hat{K})$ is dense in $\mathrm{Grass}^{p}(\hat{K})$.

(2) $\Lambda^{m}(F^{p-m}\hat{K}) = \varprojlim_{n} \Lambda^{m}(F^{p-m}/F^{n})$ induces

$$\mathrm{Grass}(m, F^{p-m}\hat{K}) = \varprojlim_{n} \mathrm{Grass}(m, F^{p-m}/F^{n})$$

as schemes.

<u>Infinitesimal structure</u>

2.6 We have a usual description of tangent spaces of Grassmannians. For $V \in \mathrm{Grass}(m, F^{p-m}\hat{K})$,

$$T_{V}\mathrm{Gras}(m, F^{p-m}\hat{K}) = \mathrm{Hom}(V, F^{p-m}\hat{K}/V)$$

$$\begin{aligned} T_{\iota(V)}\mathrm{Grass}(m+1, F^{p-(m+1)}\hat{K}) &= \mathrm{Hom}(V\oplus\mathbb{C}z^{p-(m+1)}, F^{p-(m+1)}\hat{K}/V\oplus\mathbb{C}z^{p-(m+1)}) \\ &= \mathrm{Hom}(V\oplus\mathbb{C}z^{p-(m+1)}, F^{p-m}\hat{K}/V) \\ &= \mathrm{Hom}(V, F^{p-m}\hat{K}/V)\oplus\mathrm{Hom}(\mathbb{C}z^{p-(m+1)}, F^{p-m}\hat{K}/V) \end{aligned}$$

Then $d\pi_{\iota(V)}$ is the projection to the first factor and $d\iota_{V}$ is the injection onto the first factor. If $\ell = \mathrm{Pl}(V) \in \mathbb{P}(\hat{\mathcal{F}}_{p}^{(m)})$, $T_{\ell}\mathbb{P}(\hat{\mathcal{F}}_{p}^{(m)}) = \hat{\mathcal{F}}_{p}^{(m)}/\ell$ and we have an obvious description for $d\pi_{\iota(\ell)}$ and $d\iota_{\ell}$.

§3 Truncated Krichever map

3.1 The Krichever map is the correspondence

$$\mathfrak{X} = (C,Q,z) \longmapsto U(\mathfrak{X}) = tH^0(C,\mathcal{L}(*Q)) \subset \mathbb{C}((z))=\hat{K} .$$

Here t means the Laurent expansion at Q using the coordinate z. $U(\mathfrak{X})$, as a subspace of $\hat{K}$, is naturally filtered :

$$U_m(\mathfrak{X}) := F_mU(\mathfrak{X}) = U(\mathfrak{X})\cap F_m\hat{K} = tH^0(C,\mathcal{L}(mQ)) \subset F_m\hat{K} .$$

For $m \geq g$, $F_mU(\mathfrak{X})$ is m-dimensional and is a point of $\mathrm{Grass}(m,F_m\hat{K})$. Thus we have a morphism, called m-truncated Krichever map :

$$U_m : \hat{\mathcal{M}}_g \longrightarrow \mathrm{Grass}(m,F_m\hat{K}) .$$

Since $U_m(\mathfrak{X})\cap F_\ell\hat{K} = U_\ell(\mathfrak{X})$ for $\ell \leq m$, we have a projective system of morphisms

$$\begin{array}{ccl} \hat{\mathcal{M}}_g & \xrightarrow{U_m} & \mathrm{Grass}(m,F_m\hat{K}) \\ & \searrow^{U_\ell} & \downarrow \text{ restriction} \\ & & \mathrm{Grass}(\ell,F_\ell\hat{K}) \qquad (\ell \leq m). \end{array}$$

Passing to the projective limit,we recover the original Krichever map $\hat{\mathcal{M}}_g \longrightarrow \mathrm{Grass}^0(\hat{K})$.

By the truncated version of Plücker embedding, bosonization in §2, we obtain the following proposition.

<u>Proposition</u> We have a projective system of morphisms

$$\hat{\mathcal{M}}_g \xrightarrow{U_m} \mathrm{Grass}(m,F_m\hat{K}) \xhookrightarrow{Pl} \mathbb{P}(\hat{\mathcal{F}}_p^{(m)}) \xrightarrow[\sim]{B} \mathbb{P}(\hat{\mathcal{H}}_p^{(m)}) \qquad (m\geq g).$$

3.2 <u>Remark</u> 1) If we use $\mathcal{L}^{\otimes j}$ instead of $\mathcal{L}$ for any integer j, we have a similar diagram for sufficiently large m. ($F_m\hat{K}$ remains the same, but the dimension m is replaced by $m+(j-1)(g-1)$ while the charge is $(j-1)(g-1)$.)

2) We hoped to have a morphism from $\hat{\mathcal{M}}_g^{(m)}$ into some finite dimensional Grassmanian. But the identification of the fraction

field $\hat{K}_Q$ of $\hat{\mathcal{O}}_{C,Q}$ and $\hat{K}=\mathbb{C}((z))$ is so important that we cannot produce a non-trivial morphism of this kind. In this respect, notice the following :

<u>Lemma</u> The isomorphism $F^{p-m}\hat{K}_Q/F^p\hat{K}_Q \xrightarrow{\sim} F^{p-m}\hat{K}/F^p\hat{K}$ induced by z, i.e., $t : \hat{K}_Q \to \hat{K}$, is invariant under the action of the group $D^{(m)} = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[z]/(z^{m+1}))$, but not under $D^{(m-1)}$.

3.3 The interest in the truncated Krichever maps lies in the connection with the truncated KP hierarchy of Harada, Noumi, etc. [H],[N]. They have established the correspondence between points of $\mathrm{Grass}(m,F^0\hat{K})$ and wave functions, Hirota's bilinear equations for τ-functions, among others.

In principle, over $\mathbb{C}$, we can define a truncated version of the τ-function associated to a point of $\hat{\mathcal{M}}_g$, starting from the τ-function defined in [KNTY] via the Riemann theta function: $\tau_m(t,\mathfrak{X}) \in \hat{\mathcal{H}}_0^{(m)}$ (restiction of $\tau(t,\mathfrak{X}) \in \hat{\mathcal{H}}_0$ to $\hat{\mathcal{H}}_0^{(m)}$). By the results of Harada, Noumi, etc., $\tau_m(t,\mathfrak{X})$ satisfies Hirota's bilinear equations : for $0\le k_0<\ldots<k_{m-2}$, $0\le \ell_0<\ldots<\ell_m$, $k_i,\ell_j\in\mathbb{Z}$,

$$\sum_{i=0}^{m}(-1)^i\chi_{k_0\ldots k_{m-2}\ell_i}(\tfrac{1}{2}\tilde{D}_t)\chi_{\ell_0\ldots\hat{\ell}_i\ldots\ell_m}(-\tfrac{1}{2}\tilde{D}_t)\tau_m(t)\tau_m(t) = 0$$

where $\chi_{k_0\ldots k_{m-1}}(t)$ denotes $\chi_\lambda(t)$ for the partition $\lambda = (k_{m-1}-(m-1), \ldots, k_1-1, k_0)$ and $\tilde{D}_t$ means Hirota differentiation cf.[DJKM].

3.4 At present there are several defects compared with the original version. First we haven't developped the operator formalism for the truncated KP hierarchy (or it may have been developped but does not appear in the literature). Secondly we haven't explicitly written down $\tau_m(t,\mathfrak{X})$ in terms of the theta function.

§4 Another approach to the truncated Krichever map

We sketch a reformulation of the Krichever map and a way to truncate it.

4.1 <u>Definition</u> For a pointed $\mathbb{C}$-scheme (X,Q), i.e., $Q\in X(\mathbb{C})$, we put

$$C_{(m)}(X,Q) = \{\text{morphism Spec } \mathbb{C}[z]/(z^{m+1}) \longrightarrow X \text{ , which sends the closed point to } Q\}$$

Elements of $C_{(m)}(X,Q)$ are called m-truncated curves at Q in X.

$C_{(m)}$ is a covariant functor on the category of pointed $\mathbb{C}$-schemes. $C_{(m)}$ $(m\in\mathbb{N})$ form a projective system and put $C := \varprojlim_m C_{(m)}$, which is also a covariant functor. Elements of $C(X,Q)$ are called curves at Q in X. When X is a group scheme and Q is its identity element, we recover the covariant Dieudonné module (Cartier module).

4.2 Let $\hat{\ }^{(m)}_{g}$ denote the collection of pointed curves with m-truncated formal coordinate cf.[KNTY]. Then,

<u>Lemma</u> For $(C,Q) \in \mathcal{M}_g^{(0)}$; $C_{(m)}(C,Q)$ equals the fiber of $\mathcal{M}_g^{(m)} \longrightarrow \mathcal{M}_g^{(0)}$ at (C,Q).

If we apply $C_{(m)}$ to the "universal" family of pointed curves $\mathcal{M}_g^{(0)} \longrightarrow \mathcal{M}_g$, then $C_{(m)}(\mathcal{M}_g^{(0)}/\mathcal{M}_g) = \mathcal{M}_g^{(m)}$.

Now consider the "universal" family of pointed principally polarized abelian varieties $\mathcal{A}_g^{(0)} \longrightarrow \mathcal{A}_g$ and put $C(\mathcal{A}_g^{(0)}/\mathcal{A}_g) =: \tilde{\mathcal{A}}_g$, which is an $\mathrm{Aut}_{\mathbb{C}}\mathbb{C}[[z]]\times\mathcal{A}_g^{(0)}$-torsor over $\mathcal{A}_g^{(0)}$.

4.3 Albanese morphism α : $(C,Q) \longrightarrow (\mathrm{Jac}(C),[Q])$, $[Q]=\mathcal{O}_C(Q)$, induces

$$C_{(m)}(C,Q) \longrightarrow C_{(m)}(\mathrm{Jac}(C),[Q]) \text{ .}$$

If we consider Albanese morphism for the family $\mathcal{M}_g^{(0)}/\mathcal{M}_g \longrightarrow \mathcal{A}_g^{(0)}/\mathcal{A}_g$, we get

$$\hat{\mathcal{M}}_g \longrightarrow \tilde{\mathcal{A}}_g$$

and its truncations

$$\mathcal{M}_g^{(m)} \longrightarrow \mathcal{A}_g^{(m)} := C_{(m)}(\mathcal{A}_g^{(0)}/\mathcal{A}_g) .$$

Remark Notice a theorem of Cartier :

$$C(Jac(C),[Q]) \rightarrow Hom_{f.gp}(\hat{W}, Jac(C)\hat{}_{/[Q]}) .$$

For the notation and its content, cf.[KSU2].

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Deformations of Complex Analytic Subspaces with Locally Stable Parametrizations of Compact Complex Manifolds

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Introduction. In this paper we shall give a definition of *complex analytic subspaces with locally stable parametrizations* of compact complex manifolds, which is a generalization of closed complex analytic subsets of *simple normal crossing* in [3] and analytic subvarieties with *ordinary singularities* in [9], and we show that their *logarithmic deformations* and *locally trivial displacements* (cf. Definition 1.5 below) are equivalent to deformations of *locally stable holomorphic maps* (cf. Definition 1.1 below). From this equivalence and Miyajima-Namba-Flenner's theorem on the existence of the Kuranishi family of deformations of holomorphic maps, it follows that there exist the Kuranishi family of logarithmic deformations and the maximal family of locally trivial displacements of a complex analytic subspace with a locally stable parametrization. These are a unification and a generalization of the results in [3] and [9]. Throughout this paper all complex analytic spaces are assumed to be reduced, second countable, and finite dimensional. For notation and terminology concerning logarithmic deformations , locally trivial displacements of a complex analytic subspace and deformations of a holomorphic map, we refer to [3], [9] and [2], respectively.

§1 Complex analytic subspaces with locally stable parametrizations and their deformations

Let X and Y be complex manifolds, and S and T finite subsets of X and Y, respectively. A multi-germ $f:(X,S)\longrightarrow(Y,T)$ of a holomorphic map at S is an equivalence class of holomorphic maps $g:U\longrightarrow Y$ with $g(S)=T$, where U are open neighborhoods of S in X. Throughout this paper we shall interchangeably use a multi-germ of f and a representative g of f. A germ of a parametrized family of multi-germs of holomorphic maps is a multi-germ $F:(X\times\mathbb{C}^r, S\times 0)\longrightarrow(Y\times\mathbb{C}^r, T\times 0)$ of a holomorphic map such that $F(X\times t)\subset Y\times t$ for any t in some open neighborhood of 0 in $\mathbb{C}^r$. An unfolding of a multi-germ $f:(X,S)\longrightarrow(Y,T)$ of a holomorphic map is a germ of a parametrized family of multi-germs of holomorphic maps $F:(X\times\mathbb{C}^r, S\times 0)\longrightarrow(Y\times\mathbb{C}^r, T\times 0)$ such that $F(x,0)=(f(x),0)$ for $x\in X$. We say that an unfolding $F:(X\times\mathbb{C}^r, S\times 0)\longrightarrow(Y\times\mathbb{C}^r, T\times 0)$ of a multi-germ $f:(X,S)\longrightarrow(Y,T)$ of a holomorphic map is *trivial* if there exist germs of t-levels ($t\in\mathbb{C}^r$) preserving analytic automorphisms $G:(X\times\mathbb{C}^r, S\times 0)\longrightarrow(X\times\mathbb{C}^r, S\times 0)$ and $H:(Y\times\mathbb{C}^r, T\times 0)\longrightarrow(Y\times\mathbb{C}^r, T\times 0)$ with $G_{|X\times 0}=id_X$, $H_{|Y\times 0}=id_Y$, such that $H\circ F\circ G^{-1}=f\times id_{\mathbb{C}^r}$. We say that a multi-germ $f:(X,S)\longrightarrow(Y,T)$ of a holomorphic map is *simultaneously stable* if any unfolding of f is trivial.

1.1 Definition. A holomorphic map $f:X\longrightarrow Y$ between complex manifolds is said to be *locally stable* if, for any point $y\in Y$ and any finite subset $S\subset f^{-1}(y)$, a multi-germ $f:(X,S)\longrightarrow(Y,y)$ is simultaneously stable.

1.2 Definition. A complex anlytic subspace Z of a complex manifold Y is said to be *with a locally stable parametrization* if

(i) the normal model X of Z is non-singular, and

(ii) the composite map $f:=\iota\circ n: X\longrightarrow Y$ is locally stable, where $n:X\longrightarrow Z$ is the normalization map and $\iota:Z\subset Y$ is the inclusion map.

1.3 Example. A closed complex analytic subset Z of *simple normal crossing* of a complex manifold Y is a complex analytic subspace with a locally stable parametrization. This follows from Proposition(7.1) in [9]. Here we say that a closed complex analytic subset Z of a complex manifold Y is of *simple normal crossing* if the following conditions are satisfied:

(i) $Z=\bigcup_{i=1} Z_i$,where Z_i $(1\le i\le k)$ are complex submanifolds of Y.

(ii) For any point $p\in Z$, if we let Z_{i_1}, Z_{i_2}, ..., Z_{i_k} be all irreducible components of Z which are through p , then there exists a local coordinate system

$(z_1,\ldots,z_{r_1},z_{r_1+1},\ldots,z_{r_2},\ldots,z_{r_{k-1}+1},\ldots,z_{r_k},z_{r_k+1},\ldots,z_n)$ of Y with center p such that each Z_{r_α} $(1\le\alpha\le k)$ is defined by $z_{r_{\alpha-1}+1}=\ldots=z_{r_\alpha}=0$, where we understand $r_0=0$.

1.4 Example. Suppose that $(dimY,\ dimZ)$ belongs to the *nice range* in the sense of J.N.Mather ([4], or [9, Definition(3.3)]), then an analytic subvariety Z with *ordinary singularities* in a complex manifold Y is with a locally stable parametrization. Here *ordinary singularities* are defined to be the ones which occur in the image of a manifold by a *generic* linear projection ([9]).

From now on let Z be a complex analytic subspace with a locally stable parametrization of a compact complex manifold Y.

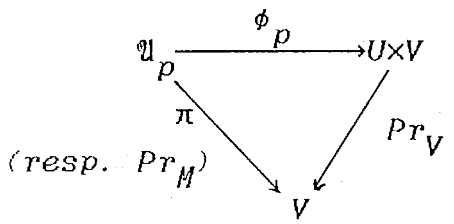
1.5 Definition. An analytic family of logarithmic deformations of a pair (Y, Z) (resp. of locally trivial displacements of Z in Y) parametrized by a complex analytic space M is a sextuplet $\mathcal{F}=(\mathcal{Y}, \mathcal{X}, \pi, M, 0, \psi)$ (resp. a quintuplet $\mathcal{F}=(Y\times M, \mathcal{X}, \pi, M, 0))$ satisfying the following conditions:

(i) $\pi:\mathcal{Y}\longrightarrow M$ is a proper smooth holomorphic map between complex spaces $\mathcal{Y}$ and M (resp. $\pi:\mathcal{X}\longrightarrow M$ is the restriction to $\mathcal{X}$ of the canonical projection $Pr_M:Y\times M\longrightarrow M$),

(ii) $\mathcal{X}$ is a closed complex analytic subspace of $\mathcal{Y}$ (resp. of $Y\times M$),

(iii) 0 is an assigned point of M and $\psi:Y\longrightarrow\pi^{-1}(0)$ is an isomorphism such that $\psi(Z)=\pi^{-1}(0)\cap\mathcal{X}$ (resp. $\pi^{-1}(0)=Z\times 0$), and

(iv) π is locally a projection of a product space and the restriction of π to $\mathcal{X}$ is so (resp. π is locally a projection of a product space); that is, for each point $p\in\mathcal{Y}$ (resp. $p\in Y\times M$), there exist an open neighborhood $\mathfrak{U}_p\subset\mathcal{Y}$ (resp. $\mathfrak{U}_p\subset Y\times M$) of p and an isomorphism $\phi_p:\mathfrak{U}_p\longrightarrow U\times V$, where $U=\mathfrak{U}_p\cap\pi^{-1}(\pi(p))$ and $V=\pi(\mathfrak{U}_p)$ (resp. $V=Pr_M(\mathfrak{U}_p)$), such that (a) the diagram



is commutative, (b) $\phi_p(\mathfrak{U}_p\cap\mathcal{X})=(U\cap\mathcal{X})\times V$, and (c) $\phi_{p\,|U\times\pi(p)}= id_{U\times\pi(p)}$.

For a pair (Y, Z) we denote by $f:=\iota\circ n:X\longrightarrow Y$ the composite of the normalization map $n:X\longrightarrow Z$ and the inclusion map $\iota:Z\subset Y$,and by $\mathcal{D}(f, X, Y)$ (resp. $\mathcal{D}(f, X))$ the category of germs of families of deformations of $f:X\longrightarrow Y$ with Y varied (resp. with Y fixed), and by $\mathcal{L}(Y,Z)$ (resp. $\mathcal{L}(Z)$) the category of germs of families of logarithmic deformations of (Y,Z) (resp. of locally trivial displacements of Z in Y).

1.6 Theorem. *$\mathcal{D}(f,X,Y)$ and $\mathcal{L}(Y,Z)$ (resp. $\mathcal{D}(f,X)$ and $\mathcal{L}(Z)$) are isomorphic as categories.*

Proof. The proof is almost identical with that of Theorem(11.1) in [9] (= Main theorem in [8]). Although in [9] we consider only locally trivial displacements of Z in a fixed ambient manifold Y and deformations of $f:X\longrightarrow Y$ with Y fixed, the proof of Theorem(11.1) in [9] is also valid for logarithmic deformations of a pair (Y,Z) and for deformations of $f:X\longrightarrow Y$ with Y varied. Q.E.D.

§2 Comparison of infinitesimal deformation spaces

As in the preceding section, let Z be an analytic subspace with a locally stable parametrization in a compact complex manifold Y, and let $f:=\iota\circ n$: $X\longrightarrow Y$ be the composite of the normalization map $n:X\longrightarrow Z$ and the inclusion map $\iota:Z\subset Y$. We denote by T_Y the sheaf of holomorphic tangent vector fields on Y, and by $T_Y(\log Z)$ the sheaf of *logarithmic tangent vector fields along Z in Y*, that is, the subsheaf of T_Y consisting of the derivations of $\mathcal{O}_Y$ which send the ideal sheaf of Z in $\mathcal{O}_Y$ into itself. We define a sheaf $\mathcal{N}_{Z/Y}$ by the following exact sequence:

(2.1) $$0 \longrightarrow T_Y(\log Z) \longrightarrow T_Y \longrightarrow \mathcal{N}_{Z/Y} \longrightarrow 0 ,$$

and $\mathcal{T}_{X/Y}$ by the following one:

$$(2.2) \qquad 0 \longrightarrow T_X \xrightarrow{Jf} f^*T_Y \longrightarrow \mathcal{T}_{X/Y} \longrightarrow 0.$$

The infinitesimal deformation spaces of logarithmic deformations of a pair (Y,Z), locally trivial displacements of Z in Y and deformations of a map $f:X\longrightarrow Y$ with Y fixed, are $H^1(Y,T_Y(\log Z))$, $H^0(Z,\mathcal{N}_{Z/Y})$ and $H^0(X,\mathcal{T}_{X/Y})$, respectively. Their obstruction classes belong to $H^2(Y,T_Y(\log Z))$, $H^1(Z,\mathcal{N}_{Z/Y})$ and $H^1(X,\mathcal{T}_{X/Y})$, respectively. As to the infinitesimal deformation space of a map $f:X\longrightarrow Y$ with Y varied, there are two spaces; $H^1(T_X,T_Y,f^*T_Y)$ defined by Namba in [6] and $Ext^1_{\mathcal{C}}((\Omega^1_X,\Omega^1_Y),(\mathcal{O}_X,\mathcal{O}_Y))$ defined by Flenner in [1]. Here $\mathcal{C}$ is an abelian category whose objects are triplets $(\mathcal{F},\mathcal{G},\varphi)$, where $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, $\mathcal{G}$ a coherent $\mathcal{O}_Y$-module and $\varphi\in Hom_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F})$, and for $(\mathcal{F},\mathcal{G},\varphi)$, $(\mathcal{F}',\mathcal{G}',\varphi')\in\mathcal{C}$, a morphism from $(\mathcal{F},\mathcal{G},\varphi)$ to $(\mathcal{F}',\mathcal{G}',\varphi')$ is $(\alpha,\beta)\in Hom_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}')\times Hom_{\mathcal{O}_Y}(\mathcal{G},\mathcal{G}')$ such that the diagram

$$\begin{array}{ccc} f^*\mathcal{G} & \xrightarrow{f^*\beta} & f^*\mathcal{G}' \\ \varphi\downarrow & & \downarrow\varphi' \\ \mathcal{F} & \xrightarrow{\alpha} & \mathcal{F}' \end{array}$$

is commutative. The obstruction classes to deformations of $f:X\longrightarrow Y$ with Y varied belong to $H^2(T_X,T_Y,f^*T_Y)$.

2.1 Proposition.

(i) $\mathcal{N}_{Z/Y}\simeq f^*\mathcal{T}_{X/Y}$ *, and so there exists an isomorphism*

$$H^i(Z,\mathcal{N}_{Z/Y})\xrightarrow[\sim]{f^i} H^i(X,\mathcal{T}_{X/Y}) \quad \textit{for } i\geq 0.$$

(ii) *There exist isomorphisms*

$$H^i(T_X, T_Y, f^*T_Y) \overset{g^i}{\xleftarrow{\sim}} H^i(Y, T_Y(\log Z)) \overset{h^i}{\xrightarrow{\sim}} Ext^i_{\mathscr{C}}((\Omega^1_X, \Omega^1_Y), (\mathcal{O}_X, \mathcal{O}_Y)) \quad for\ i \geq 0.$$

Proof. For the proof of (i) we refer to Proposition(9.1) in [9]. Here we prove (ii). By (2.1),(2.2) and (i) of the proposition, we have the following diagram of exact cohomology sequences;

$$(2.3)\quad \begin{array}{ccccccccc} \longrightarrow H^{i-1}(X, \mathscr{T}_{X/Y}) & \xrightarrow{\delta_1} & H^i(X, T_X) & \xrightarrow{Jf} & H^i(X, f^*T_Y) & \xrightarrow{u} & H^i(X, \mathscr{T}_{X/Y}) & \text{—} \\ \wr\| \uparrow f^{i-1} & & \uparrow \alpha & & \uparrow f^* & & \wr\| \uparrow f^i & \\ \longrightarrow H^{i-1}(Z, \mathscr{N}_{Z/Y}) & \xrightarrow{\delta_2} & H^i(Y, T_Y(\log Z)) & \xrightarrow{\beta} & H^i(Y, T_Y) & \xrightarrow{v} & H^i(Z, \mathscr{N}_{Z/Y}) & \text{—} \end{array}$$

Since $f: X \longrightarrow Y$ is an immersion outside a two-codimensional subset of X (cf. Corollary(4.2)in [9]), it naturally induces a homomorphism $H^i(Y, T_Y(\log Z)) \longrightarrow H^i(X, T_X)$. This is the map α in (2.3). By (2.3) we have an exact sequence of cohomologies

$$(2.4)\quad \begin{aligned} &\longrightarrow H^{i-1}(X, f^*T_Y) \xrightarrow{\delta} H^i(Y, T_Y(\log Z)) \xrightarrow{\alpha \oplus \beta} H^i(X, T_X) \oplus H^i(Y, T_Y) \\ &\xrightarrow{Jf - f^*} H^i(X, f^*T_Y) \longrightarrow \end{aligned}$$

Here δ is the composite of the homomorphisms:

$$H^{i-1}(X, f^*T_Y) \xrightarrow{u} H^{i-1}(X, \mathscr{T}_{X/Y}) \xrightarrow{(f^{i-1})^{-1}} H^{i-1}(Z, \mathscr{N}_{Z/Y}) \xrightarrow{\delta_2} H^i(Y, T_Y(\log Z))$$

On the other hand there are exact sequences of cohomologies;

$$(2.5)\quad \begin{aligned} &\longrightarrow H^{i-1}(X, f^*T_Y) \longrightarrow H^i(T_X, T_Y, f^*T_Y) \longrightarrow H^i(X, T_X) \oplus H^i(Y, T_Y) \\ &\longrightarrow H^i(X, f^*T_Y) \longrightarrow \end{aligned}$$

([6, Proposition(3.6.9)], and

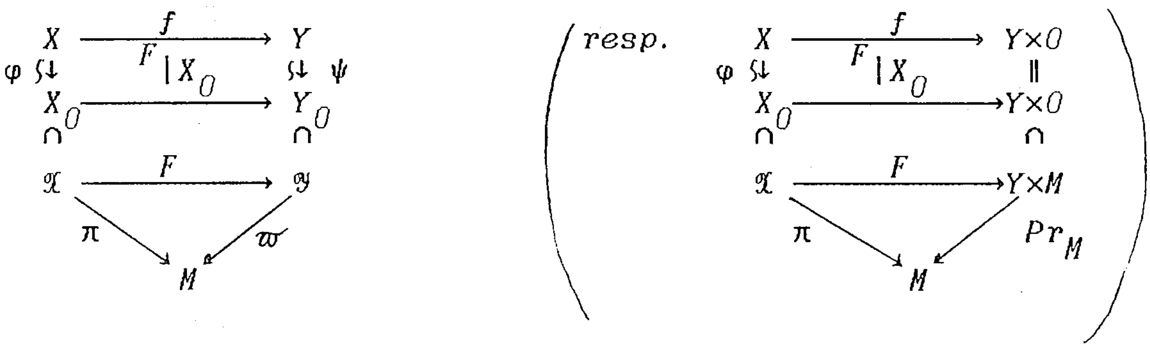
$$\longrightarrow Ext^{i-1}_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X) \longrightarrow Ext^i_{\mathscr{C}}((\Omega^1_X, \Omega^1_Y), (\mathcal{O}_X, \mathcal{O}_Y))$$

(2.6)

$$\longrightarrow Ext^i_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \oplus Ext^i_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow Ext^i_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X) \longrightarrow$$

([7, (2.2)]). By comparing (2.4) with (2.5) and (2.6), we have the assertion (ii). Q.E.D.

Suppose we are given an analytic family $\mathcal{F}=(\mathcal{Y}, \mathfrak{X}, \pi, M, 0, \psi)$ of logarithmic deformations of a pair (Y, Z) (resp. $\mathcal{F}=(Y\times M, \mathfrak{X}, \pi, M, 0)$ of locally trivial displacements of Z in Y). Then, by normalization (cf. [9, Theorem(10.1)], or [8, Theorem2]) from $\mathcal{F}$ we get a family $\hat{\mathcal{F}}=(\mathfrak{X}, F, \mathcal{Y}, \pi, \varpi, M, 0, \varphi, \psi)$ (resp. $\hat{\mathcal{F}}=(\mathfrak{X}, F, \pi, M, 0, \varphi))$ of deformations of $f: X\longrightarrow Y$ with Y varied (resp. with Y fixed):



2.2 Proposition. *The characteristic maps of* $\mathcal{F}=(\mathcal{Y}, \mathfrak{X}, \pi, M, 0, \psi)$ *and* $\hat{\mathcal{F}}=(\mathfrak{X}, F, \mathcal{Y}, \pi, \varpi, M, 0, \varphi, \psi))$ *(resp.* $\mathcal{F}=(Y\times M, \mathfrak{X}, \pi, M, 0)$ *and* $\hat{\mathcal{F}}=(\mathfrak{X}, F, \pi, M, 0, \varphi))$ *are related as follows;*

$$
\begin{array}{ccc}
 & & H^1(T_X, T_Y, f^*T_Y) \\
 & \overset{\tau_0}{\nearrow} & \uparrow g^1 \\
T_0(M) & & \\
 & \underset{\rho_0}{\searrow} & \\
 & & H^1(Y, T_Y(\log Z))
\end{array}
\qquad
\left(\begin{array}{cccc}
\text{resp.} & & & H^0(X, \mathcal{T}_{X/Y}) \\
 & & \overset{\tau_0'}{\nearrow} & \uparrow -f^0 \\
 & T_0(M) & & \\
 & & \underset{\sigma_0}{\searrow} & \\
 & & & H^0(Z, \mathcal{N}_{Z/Y})
\end{array}\right)
$$

Proof. By Direct calculation of the characteristic maps. Q.E.D.

By Miyajima-Namba-Flenner's theorem on the existence of the Kuranishi family of deformations of holomorphic maps ([5, Main Theorem]),[6, Theorem(3.6.10)],[1, Theorem(8.5)]), Proposition 2.1 and Proposition 2.2, we obtain the following.

2.3 Theorem. *For an analytic subspace Z with a locally stable parametrization of a compact complex manifold Y, there exists an analytic family $\mathcal{F}=(\mathcal{Y}, \mathfrak{X}, \pi, M, 0, \psi)$ of logarithmic deformations of a pair (Y, Z) (resp. $\mathcal{F}=(Y\times M, \mathfrak{X}, \pi, M, 0)$ of locally trivial displacements of Z in Y) such that;*

(i) the characteristic map $\rho_0 : T_0(M) \longrightarrow H^1(Y, T_Y(\log Z))$ (resp. $\sigma_0 : T_0(M) \longrightarrow H^0(Z, \mathcal{N}_{Z/Y})$) is injective,

(ii) it is complete at any point $t\in M$ (resp. it is maximal at any point $t\in M$)), and

(iii) it is semi-universal at 0 (resp. it is universal at 0).

Furthermore, if $H^2(Y, T_Y(\log Z))=0$ (resp. $H^1(Z, \mathcal{N}_{Z/Y})=0$), then the parameter space M is non-singular and the characteristic map *$\rho_0 : T_0(M) \longrightarrow H^1(Y, T_Y(\log Z))$ (resp. $\sigma_0 : T_0(M) \longrightarrow H^0(Z, \mathcal{N}_{Z/Y})$) is bijective.*

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Non-Galois triple coverings and double coverings

by Hiro-o TOKUNAGA

§0. Introduction

To study non-Galois triple coverings of smooth projective varieties, " the Cardano formula" plays an important role. In preceding two papers [7], [8], we have studied non-Galois triple coverings of algebraic surfaces by using the Cardano formula. The method used there was different from R. Miranda's one which was developed in his paper [4]. This article can be regarded as a research by the same method as preceding papers [7], [8]. But, our view point is different from them in the following sense :

Let $p : X \longrightarrow Y$ be a finite normal non-Galois triple covering of a smooth projective variety Y. In [7], we have defined " the discriminant variety, $D(X/Y)$" and " the minimal splitting variety, $\tilde{X}$." (Cf. [7], Definition1.1 and Definition 1.2.) Both of them are Galois coverings of Y, and satisfy the diagram :

(diagram 1)

$p_1 : \tilde{X} \longrightarrow Y$: a Galois covering with Galois group $\mathfrak{S}_3$.
$\beta_1 : D(X/Y) \longrightarrow Y$: a double covering,
$\beta_2 : \tilde{X} \longrightarrow D(X/Y)$: a cyclic triple covering,
$\alpha : \tilde{X} \longrightarrow X$: a double covering.

In [7], [8], we have studied the structure of a given triple covering $p : X \longrightarrow Y$ and a concrete construction for a given Y and a given non-Galois cubic extension of the rational function field of Y. In both cases, our main problem was concerned with the base variety and its non-Galois triple coverings. In this paper, we will attend the double covering $D(X/Y)$. From the above diagram, we can consider the following natural correspondence :

(diagram 2)

We would like to consider something like an "inverse" of this correspondence, that is, to give some conditions for a double covering Z of Y such that there exists a non-Galois triple covering of Y whose discriminant variety is Z. Concerning with this problem, we have a partial answer as follows :

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Proposition 1.1. Let $f: Z \longrightarrow Y$ be a smooth finite double covering over Y. Let D_1 be a reduced effective divisor and D_2, D_3 be effective divisors. We denote the involution induced by the covering transformation by σ, and σ^*D_1 is the transformed divisor of D_1 by σ. Assume that

(i) $D_1 \cap \sigma^*D_1$ has codimension at least 2,

(ii) $D_1 + 3D_2$ is linearly equivalent to $\sigma^*D_1 + 3D_3$.

Then, there exitsts a non-Galois triple covering of Y whose discriminant variety is Z, and $f(D_1)$ is just the totally branched divisor of the non Galois triple covering.

The conditions in the above proposition can be reasonable, because we have the following proposition.

Proposition 1.2. Let $p: X \longrightarrow Y$ be a finite normal non-Galois triple covering. Assume that the discriminant variety $D(X/Y)$ is smooth. Then there exist three effective divisors D_1, D_2, D_3 which satisfy the conditions as follows:

(i) D_1 is reduced and the intersection of D_1 and σ^*D_1 has codimension at least 2, where σ denotes the involution on $D(X/Y)$,

(ii) $D_1 + 3D_2$ is linearly equivalent to $\sigma^*D_1 + 3D_3$,

(iii) $D_1 + \sigma^*D_1$ is just the branch divisor of β_2.

There are some application of the above propositions. For the first, we will consider non-Galois totally ramified triple coverings of abelian varieties, and we have :

Therorem 2.4. Let $p: X \longrightarrow A$ be a smooth, finite, non-Galois totally ramified triple covering of an abelian variety A. Then, there exist an elliptic curve E and a non-Galois totally ramified triple covering $p: C \longrightarrow E$ such that the following diagram commutes :

(diagram 3)

where q has a connected fibre.

Next, we will study non-Galois triple coverings of algebraic surfaces with Picard number = 1. Let Σ be such a surface, and C be a smooth curve on Σ. Then, we have :

Theorem 3.1. Let (Σ, C) be as above, D be a smooth curve on Σ, and $p : S \longrightarrow \Sigma$ be a finite normal triple covering of Σ which satisfies :

(i) the branch locus $\Delta(S/\Sigma) = C + D$,

(ii) for any $x \in C$, $p^{-1}(x)$ consists of two points,

(iii) for any $y \in D$, $p^{-1}(y)$ consists of one point.

Assume that $C \approx 2n\ell$ where ℓ is a generator of $NS(\Sigma)$.

Then, we have :

a) $D \approx m\ell$ $m \leq 2n$.

b) for each $x \in C \cap D$, $p^{-1}(x)$ is a singular point of A_{3k-1} type, where k depends on x.

Moreover, if Σ is simply connected, and the equality in the above statement a) holds, we can study their structure in detail, and we will obtain Theorem 3.4.

This article consists of three sections. In §1, we prove Proposition 1.1, and Proposition 1.2. In §2, we will study non-Galois totally ramified triple coverings of abelian varieties, and prove Theorem 2.4. In the last section, we will study non-Galois triple coverings of algebraic surfaces with $\rho(\Sigma) = 1$, and prove some statements.

Notations and conventions.

In this article, the ground field is always the complex number field, $\mathbf{C}$.

For a finite normal non-Galois triple covering $p : X \longrightarrow Y$, varieties $D(X/Y)$ and $X^{\sim}$ always mean " the discriminant variety" and "the minimal splitting variety, respectively, and morphisms p_1, β_1, and β_2 always mean the morphisms satisfying the first diagram in § 0.

$\Delta(X/Y)$: the branch locus of a triple covering $p : X \longrightarrow Y$

$\mathbf{C}(X)$: the rational function field of a variety X.

Let ϕ be an element of $\mathbf{C}(X)$. we denote its zero divisor and polar divisor by $(\phi)_0$ and $(\phi)_\infty$ respectively.

Let D be a component of the branch locus of a triple covering $p : X \longrightarrow Y$. We call D is a totally branched divisor if for any p D, $p^{-1}(p)$ consists of one point.

For a line bundle L over a smooth variety X, we call L is numerically effective (*nef* for short) if $L.C \geq 0$ for and irreducible cuuve on C.

For a line bundle L over a smooth variety X, we call L is big if $\kappa(D, X) = \dim X$.

Let D_1, D_2 be divisors on X. $D_1 \sim D_2$ means linearl equivalence for two divisors, and $D_1 \approx D_2$ means algebraic equivalence for two divisors.

NS(S) : Neron-Severi group of a surface S.

$\rho(S)$: Picard number of a surface S .

§1. Triple coverings and double coverings

Let Y be a smooth projective variety. In this section, we will consider somthing like an "inverse" of the natural correspondence from non-Galois triple coverings of Y to double coverings of Y. For the first, we will construct a finite normal non-Galois triple covering of Y from a smooth double covering of Y.

Propositions 1. 1. Let $f : Z \longrightarrow Y$ be a smooth finite double covering of Y. Let D_1 be a reduced effective divisor, and D_2, D_3 be effective divisors. We denote the involution induced by the covering transformation by σ, and σ^*D_1 is the transformed divisor of D_1 by σ. Assume that

(i) $D_1 \cap \sigma^*D_1$ has codimension at least 2,

(ii) $D_1 + 3D_2$ is linearly equivalent to $\sigma^*D_1 + 3D_3$

Then, there exists a non-Galois triple covering of Y whose discriminant variety is Z, and $f(D1)$ is just the totally branched divisor of the non-Galois triple covering.

Proof. From the above assumptions, there exists a rational function $\phi \in C(Z)$ which satisfies :

$$(\phi)_0 = D_1 + 3D_2 ,$$

$$(\phi)_\infty = \sigma^*D_1 + 3D_3 .$$

Let g_1, g_2, and g_3 be defining equations of D_1, D_2, and D_3 respectively. Then,

$$\phi\sigma^*\phi = (g_1 g_2^3)/(\sigma^*g_1 g_3^3)\cdot(\sigma^*g_1\sigma^*g_2^3)/(g_1\sigma^*g_3^3)$$
$$= ((g_2\sigma^*g_2)/(g_3\sigma^*g_3))^3 .$$

Note that there are two possibilities :

(i) $D_2 + \sigma^*D_2$ is linearly equivalent to $D_3 + \sigma^*D_3$,

(ii) $D_2 + \sigma^*D_2$ is not linearly equivalent to $D_3 + \sigma^*D_3$.

Now, we will define a cyclic cubic extension K of $C(Z)$ for each case as follows :

Case (i) $\quad K = C(Z)(\xi),$

$$\xi^3 = \phi .$$

Case (ii) $\quad K = C(Z)(\xi),$

$$\xi^3 = \tilde{\phi} ,$$

where $\phi^{\sim} = a\phi$, $a = ((g_2\sigma^*g_2)/(g_3\sigma^*g_3))^3$ ($= \phi\sigma^*\phi$) .

Claim. Let K be the field defined as above. Then, for both case (i) and (ii), K is a Galois extension of $C(Y)$, and its Galois group is $\mathfrak{S}_3$ (third symme tric group).

Proof of Claim. Case (i) : Since $\sigma^*\phi = \psi^3/\phi$, $\psi \in C(Y)$, our claim is clear. Case (ii) : Since $\sigma^*\phi^{\sim} = a\sigma^*\phi = a^2/\phi = a^3/\phi^{\sim}$ and $a \in C(Y)$, our claim is clear.

Let Z' be the K - normalization of Y. Then, by the above claim, Z' is a Galois covering of Y whose Galois group is $\mathfrak{S}_3$ for each case. Let σ be a birational involution on Z' induced by an element of the Galois group. By Proposition 1. 3, [7], it is an automorphism. Let X be the quotient variety of Z' by σ Then, it is clear that X is a non-Galois triple covering of Y whose discriminant variety is Z.

Q. E. D.

Remark. 1. Let D be the totally branch divisor of the triple covering defined in the above proposition. Then, $\beta_1{}^*D$ (= f^*D) is contained in $D_1 + \sigma^*D_1$.

2. Since there are three involutions in $\mathfrak{S}_3$, there are three non-Galois triple coverings of Y corresponding to each involution. But, they are all isomorphic to each other.

Proposition 1.2. Let $p : X \longrightarrow Y$ be a finite normal non-Galois triple covering. Assume that the discriminant variety $D(X/Y)$ is smooth. Then there exist three effectiive divisors D_1, D_2, D_3 which satisfy the conditions as follows :

(i) D_1 is reduced and the intersection of D_1 and σ^*D_1 has codimension at least 2, where σ denotes the involution on $D(X/Y)$,

(ii) $D_1 + 3D_2$ is linearly equivalent to $\sigma^*D_1 + 3D_3$,

(iii) $D_1 + \sigma^*D_1$ is just the branch divisor of β_2.

Remark. Let D be the totally branch divisor of p . Then, $\beta_1{}^*D = D_1 + \sigma^*D_1$.

Proof. Since X is a non-Galois triple covering over Y, we may assume that $C(X)$ is a cubic extension of $C(Y)$ as follows :

$$C(X) = C(Y)(\theta),$$

where θ satisfies a cubic equation :

$$X^3 + 3aX + 2b = 0, \quad a, b \quad C(Y).$$

Under the above notation, D(X/Y) is the $C(Y)(\zeta)$-normalization of Y, where ζ satisfies a quadratic equation :

$$X^2 = a^3 + b^2.$$

Let K be the minimal splitting field of C(X). Then, by the Cardano's formla, we have :

$$K = C(D(X/Y))(\xi),$$

$$\xi^3 = -\beta_1{}^* b + \zeta.$$

Put $\phi = -\beta_1{}^* b + \zeta$, and we denote its zero divisor and polar divisor by $(\phi)_0$, $(\phi)_\infty$ respectively, and its decomposition into their irreducible components as follows :

$$(\phi)_0 = \sum_i \mu_i D_i^{(0)}, \quad (\phi)_\infty = \sum_j v_j D_j^{(\infty)} \quad ,\text{where} \quad \mu_i, v_j \geq 0$$

In the above notation, we rewrite μ_i and v_j as follows :

$\mu_i = \mu'_i + 3\mu''_i$, $v_j = v'_j + 3v''_j$, where μ'_i , $v_j = 1$ or 2 and μ''_i , $v''_j \geq 0$.

Then we can write $(\phi)_0$ and $(\phi)_\infty$ as follows :

$$(\phi)_0 = \sum_i{}' \mu'_i D_i^{(0)} + 3\sum_i{}' \mu''_i D_i^{(0)},$$

$$(\phi)_\infty = \sum_j{}' v'_j D_j^{(\infty)} + 3\sum_j{}' v''_j D_j^{(\infty)},$$

where Σ'_i and Σ'_j denote that the sums are taken for non-zero μ'_i, μ''_i, v'_j, and v''_j. Let $D_i^{(0)}$ be an irreducible component of $(\phi)_0$ for which μ'_i is not equal to zero. Since $\phi\,\sigma^*\phi = a^3$, $\sigma^* D_i^{(0)} = D_i^{(0)}$ and there are two possibilities :

Case (1) $\sigma^* D_i^{(0)}$ is one of the irreducible components of $(\phi)_0$

μ'_i of $D_i^{(0)} = 1$, and μ'_i of $\sigma^* D_i^{(0)} = 2$,

or

μ'_i of $D_i^{(0)} = 2$, and μ'_i of $\sigma^* D_i^{(0)} = 1$.

Since $\sigma^2 = id$, we may assume that μ'_i of $D_i^{(0)} = 1$, and μ'_i of $\sigma^* D_i^{(0)} = 2$.

Case (2) $\sigma^* D_i^{(0)}$ does not appear in the irreducible components of $(\phi)_0$

In this case, $\sigma^* D_i^{(0)}$ must appear in the irreducble components of $(\phi)_\infty$, and $\mu'_i = v'_j$.

For any $D_j^{(\infty)}$ for which v'_j is not equal to zero, we have the same results as above. Hence, we can rewrite $(\phi)_0$ and $(\phi)_\infty$ as follows :

$$(\phi)_0 = \sum_{\mu'_i \neq 0, \sigma^* D_i^{(0)} \not\subset (\phi)_0} \mu'_i D_i^{(0)} + \sum_{\mu'_i = 1, \sigma^* D_i^{(0)} \subset (\phi)_0} (D_i^{(0)} + 2\sigma^* D_i^{(0)}) + 3 \sum_{\mu''_i \neq 0} \mu''_i D_i^{(0)}$$

$$(\phi)_\infty = \sigma^* \Big(\sum_{\mu'_i \neq 0, \sigma^* D_i^{(0)} \not\subset (\phi)_0} \mu'_i D_i^{(0)} \Big) + \sum_{v'_j = 1, \sigma^* D_j^{(\infty)} \subset (\phi)_\infty} (D_j^{(\infty)} + 2\sigma^* D_j^{(\infty)}) + 3 \sum_{v''_j \neq 0} v''_j D_j^{(\infty)}$$

Now, we will define three reduced divisors D_1, D_2, and D_3 as follows :

$$D_1 := \sum_{\mu_i' = 1} D_i^{(0)} + \sum_{\mu'_i = 2, \sigma^* D_i^{(0)} \not\subset (\phi)_0} \sigma^* D_i^{(0)} + \sum_{v'_j = 1, \sigma^* D_j^{(\infty)} \subset (\phi)_\infty} \sigma^* D_j^{(\infty)},$$

$$D_2 := \sum_{\mu_i' = 2, \sigma^* D_i^{(0)} \not\subset (\phi)_0} D_i^{(0)} + \sum_{\mu'_i = 1, \sigma^* D_i^{(0)} \subset (\phi)_0} \sigma^* D_i^{(0)} + \sum_{\mu''_i \neq 0} \mu''_i D_i^{(0)},$$

$$D_3 := \sum_{v_j' = 1, \sigma^* D_j^{(\infty)} \subset (\phi)_\infty} D_j^{(\infty)} + \sum_{\mu'_i = 2, \sigma^* D_i^{(0)} \not\subset (\phi)_0} \sigma^* D_i^{(0)} + \sum_{v''_j \neq 0} v''_j D_j^{(\infty)},$$

It is clear that divisors D_1, D_2, D_3 satisfy the statement (i), (ii) in Proposition 1. 2, so we get the desired divisors on $D(X/Y)$.

Q. E. D.

Corollary 1. 3. Let X be a finite normal non-Galois triple covering of a smooth projective variety Y, and B be one of the irreducible components of totally branched divisors on Y. Assume that D(X/Y) is smooth. Then, the divisor $\beta_1{}^*B$ is a divisor on D(X/Y) which consists of two irreducible components which are isomorphic to each other.

Proof. Assume that $\beta_1{}^*B$ is irreducible. Then, it is clear that $\beta_1{}^*B = \sigma^*(\beta_1{}^*B)$. However, by the assumption, $\beta_1{}^*B$ is an irreducible component of the branch divisor of β_2. This contradicts to Proposition 1. 2, (i). Therefore, $\beta_1{}^*B$ consists of two irreducible components, and they are isomorphic to each other.

Q. E. D.

§2. Totally ramified triple coverings of abelian varieties.

In [7], we have definend " a totally ramified triple covering" as follows :

Definition 2.1. Let $p : X \longrightarrow Y$ be a normal finite triple covering over a smooth projective variety Y. We say that p is totally ramified if for all irreducible components of ramification divisors, its ramification index is equal to 3.

Remark 2.2. Note that if p is cyclic, then p is totally ramified.

It is easy to show that if Y is simply connected, then p is always cyclic. (Cf. Proposition 3.1, [7]) In particular, any totally ramified triple coverings of $\mathbf{P}^n$ are cyclic. In the rest of this section, we will study non-Galois totally ramified triple coverings over abelian varieties. First of all, we will give an example of non-Galois totally ramified triple coverings. It is an easy one but essential.

Example 2.3. Let $E = \mathbf{C}/(1,\tau)$, $\mathrm{Im}(\tau) > 0$ be an elliptic curve, and σ be an isomorphism of E such that

$$\begin{array}{ccc} \sigma : E & \longrightarrow & E \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ z & \longmapsto & z+\eta \end{array}$$

where η is a two torsion point of E.
Let p, q be points on E such that $q = \sigma(p)$. Then, since $2p$ is linearly equivalent to $2q$, there exists a rational function ϕ satisfying

$$(\phi) = 2p - 2q.$$

Moreover, it is clear that

$$\sigma'\phi = 1/\phi.$$

Let C be the $\mathbf{C}(E)(\xi)$-normalization of E, where $\xi^3 = \phi$, and $\overline{E}$ be the quotient curve of E by σ. Then, it is easy to show that C is a Galois covering over E whose Galois group is $\mathfrak{S}_3$ (the third symmetric group). Let σ' be an element of order 2, and $\overline{C}$ be the quotient curve of C by σ . Then, $p : \overline{C} \longrightarrow \overline{E}$ is a non-Galois totally ramified triple covering over E, and its brach locus is one point.

Figure 1

Now we will state our theorem.

Theorem 2.4. Let $p : X \longrightarrow A$ be a smooth, finite, non-Galois totally ramified triple covering over an abelian variety A. Then, there exists an elliptic curve E and a non-Galois totally ramified triple covering $p : C \longrightarrow E$ such that the following diagram commutes :

(diagram 3)

i. e. p is induced by p .

To prove Theorem 2.4, we need a lemma.

Lemma 2.6. Let $p : X \longrightarrow A$ be as above. Then, β_1 is unramified. (Hence, $D(X/A)$ is also an abelian varietiy.) Moreover, $\beta_1{}^*(\Delta(X/A)) = D + T_\eta{}^* D$, where η is a two torsion point of $D(X/A)$ and T_η is the translation induced by η.

Proof. Under the assumptions, $\alpha : X^{\sim} \longrightarrow X$ is unramified in codimesion 1. This fact implies that β_1 is also unramified in codimension 1. So, by the purity of the branch locus (see Zariski [9], or Fischer [2]), β_1 is unramified. Hence, by theorem of Serre-Lang (see Mumford [5], p167), $D(X/A)$ has a structure of abelian variety such that β_1 is isogeny. Since there is a one to one correspondence between finite subgroups and isogeneis (see Mumford [5], p72), there is the discrete subgroup K such that

$$D(X/A)/K = A.$$

Therefore, we may assume that $K = \langle \eta \rangle$, $2\eta = 0$. By Corollary 1. 3 , $\beta_1{}^*(\Delta(X/A))$ has a form such as $D + \sigma^* D$, where σ denotes the involution on $D(X/A)$, hence, we have

$$\beta_1{}^*(D(X/A)) = D + T_\eta{}^* D.$$

Q. E. D.

To prove our theorem, we quote the following fact.

Fact 2.7. Let D be an effective divisor on an abelian variety A. Then, there exists an abelian variety A_1, a morphism with a connected fibre q : $A \longrightarrow A_1$, and an ample divisor D_1 on A_1 such that $D = q' D_1$

For a proof, see Mumford [5], p88.

Proof of Theorem 2.4. Since X is smooth, and a is unramified in codimension 1, a is unramified. Hence, $X^{\sim}$ is also smooth, and $\beta_1{}^*(\Delta(X/A))$ is a smooth divisor. By the results in §1, $\beta_1{}^*(\Delta(X/A))$ is can not be irreducible. For this divisor, we have the next claim :

Claim 2.8. There exist an elliptic curve E, and morphism $q : A \longrightarrow E$, and a divisor δ on E such that :

$$D(X/A) = q^*(\delta) .$$

Hence, all irreducible components are abelian subvarieties, and they do not inetersect each other.

Proof of Claim 2.8. By Fact 2.7, there exists an abelian variety A_1, a morphism $q_1 : D(X/A) \longrightarrow A_1$ with a connected fibre, and an ample divisor D_1 on A_1 such that $q_1{}^*(D_1) = \beta_1{}^*(\Delta(X/A))$. Assume that $\dim A_1 > 1$. Then, since D_1 is ample, D_1 is connected, so $\beta_1{}^*(\Delta(X/A))$ is also connected. Since $\beta_1{}^*(\Delta(X/A))$ is smooth, $\beta_1{}^*(\Delta(X/A))$ must be irreducible. This is contradiction. Therefore, $\dim A_1 = 1$. Moreover, an arbitrary irreducible component of $\beta_1{}^*(\Delta(X/A))$ is transformed to another different irreducible component by the involution on $D(X/A)$, so, $\Delta(X/A)$ has the same form such as $\beta_1{}^*(\Delta(X/A))$, and our claim holds.

Q. E. D.

Now, we will continue a proof of Theorem 2.4. It is sufficient to show that the following statement.

The diagram :

(diagram 4)

is commutes, where $C^{\sim}$ is a Galois covering of the elliptic curve E whose Galois group is isomorphic to $\mathfrak{S}_3$ (the third symmetric group), and q, q_1, q_2 have connected fibres.

Step 1 : The diagram

(diagram 5)

commutes, and q, q_1 have connected fibres.

This statement follows the proof of claim 2.8.

Step 2 : There exists a Galois covering of the elliptic curve E whose Galois group is isomorphic to $\mathfrak{S}_3$, and the diagram :

(diagram 6)

commutes, and q_2 has a connected fibre.

Let $q_2 : X' \longrightarrow C$ be the stein factorization of the morphism $(q_1 \circ \beta_2)$: $X' \longrightarrow E'$. We will show that C' has desired properties. For the first, we prove the following claim :

Claim 2.9. Let y be a general point on E'. Then $(q_1 \circ \beta_2)^{-1}(y)$ consists of three irreducible components.

Proof of Claim 2.9. By the results of §1, we have

$$C(X) = C(D(X/A))(\xi),$$
$$\xi^3 = \phi,$$
$$(\phi) = (D_1 + 3D_2) - (\sigma' D_1 + 3D_3),$$

where D_1, D_2, D_3 are effective divisors, and D_1 is reduced and consists of just branch divisors. Put

$$L = D_2 - D_3 .$$

It is sufficient to show that

$$L\Big|_{q_1^{-1}(y)}$$

is trivial. Since $3L \sim \sigma' D_1 - D_1$, and all components of $\sigma' D_1$ and D_1 are the branch divisors of β_2,

$$3L\Big|_{q_1^{-1}(y)}$$

is trivial. On the other hand,

$$\sigma'\phi\ \phi = a^3, \text{ for some } a \in C(A).$$

Therefore,

$$\sigma^* D_2 + D_2 - (\sigma^* D_3 + D_3) = (a)_0 - (a)_\infty \sim 0.$$

So we have :

$$\sigma^* D_2 - \sigma^* D_3 \sim -(D_2 - D_3) \sim -L.$$

By Lemma 2.6, we have $\sigma^* D_2 = T_\eta^* D_2$, $\sigma^* D_3 = T_\eta^* D_3$, hence by Theorem of the square (see Mumford [ ], p59),

$$2\sigma^* D_2 \sim T_{\eta+\eta}^* D_2 + D_2 \sim 2D_2,$$

$$2\sigma^* D_3 \sim T_{\eta+\eta}^* D_3 + D_3 \sim 2D_3 .$$

By these equivalences, we have :

$$\begin{aligned} -2L &\sim 2(\sigma^* D_2 - \sigma^* D_3) \\ &\sim 2(D_2 - D_3) \\ &\sim 2L . \end{aligned}$$

Since

$$3L|_{q_1^{-1}(y)} \sim 0$$

is trivial, we have :

$$0 \sim 4L|_{q_1^{-1}(y)} \sim L|_{q_1^{-1}(y)}$$

as desired.

By Claim 2.9, $C' \longrightarrow E$ is a covering of E of degree 6. To show that the covering is a Galois covering as desired type, notice that since q_2 has a connected fibre, $C(C')$ is the algebraic closure of $C(E)$ in $C(X')$.

By Step1 and Step 2, we obtain Theorem 2.4.

Q. E. D.

Remark 2.10. Let $p : X \longrightarrow A$ be a cyclic triple covering of an abelian variety A. Assume that X is smooth and the branch divisor of p is the same form as in Claim 2.8. Under these conditions, it seems that the analogous statement such as Theorem 2.4 holds, that is X is induced by a cyclic triple covering of an elliptic curve E. But this is false. We will give an example :

Let E_1, E_2 be an elliptic curves, and put $A = E_1 \times E_2$. Let δ_1 (resp. δ_2) be principal divisors on E_1 (resp. E_2) such that

$\delta_1 = p_1 + p_2 + p_3 - 3p_4$, where p_i are all distinct four points,

$\delta_2 = 3q_1 - 3q_2$, where $q_2 = q_1 + \eta$, η : a 3-torsoin point.

Let ϕ be an element of $C(A)$ satisfying

$$(\phi) = p_1^*(\delta_1) + p_2^*(\delta_2),$$

where p_i denotes the projection $A = E_1 \times E_2 \longrightarrow E_i$.

Let $p : X \longrightarrow A$ be a cyclic triple covering of A such that

$$C(X) = C(A)(\xi), \ \xi^3 = \phi .$$

Then, X satisfies the conditions as above, but it is not induced by a cyclic triple covering of E_1.

§3. A non-Galois triple covering of an algebraic surface with $\rho(S) = 1$

Let Σ be an algebraic surface with $\rho(\Sigma) = 1$. In this section, we will study a finite normal non-Galois triple covering S of Σ whose branch locus consists of two smooth curves. Let C be a smooth curve on Σ. Then we have the following theorem :

Theorem 3.1. Let (Σ, C) be as above, D be a smooth curve on Σ, and $p : S \longrightarrow \Sigma$ be a finite normal triple covering of Σ which satisfies :

(i) the branch locus $\Delta(S/\Sigma) = C + D$,

(ii) for any $x \in C$, $p^{-1}(x)$ consists of two points,

(iii) for any $y \in D$, $p^{-1}(y)$ consists of one point.

Assume that $C \approx 2nl$ where l is a generator of $NS(\Sigma)$.

Then, we have :

a) $D \approx ml$, $m \leq 2n$,

b) for each $x \in C \cap D$, $p^{-1}(x)$ is a singular point of A_{3k-1} type, where k depends on x.

To prove Theorem 3.1, we need the following lemma :

Lemma 3.2. Let C, D be the curves in Theorem 3.1. Let p be a point in $C \cap D$. Then, by taking a suitable local coordinate system (x, y), we can represent local equations of C and D as follows :

$p = (0,0)$,

$C : y = 0$,

$D : f(x, y) = y + ax^{2k} + \cdots = 0, \quad k \in N, \quad a \neq 0.$

Proof. By taking a suitable local coordinate system, we may assume that

$p = (0,0)$

$C : y = 0$,

$D : f(x, y) = 0$.

Since D is smooth, $(\partial f/\partial x\,(0,0), \partial f/\partial y(0,0)) \neq (0,0)$. Assume that

$$\begin{vmatrix} \partial f/\partial x\,(0,0) & \partial f/\partial y\,(0,0) \\ 0 & 1 \end{vmatrix} \neq 0$$

Then, we can rewrite $f(x,y)$ such as :

$$f(x,y) = x\,f_1(x,y) + f_2(y), \quad f_1(0,0) \neq 0, \quad f_2(0) = 0.$$

By using the local coordinate system (x,y), $\beta_1 : D(S/\Sigma) \longrightarrow \Sigma$ is represented as $\beta_1 : (x, z) \longrightarrow (x, y) = (x, z^2)$. Hence, by using the local coordinate system (x, z), the local equation of $\beta_1{}^*D$ (we denote it by $f^{\sim}$) is as follows :

$$f^{\sim}(x, z) = x\, f_1(x, z^2) + f_2(z^2),$$

where f_1 and f_2 are as above.

Therefore, $\beta_1{}^*D$ is smooth and p is a branch point of $\beta_1{}^*D \longrightarrow D$. Hence, $\beta_1{}^*D$ is irreducible around $\beta_1{}^{-1}(p)$, and this contradicts to the result in Corollary 1.3. So, $f(x, y)$ satisfies $\partial f/\partial x(0,0) = 0$, and we can write $f(x, y)$ as :

$$f(x, y) = y\, g_1(x, y) + g_2(x), \quad g_1(0,0) \neq 0,$$
$$g_2(x) = x^h g_2(x), \quad g_2(0) \neq 0, \quad h > 0.$$

So, $f(x, y)/g_1(x, y)$ is expanded as :

$$f(x, y)/g_1(x, y) = y + a(x, y)x^h + \cdots, \quad a(0,0) \neq 0.$$

If h is odd, then by using the same local coordinate system (x, z) as above :

$$f(x, z^2)/g_1(x, z^2) = z^2 + a(x, z^2)\, x^h + \cdots .$$

But this expansion means that $\beta_1{}^*D$ is irreducible around $\beta_1{}^{-1}(p)$, and this contradicts to Corollary 1. 3. Hence, h must be even, and we have the desired result.

Proof of Theorem 3. 1. a) Put $\beta_1{}^*D = D_1 + \sigma^*D_1$, where σ is the involution of $D(S/\Sigma)$. Since D is smooth, $D \cap \sigma^*D \subset (\beta_1{}^*C)_{red}$, and by Lemma 3.3, we can derive equalities :

$$D_1\, \sigma^*D_1 = D_1\, (\beta_1{}^*C)_{red} = \sigma^*D_1\, (\beta_1{}^*C)_{red} .$$

Hence, we have $\beta_1{}^*C\,(D_1 - \sigma^*D_1) = 0$, and the Hodge index theorem, we have $D_1{}^2 \leq D_1\, \sigma^*D_1$. Put $D \approx m\ell$ Then, by the above equalities, we have :

$$D_1(\beta_1{}^*C)_{red} = \sigma^*D_1(\beta_1{}^*C)_{red} = nm\ \ell^2.$$

Therefore, we have :

$$\begin{aligned} 2m^2\ \ell^2 &= (\beta_1{}^*D)^2 \\ &= (D_1 + \sigma^*D_1)^2 \\ &\leq 4D_1\, \sigma^*D_1 \\ &= 4D_1(\beta_1{}^*C)_{red} = 4nm\ \ell^2 . \end{aligned}$$

Finally, we have the desired inequalty :

$$m \leq 2n .$$

b) Let p be a point in $C \cap D$. By taking suitable local cooridinate systems $U : (x, y)$ and $\beta_1^{-1}(U) : (x, z)$ as in Lemma 3. 2, we may assume that :

$p = (0, 0)$,
$C : y = 0$,
$D : y + x^{2k} = 0$,

and by using this local coordinate, β_1 is represented as :

$\beta_1 : (x, z) \longrightarrow (x, y) = (x, z^2)$.

So, we have :

$(\beta_1{}^*C)_{red} \; : \; z = 0$,
$D_1 : z + \sqrt{-1} \; x^k$
$\sigma^*D_1 : z - \sqrt{-1} \; x^k$

Let $S^{\sim}$ be the minimal splitting surface of S. By Proposition 1. 2,

$D_1 - \sigma^*D_1 \in 3NS(D(S/\Sigma))$.

Hence, over $\beta_1^{-1}(U)$, $S^{\sim}$ is obtained as the normalization of a hypersurface defined by an equation :

$w^3 - (z + \sqrt{-1} \; x^k)(z - \sqrt{-1} \; x^k)^2 = 0$. (See [6], Proposition 1. 1.)

To obtain a minimal resolution of the singularity $p_1^{-1}(p)$, consider a k-times succession of blowing-ups $\pi : V \longrightarrow \beta_1^{-1}(U)$ such that the strict transformations of $(\beta_1{}^*C)_{red}$, D_1 and σ^*D_1 (we dnote them by $\overline{R}$, $\widetilde{D}_1$, and $\widetilde{\sigma^*D}_1$, respectively) satisfy :

$\overline{R} \cap \widetilde{D}_1 \cap \widetilde{\sigma^*D}_1 = \phi$ in V.

(Figure 2)

Then, we can construct a minimal resolution of $p_1^{-1}(p)$ as cyclic triple covering of V branched along $D_1 + \sigma^*D_1$, and its configration of the exceptional curves is as follows :

(Figure 3)

To obtain a minimal resolution of the singularity $p^{-1}(p)$, we must consider the action of an involution induced by σ. We denote the indcuced involution by $\sigma^{\sim}$. Then, it is easy to show that $\sigma^{\sim}$ has isolated fixed points as follows :

(Figure 4)

Therefore, we have a minimal resolution of $p^{-1}(p)$, and its configulation of the exceptional curves is as follows :

(Figure 5)

Since all exceptional curves are (-2)- curves, we know that $p^{-1}(p)$ is an A_{3k-1} - singularity, and this is our statement.

Q. E. D.

Example 3. 3. Let $\Sigma = P^2$, and C be a smooth plane curve of degree 2n. We will construct an example of a non-Galois triple covering S satisfying the conditions in Theorem 3. 1. Let $W = P(\underline{O}_{P^2} \oplus \underline{O}_{P^2}(n\ l))$ be a P^1-bundle over P^2, where l denotes a line. We denote the tautological line bundle by L. Then, the double covering $D(S/\Sigma)$ can be regarded as a member of 2L . Since L is generated by global sections, $L|_{D(S/\Sigma)}$ is also generated by global sections. Moreover, by the Kawamata-Viehweg vanishing theorem [3], we have $h^0(W, \underline{O}(-L)) = 0$. Hence, the restriction map $H^0(W, \underline{O}(L)) \longrightarrow H^0(D(S/\Sigma), \underline{O}(L|_{D(S/\Sigma)})$ is surjective. This means that any member of $L_{D(S/\Sigma)}$ is the restriction of a member of L. On the other hand, we can regard P^2 as a member of L, so, $\beta_1{}^*(C)_{red} = P^2.$ $D(S/\Sigma) = L.\ D(S/\Sigma)$. Therefore, for a general member D_1 of $L|_{D(S/\Sigma)}$, both D_1 and σ^*D_1 intersect $\beta_1{}^*(C)_{red}$ transeversaly. Since σ is induced by an involution $\sigma^\sim$ on W, that is multiplication by -1 on the fibre, if $D_1 = \widetilde{D}_1|_{D(S/\Sigma)}$, $\widetilde{D}_1 \in L$, $\sigma^*D_1 = \sigma^*\widetilde{D}_1|_{D(S/\Sigma)}$. Hence, σ^*D_1 is a member of $L|_{D(S/\Sigma)}$, and we have $D_1 \sim \sigma^*D_1$. Therefore, by Proposition 1.1, we can obtain the desired non-Galois triple covering S of P^2. In the above construction, D_1 and σ^*D_1 intersect transversaly. So, it follows that all singularities of S are A_2 - singularities by the proof of Theorem 3. 1. We can calculate some numerical invariants of the minimal resolution $S^\sim$ of S. By similar calculation as Example 4.3, [ ], they are as follows :

$c_1{}^2(S) = 12n^2 - 36n + 27$, $c_2(S) = 18n^2 - 18n + 9$,

$p_g(S) = 1/2\ (5n^2 - 9n) + 2$, $q(S) = 0$, $K_S \approx \tilde{p}^*((2n-3)\ l)$, $\tilde{p} := \mu \circ p$

μ : the resolution.

In Example 3. 3, we have constructed an example of a non-Galois triple covering satisfying the conditions in Theorem 3. 1. It is very special one, but, if S is simply connected and the equality m = 2n holds, this example is essential because of the following theorem holds.

Theorem 3. 4. Let (Σ, C) be the pair as in Theorem 3. 1. Assume that :
(i) Σ is simply connected,
(ii) $n\ /$ is generated by global sections,
(iii) $D \approx 2n\ /$.

Then, any non-Galois triple covering S of Σ satisfying the conditions in Theorem 3. 1 is obtained by a similar method in Example 3. 4.

Proof. Let $W = P(O_\Sigma \oplus O_\Sigma(n\ /))$, $q : W \longrightarrow \Sigma$ be the projection, and L be the tautological line bundle of W. Note that L is free from base points and fixed components. Let $D(S/\Sigma)$ be the discriminant surface of $p : S \longrightarrow \Sigma$. Then, as in Example 3. 3, we can regard $D(S/\Sigma)$ as a member of $2L$. Since D is the totally branched divisor, $\beta_1{}^*D$ consists of two irreducible components D_1 and σ^*D_1. To prove Theorem 3. 4, it is sufficient to show the following claim :

Claim 3. 5. Both D_1 and σ^*D_1 are linearly equivalent to $L|_{D(S/\Sigma)}$. Moreover, there exists a member of L (we denote it by $\widetilde{D}_1$) such that $\widetilde{D}_1|_{D(S/\Sigma)} = D_1$, $\sigma^{\sim *}\widetilde{D}_1|_{D(S/\Sigma)} = \sigma^*D_1$, where $\sigma^{\sim}$ is an involution on W which induces σ on $D(S/\Sigma)$.

Proof of Claim 3. 5. By the assumption (iii), we have :

$$D_1{}^2 = (\sigma^*D_1)^2 = \sigma^*D_1 . D_1 = (\beta_1{}^*C)_{red} . D_1 = (\beta_1{}^*C)_{red} . \sigma^*D_1 .$$

Hence, by the algebraic index theorem, σ^*D_1 is algebraically equivalent to D_1. On the other hand, by the assumptions (i) and (ii), $D(S/\Sigma)$ is also simply connected(see, Catanese [1], Proposition 1. 8). Therefore, σ^*D_1 is linearly equivalent to D_1. Next, we will prove that D_1 is linearly equivalent to $L|_{D(S/\Sigma)}$. For the first, notice that :

$$2D_1 \sim \beta_1{}^*D = q^*D\ D(S/\Sigma) = q^*D .\ 2L = 4n\ q^*\ / .\ L.$$

By the Hirsch formula, $L^2 = nq^*\ / .\ L$. Therefore, on $D(S/\Sigma)$, we have :

$$L|_{D(S/\Sigma)} \ . (D_1 - L|_{D(S/\Sigma)})$$
$$= 2L . L . (q^*\ / - L) = 2L . (2L . q^*(n\ /) - L . q^*(n\ /)) = 0 .$$

Hence, by the algebraic index theorem, and the assumption (i), D_1 is linearly equivalent to $L|_{D(S/\Sigma)}$. Now, we will prove the rest. Since L is nef and big line bundle, $h^1(W, \underline{O}_W(-L)) = 0$ by the Kawamata-Viehweg vanishing theorem. Therefore, there is a divisor D_I on W with $\widetilde{D}_I|_{D(S/\Sigma)} = D_I$. Since there exists an involutions σ on W which induces σ on $D(S/\Sigma)$, we have the desired result.

Q. E. D.

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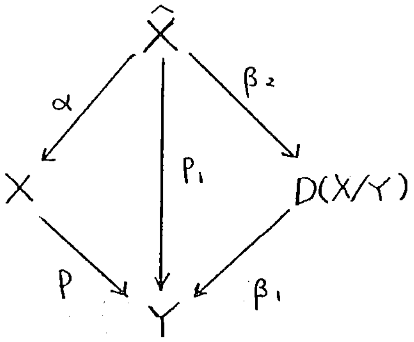
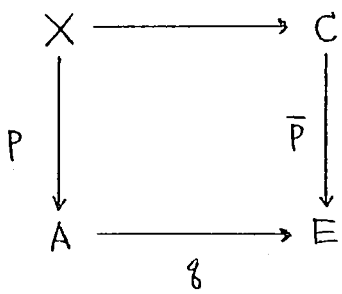
Diagrams :



(diagram 1)

[non-Galois triple coverings of Y] $\Longrightarrow$ [double coverings of Y] (diagram 2)



(diagram 3)

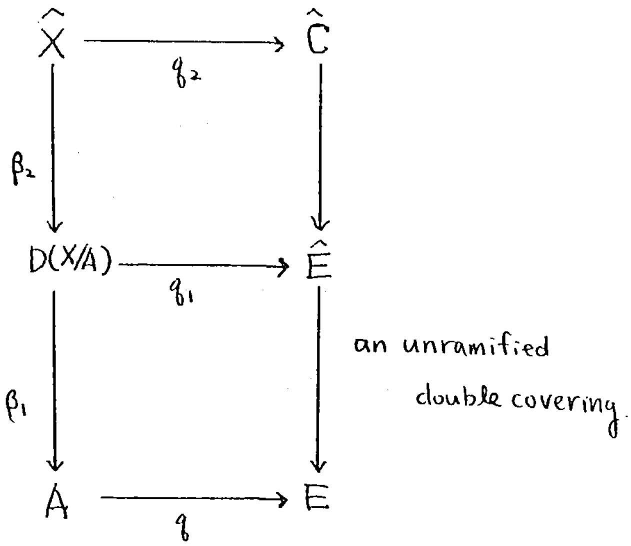


diagram 4

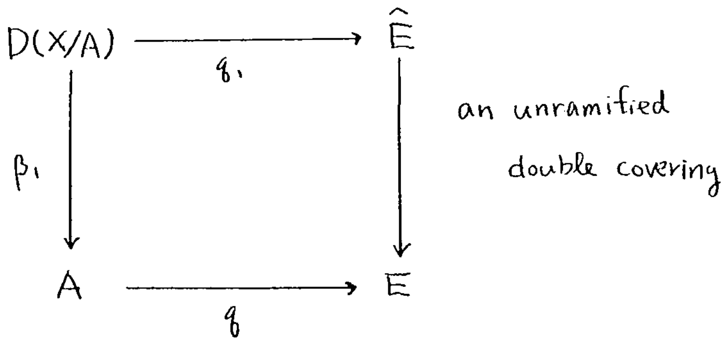


diagram 5

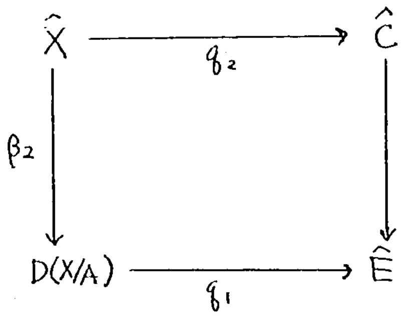


diagram 6

Figures:

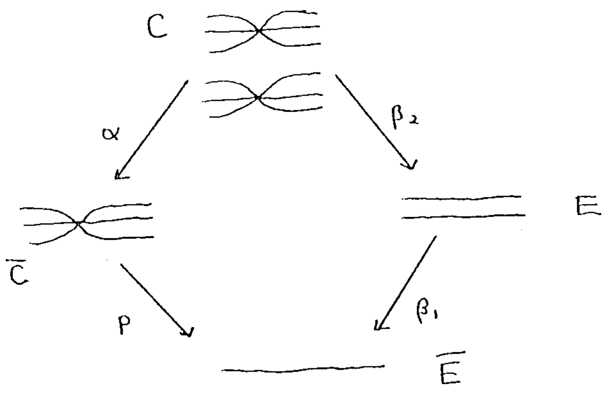


Figure 1

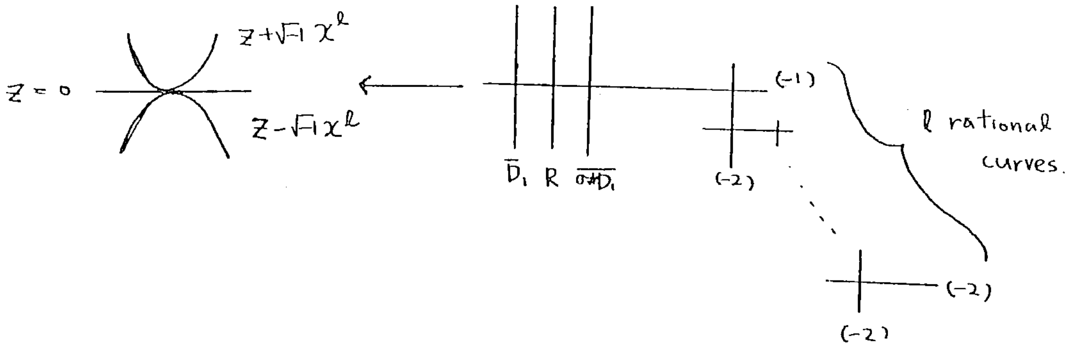


Figure 2

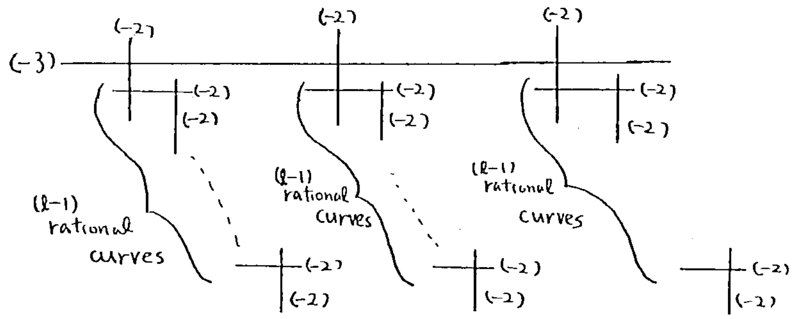


Figure 3

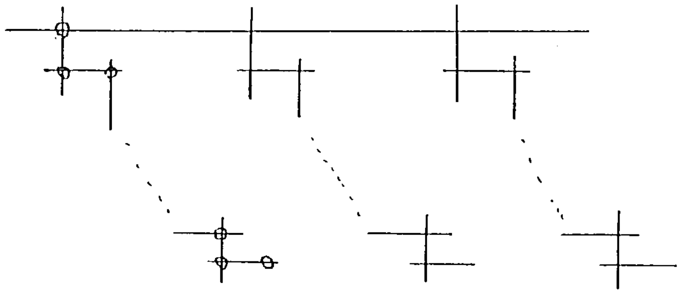


Figure 4

o ---- fixed points of σ

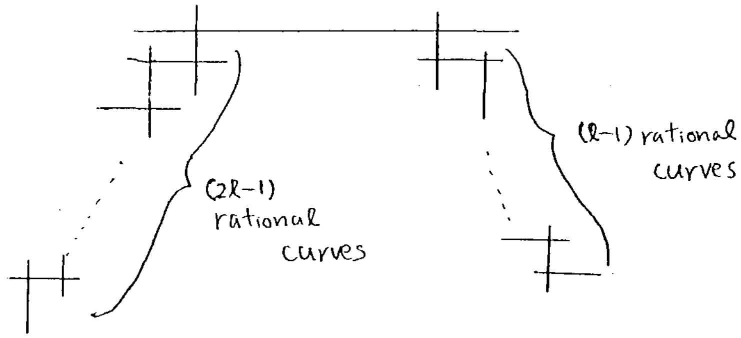


Figure 5

Partial Structures

of

a System of Projective Equations

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(Ver.1.5 – 3/20,1990)

§ 0. Introduction

It is usually difficult to see something good on the relation between the properties of a projective manifold and the geometric features of its projective embeddings by observing their defining equations. However, if we restrict ourselves to the case of arithmetically normal embeddings, we may find some subtle relations through the defining equations. Moreover, based on the results of [U-2] and [U-3], we can also study those equations regarding as (obstructed) rational sections of the conormal bundle. Thus, in this article, we intend to provide an approach to the theory of defining equations for arithmetically normal embeddings or more generally of rational sections of vector bundles.

To be more precise, we settle that "equations" means "a system of projective equations (S.P.E.)", namely, a system of minimal generators of the largest (under containment relation) homogeneous ideal defined by the embedding (see [U-2]). Then, one of the questions which first cross our mind is whether or not S.P.E.'s (or rational sections of the vector bundle) have own partial structures which reflect the geometric properties in some way, and our main interest is in this question.

Our answer to this question is fairly affirmative. In fact, S.P.E.'s (or rational sections of vector bundles) have own partial structures, and in some cases, it is possible to verify that those partial structures reflect geometric features. To obtain a partial structure of an S.P.E.(resp. of rational sections of a vector bundle), we use Lefschetz operator acting on vector bundle valued cohomology groups for measuring "the order of penetration" (cf. (2.1) Definition) of each element of the S.P.E.(resp. of each rational sections of the vector bundle), and decompose the S.P.E. (resp. the set of rational sections of the vector bundle) into several parts. We also examine the orders of penetration of the projective equations (namely, elements of S.P.E.'s) in several simple cases. And we raise several problems which will give missing links between the order of penetration and its geometric property.

Throughout this paper, we use notation and convention employed in [U-2] and [U-3], our base field is the complex number field $\mathbb{C}$, and X denotes always a non-singular projective variety, otherwise mentioned explicitly.

※ In the next version, we shall add some new sections which treat several examples relating to middle penetration order.

§1. Observations

In this section, we shall collect several well-known examples and make remarks on them in connection with defining equations. Those examples will give us an insight which support the validity of the new concept introduced in the next section.

(1.1) Example. Let X be a complete intersection (C.I.) defined by homogeneous polynomials $F_1, \cdots, F_r$ of type $(m_1, \cdots, m_r)$ in $\mathbb{P}^N(\mathbb{C}) = P$ and $j_1 : X \hookrightarrow P$ be a natural closed immersion. Assume that $\dim X \geqq 3$. Then $\operatorname{Pic} X \cong \mathbb{Z}\, j_1{}^* O_P(1)$. Hence every arithmetically normal embedding is composed of a Veronesean embedding Φ_m for P and the embedding j_1.

(1.2) Remark. Let us consider to get some information on some holomorphic invariants of X in (1.1)Example by choosing a suitable embedding of X and constructing a free resolution of the homogeneous coordinate ring of the embedding. Then there must be no objection to choosing the embedding j_1 since we automatically get a good minimal free resolution, namely Koszul resolution. Hence, we may say that projective equations of $j_1(X)$ is better than projective equations of $\Phi_m \cdot j_1(X)$ $(m \geqq 2)$.

(1.3) Example. Let $j_1 : X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = W(1)$ be a given closed immersion, m a sufficiently large positive integer, and $j_m := \Phi_m \cdot j_1$ a composed closed immersion, where $\Phi_m : W(1) \hookrightarrow \mathbb{P}^{N(m)}(\mathbb{C}) = W(m)$ $(N(m) := {}_{N+m}C_N - 1)$ denotes an m-th Veronesean embedding of

$W(1)$. Then every projective equation of $j_m(X)$ is of degree 2 except linear equations(cf. [M-1]).

(1.4) Remark. Serre's vanishing theorem tells us that a sufficiently large twisting of a coherent sheaf by an ample line bundle makes its higher cohomologies vanish. Since in some cases, the higher cohomologies contain obstructions or delicate information for geometric problems on X, we might say that the line bundle $j_1{}^*\mathcal{O}_{W(1)}(1)$ brings us more abundant geometric information than $j_m{}^*\mathcal{O}_{W(m)}(1)$, or that projective equations of $j_m(X)$ are rougher than those of $j_1(X)$ for a close study of precise structures of X.

(1.5) Example. (Mumford [M-1]) Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be a closed immersion of a projective manifold X. Then, as a closed subscheme of P, $j(X)$ is defined by the equations whose degree are equal to the degree of $j(X)$. Nevertheless, we do not know whether or not the degree of projective equations is bounded by the degree of $j(X)$.

(1.6) Remark. Even if $j(X)$ has a projective equation F whose degree is greater than the degree of $j(X)$, we may say that the equation F does not have deep relation with the intrinsic properties of X.

(1.7) Examples. (cf. (3.2) Lemma [U-2]) Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an arithmetically normal embedding, S a hypersurface defined by a homogeneous polynomial F, and W a closed subvariety of P which satisfies $j(X) = S \cap W$ (transversal). Then, the set-theoretic union of $\{F\}$ and any S.P.E. of W forms an S.P.E. of X.

(1.8) Remark. Let us denote an S.P.E. of W by $\{G_1 \cdots, G_k\}$. Then we may say that an S.P.E. $\{F, G_1, \cdots, G_k\}$ of $j(X)$ is divided into two parts $\{F\}$ and $\{G_1, \cdots, G_k\}$, and that $\{G_1, \cdots, G_k\}$ gives a "partial structure" of the S.P.E. $\{F, G_1, \cdots, G_k\}$ which reflects a decomposition of the embedding j such as $j: X \hookrightarrow W \hookrightarrow P$.

Through the examples above, we fancy that there are "useful" equations and "useless" equations. We also guss that an S.P.E. has a "partial structure" as we described in (1.8) Remark. To clarify the meaning of "partial structure", we give a definision as follows.

(1.9) Definition. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ a closed immersion of a projective maniflod X, $\mathscr{P}$ an S.P.E. of $j(X)$, and $\mathfrak{M}$ a sub set of $\mathscr{P}$. We say that $\mathfrak{M}$ is a partial structure of $\mathscr{P}$ if $\mathfrak{M}$ is an S.P.E. of a closed subvariety W of P which is non-singular along $j(X)$.

(1.10) Remark. Under the circumstances of (1.9)Definition, if $\mathfrak{M}^*$ is another S.P.E. of W. Then there exists an S.P.E. $\mathscr{P}^*$ of $j(X)$ such that $\mathfrak{M}^*$ is a partial structure of $\mathscr{P}^*$.

§2. The Order of Penetration

In [U-3], we obtained several results through the action of Lefschetz operator on the cohomologies with coefficients in a vector bundle. Now we provide a scale for measuring the "usefulness" of a given equation or of a given section of a vector bundle by using this operator for examining their capacity to penetrate the barrier of the cohomologies.

(2.1) Definition. Let X be a complex projective manifold, E a holomorphic vector bundle on X, $h \in A^1(X)$ a hyperplane class, and $\omega \in H^1(X, \Omega^1{}_X)$ the Hodge-Kahler class corresponding to the class h. Suppose that a global section $\sigma \in \Gamma(X,E)$ is given. Take the Lefschtz operator L (cf. [U-3]) defined by the class ω.

(i) For a non-negative integer p, if the cohomology class $L^p(\sigma) = \sigma \otimes \omega^p \in H^p(X, \Omega^p{}_X(E))$ is not zero and the class $L^{p+1}(\sigma) = \sigma \otimes \omega^{p+1} \in H^{p+1}(X, \Omega^{p+1}{}_X(E))$ is zero, then we say that the order of penetration (or penetration order) of the section σ with respect to the hyperplane class h is p, which is denoted by $\mathrm{pent}(\sigma; h) = p$.

Whenever we consider an equation associated with an embedding $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$, we take the hyperplane class induced by the embedding j as the class h.

(ii) For an equation $F \in H^0(P, I_{j(X)}(m))$ of $j(X)$, if the penetration order of the section $[F] \in H^0(X, N^{\vee}{}_{X/P}(m))$ is p, we say that the penetration order of the equation F is p, which is denoted by pent$(F;j(X))$ = pent(F) = p, where $N^{\vee}_{X/P}$ and $[F]$ denote the conormal bundle of $j(X)$ in P and the class of the equation F by the canonical map: $H^0(P, I_{j(X)}(m)) \to H^0(X, N^{\vee}_{X/P}(m))$, respectively.

(2.2) Remark. By the definiton, obviously: $0 \leq$ pent$(\sigma; h) \leq n = \dim X$.

In the sequel, we seek for geometric explanations of this concept "order of penetration". The first easy result is given as follows.

(2.3) Proposition. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an embedding of a projective manifold X. Assume that $j(X)$ is a complete intersection defined by homogeneous equations $F_1, \cdots, F_r$ whose degrees are $m_1, m_2, \cdots m_r$, respectively. Then,

$$\text{pent}(F_a) = n \qquad (n = \dim X,\ a = 1, \cdots, r).$$

Proof. Let us put $\sigma = [F_a] \in H^0(X, N^{\vee}_{X/P}(m_a))$. Since $N^{\vee}_{X/P} \cong \oplus O_X(-m_b)e_b$, the section σ corresponds to e_a. Then $N^{\vee}_{X/P}(m_a) \cong O_X(m_a - m_1)e_1 \oplus \cdots \oplus O_X e_a \oplus \cdots \oplus O_X(m_a - m_r)e_r$. Hence, $L^n(\sigma) = \omega^n e_a \in H^n(X, \Omega^n{}_X(N^{\vee}_{X/P}(m_a))) \cong H^n(X, \Omega^n{}_X)e_a \oplus (\oplus H^n(X, \Omega^n{}_X(m_a - m_b))e_b)$. Thus $L^n(\sigma) \neq 0$ because ω^n is a volume form of X up to multiplication by a non-zero constant. ‖

The next proposition, which is only a translation of our old result, will show one of the fundamental roles of the concept "order of penetration" in our study of equations.

(2.4) Proposition. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an arithmetically normal embedding of a projective manifold X. Take a section $\sigma \in \Gamma(X, N^{\vee}_{X/P}(m))$. Suppose that $\mathrm{pent}(\sigma ; j^*H) \geqq 1$, where $H \in A^1(P)$ denotes the unique hyperplane class of $\mathbb{P}^N(\mathbb{C}) = P$. Then there exists a projective equation F of X in degee m (cf. [U-2]) such that $L(\sigma) = L([F])$.

Proof. See (2.2) Corollary of [U-3]. ||

As for a section of a vector bundle with the highest penetration order, we have the following result.

(2.5) Proposition. Let X be a projective manifold, E a holomorphic vector bundle on X, and ω Hodge-Kahler class corresponding to a hyperplane class h. Take a section σ of E. The penetration order of the section σ with respect to the hyperplane class h is equal to $n = \dim X$, if and only if the following exact sequence splits.

$$(2.5.1) \qquad 0 \longrightarrow O_X \xrightarrow{\ \sigma\ } E \longrightarrow E/\sigma O_X \longrightarrow 0$$

Proof. It is easy to prove the "if" part by applying the same method in the proof of (2.3)Proposition. We prove the converse. Tensoring $\Omega^n{}_X$ to the sequence (2.5.1) and taking its cohomology groups, we have a map $H^n(\sigma): H^n(X,\Omega^n{}_X) \to H^n(X, \Omega^n{}_X(E))$. Since $H^n(X,\Omega^n{}_X) \cong \mathbb{C}\cdot\omega^n$, the assumption of $\sigma\otimes\omega^n \neq 0$ implies the injectivity of the map $H^n(\sigma) = \sigma$. Considering their Serre duals, we obtain that the map $H^n(\sigma)^\vee = \sigma : H^0(X, E^\vee) \to H^0(X,O_X)$ is surjective. Hence there exists a section $\tau \in H^0(X, E^\vee)$ such that $\sigma(\tau) = 1$, or equivalently τ is a splitting map of the sequence (2.5.1). ‖

To give the next result, we introduce several invariants associated to a triplet (X,D,E) where X, D, and E denote a complex projective variety, an ample divisor (or line bundle) on X, and a holomorphic vector bundle on X, respectively.

(2.6)Definition. Let (X,D,E) be a triplet as above. Taking the Lefschetz operator L associated to the class $cl(D) \in A^1(X)$, we define invariants $\lambda(p;D,E)$ ($p=1,\cdots,n=\dim X$) with respect to the triplet (X,D,E) by;

$$\lambda(p;D,E) := \dim \mathrm{Im}[L^p : H^0(X,E(*D)) \to H^p(X,\Omega^p{}_X(E(*D)))].$$

For also an embedding $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ of the projective manifold X, we put (X,D,E) to be $(X, j^*O_P(1), N^\vee{}_{X/P})$ and similarly define invariants $\lambda(p;j)$ ($p=1,\cdots,n$) with respect to the embedding j by: $\lambda(p;j) := \lambda(p; j^*O_P(1), N^\vee{}_{X/P})$.

From (2.5)Proposition, using the assumption of arithmetic normality of the embedding j, we can obtain a criterion for $j(X)$ to be a complete intersection in terms of the penetration order of the equations.

(2.7) Theorem. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an arithmetically normal embedding of a projective manifold X of dimension n. Then the following four conditions are equivalent.

(i) $j(X)$ is a complete intersection.

(ii) The penetration order of every projective equation (cf. (2.8) Remark) of $j(X)$ is equal to n.

(iii) The equality $\lambda(n;j) = r := N - n$ holds.

(iv) The inequality; $\lambda(n;j) \geq r$ holds.

Proof. Obviously (iii) implies (iv). Now we suppose (i) and prove (ii). Take an arbitrary projective equation G of $j(X)$. Then there is an S.P.E. of $j(X)$ which contains the equation G. Since the number of elements of S.P.E. is determined only by $j(X)$, the S.P.E. containing G also consists of r elements $G_1 = G, G_2, \cdots, G_r$, which give a splitting $N^\vee_{X/P} \cong \oplus O_X(-m_b)[G_b]$. Then (2.3)Proposition brings (ii), because $H^n(X, \Omega^n_X(N^\vee_{X/P}(*))) \cong \oplus H^n(X, \Omega^n_X(*))[G_b]$. This argument also shows that (i) implies (iii). Next we assume (ii) and see (iv) holding. Take an S.P.E. $\{F_1, \cdots, F_k\}$ of

$j(X)$. Then obviously $k \geqq r$. Now we suppose that $L^n([F_1]), \cdots, L^n([F_k])$ are linearly dependent in the vector space $H^n(X, \Omega^n{}_X(N^{\vee}{}_{X/P}(*)))$, for example,

$$a_1 L^n([F_{t(1)}]) + a_2 L^n([F_{t(2)}]) + \cdots + a_u L^n([F_{t(u)}]) = 0$$
$$\text{in} \quad \oplus H^n(X, \Omega^n{}_X(N^{\vee}{}_{X/P}(m))).$$

Then we may assume that $\deg(F_{t(1)}) = \deg(F_{t(2)}) = \cdots = \deg(F_{t(u)}) = m$ because every $F_{t(a)}$ is homogeneous and the map L^n preserves their degrees. Putting $G = a_1 F_{t(1)} + a_2 F_{t(2)} + \cdots + a_u F_{t(u)}$, we obtain a projective equation G with $L^n(G) = 0$, since we can easily construct an S.P.E. of $j(X)$ which contains the equation G from the S.P.E. $\{F_1, \cdots, F_k\}$. Thus the condition (ii) implies that $\lambda(p;j) = k (\geqq r)$, namely the condition (iv). To finish our proof, we have only to show that (iv) implies (i). By (2.4)Proposition, there exist homogeneous polynomials $F_1, \cdots, F_r$ of degree $m_1, \cdots, m_r$, respectively such that $L^n([F_1]) \cdots L^n([F_r])$ are linearly independent in $H^n(X, \Omega^n{}_X(N^{\vee}{}_{X/P}(*)))$. Using (2.5)Proposition successively, $N^{\vee}{}_{X/P} \cong O_X(-m_1)[F_1] \oplus \cdots \oplus O_X(-m_r)[F_r]$. Then the connecteddness of the zero locus of $F_1 = \cdots = F_r = 0$ shows the condition (i) holding. ||

<u>(2.8) Remark</u>. In (2.7)Theorem, the word "every" never means that every elements of <u>an</u> S.P.E. as we saw in the proof.

By expressing (2.7)theorem in other words, we obtain an inequality for general arithmetically normal embeddings.

<u>(2.8) Corollary</u>. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an arithmetically normal embedding of a projective manifold X of dimension n. Then the inequality $\lambda(n;j) \leqq r := N-n$ always holds. Moreover the equality holds if and only if $j(X)$ is a complete intersection.

<u>(2.9) Remark</u>. (i) For a triplet (X,D,E) as in (2.6)Definition, we can easily show the following inequality by the similar argument in the proof of (2.7)Theorem.

(2.9.1) $\lambda(n;D,E) \leqq \mathrm{rank}\, E$ (the equality holds if and only if E splits as $E \cong \oplus O_X(m_a D)$)

(ii) If (4.1)Problem of [U-3] is affirmative, then we have the following claim.

<u>Claim</u> Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an arithmetically normal embedding of a projective manifold X. Then, there exist a closed subvariety W and hypersurfaces $S_1, \cdots, S_t$ such that $j(X) = W \cap S_1 \cap \cdots \cap S_t$ (transversal) if and only if $\lambda(n;j) \geqq t$.

With relation to the (2.9) Remark (ii), we give an easy fact on linear equations.

(2.10) Proposition. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an embedding of a projective manifold X of dimension n, and $\sigma \in H^0(X, N^\vee_{X/P}(1))$ a non-zero section. Then there is a (linear) projective equation $F \in H^0(P, I_{j(X)}(1))$ such that $[F] = \sigma$. Moreover,

$$\mathrm{pent}(\sigma; j^*O_P(1)) = \mathrm{pent}(F) = n.$$

Proof. Regarding the section σ as that of $H^0(X, \Omega^1_P|_X(1))$, we consider the exact sequence:

$$(2.10.1) \qquad 0 \to H^0(X, \Omega^1_P|_X(1)) \xrightarrow[\overline{\alpha}_E]{} \bigoplus_{a=0}^{N} H^0(X, O_X)e_a \xrightarrow[\overline{\beta}_E]{} H^0(X, O_X(1)),$$

which is induced from the Euler sequence. Since the map $\overline{\beta}_E$ sends the non-zero element $\overline{\alpha}_E(\sigma) = \sum_{a=0}^{N} k_a e_a$ to $0 = \sum_{a=0}^{N} k_a z_a$, where $z_0, \cdots, z_N$ denots the images of homogeneous coordinates $Z_0, \cdots, Z_N$ of P. Hence we obtain a linear equation $F = \Sigma k_a Z_a$. It is easy to see that $[F] = \sigma$ and F is actually a projective equation. Now let us consider the hyperplane H defined by the equation $F = 0$ and a point v_0 outside of H. Putting W to be the projective cone of $j(X)$ with the vertex v_0, we see that $j(X) = W \cap H$, and therefore $N^\vee_{X/P} \cong O_X(-1)[F] \oplus N^\vee_{W/P}|_X$. Thus we have $\mathrm{pent}(F) = n$ as we saw in (2.5)Proposition. ||

(2.11) Remark. In the case $m \geq 2$, for a given section $\sigma \in H^0(X, N^\vee_{X/P}(m))$, the above argument can not be applied to proved the existence of a projective equation corresponding to the section σ.

§3. Veronesean embeddings of $\mathbb{P}^n(\mathbb{C})$

In (1.4)Remark, we regarded projective equations of an m-th Veronesean embedding $\Phi_m : X = P(n) = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P(N) = P$ $(m \geqq 2)$ as typical examples of "useless" equations. Hence we shall study those examples precisely. To distinguish $O_{P(n)}(1)$ from $O_{P(N)}(1) \otimes O_X$, here we denote $O_{P(N)}(m)$, $O_{P(N)}(m) \otimes O_X$ and $O_{P(n)}(m)$ by $O_P(mH)$, $O_X(mH)$ and $O(m)$ respectively. To study the penetration order of each projective equation of $\Phi_m(X)$, we need some easy preparations which simplify our calculation of $H^p(X, \Omega^p{}_X(N^\vee{}_{X/P}(*)))$. The following proposition is often used in several papers in some special versions. For later use and convenience of the readers, we give also a proof.

<u>(3.1) Proposition</u>. Let $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an m-th Veronesean embedding of X $(m \geqq 2)$. Put $(T_0 : \cdots : T_n)$ and $(Z_0 : \cdots : Z_N)$ to be homogeneous coordinates of X and P respectively. Then, Φ_m is expressed by $\Phi_m : X \ni (T_0 : \cdots : T_n) \to (Z_0 : \cdots : Z_N) = (M_0 : \cdots : M_N) \in P$, where M_a's denote all monomials of $T_0, \cdots, T_n$ in degree m. We have an exact commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \to & \Omega^1{}_P|_X & \xrightarrow{q} & \bigoplus_{a=0}^{N} O(-m)[Z_a] & \xrightarrow{p} & O_X & \to & 0 \\
 & & \text{canonical}\downarrow J_0 & & \downarrow J_1 & & \downarrow \times m & & \\
0 & \to & \Omega^1{}_X & \xrightarrow[q']{} & \bigoplus_{b=0}^{n} O(-1)[T_b] & \xrightarrow[p']{} & O_X & \to & 0,
\end{array}
\qquad (3.1.1)
$$

where $[Z_a]$'s and $[T_b]$'s denote free basis corresponding to the free basis $\{Z_0, \cdots, Z_N\}$ of $H^0(P, O_P(1))$ and $\{T_0, \cdots, T_n\}$ of $H^0(X, O(1))$ respectively, and J_1 sends $[Z_a]$ to $\sum_{b=0}^{n} (\partial M_a / \partial T_b)[T_b]$.

Proof. On the first n monomials, we may assume that

(3.1.2) $M_0 = T_0{}^m, \quad M_1 = T_1{}^m, \quad \cdots, \quad M_n = T_n{}^m.$

First we shall see the commutativity of the right hand side of the diagram (3.1.1). After tensoring O(m), we chase $[Z_a]$.

$$m \cdot p([Z_a]) = m\, M_a$$

$$p' \cdot J_1([Z_a]) = p'(\Sigma\ (\partial M_a / \partial T_b)[T_b]) = \Sigma\ (\partial M_a / \partial T_b) T_b$$

$$= m\, M_a$$

Thus we obtain $m \cdot p = p' \cdot J_1$.

Next we shall check the left hand side of the diagram (3.1.1). Since we have a global homomorphisms, we may check locally the commutativity. For $a = 0, 1, \cdots, n$, we put U_a to be an open set $D_+(Z_a)$ of P. By the assumption (3.1.2), we have :

$$X \subset \bigcup_{a=0}^{n} U_a \ .$$

Thus we have only to chase $\{ d(Z_b/Z_a) \}$, which is a free basis of $(\Omega^1{}_P|_X)|U_a$ $(a=0, 1, \cdots, n)$ corresponding to that of $\Omega^1{}_P|U_a$. Set M_b to be $T_0{}^{e(0)} \cdots T_n{}^{e(n)}$ $(e(0) + \cdots + e(n) = m)$. Then,

$$q(\ d(Z_b/Z_a)) = (1/M_a)[Z_b] - (M_b/M_a{}^2)[Z_a]$$

$$J_1(q(d(Z_b/Z_a))) = (1/M_a)\{e(0)(M_b/T_0)[T_0] + \cdots + e(a)(M_b/T_a)[T_a]$$

$$+ \cdots + e(n)(M_b/T_n)[T_n]\} - m(M_b/M_a{}^2)T_a{}^{m-1}[T_a]$$

$$= (M_b/M_a)\{ \sum (e(k)/T_k)[T_k] - (m/T_a)[T_a] \}.$$

On the other hand,

$$q'J_0(d(Z_b/Z_a)) = q'd\{ (T_0/T_a)^{e(0)} \cdots \hat{a} \cdots (T_n/T_a)^{e(n)} \}$$

$$= q'(\sum_{k=0,\ k\neq a}^{n} e(k)(M_b/M_a)(T_a/T_k)\ d(T_k/T_a)\ )$$

$$= \sum_{k=0,\ k\neq a}^{n} e(k)(M_b/M_a)(T_a/T_k)\{ (1/T_a)[T_k] - (T_k/T_a^2)[T_a]\}$$

$$= \sum_{k=0}^{n} e(k)(M_b/M_a)\{ (1/T_k)[T_k] - (1/T_a)[T_a] \}$$

$$= (M_b/M_a)\{\sum_{k=0}^{n} (e(k)/T_k)[T_k] - m\ (1/T_a)[T_a] \}$$

Thus we obtain $q'J_0 = J_1 q.$ ||

By the proposition above, we obtain the following commutative diagram.

(3.1.3)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & O_X & \overset{\times m}{=\!=} & O_X & & \\
 & & & & \uparrow & & \uparrow & & \\
0 \to & N^{\vee}{}_{X/P} & \longrightarrow & & \bigoplus_{a=0}^{N} O(-m)[Z_a] & \overset{J_1}{\longrightarrow} & \bigoplus_{b=0}^{n} O(-1)[T_b] & \longrightarrow & 0 \\
 & \| & & & \uparrow & & \uparrow & & \\
0 \to & N^{\vee}{}_{X/P} & \longrightarrow & & \Omega^1{}_P|_X & \longrightarrow & \Omega^1{}_X & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Thus we have an useful exact sequence of the conormal bundle of the m-th Veronesean embedding as follows.

(3.2) Proposition. For an m-th Veronesean embedding $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C}) = P$, there exists an exact sequence:

$$(3.2.1) \quad 0 \to N^{\vee}_{X/P} \to \bigoplus_{a=0}^{N} O(-m)[Z_a] \xrightarrow[J_1]{} \bigoplus_{b=0}^{n} O(-1)[T_b] \to 0,$$

where the map J_1 is given as in (3.1) Proposition.

To calculate the cohomologies of the conormal bundle corresponding to Φ_m, we also need the following proposition.

(3.3) Proposition. For an m-th Veronesean embedding Φ_m, if $k \geqq m+1$, then we have an exact sequence:

$$(3.3.1) \quad 0 \to H^0(N^{\vee}_{X/P}(k)) \to \bigoplus_{a=0}^{N} H^0(O(k-m))[Z_a] \xrightarrow[J_1]{} \bigoplus_{b=0}^{n} H^0(O(k-1))[T_b] \to 0.$$

Proof. Since $\Phi_m(X)$ is arithmetically normal, we have only to show the surjectivity of the map J_1 in the case $k=m+1$. For a given monomial B of $T_0, \cdots, T_n$ in degree m, the map J_1 sends the element:

$$(3.3.2) \quad (1/m)\{ T_b[B] + \sum_{k=0}^{n} T_k[(\partial B/\partial T_k)T_b] - \sum_{k=0}^{n} T_b[(\partial B/\partial T_k)T_k] \}$$

of $\oplus H^0(O(1))[Z_a]$ to the element $B[T_b]$ of $\oplus H^0(O(m))[T_b]$. ∥

By using (3.2) Proposition and (3.3) Proposition, we get a list of the cohomologies of the conormal bundle of Φ_m.

(3.4) <u>Corollary</u>. Let $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an m-th Veronesean embedding. Then, for integers k and e $(0 \leq e \leq n)$, the cohomology groups $H^e(N^\vee_{X/P}(k))$ satisfy the following properties.

(3.4.1) $2 \leq e \leq n-1 \Longrightarrow H^e(X, N^\vee_{X/P}(k)) = 0$,

(3.4.2) e=0

$$\begin{cases} k \leq m \Longrightarrow H^0(X, N^\vee_{X/P}(k)) = 0 \\ k \geq m+1 \Longrightarrow 0 \to H^0(N^\vee_{X/P}(k)) \to \bigoplus^{N+1} H^0(O(k-m)) \\ \qquad \to \bigoplus^{n+1} H^0(O(k-1)) \to 0 \text{ (exact)}, \end{cases}$$

(3.4.3) e=n $\dim X \geq 2 \Rightarrow$

$$\begin{cases} 0 \to H^n(N^\vee_{X/P}(k)) \longrightarrow \bigoplus^{N+1} H^n(O(k-m)) \\ \longrightarrow \bigoplus^{n+1} H^n(O(k-1)) \longrightarrow 0 \quad \text{(exact)}, \end{cases}$$

(3.4.4) e=1 If $\dim X \geq 2$,

$$\Rightarrow \begin{cases} k \leq 0 \Longrightarrow H^1(N^\vee_{X/P}(k)) = 0, \\ 1 \leq k \leq m-1 \Longrightarrow H^1(N^\vee_{X/P}(k)) \cong \bigoplus^{n+1} H^0(O(k-1)), \\ k = m \Longrightarrow H^1(N^\vee_{X/P}(m)) \cong H^0(\Omega^1_X(m)), \\ k \geq m+1 \Longrightarrow H^1(N^\vee_{X/P}(k)) = 0. \end{cases}$$

(3.4.5) e=n=1 If $\dim X = 1$,

$$\Rightarrow \begin{cases} k \leq 0 \Longrightarrow \text{the same as in (3.4.3)}, \\ 1 \leq k \leq m-1 \Longrightarrow 0 \to \bigoplus^{n+1} H^0(O(k-1)) \to H^1(N^\vee_{X/P}(k)) \\ \qquad \to \bigoplus^{N+1} H^1(O(k-m)) \to 0 \text{ (exact)}, \\ k \geq m \Longrightarrow \text{the same as in (3.4.4)}. \end{cases}$$

Proof. The exact sequence (3.2.1) immediately shows: (3.4.2) in the case $k \leqq m-1$; (3.4.4) in the cases $k \leqq 0$ or $1 \leqq k \leqq m-1$; (3.4.1); (3.4.3). On the other hand, (3.3) Proposition brings us : (3.4.2) in the case $k \geqq m+1$; (3.4.4) in the case $k \geqq m+1$. As for the remainder $k = m$ of (3.4.2), we use (2.10) Proposition and two facts: (i) $j^*O_P(H) \cong O(m)$; (ii) the image $\Phi_m(X)$ is not contained by any hyperplane of P. The fact (ii) above also gives that $H^0(X, \Omega^1{}_P|_X(m)) = H^0(X, \Omega^1{}_P|_X \otimes j^*O_P(H)) = 0$ as we saw in the proof of (2.10) Proposition. Moreover, considering the long cohomology exact sequence (2.10.1) and two facts: surjectivity of the map β_E; $H^1(X, O_X) = 0$, we see also that $H^1(X, \Omega^1{}_P|_X(m)) = 0$. To show that (3.4.4) in the case $k = m$, we have only to apply the facts $H^0(X, \Omega^1{}_P|_X(m)) = H^1(X, \Omega^1{}_P|_X(m)) = 0$ to the long cohomology exact sequence induced from the familiar exact sequence: $0 \to N^\vee{}_{X/P} \to \Omega^1{}_P|_X \to \Omega^1{}_X \to 0$. The remainder (3.4.5) can be treated by the almost similar arguments above and is easy to see. ||

By this corollary, the following well-knowm fact is also easy to prove.

(3.5) Corollary. Let $\Phi_m : X = \mathbb{P}^1(\mathbb{C}) \hookrightarrow \mathbb{P}^{m+1}(\mathbb{C}) = P$ be an m-th Veronesean embedding of a rational non-singular curve. Then $N^\vee{}_{X/P} \cong O(-(m+2)) \oplus \cdots \oplus O(-(m+2))$.

Proof Using $\det N^\vee{}_{X/P} = O(-m(m+2))$ (cf.(3.2.1)), rank $N^\vee{}_{X/P} = m$, and $H^0(N^\vee{}_{X/P}(m+1)) = 0$ it is easy to prove. ||

Here we recall the definition of "k-regular" and its fundamental properties which are precisely explained in §14 of the text [M-2].

(3.6) Definition and Proposition. Let M be a coherent sheaf on the projective space $\mathbb{P}^n(\mathbb{C})$. We say that the sheaf M is k-regular if $H^e(\mathbb{P}^n(\mathbb{C}), M(k-e))=0$ for all $e>0$. Moreover if the sheaf M is k-regular, it enjoys the following properties.

(3.6.1) The sheaf $M(k)$ is generated by its global sections.

(3.6.2) The map $H^0(M(k)) \otimes H^0(O(1)) \rightarrow H^0(M(k+1))$ is surjective.

(3.6.3) The sheaf M is also $(k+1)$-regular.

Now we give a result on "k-regularity" of $N^{\vee}{}_{X/P}$ for an m-th Veronesean embedding of $X=\mathbb{P}^n(\mathbb{C})$.

(3.7) Corollary. Let $\Phi_m: X=\mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C})=P$ be an m-th Veronesean embedding.

(3.7.1) $m \geqq 3 \implies H^e(N^{\vee}{}_{X/P}(2m-1-e))=0 \quad (e>0)$,

namely $N^{\vee}{}_{X/P}$ is $(2m-1)$-regular.

(3.7.2) $m=2 \implies H^e(N^{\vee}{}_{X/P}(2m-e))=0 \quad (e>0)$,

namely $N^{\vee}{}_{X/P}$ is $2m$-regular.

Proof. It is easy to prove by careful checking on the list in (3.4) Corollary. ||

Thus we obtain the following result.

(3.8) Corollary. Let $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an m-th Veronesean embedding.

(3.8.1) $m \geqq 3 \Longrightarrow$ $H^0(O(1)) \otimes H^0(N^\vee{}_{X/P}(k)) \to H^0(N^\vee{}_{X/P}(k+1))$ is surjective, if $k \geqq 2m-1$.

(3.8.2) $m = 2$

$$\Longrightarrow \begin{cases} H^0(O(1)) \otimes H^0(N^\vee{}_{X/P}(k)) \to H^0(N^\vee{}_{X/P}(k+1)) \\ \qquad\qquad \text{is surjective, if } k \geqq 2m. \\ H^0(N^\vee{}_{X/P}(2m-1)) = 0. \end{cases}$$

Proof. Almost all will be deduced from (3.6.2), (3.6.3), and (3.7) Corollary. We have only to show that $H^0(N^\vee{}_{X/P}(2m-1)) = 0$ in the case of $m = 2$. Let us recall (3.3)Proposition. Since $2m-1 = m+1$ in this case, we have the exact sequence (3.3.1) for $k = 2m-1 = 3$. Then we can get the result through a computation of $\dim H^0(X, N^\vee{}_{X/P}(3))$ by using the fact: $N = {}_{n+2}C_n - 1$. ||

Now we have finished our preparations and come to a position for giving our results on the penetration orders of projective equations associated to an m-th Veronesean embedding of $X = \mathbb{P}^n(\mathbb{C})$. First we see the case $m \geqq 3$.

(3.9) Theorem. Let $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an m-th Veronesean embedding. If $m \geqq 3$, then the penetration order of every projective equation of $\Phi_m(X)$ is zero.

Proof. Let us consider an exact sequence;

(3.9.1)

$$0 \to \Omega^1_X \otimes N^\vee_{X/P}(2m) \to \bigoplus_{b=0}^{n} N^\vee_{X/P}(2m-1)[T_b] \to N^\vee_{X/P}(2m) \to 0,$$

which is obtained by tensoring $N^\vee_{X/P}(2m)$ to the Euler sequence of X. Taking the cohomologies of (3.9.1), we have:

(3.9.2)

$$\bigoplus^{n+1} H^0(N^\vee_{X/P}(2m-1)) \xrightarrow{\beta} H^0(N^\vee_{X/P}(2H)) \xrightarrow{\delta} H^1(\Omega^1_X(N^\vee_{X/P}(2H))).$$

On the other hand, by comparing the diagram (3.1.3) with (1.1.2) in [U-3], we find that the sheaf Π in (1.1.2) of [U-3] coincides with $\bigoplus_{b=0}^{n} O(-1)[T_b]$ in our case. By (1.2) Lemma of [U-3], we see that the map δ coinsides with Lefschetz operator L up to multiplying by a non-zero constant. Since $m \geqq 3$, (3.8.1) implies the surjectivity of the map β, that is to say, the map $L = \delta$ is zero. As is well-known, every projective equation of $\Phi_m(X)$ is quadratic, we obtain the result above. ||

Next we treat the remainder case $m = 2$, which is a little different from the case $m \geqq 3$.

<u>(3.10) Theorem</u>. Let $\Phi_2 : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be a second Veronesean embedding. Then,

(i) for an arbitrary projective equation F of $\Phi_2(X)$, pent(F) = 1,

(ii) $H^0(P, I(2H)) \longrightarrow H^0(X, N^\vee_{X/P}(2H))$ is surjective.

<u>Proof</u>. Let us recall the sequence (3.9.2). As we saw in (3.8) Corollary, $H^0(N^\vee_{X/P}(2m-1)) = 0$. Then the map β is zero, which means the (first) Lefschetz operation $L = \delta$ is injective.

Since also in this case, every projective equation of $\Phi_2(X)$ is quadratic, we have $L([F]) \neq 0$ for an arbitrary projective equation F of $\Phi_2(X)$. To show that $L^2([F]) = 0$, we have only to show that $H^2(X, \Omega^2{}_X(N^\vee{}_{X/P}(2H))) = 0$. Let us consider an exact sequence :

(3.10.1)

$$0 \to N^\vee{}_{X/P} \otimes \Omega^2{}_X(2H) \to \overset{N+1}{\oplus} \Omega^2{}_X(2) \to \overset{n+1}{\oplus} \Omega^2{}_X(3) \to 0,$$

which is obtained by tensoring $\Omega^2{}_X(2H)$ to the sequence (3.2.1) in the case $m = 2$. Taking their cohomologies, we see that $H^2(X, \Omega^2{}_X(N^\vee{}_{X/P}(2H))) = 0$. For the remainder (ii), we recall that the map $L: H^0(N^\vee{}_{X/P}(2H)) \to H^1(\Omega^1{}_X(N^\vee{}_{X/P}(2H)))$ is injective. Since Φ_2 is an arithmetically normal embedding, we can use (2.4) Proposition to show that for any section $\sigma \in H^0(X, N^\vee{}_{X/P}(2H))$, there is a projective equation $F \in H^0(P, I_X(2H))$ such that $L([F]) = L(\sigma)$. Then the injectivity of L gives $[F] = \sigma$. ‖

Here we give an easy corollary for this theorem.

<u>(3.11) Corollary</u>. Let $\Phi_2 : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be a second Veronesean embedding. Then, $H^1(P, I^2{}_X(*H)) = 0$.

<u>Proof</u>. We have only to prove that each map $H^0(P, I(tH)) \longrightarrow H^0(X, N^\vee{}_{X/P}(2t))$ is surjective for all $t \in \mathbb{Z}$. For $t \geqq 2$, we use (3.8.2) (3.10)Theorem and arithmetic normality of Φ_2. For $t \leqq 1$, (3.7.2) gives the surjectivity. ‖

As we saw in the above, there exists a difference between the case $m = 2$ and the case $m \geqq 3$. Then one question arises in our mind: What kind of geometric phenomena cause this difference?

In other words, first we want to find geometric phenomena which concur with the difference, and next to choose suitable ones for explaining the difference. At the present stage, we know only two kinds of phenomena which concur with the difference. The first one is as follows(cf. (3.4)Corollary in [U-3]).

<u>(3.12) Proposition</u>. Let $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an m-th Veronesean embedding. Then,

(i) $m = 2 \Rightarrow H^1(P, I^2{}_X(*H)) = 0$,

(ii) $m \geq 3 \Rightarrow H^1(P, I^2{}_X(*H)) \neq 0$.

<u>Proof</u>. The claim (i) was already proved in (3.11) Corollary. We have only to show that $H^1(P, I^2{}_X(2H)) \neq 0$ for all $m \geq 3$. By the sequence : $0 \to I^2{}_X \to I_X \to N^\vee{}_{X/P} \to 0$, we get an exact sequence:

$$(3.12.1) \quad H^0(P, I_X(2H)) \to H^0(X, N^\vee{}_{X/P}(2m)) \to H^1(P, I^2{}_X(2H)) \to 0.$$

Hence,

$$(3.12.2) \quad \dim H^0(P, I^2(2H)) \geq (1/2)({}_{n+m}C_n)^2 - (1/2)({}_{n+m}C_n) - (2m-1)({}_{n+2m-1}C_{2m}).$$

It is enough to see that the right hand side of the inequality (3.12.2) is positive for all $m \geq 3$ and $n \geq 1$. Now we set

$$R(m,n) := (1/2)({}_{n+m}C_n)^2 - (1/2)({}_{n+m}C_n) - (2m-1)({}_{n+2m-1}C_{2m}),$$

$$P(m,n) := ({}_{m+n}C_m)({}_{2m}C_m) - ({}_{2m}C_m) - 4n({}_{n+2m-1}C_{m-1}).$$

Then, a direct computation shows us that

$$R(m,n) = (1/2)({}_{m+n}C_m)({}_{2m}C_m)^{-1}\{P(m,n) + (2n/m)({}_{n+2m-1}C_{m-1})\}.$$

Since $(2n/m)({}_{n+2m-1}C_{m-1}) > 0$, we may show that $P(m,n) \geqq 0$ for all $m \geqq 3$ and $n \geqq 1$. If $m=3$, then $P(3,n) = (4/3)(n^3 - n) + 2(n^2 - n) \geqq 0$ for $n \geqq 1$. For $m \geqq 4$, we shall prove it by showing that each coefficient S_k $(k=0,1,\cdots,m)$ of the polynomial:

$$\begin{aligned} P(m,T) &= (m!)^{-1}({}_{2m}C_m)(T+m)(T+m-1)\cdots(T+1) \\ &\quad - 4\ ((m-1)!)^{-1}\ T(T+2m-1)\cdots(T+m+1) - ({}_{2m}C_m) \\ &=: S_m T^m + S_{m-1}T^{m-1} + \cdots + S_0 \end{aligned}$$

is non-negative. For $k=0$,

$$S_0 = (1/m!)({}_{2m}C_m)m! - ({}_{2m}C_m) = 0.$$

Next for k satisfying: $m \geqq k \geqq 1$,

$$\begin{aligned} S_k &= (1/m!)({}_{2m}C_m)(\ \Sigma\ a_{u(1)}\cdots a_{u(m-k)}) \\ &\quad - (4/(m-1)!)\ (\ \Sigma\ (m+b_{w(1)})\cdots(m+b_{w(m-k)})\ ), \end{aligned}$$

where the $(m-k)$-tuple of integers $(a_{u(1)},\cdots,a_{u(m-k)})$ and $(b_{w(1)},\cdots,b_{w(m-k)})$ run over the ranges $m \geqq a_{u(m-k)} > \cdots > a_{u(1)} \geqq 1$ and $m-1 \geqq b_{w(m-k)} > \cdots > b_{w(1)} \geqq 1$ respectively. Hence, we shall see that $m!\ S_k \geqq 0$. Putting $p=m-k$, our problem is to show that

$$\begin{aligned} Q_p(m) &= ({}_{2m}C_m)(\ \Sigma\ a_{u(1)}\cdots a_{u(p)}) - 4m\ (\ \Sigma\ (m+b_{w(1)})\cdots(m+b_{w(p)})\ ) \\ &\geqq 0, \end{aligned}$$

for $m \geqq 4$, $m \geqq a_{u(p)} > \cdots > a_{u(1)} \geqq 1$, $m-1 \geqq b_{w(p)} > \cdots > b_{w(1)} \geqq 1$, $m-1 \geqq p \geqq 0$. Moreover, we may assume $p \geqq 1$, because

$$Q_0(m) = ({}_{2m}C_m) - 4m \geqq 4(m+1) - 4m \geqq 0.$$

For $p \geqq 1$,

$$Q_p(m) = ({}_{2m}C_m)(m \sum b_{w(1)} \cdots b_{w(p-1)} + \sum b_{v(1)} \cdots b_{v(p)})$$
$$- 4m\sum (m + b_{v(1)}) \cdots (m + b_{v(p)}).$$

Hence we may show that

$$({}_{2m}C_m) \{ (m/(m-p)) \sum_{e=1}^{p} b_{w(1)} \cdots \hat{e} \cdots b_{w(p)} + b_{w(1)} \cdots b_{w(p)} \}$$
$$- 4m(m + b_{w(1)}) \cdots (m + b_{w(p)}) \geqq 0$$

for $m \geqq 4$, $m-1 \geqq p \geqq 1$, $m-1 \geqq b_{w(p)} > \cdots > b_{w(1)} \geqq 1$. Thus, it is sufficient to prove that

$$F_p(z_1, \cdots, z_p) := ({}_{2m}C_m) \{ (m/(m-p)) \sum_{e=1}^{p} z_1 \cdots \hat{e} \cdots z_p + z_1 \cdots z_p \}$$
$$- 4m(m + z_1) \cdots (m + z_p) \geqq 0$$

for any $(z_1, \cdots, z_p) \in \{ (z_1, \cdots, z_p) \in \mathbb{R}^p \mid z_1 \geqq 1, \cdots, z_{k+1} \geqq z_k + 1, \cdots, z_p \geqq z_{p-1} + 1 \}$. Now we use induction on p.

For $p = 1$, $(\partial F_1 / \partial z_1) = {}_{2m}C_m - 4m \geqq 0$ and $F_1(1) > 0$. Thus we have $F_1(z_1) > 0$ for all $z_1 \geqq 1$.

For $p \geqq 2$,

$$(\partial F_p / \partial z_a)$$

$$= ({}_{2m}C_m) \{ (m/(m-p)) \sum_{e=1 (e \neq a)}^{p} z_1 \cdots \hat{a} \cdots \hat{e} \cdots z_p + z_1 \cdots \hat{a} \cdots z_p)$$
$$- 4(z_1 + m) \cdots \hat{a} \cdots (z_p + m)$$

$$\geqq ({}_{2m}C_m) \{ (m/(m-p+1)) \sum_{e=1 (e \neq a)}^{p} z_1 \cdots \hat{a} \cdots \hat{e} \cdots z_p + z_1 \cdots \hat{a} \cdots z_p \}$$
$$- 4(z_1 + m) \cdots \hat{a} \cdots (z_p + m)$$

$= F_{p-1}(z_1, \cdots \hat{a} \cdots, z_p) \geqq 0.$

Thus we have only to show that $F_p(1, 2, \cdots, p) \geqq 0$. For $p = m-1$, $F_{m-1}(1, 2, \cdots, m-1) = 2m(2m-1)\cdots(m+1)(\sum_{e=1}^{m}(1/e) - 2) \geqq 0$. Then we may assume that $m-1 \geqq p \geqq 2$. Since $({}_{m+p}C_m) \leqq (1/2)^{m-p}({}_{2m}C_m)$, we obtain:

$$F_p(1, \cdots, p) \geqq m(p!)({}_{2m}C_m)\{(1/(m-p)\sum_{e=1}^{p}(1/e) + 1/m - 4(1/2)^{m-p}\}.$$

Thus it is enough to see that

$$G_p(m) := (1/(m-p))\sum_{e=1}^{p}(1/e) + 1/m - 4(1/2)^{m-p} \geqq 0.$$

for $m-2 \geqq p \geqq 2$. If $m-4 \geqq p \geqq 2$, then

$$G_p(m) \geqq (1/(m-p))(3/2) - (1/2)^{m-p-2} \geqq 1/q - (1/2)^{q-2} \geqq 0,$$

where $q = m-p \geqq 4$. Thus our problem reduces to see that $G_p(m) \geqq 0$ for $p = m-3$ or $p = m-2$. If $p = m-3$, then $G_{4-3}(4) > 0$ and $G_{m+1-3}(m+1) - G_{m-3}(m) > 0$ $(\forall m \geqq 4)$, which implies $G_{m-3}(m) > 0$ for $m \geqq 4$. If $p = m-4$, then $G_{4-4}(4) = 0$ and $G_{m+1-4}(m+1) - G_{m-4}(m) > 0$ $(\forall m \geqq 4)$. Thus we obtain $G_{m-4}(m) \geqq 0$ for $m \geqq 4$ as required. ‖

<u>(3.13) Remark</u> Through this proposition, we may say that a gap appearing between $\Phi_m(X)$ and its ambient space $P = \mathbb{P}^N(\mathbb{C})$ for $m \geqq 3$ is wider than that for $m=2$. In the former case, $\Phi_m(X)$ has only projective equations with penetration order zero. On the

other hand, in the latter case $\Phi_m(X)$ has projective equations with penetration order one. Hence our concept "penetration order" is compatible with our intuition descibed in (1.4) Remark.

The next theorem also backs up our intuition and the validity of the concept "penetration order".

<u>(3.14) Theorem</u>. Let $j_1: Y \hookrightarrow \mathbb{P}^n(\mathbb{C}) = X$ be a closed immersion of a projective manifold Y. For a sufficiently large m, consider an m-th Veronesean embedding $\Phi_m: X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^{n(m)}(\mathbb{C}) = W(m)$. Then,

(3.14.1) penetration order of every projective equation (except linear equations) for $\Phi_m j_1(Y)$ is zero,

(3.14.2) $H^1(W(m), I_m{}^2(*)) \neq 0$, where I_m denotes the sheaf of ideals which defines $\Phi_m j_1(Y)$ in $W(m)$.

<u>Proof</u>. First we consider the claim (3.14.1). As we saw in (1.3) Example, every projective equation of $\Phi_m j_1(Y)$ is quadratic for $m \gg 0$. This fact can be proved by a slight generalization of the argument appeared in [M-1]. Moreover the same argument shows also simultaneously that all quadratic projective equations of $\Phi_m j_1(Y)$ are induced by those of $\Phi_m(X)$ for $m \gg 0$. Hence it is sufficient to show the following easy lemma.

<u>(3.15) Lemma</u>. Let X and Y be closed submanifols of $\mathbb{P}^N(\mathbb{C}) = P$ which satisfies: $Y \subset X$. Assume that both X and Y have an equation F of degree$= k$ as their own projective equations. Then,

(3.15.1) $\quad \mathrm{pent}(F;X) \geqq \mathrm{pent}(F;Y)$.

Proof of (3.15) Lemma. Let us consider a commutative diagram:

$$(3.15.2)\qquad \begin{array}{ccc} H^0(P, I_X(k)) \ni F & \longrightarrow & [F]_X \in H^0(X, N^\vee{}_{X/P}(k)) \\ & & \downarrow \\ \downarrow & & [F]_X|_Y \in H^0(Y, N^\vee{}_{X/P}|_Y(k)) \\ & & \downarrow \\ H^0(P, I_Y(k)) \ni F & \longrightarrow & [F]_Y \in H^0(Y, N^\vee{}_{Y/P}(k)). \end{array}$$

Put $\omega \in H^1(X, \Omega^1{}_X)$ to be the Hodge-Kahler class of X associated to the inclusion. Then the Hodge-Kahler class of Y associated to its inclusion coincides with the restriction class $\omega|_Y \in H^1(Y, \Omega^1{}_Y)$ of the class ω. Hence we have a commutative diagram on the actions of Lefschetz operators L_X and L_Y:

(3.15.3)

$$\begin{array}{ccc} H^b(X, \Omega^a{}_X(N^\vee{}_X(k))) \ni \phi_X & \xrightarrow{L_X} & \phi_X \otimes \omega \in H^{b+1}(X, \Omega^{a+1}{}_X(N^\vee{}_X(k))) \\ \downarrow & & \downarrow \\ H^b(Y, \Omega^a{}_Y(N^\vee{}_Y(k))) \ni \phi_Y & \xrightarrow[L_Y]{} & \phi_Y \otimes \omega|_Y \in H^{b+1}(Y, \Omega^{a+1}{}_Y(N^\vee{}_Y(k))). \end{array}$$

Here we set $p = \mathrm{pent}(F:X)$. Then $L_X{}^{p+1}([F]_X) = 0$. Hence by using the commutative diagrams (3.15.2) and (3.15.3) it is easy to see that $L_Y{}^{p+1}([F]_Y) = 0$, namely $\mathrm{pent}(F;Y) \leqq p = \mathrm{pent}(F;X)$. ||

Now let us go back to our proof of (3.14) Theorem. We have to show the remainder (3.14.2). First, put $s(m) := n(m) + 1 - h^0(O_Y(m))$. Since $\Phi_m j_1 : Y \hookrightarrow W(m)$ is arithmetically normal for $m \gg 0$, we may put $h^0(I_m(H_m)) = s(m)$, where $O_{W(m)}(H_m)$ denotes the hyperplane bundle of $W(m)$. Then we get an inequality:

$$(3.15.4)\qquad h^0(I_m{}^2(2H_m)) \geqq s(m)(s(m)+1)/2.$$

Let us compare the sequence (3.2.1) tensored by $O_X(2m)$ with that tensored by $O_Y(2m)$. Then (3.3) Proposition gives us an exact commutative diagram:

(3.15.5)

$$\begin{array}{ccccccc}
0 \to H^0(N^\vee_{X/W(m)}(2H_m)) & \to & \overset{n(m)+1}{\oplus} H^0(O_X(m)) & \overset{J_1}{\to} & \overset{n+1}{\oplus} H^0(O_X(2m-1)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 \to H^0(N^\vee_{X/W(m)}|_Y(2H_m)) & \to & \overset{n(m)+1}{\oplus} H^0(O_Y(m)) & \underset{J_1|_Y}{\to} & \overset{n+1}{\oplus} H^0(O_Y(2m-1)), & \\
& & \downarrow & & \downarrow & \\
& & 0 & & 0 &
\end{array}$$

which brings us the sujectivity of the map $J_1|_Y$ and an equality:

(3.15.6) $$h^0(N^\vee_{X/W(m)}|_Y(2H_m)) = (n(m)+1)\, h^0(O_Y(m)) - (n+1)\, h^0(O_Y(2m-1))$$

Let us consider an exact sequence induced by that of conormal bundles:

(3.15.7) $$0 \to H^0(N^\vee_X|_Y(2m)) \to H^0(N^\vee_Y(2m)) \to H^0(N^\vee_{Y/X}(2m)) \to H^1(N^\vee_X|_Y(2m)) = 0,$$

where $N^\vee_X = N^\vee_{X/W(m)}$ and $N^\vee_Y = N^\vee_{Y/W(m)}$. Then the sequence (3.15.7) and the equality (3.15.6) gives an equality:

(3.15.8) $$h^0(N^\vee_{Y/W(m)}(2H_m)) = h^0(N^\vee_{Y/X}(2m)) + (n(m)+1)h^0(O_Y(m)) - (n+1)h^0(O_Y(2m-1)).$$

On the other hand an exact sequence:

(3.15.9) $$0 \to H^0(I_m{}^2(2H_m)) \to H^0(I_m(2H_m)) \to H^0(N^\vee_{Y/W(m)}(2H_m)) \to H^1(I_m(2H_m)) \to 0$$

shows that

(3.15.10) $$h^1(I_m{}^2(2H_m)) = h^0(N^\vee_{Y/W(m)}(2m)) - h^0(I_m(2H_m)) + h^0(I_m{}^2(2H_m))$$

$= h^0(N^{\vee}{}_{Y/W(m)}(2m)) - h^0(O_{W(m)}(2H_m)) + h^0(O_Y(2m)) + h^0(I_m{}^2(2H_m))$.

Then we gather up the equalities (3.15.8), (3.15.10), and the inequality (3.15.4) and compute as follows.

$$h^1(I_m{}^2(2H_m)) \geqq h^0(N^{\vee}{}_{Y/X}(2m)) + (n(m)+1)h^0(O_Y(m)) - (n+1)h^0(O_Y(2m-1))$$
$$- h^0(O_{W(m)}(2H)) + h^0(O_Y(2m)) + (1/2)s(m)(s(m)+1)$$

$$= h^0(N^{\vee}{}_{Y/X}(2m)) - (n+1)h^0(O_Y(2m-1)) + h^0(O_Y(2m))$$
$$- (1/2)h^0(O_Y(m)) + (1/2)h^0(O_Y(m))^2,$$

using that the sheaf $N^{\vee}{}_{Y/X}$ does not depend on m,

$$= (1/2)h^0(O_X(m))^2 + (\text{lower term}) > 0 \qquad \|$$

(3.16) Remark. In general $\Phi_m j_1(Y)$ is contained by a linear space L of $W(m)$. Nevertheless, the claims (1.14.1) and (1.14.2) are also affirmative after replacing $W(m)$ by L.

Now let us return to the original case $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$. The second phenomenon which shows us the difference between the case $m = 2$ and $m \geqq 3$ is given in the sequel. First we see the following example which treats the case of $m = 2$.

(3.17) Example. Let $\Phi_2 : X = \mathbb{P}^n(\mathbb{C}) \ni [T_0 : \cdots : T_n] \to [T_0{}^2 : T_0T_1 : \cdots : T_0T_n : \cdots : T_aT_b : \cdots : T_n{}^2] = [Z_{00} : Z_{01} : \cdots : Z_{0n} : \cdots : Z_{ab} : \cdots : Z_{nn}] \in \mathbb{P}^N(\mathbb{C}) = P$ be a second Veronesean embedding. Then, for every projective equation F of $\Phi_2(X)$, there exists a line Y in X such that $\Phi_2(Y)$ is a complete intersection on the hypersurface $\{F = 0\}$. Since a "typical" projective equation is given by $Z_{ab}Z_{cd} - Z_{ac}Z_{bd}$ (a, b, c, d are distinct index), we may assume that $n = 3$ and $F = Z_{01}Z_{23} - Z_{02}Z_{13}$. Then we put $Y =$ the image of $\mathbb{P}^1(\mathbb{C}) \ni [U_0 : U_1] \to$

$[U_0 : U_1 : U_0 + U_1 : U_0 - U_1] = [T_0 : T_1 : T_2 : T_3] \in X = \mathbb{P}^3(\mathbb{C})$. Then $\Phi_2(Y)$ is defined by the equations: $F = 0$, $Z_{02} - Z_{00} - Z_{01} = Z_{03} - Z_{00} + Z_{01} = Z_{12} - Z_{01} - Z_{11} = Z_{13} - Z_{01} + Z_{11} = Z_{22} - Z_{00} - 2Z_{01} - Z_{11} = Z_{23} - Z_{00} + Z_{11} = Z_{33} - Z_{00} + 2Z_{01} - Z_{11} = 0$, which means $\Phi_2(Y)$ is a complete intersection on $\{F = 0\}$.

On the other hand, for the case $m \geqq 3$, we have the following result, which also shows the utility value of the concept "penetration order".

(3.18) Theorem. Let $\Phi_m : X = \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ be an m-th Veronesean embedding and F a projective equation of $\Phi_m(X)$. Assume that $m \geqq 3$. Then there does not exist a submanifold Y of positive dimension in X such that $\Phi_m(Y)$ is a complete intersection on the hypersurface $\{F = 0\}$.

Proof. Let us assume that there is a submanifold Y of positive dimension in X which satisfies: $\Phi_m(Y)$ is a complete intersection on the hypersurface $\{F = 0\}$. Then the equation F is a projective equation of $\Phi_m(Y)$. Moreover, by (2.7) Theorem, $\mathrm{pent}(F; \Phi_m(Y)) = \dim Y > 0$. On the other hand, (3.9) Theorem and (3.15) Lemma show that $\mathrm{pent}(F; \Phi_m(Y)) \leqq \mathrm{pent}(F; \Phi_m(X)) = 0$, which gives a contradiction. ‖

(3.19) Remark. Through (3.17) Example and (2.7) Theorem, we may suppose that the penetration order of a projective equation is affected by the existence of a special subvariety (for example, in (3.17) Example above, a special subvariety is obtained by choosing a suitable line Y in X with respect to the given projective equation.

§4 Elementary Properties of penetration order

Stimulated by the obsevations in (3.19) Remark, we start to study the relation between the penetration order of a given section of a holomorphic vector bundle on a projective manifold and its submanifolds. First we give a definition as follows.

(4.1) Definition Let X be a projective manifold, Y a subvariety(resp. manifold) of X, E a holomorphic vector bundle on X and σ a global section of E. Consider an exact sequence:

$$(4.1.1) \qquad 0 \to O_X \xrightarrow{\sigma} E \to E/\sigma O_X \to 0.$$

Then, we say that Y is a splitting subvariety(resp. manifold) of (E, σ) if the restriction of the sequence (4.1.1.) to Y is also exact and splits.

Then, we immediately obtain the following result.

(4.2) Proposition. Let X be a projective manifold, E a holomorphic vector bundle on X, σ a global section of E. Take a hyperplane class $h \in A^1(X)$ and the Hodge-Kahler class ω_X corresponding to the class h. Assume that there exists a splitting submanifold Y of (E, σ) whose dimension is p. Then,

$$\mathrm{pent}(\sigma, h) \geqq p.$$

Proof. Since $E|_Y \cong O_Y\sigma \oplus [(E/\sigma O_X) \otimes O_Y]$, we have a diagram;

$$\begin{array}{ccccc} H^p(X, \Omega^p{}_X(E)) & \to & H^p(Y, \Omega^p{}_Y(E|_Y)) & \xrightarrow{pr} & H^p(Y, \Omega^p{}_Y) \\ \cup & & \cup & & \cup \\ L^p(\sigma) = \sigma \otimes \omega^p{}_X & \longrightarrow & \sigma|_Y \otimes \omega^p{}_Y & \longrightarrow & \omega^p{}_Y \neq 0, \end{array}$$

where "pr" denotes the projection to the direct summand. Thus we get $L^p(\sigma) \neq 0$, namely $\mathrm{pent}(\sigma : h) \geqq p$. ||

We also have an easy result which is proved by the similar argument of (3.15) Lemma.

(4.3) Lemma. Let X be a projective manifold, Z a submanifold of X, E a holomorphic vector bundle on X and σ a global section of E. Then,

$$\mathrm{pent}(\sigma : h) \geqq \mathrm{pent}(\sigma|_Z : h|_Z)$$

(4.4) Remark. We can often easily modify the claims and their prooves on penetration order of a projective equation into those of a secion. To avoid confusions, we must however distinguish them. For example, there is no implication between (3.15) Lemma and (4.3) Lemma because $N^{\vee}{}_{X/P}|_Y \neq N^{\vee}{}_{Y/P}$.

On the other hand, we can give a lower bound for the behavior of penetration order with respct to restrictions in the following special case.

(4.5) Proposition. Let X be a projective manifold, $h \in A^1(X)$ a hyperplane class, E a holomorphic vector bundle on X, and σ a global section of E. Take a smooth irreducible ample divisor D of X whose fundamental class $\eta_D \in H^2(X, \mathbb{C})$ satisfies the condition: $\omega_X \in \mathbb{Q}\,\eta_D$, where ω_X denotes the Hodge-Kahler class associated to the class h. Then,

$$\mathrm{pent}(\sigma, h) - 1 \leqq \mathrm{pent}(\sigma|_D, h|_D).$$

Proof. Putting $p = \mathrm{pent}(\sigma ; h)$, let us recall an exact sequence:

$$0 \to \Omega^p{}_X \to \Omega^p{}_X(\log D) \to \Omega^{p-1}{}_D \to 0,$$

which induces a diagram:

$$\begin{array}{ccccc}
\cdots \to H^{p-1}(D, \Omega^{p-1}{}_D(E|_D)) & \xrightarrow{\delta} & H^p(X, \Omega^p{}_X(E)) \to & H^p(X, E \otimes \Omega^p{}_X(\log D)) \to \cdots \\
\text{restriction} \uparrow & \circlearrowleft & & \nearrow \cup \eta_D = \cup m \omega_X . \\
H^{p-1}(X, \Omega^{p-1}{}_X(E)) & & &
\end{array}$$

By the assumption, $m L_X{}^p(\sigma) \neq 0$, which means $\delta(L_D{}^{p-1}(\sigma|_D)) = \delta(\sigma|_D \otimes \omega_D) \neq 0$. Hence we have $L_Y{}^{p-1}(\sigma|_D) \neq 0$. ‖

Moreover we have one more interesting fact on the behavior of penetration order with respect to restrictions on divisors.

(4.6) Proposition. Let X, E, σ, ω_X h be as above. Assume that $\mathrm{pent}(\sigma ; h) < (1/2) \dim X$. Then there exists a smooth ample irreducible divisor D such that

$$\mathrm{pent}(\sigma ; h) = \mathrm{pent}(\sigma|_D ; h|_D).$$

Proof. It is easy to prove by applying Serre vanishing theorem to several cases. Hence we omit the precise explanation.

Through the result of (4.2) Proposition, a simple question comes to our mind.

(4.7) Question. Let X, E, σ, ω_X h be as above. Assume that $\mathrm{pent}(\sigma, h) = p$. Then does there exists a splitting subvariety of dimension p ?

Our answer to this question is negative.

<u>(4.8) Example</u>. Put X_0 to be $\mathbb{P}^2(\mathbb{C})$. Take a nonsingular cubic curve Y_0 and generaric 10 points $p_1, \cdots, p_{10}$ on Y_0. Then set X to be the blowing up at 10 points $p_1 \cdots, p_{10}$ of X_0, Y to be the strict transformation of Y_0, E to be the line bundle $O_X(Y)$ associated to the divisor Y, and σ to be the section which defines Y. Then there exists a hyperplane class h such that pent(σ : h)= 1. On the other hand if there exists a splitting curve C, then C can never meet Y. However, this is a contradiction, since such a curve does not exist by the reason of intersection calculus.

<u>(4.9) Remark</u>. In the example above, there exists an algebraic 1-cycle ζ which is not numerically equivalent to zero and $Y \cdot \zeta = 0$.

The following example shows us that penetration order is also a considerably good tool for finding partial structures of a system of projective equations.

<u>(4.10) Example</u>. Let $W \subseteq \mathbb{P}^N(\mathbb{C}) \simeq P$ be the image of an m-th Veronesean embedding of $\mathbb{P}^{n+1}(\mathbb{C})$ $(n \geqq 2)$, S a non-singular hypersurface of P which meets W transversely. Put X to be $W \cap S$. Then X is arithmetically normal and has an S.P.E. $\{F, G_1, \cdots, G_k\}$, where $\{G_1, \cdots, G_k\}$ is an S.P.E. of W and $\{F\}$ is an S.P.E.of S. Then,

$$\text{pent}(F) = n \geqq 2$$

$$\text{pent}(G_a) \leqq 1 \quad (a=1, \cdots k).$$

Hence, using penetration order, we can find a partial structure $\{G_1, \cdots, G_k\}$, which is an S.P.E. of W, in the S.P.E. of X.

§5. Problems

From the arguments of this article, we shall raise several problems which will be our working problems.

(5.1) Problem. Let F be a projective equation of j(X) for a closed immersion $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ of a projective manifold X. Assume that $\deg F > \deg j(X)$. Then prove that pent(F)= 0 or that such a projective equation F does not exist(cf.[G-L-P], [S-V]).

(5.2) Problem. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C})$ a closed immersionof a projective manifold X. Suppose that the penetration order of every projective equation of j(X) is zero. Then $H^1(P, I^2_{j(X)}(*)) \neq 0$?

(5.3) Problem. Let $j: X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ a close immersion of a projective manifold. Assume that $H^1(P, I^2_{j(X)}(*)) \neq 0$. Then does there exist a projective equation of j(X) whose penetration order is zero ?

(5.4) Problem. Besides Lefschetz operator, seek other tools which can be used in precise studying of partial structures.

(5.5) Problem. Let X be a projective manifold, E a holomorphic vector bundle of rank r on X, h a hyperplane class in $A^1(X)$, and σ a global section of E with $\mathrm{pent}(\sigma; h) = p > 0$. Then find a necessary and sufficient condition on the section σ for the existence of a splitting subvariety of dimension p.

(5.6) Problem. Let X, E, σ, and h be as in (5.5) Problem above. Assume that the zero locus $(\sigma)_0$ of the section σ is of pure codimension r. Then, does there exist an algebraic p-cycle ζ such that the cycle ζ is not numerically equivalent to zero and $\zeta \cdot (\sigma)_0$ is numerically equivalent to zero (cf.[U-4]).

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Mixed Torelli problem for Todorov surfaces

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Introduction

There is an approach to Torelli problem by using degenerate loci. Namikawa and Friedman succeeded to prove the generic Torelli theorem for curves [Nam] and the Torelli theorem for algebraic K3 surfaces [F2] respectively in this direction.

In case of Todorov surfaces X, since the period map corresponding X to the Hodge structure on $H^2(X)$ has positive dimensional fibers ([T1], [T2], [U1], [U2], [U3]) it is necessary to consider the mixed period map which corresponds X to the mixed Hodge structure on $H^2(X - C)$, where C is the unique canonical curve of X ([U4], [SSU]). On the other hand, we can observe that Todorov surfaces are connected by "tame" degenerations and smooth deformations. It is the purose of the present paper to try to solve mixed Torelli problem for Todorov surfaces by using the "tame" degenrations. At present we have formulated the problem inductively and obtained some results but we have not yet arrived a final destination.

We give examples of "tame" degenrations of double covers of surfaces as Table 0 on the last page of this article. Degenerations of type (I_1) in Table 0 are observed for Todorov surfaces and surfaces with $c_1^2 = 2p_g - 3$, type (I_2) are obvserved for Kunev surfaces, and (II_1) are observed for surfaces on the Noether line ([U6], [U7]). Recently these phenomena are observed more widely ([K], [AK1], [AK2], [A1], [A2]). So our present trial can be seen as a miniature of a more ambitious attempt, namely, to attack (mixed) Torelli problem for surfaces of general type via degenerate loci.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1 is a Hodge theoretic prelininary. We recall, after [SZ], the constructions of (filtered) cohomological mixed Hodge complexes whose hypercohomologies yield the terms in a mixed version of the Clemens-Schmid sequence. We distinguish filtrations corresponding to the openness of the varieties in question and to their singularity and see their relationships. We prove partial results on the exactness of the mixed Clemens-Schmid sequence.

§2 contains an observation that the moduli spaces of Todorov surfaces are connected by "tame" degeneration, i.e., type (I_1) in Table 0. We use the results in [M].

In §3, we recall the moduli spaces of Todorov surfaces constructed in [M] and the formulation of a mixed period map in [U4]. We give a candidate of a global monodromy.

In §4, we prove the splitting of the local monodromy over **Z** by using the result in §1 and extend the mixed period map over the "tame" degenerations.

§5 contains a useful result in the induction step of our framework. We also prove partially the infinitesimal mixed Torelli theorem for the extended mixed period map.

§1. Mixed version of Clemens-Schmid sequence.

(1.1) Let

(1.1.1)
$$f : (\mathcal{X}, \mathcal{Y}) \longrightarrow \Delta$$

be a semi-stable degeneration of pairs, i.e., $\mathcal{X}$ is a submanifold of $\mathbf{P}^N \times \Delta$, the restriction of the projection $f : \mathcal{X} \to \Delta$ is a flat morphism over a disc Δ whose fiber $X_t := f^{-1}(t)$ over $t \in \Delta$ is smooth for $t \neq 0$ and X_0 is a reduced divisor with normal crossings, $\mathcal{Y}$ is a reduced divisor of $\mathcal{X}$ flat with respect to f, and $X_0 + \mathcal{Y}$ is of simple normal crossings. For any 1-parameter degeneration of pairs, we can reduce to this case (cf. [KKMS,II], [SSU,I.9]).

We use the following notation:

(1.1.2)
$$\begin{aligned}
&X_t := f^{-1}(t) \quad (t \in \Delta), && Y_t := \mathcal{Y} \cap X_t, \\
&\mathcal{X}^* := \mathcal{X} - X_0, && \mathcal{Y}^* := \mathcal{Y} - Y_0, \\
&\overset{\circ}{\mathcal{X}} := \mathcal{X} - \mathcal{Y}, && \overset{\circ}{\mathcal{X}}{}^* := \overset{\circ}{\mathcal{X}} \cap \mathcal{X}^*, \\
&\Delta^* := \Delta - \{0\}, && \widetilde{\Delta}^* \to \Delta^* \quad \text{universal cover } u \mapsto \exp(2\pi i u), \\
&X_\infty := \mathcal{X}^* \times_{\Delta^*} \widetilde{\Delta}^*, && Y_\infty := \mathcal{Y}^* \times_{\Delta^*} \widetilde{\Delta}^*, \\
&\overset{\circ}{X}_\infty := X_\infty - Y_\infty.
\end{aligned}$$

We consider a diagram

(1.1.3)
$$\begin{array}{ccc}
\overset{\circ}{\mathcal{X}}{}^* & \xrightarrow{\overset{\circ}{j}} & \overset{\circ}{\mathcal{X}} \\
\downarrow{\scriptstyle \overset{\circ}{\ell}} & & \downarrow{\scriptstyle \ell} \\
\mathcal{X}^* & \xrightarrow{j} & \mathcal{X}
\end{array}$$

$$\overset{\circ}{f} := f\,|_{\overset{\circ}{\mathcal{X}}} : \overset{\circ}{\mathcal{X}} \longrightarrow \Delta$$

Since (1.1.1) is locally C^∞-trivial over Δ^*, $R^n\overset{\circ}{f}_*\mathbf{Q}_{\overset{\circ}{\mathcal{X}}{}^*}$ is a local system and the Gysin filtration G induced from the canonical filtration τ (see [D2,II.(1.4.6)]) of $R\overset{\circ}{\ell}_*\mathbf{Q}_{\overset{\circ}{\mathcal{X}}{}^*}$ (i.e., the Leray filtration for $\overset{\circ}{f} : \overset{\circ}{\mathcal{X}}{}^* \xrightarrow{\overset{\circ}{\ell}} \mathcal{X}^* \xrightarrow{f} \Delta^*$) consists of local subsystems. We denote by

(1.1.4) $$(\mathcal{V}, G, \nabla), \qquad \mathcal{V} := R^n\overset{\circ}{f}_*\mathbf{Q}_{\overset{\circ}{\mathcal{X}}{}^*} \otimes_{\mathbf{Q}} \mathcal{O}_{\Delta^*}$$

the associated filtered vector bundle with the Gauss-Manin connection.

Since $\overset{\circ}{f}{}^{-1}\mathcal{O}_{\Delta^*} \to \Omega^{\cdot}_{\overset{\circ}{\mathcal{X}^*}/\Delta^*}$ is a resolution and $\overset{\circ}{\ell}$ in (1.1.3) is Stein, $R\overset{\circ}{\ell}_*\overset{\circ}{f}{}^{-1}\mathcal{O}_{\Delta^*}$ is represented by $\overset{\circ}{\ell}{}^m_*\Omega^{\cdot}_{\mathcal{X}^*/\Delta^*}$ hence by $\Omega^{\cdot}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*)$ [D2,II.(3.3.1),(3.14.i)], which together with the canonical filtration τ and with the weight filration $W(\mathcal{Y}^*)$ are filtered quasi-isomorphic [D2,II.(3.1.8)]. Therefore

$$(1.1.5) \qquad (\mathcal{V}, G) \simeq (R^n f_* \Omega^{\cdot}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*), W(\mathcal{Y}^*)).$$

By the same reasoning, an exact sequence

$$(1.1.6) \qquad 0 \to R\overset{\circ}{\ell}_*\overset{\circ}{f}{}^{-1}\Omega^1_{\Delta^*}[-1] \to R\overset{\circ}{\ell}_*\overset{\circ}{f}{}^{-1}\Omega^{\cdot}_{\Delta^*} \to R\overset{\circ}{\ell}_*\overset{\circ}{f}{}^{-1}\mathcal{O}_{\Delta^*} \to 0$$

is represented by

$$(1.1.7) \qquad \begin{aligned} 0 \to f^{-1}\Omega^1_{\Delta^*} \otimes_{f^{-1}\mathcal{O}_{\Delta^*}} \Omega^{\cdot}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*)[-1] \to \\ \Omega^{\cdot}_{\mathcal{X}^*}(\log \mathcal{Y}^*) \to \Omega^{\cdot}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*) \to 0, \end{aligned}$$

hence we see that the Gauss-Manin connection ∇ of $\mathcal{V}$ is induced as the connecting homomorphism of the hypercohomology sequence of (1.1.7) [KO].

The following lemma can be found in [D1,II.(5.2),(7.11)], [St,(2.16)] and [SZ, (5.3)].

Lemma (1.1.8). *$\widetilde{\mathcal{V}} := R^n f_*\Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))$ is the canonical extension of $(\mathcal{V}, G, \nabla)$, i.e., the following hold:*

(i) *$\widetilde{\mathcal{V}}$ is a vector bundle on Δ with $\widetilde{\mathcal{V}}\,|_{\Delta^*} = \mathcal{V}$.*

(ii) *G on $\mathcal{V}$ extends uniquely to a filtration of $\widetilde{\mathcal{V}}$ by subbundles, also denoted by G.*

(iii) *∇ extends to a connection of $\widetilde{\mathcal{V}}$ with logarithmic pole at $0 \in \Delta$ with $Res_0(\nabla)$ nilpotent.*

Idea of Proof. (i) and (ii) follow from a fundamental observation: For $X_\infty \xrightarrow{k} \mathcal{X} \xhookleftarrow{i} X_0$ and $u = \log(t/2\pi i)$,

$$\textbf{(1.1.9)} \qquad \begin{aligned} \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0)) \otimes \mathcal{O}_{X_0} &\underset{QIS}{\overset{\psi_t}{\longleftarrow}} i^{-1}\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[u] \\ &\underset{QIS}{\longrightarrow} i^{-1}k_*\Omega^{\cdot}_{X_\infty}(\log Y_\infty), \end{aligned}$$

where

$$\textbf{(1.1.10)} \qquad \psi_t\left(\sum \omega_j u^j\right) := (\text{image of } \omega_0).$$

(iii) follows from an exact sequence

$$\textbf{(1.1.11)} \qquad \begin{aligned} 0 \to f^{-1}\Omega^1_\Delta(\log 0) \otimes_{f^{-1}\mathcal{O}_\Delta} \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))[-1] \to \\ \Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) \to \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0)) \to 0, \end{aligned}$$

which is an extension of (1.1.7), and a direct computation of the residue. For details, see the above references. ∎

(1.2) We recall the construction of the mixed version of the Steenbrink complex $A^{\cdot}$ in [SZ,§5] (see also [Nav,§14], [E2]). In the situation and the notation in (1.1), we consider a diagram

$$\textbf{(1.2.1)} \qquad \begin{array}{ccccc} \overset{\circ}{X}_\infty & \xrightarrow{\overset{\circ}{k}} & \overset{\circ}{\mathcal{X}} & & \\ \downarrow & & \downarrow \ell & & \\ X_\infty & \xrightarrow{k} & \mathcal{X} & \xleftarrow{i} & X_0 \end{array}$$

$$k' := \ell \overset{\circ}{k} : \overset{\circ}{X}_\infty \longrightarrow \mathcal{X}$$

By the Eilenberg-Zilber theorem [Sp,p.232], we see

$$k'_* \Delta^\cdot(\overset{\circ}{X}_\infty) \underset{QIS}{\simeq} \ell_* \Delta^\cdot(\overset{\circ}{\mathcal{X}}) \otimes_{\mathbf{Q}} k_* \Delta^\cdot(X_\infty)$$

[SZ,(5.20)], where $\Delta^\cdot(Z)$ is the complex of sheaves of germs of singular $\mathbf{Q}$-cochains on a topological space Z. Since $\Delta^\cdot(Z)$ is a fine resolution of $\mathbf{Q}_Z$, we see by the above result, that

$$\text{(1.2.2)} \qquad \begin{aligned} &I^\cdot(\overset{\circ}{\mathcal{X}}) := i^{-1}\ell_*\Delta^\cdot(\overset{\circ}{\mathcal{X}}), \qquad I^\cdot(X_\infty) := i^{-1}k_*\Delta^\cdot(X_\infty) \qquad \text{and} \\ &I^\cdot(\overset{\circ}{X}_\infty) := s^\cdot(I^\cdot(\overset{\circ}{\mathcal{X}}) \otimes_{\mathbf{Q}} I^\cdot(X_\infty)) \end{aligned}$$

are representatives of $i^{-1}R\ell_*\mathbf{Q}_{\overset{\circ}{\mathcal{X}}}$, $i^{-1}Rk_*\mathbf{Q}_{X_\infty}$ and $i^{-1}Rk'_*\mathbf{Q}_{\overset{\circ}{X}_\infty}$ respectively.

$I^\cdot(\overset{\circ}{X}_\infty)$ is of course a candidate of the $\mathbf{Q}$-structure but the monodromy logarithm $\log T$ can not be lifted on this complex. In order to rescue this situation, we need a rather complicated construction of $A^\cdot_{\mathbf{Q}}$ in the following way.

The automorphism $(x, u) \mapsto (x, u-1)$ on $X_\infty := \mathcal{X}^* \times_{\Delta^*} \widetilde{\Delta}^*$ induces an automorphism T of $I^\cdot(X_\infty)$. Define

$$\text{(1.2.3)} \qquad \begin{aligned} &B^\cdot(X_\infty) := \bigcup_{m \geqslant 0} \operatorname{Ker}(T-1)^{m+1} \subset I^\cdot(X_\infty), \\ &B^\cdot := B^\cdot(\overset{\circ}{X}_\infty) := I^\cdot(\overset{\circ}{\mathcal{X}}) \otimes_{\mathbf{Q}} B^\cdot(X_\infty) \subset I^\cdot(\overset{\circ}{X}_\infty). \end{aligned}$$

Then these inclusions are quasi-isomorphisms [SZ,(5.9)] (more precisely, see [Nav, §14]), and

$$\text{(1.2.4)} \qquad \delta := \log T : B^\cdot(X_\infty) \longrightarrow B^\cdot(X_\infty)$$

is well-defind by construction.

Let

$$\rho(B)^{\cdot} := \rho(B^{\cdot}(\overset{\circ}{X}_{\infty}), 1 \otimes \delta)^{\cdot} \simeq s^{\cdot}(\Gamma^{\cdot}(\overset{\circ}{\mathcal{X}}) \otimes \rho(B^{\cdot}(X_{\infty}), \delta))$$

be the mapping cone, i.e.,

$$\rho(B)^p := B^p \oplus B^{p-1}, \qquad d(x,y) := (dx, (1 \otimes \delta)x - dy)$$

We define a morphism of complexes

(1.2.5) $$\theta : \rho(B)^{\cdot} \longrightarrow \rho(B)^{\cdot}[1] \quad \textit{by} \quad \theta(x,y) := (0,x).$$

Let

$$\tau'_q(K^{\cdot} \otimes L^{\cdot}) := (\tau_q K^{\cdot}) \otimes L^{\cdot}, \qquad \tau''_q(K^{\cdot} \otimes L^{\cdot}) := K^{\cdot} \otimes (\tau_q L^{\cdot}),$$

be the partial canonical filtrations for a tensor product of complexes $K^{\cdot}$ and $L^{\cdot}$, where τ is the canonical filtration.

A double complex $A^{\cdot\cdot}_{\mathbf{Q}} = A^{\cdot\cdot}_{\mathbf{Q}}(\overset{\circ}{X}_{\infty})$ is defined as

(1.2.6)

$$A^{\cdot q}_{\mathbf{Q}} := \begin{cases} (\rho(B)^{\cdot}/\tau''_q)[q+1] & \text{if } p \geqslant -1 \text{ and } q \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$d' : A^{pq}_{\mathbf{Q}} \to A^{p+1,q}_{\mathbf{Q}}$ is induced from $(-1)^{q+1} d_{\rho(B)^{\cdot}}$, and

$d'' : A^{pq}_{\mathbf{Q}} \to A^{p,q+1}_{\mathbf{Q}}$ is induced from θ.

The **Q**-*structure of the mixed version of the Steenbrink complex* is the associated single complex:

(1.2.7) $$A^{\cdot}_{\mathbf{Q}} := s^{\cdot}(A^{\cdot\cdot}_{\mathbf{Q}}), \qquad d := (-1)^q d' + d'' = -d_{\rho(B)^{\cdot}} + \theta \quad \text{on} \quad A^{pq}_{\mathbf{Q}}$$

It can be seen that the map $B^{\cdot} \hookrightarrow A^{\cdot}_{\mathbf{Q}}$ defined by $B^p \ni x \mapsto (0,x) \in A^{p,0}_{\mathbf{Q}}$ is a quasi-isomorphism [SZ,(5.13)].

Let $\widetilde{\delta}$ and ν be endomorphisms of the complex $A^{\cdot}_{\mathbf{Q}}$ defined by

$$(1.2.8) \qquad \begin{array}{ll} \widetilde{\delta} : A^{pq}_{\mathbf{Q}} \to A^{pq}_{\mathbf{Q}} & \widetilde{\delta}(x,y) := ((1\otimes\delta)x, (1\otimes\delta)y), \quad \text{and} \\ \nu : A^{pq}_{\mathbf{Q}} \to A^{p-1,q+1}_{\mathbf{Q}} & \text{projection.} \end{array}$$

These are homotopic [SZ,(5.14)]. In fact, it is easy to verify that the map given by

$$h : A^{p+1,q-1}_{\mathbf{Q}} \to A^{p,q-1}_{\mathbf{Q}} \qquad h(x,y) := (y,0)$$

satisfies $\nu - \widetilde{\delta} = dh + hd$. Moreover the endomorphisms $1\otimes\delta$ of $B^{\cdot}$ and $\widetilde{\delta}$ of $A^{\cdot}_{\mathbf{Q}}$ are compatible with $B^{\cdot} \hookrightarrow A^{\cdot}_{\mathbf{Q}}$. Hence ν on $A^{\cdot}_{\mathbf{Q}}$ induces $\log T$ on the hypercohomology, which is the significance of the complex $A^{\cdot}_{\mathbf{Q}}$.

Let $W(X_0)$ be the partial weight filtration of the complex $\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))$, i.e., $W_q(X_0)\Omega^p_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) := \Omega^q_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) \wedge \Omega^{p-q}_{\mathcal{X}}(\log\mathcal{Y})$. We define a double complex $A^{\cdot\cdot}_{\mathbf{C}} = A^{\cdot\cdot}_{\mathbf{C}}(\overset{\circ}{X}_\infty)$ by

$$(1.2.9) \qquad \begin{array}{l} A^{\cdot q}_{\mathbf{C}} := \begin{cases} (\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))/W_q(X_0))[q+1] & \text{if } p,q \geqslant 0, \\ 0 & \text{otherwise,} \end{cases} \\ d' : A^{pq}_{\mathbf{C}} \to A^{p+1,q}_{\mathbf{C}} \quad \text{is induced from } (-1)^{q+1}(\text{exterior differential}), \text{ and} \\ d'' : A^{pq}_{\mathbf{C}} \to A^{p,q+1}_{\mathbf{C}} \quad \text{is induced from } \theta\wedge, \end{array}$$

where

$$(1.2.10) \qquad \theta := f^* d\log t/2\pi i, \qquad t : \text{a parameter of the disc } \Delta.$$

The **C**-*structure of the mixed version of the Steenbrink complex* is the associated single complex:

(1.2.11)

$$A^{\cdot}_{\mathbf{C}} := s^{\cdot}(A^{\cdot\cdot}_{\mathbf{C}}), \qquad d := (-1)^q d' + d'' = -(\text{exterior differential}) + \theta \wedge \ \text{on } A^{pq}_{\mathbf{C}}$$

Let ν be an endomorphism of the complex $A^{\cdot}_{\mathbf{C}}$ defined by

(1.2.12) $$\nu : A^{pq}_{\mathbf{C}} \to A^{p-1,q+1}_{\mathbf{C}} \qquad \text{projection.}$$

In order to see the relation between the **Q**-structure and the **C**-structure, we set

(1.2.13)
$$\widetilde{B}^{\cdot}(X_\infty) := i^{-1}\Omega^{\cdot}_{\mathcal{X}}(\log X_0)[u], \quad \text{where } u = \log t/2\pi i, \text{ and}$$
$$\widetilde{B}^{\cdot} := \widetilde{B}^{\cdot}(\overset{\circ}{X}_\infty) := \Omega^{\cdot}_{\mathcal{X}}(\log \mathcal{Y}) \otimes_{\mathbf{C}} \widetilde{B}^{\cdot}(X_\infty)$$

and construct a complex

(1.2.14) $$\widetilde{A}^{\cdot}_{\mathbf{C}} \qquad \text{from} \qquad \widetilde{B}^{\cdot}$$

in the same way as the construction of $A^{\cdot}_{\mathbf{Q}}$ from $B^{\cdot}$. We define an endomorphism

(1.2.15) $$\delta \ \text{ of } \ \widetilde{B}^{\cdot}(X_\infty) \ \text{ by } \ \delta\left(\sum \omega_j u^j/j!\right) := -\sum \omega_j u^{j-1}/(j-1)!,$$

and denote the induced ones by

(1.2.16)
$$1 \otimes \delta \quad \text{on} \quad \widetilde{B}^{\cdot},$$
$$\widetilde{\delta} \quad \text{on} \quad \widetilde{A}^{\cdot}_{\mathbf{C}}, \qquad \widetilde{\delta}(x,y) := ((1\otimes\delta)x, (1\otimes\delta)y).$$

We denote also by

(1.2.17) ν on $\widetilde{A}^{\cdot}_{\mathbf{C}}$: the one induced from the projection $\widetilde{A}^{pq}_{\mathbf{C}} \to \widetilde{A}^{p-1,q+1}_{\mathbf{C}}$.

Then we have compatible quasi-isomorphisms:

$$
\begin{array}{llll}
 & \widetilde{\delta} & \text{on} \quad A^{\cdot}_{\mathbf{Q}} \otimes \mathbf{C} & \\
 & & \uparrow \wr \ \text{QIS} & [\text{SZ},(5.13)] \\
 & \delta & \text{on} \quad B^{\cdot} \otimes \mathbf{C} & \\
 & & \uparrow \wr \ \text{QIS} & [\text{SZ},(5.18)] \\
 & 1 \otimes \delta & \text{on} \quad \widetilde{B}^{\cdot} & \\
 & & \downarrow \wr \ \text{QIS} & [\text{SZ},(5.13)] \\
(1.2.18) & \widetilde{\delta} & \text{on} \quad \widetilde{A}^{\cdot}_{\mathbf{C}} & \\
 & & \| & [\text{SZ},(5.14)] \\
 & \nu & \text{on} \quad \widetilde{A}^{\cdot}_{\mathbf{C}} & \\
 & & \psi_t \downarrow \wr \ \text{QIS} & [\text{SZ},(5.18)] \\
 & \nu & \text{on} \quad A^{\cdot}_{\mathbf{C}} & \\
 & & (-1)^{\cdot}\theta \wedge \uparrow \wr \ \text{QIS} & [\text{SZ},(5.5)],\ [\text{St},(4.16)] \\
 & & \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0)) \otimes \mathcal{O}_{X_0}, &
\end{array}
$$

where ψ_t above is induced from a morphism of double complexes defined by

$$\textbf{(1.2.19)} \qquad \psi_t : \widetilde{A}^{pq}_{\mathbf{C}} \to A^{pq}_{\mathbf{C}}, \quad \psi_t\left(\sum x_j u^j/j!, \sum y_j u^j/j!\right) := x_0 + du \wedge y_0.$$

Taking hypercohomology, (1.2.18) induces a compatible isomorphism (cf. [St,(4.22)]):

$$
\begin{array}{llll}
 & \log T & \text{on} & H^n(\overset{\circ}{X}_\infty, \mathbf{C}) \\
(1.2.20) & & & \psi_t \downarrow \wr \\
 & -2\pi i \mathrm{Res}_0(\nabla) & \text{on} & \widetilde{\mathcal{V}}(0) = H^n(X_0, \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0)) \otimes \mathcal{O}_{X_0}),
\end{array}
$$

where ∇ is the Gauss-Manin connection in (1.1.8). In this sense, we hereafter denote

$$\textbf{(1.2.21)} \qquad N := \log T = -2\pi i \mathrm{Res}_0(\nabla).$$

Remark (1.2.22). [St,(4.24)] explains how the isomorphism ψ_t in (1.2.20) depends on the choice of the parameter t of Δ (cf. also [Nav,(14.18)]). This can be also explained in the following way (superfluous ?).

Let $\{e_1, ..., e_r\}$ be a multi-valued flat frame of $\mathcal{V}$ in (1.1.5). Modifying

$$\widetilde{e}_j := \exp(-u \log T) e_j$$

we get an invariant frame $\{\widetilde{e}_1, ..., \widetilde{e}_r\}$ which extends over Δ and induces a basis of the central fiber $\widetilde{\mathcal{V}}(0)$ of the canonical extension [D1,II.§5], also denoted by the same symbols. Let M_∇ and M_T be the matrices such that

$$(\nabla\widetilde{e}_1, \ldots, \nabla\widetilde{e}_r) = (\widetilde{e}_1, \ldots, \widetilde{e}_r) M_\nabla, \qquad \text{and}$$

$$(Te_1, \ldots, Te_r) = (e_1, \ldots, e_r) M_T$$

Then

$$\psi_t(e_j) = \widetilde{e}_j \qquad \text{for all} \quad j,$$

and under this identification we have (cf. [D1,II.(1.17),(5.6)])

$$\log M_T = -2\pi i \mathrm{Res}_0(M_\nabla). \quad \blacksquare$$

We define filtrations of $A^{\cdot}$ by

(1.2.23)

$$G_i A^{\cdot q} := \text{image}\begin{cases} (\tau'_i \rho(B)^{\cdot})[q+1] \to A^{\cdot q}_{\mathbf{Q}}, \\ W_i(\mathcal{Y}) \Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[q+1] \to A^{\cdot q}_{\mathbf{C}}, \end{cases}$$

$$L_j A^{\cdot q} := \text{image}\begin{cases} (\tau''_{j+2q+1} \rho(B)^{\cdot})[q+1] \to A^{\cdot q}_{\mathbf{Q}}, \\ W_{j+2q+1}(X_0) \Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[q+1] \to A^{\cdot q}_{\mathbf{C}}, \end{cases}$$

$$W_k A^{\cdot q} := \text{image}\begin{cases} (\tau_{k+2q+1} \rho(B)^{\cdot})[q+1] \to A^{\cdot q}_{\mathbf{Q}}, \\ W_{k+2q+1}(\mathcal{Y}+X_0) \Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[q+1] \to A^{\cdot q}_{\mathbf{C}}, \end{cases}$$

$$F^p A^{\cdot\cdot}_{\mathbf{C}} := \bigoplus_{p' \geqslant p} A^{p'\cdot}_{\mathbf{C}}.$$

A *convolution* (or *amalgam*) $F' * F''$ of two filtrations F' and F'' is defined by

$$(F' * F'')_k := \sum_{i+j=k} F'_i \cap F''_j \qquad \text{[SZ,(1.4)]}.$$

Lemma (1.2.24). (i) *$(A^{\cdot}, G * L) \to (A^{\cdot}, W)$ is a filtered quasi-isomorphism.*

(ii) *G on $A^{\cdot}$ satisfies*

$$\nu G_i \subset G_i, \qquad gr_i^G A^{\cdot} \underset{QIS}{\overset{\sim}{\longrightarrow}} A^{\cdot}(\widetilde{Y}_\infty^{(i)})[-i],$$

and induces the Gysin filtration on the hypercohomology, where $\widetilde{Y}_\infty^{(i)}$ is the normalization of the i-ple locus of Y_∞.

(iii) *$(A^{\cdot}_{\mathbf{Q}}, L) \otimes \mathbf{C} \underset{FQIS}{\overset{\sim}{=}} (A^{\cdot}_{\mathbf{C}}, L)$ and L on $A^{\cdot}_{\mathbf{Q}}$ induces the N-filtration on the hypercohomology, i.e., $NL_j \subset L_{j-2}$ and $N^j : gr_j^L \overset{\sim}{\to} gr_{-j}^L$ on $H^n(X_0, A^{\cdot}_{\mathbf{Q}}) = H^n(\overset{o}{X}_\infty, \mathbf{Q})$.*

Proof. Set $\Omega := \Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y} + X_0))$. Then

$$\begin{aligned}
&(G * L)_k A^{\cdot q}_{\mathbf{C}} \\
=&\Big(\sum_{i+j=k} ((W_i(\mathcal{Y}) + W_q(X_0)) \cap (W_{j+2q+1}(X_0) + W_q(X_0)))/W_q(X_0)\Big)\Omega[1] \\
=&\Big(\sum_{i+j=k} (W_i(\mathcal{Y}) \cap W_q(X_0) + W_{j+2q+1}(X_0))/W_q(X_0)\Big)\Omega[1] \\
=&((W_{k+2q+1}(\mathcal{Y} + X_0) + W_q(X_0))/W_q(X_0))\Omega[1] = W_k A^{\cdot q}_{\mathbf{C}}.
\end{aligned}$$

Similarly we have $(G * L)_k A^{\cdot}_{\mathbf{Q}} \subset W_k A^{\cdot}_{\mathbf{Q}}$. These together with [D2,II.(3.1.8)] yield a commutative diagram:

$$\begin{array}{ccccc}
(A^{\cdot}_{\mathbf{Q}}, G * L) \otimes \mathbf{C} & \underset{FQIS}{\overset{\sim}{=}} & (A^{\cdot}_{\mathbf{C}}, \tau' * \tau''[-2q-1]) & \underset{FQIS}{\overset{\sim}{\longrightarrow}} & (A^{\cdot}_{\mathbf{C}}, G * L) \\
\downarrow & & & & \| \\
(A^{\cdot}_{\mathbf{Q}}, W) \otimes \mathbf{C} & \underset{FQIS}{\overset{\sim}{=}} & (A^{\cdot}_{\mathbf{C}}, \tau[-2q-1]) & \underset{FQIS}{\overset{\sim}{\longrightarrow}} & (A^{\cdot}_{\mathbf{C}}, W)
\end{array}$$

From this we get the assertion for the **Q**-structure. This proves (i).

The first assertion of (ii) is immediate by definition. As for the second,

$$\begin{aligned}\mathrm{gr}_i^G A_{\mathbf{Q}}^{\cdot q} &\simeq (((\tau_i' + \tau_q'')/(\tau_{i-1}' + \tau_q''))\rho(B)^{\cdot})[q+1] \\ &\simeq ((\tau_i'/(\tau_{i-1}' + \tau_i' \cap \tau_q''))\rho(B)^{\cdot})[q+1] \\ &\simeq (\mathrm{gr}_i^{\tau'} \rho(B)^{\cdot}/\tau_q''\rho(B)^{\cdot})[q+1] \\ &\underset{QIS}{\overset{\sim}{\longrightarrow}} (a_*\mathbf{Q}_{\widetilde{\mathcal{Y}}^{(i)}}[-i] \otimes (\rho(B^{\cdot}(X_\infty))^{\cdot}/\tau_q''))[q+1] \\ &= a_*\mathbf{Q}_{\widetilde{\mathcal{Y}}^{(i)}}[-i] \otimes A_{\mathbf{Q}}^{\cdot q}(X_\infty) \underset{QIS}{\overset{\sim}{\longrightarrow}} A_{\mathbf{Q}}^{\cdot q}(\widetilde{Y}_\infty^{(i)})[-i].\end{aligned}$$

Similarly we have the second assertion for the **C**-structure. The last assertion follows from these. This proves (ii).

The first assertion of (iii) is easy by construction. We prove the second assertion.

$$\nu L_j A_{\mathbf{Q}}^{pq} = \nu \tau_{j+2q+1}'' A_{\mathbf{Q}}^{pq} = \tau_{j+2q+1}'' A_{\mathbf{Q}}^{p-1,q+1} = L_{j-2} A_{\mathbf{Q}}^{p-1,q+1}.$$

Hence $NL_j \subset L_{j-2}$ on $H^n(\overset{\circ}{X}_\infty, \mathbf{Q})$. Next we observe that

$$(A_{\mathbf{Q}}^{\cdot}, G) \underset{QIS}{\overset{\sim}{\longrightarrow}} (R\ell_{t*}\mathbf{Q}_{\overset{\circ}{X}_t}, \tau), \qquad \text{where } \ell_t : \overset{\circ}{X}_t \hookrightarrow X_t \quad (t \in \Delta^*)$$

and that the latter is a part of the functorial cohomological mixed Hodge complex for $\overset{\circ}{X}_\infty$ (see [D2,III.(8.1)]) hence the spectral sequence of $(R\Gamma R\ell_{t*}\mathbf{Q}_{\overset{\circ}{X}_t}, \tau)$ degenerates in $E_2 = E_\infty$. We also observe that under

$$\mathrm{gr}_i^G A^{\cdot\cdot}(\overset{\circ}{X}_\infty) \underset{QIS}{\overset{\sim}{\longrightarrow}} A^{\cdot - i, \cdot}(\widetilde{Y}_\infty^{(i)})$$

$L_j \mathrm{gr}_i^G A^{\cdot\cdot}(\overset{\circ}{X}_\infty)$ corresponds to $W_j A^{\cdot - i, \cdot}(\widetilde{Y}_\infty^{(i)})$ and the d_1 of the above spectral sequence are morphisms of mixed Hodge structures (actually, Gysin maps), so $L = W$

on E_1 is strict for d_1 [D2,II.(2.3.5.iii)]. It follows that taking cohomology and gr^L commute [D2,II.(1.1.11.ii)]. By [St,(5.9)] (see (A.1) below),

$$N^j : \mathrm{gr}^L_j E_1^{pq} \xrightarrow{\sim} \mathrm{gr}^L_{-j} E_1^{pq}.$$

Hence

$$N^j : \mathrm{gr}^L_j E_2^{-i,n+i} \xrightarrow{\sim} \mathrm{gr}^L_{-j} E_2^{-i,n+i}.$$

That is

$$N^j : \mathrm{gr}^L_j \mathrm{gr}^G_i H^n(\overset{\circ}{X}_\infty, \mathbf{Q}) \xrightarrow{\sim} \mathrm{gr}^L_{-j} \mathrm{gr}^G_i H^n(\overset{\circ}{X}_\infty, \mathbf{Q}).$$

This implies

$$N^j : \mathrm{gr}^L_j H^n(\overset{\circ}{X}_\infty, \mathbf{Q}) \xrightarrow{\sim} \mathrm{gr}^L_{-j} H^n(\overset{\circ}{X}_\infty, \mathbf{Q}). \quad \blacksquare$$

[El] generalized the notion of cohomological mixed Hodge complex (*CMHC*, for short) in [D2,III.(8.1)] to:

Definition (1.2.25). $(M, G) = ((M^\cdot_{\mathbf{Q}}, G, W), (M^\cdot_{\mathbf{C}}, G, W, F), \alpha)$ *is a G-filtered CMHC on a topological space Z if it satisfies the following conditions:*

(i) *M is a* $\mathbf{Q}$*-CMHC on Z.* $\alpha : (M^\cdot_{\mathbf{Q}}, G, W) \otimes \mathbf{C} \simeq (M^\cdot_{\mathbf{C}}, G, W)$ *is a bifiltered quasi-isomorphism.*

(ii) $\mathrm{gr}^G_i M$ *is a* $\mathbf{Q}$*-CMHC on Z for each i.*

(iii) *Dec W and* gr^G *commute on* $\widetilde{M}^\cdot := R\Gamma M^\cdot_{\mathbf{Q}}$.

(iv) *The spectral sequence of* $(\widetilde{M}^\cdot, G)$ *degenerates in* $E_2 = E_\infty$.

Recall that the Hodge filtration F on $\mathcal{V}$ in (1.1.5) is the one induced from the stupid filtration

$$F^p \Omega^\cdot_{\mathcal{X}^\bullet}(\log \mathcal{Y}^*) := \sum_{p' \geqslant p} \Omega^{p'}_{\mathcal{X}^\bullet}(\log \mathcal{Y}^*)$$

The following lemma can be found in [SZ,§5,(6.9),(3.13),Appendix].

Lemma (1.2.26). *$((A^{\cdot}_{\mathbf{Q}}, G, W), (A^{\cdot}_{\mathbf{C}}, G, W, F), \alpha)$ is a G-filtered CMHC on X_0, whose hypercohomology yields a limit of the variation of mixed Hodge structure arising from $\overset{\circ}{f} : \overset{\circ}{\mathcal{X}}{}^* \to \Delta^*$, that is, the following hold:*

(i) *W on $A^{\cdot}_{\mathbf{Q}}$ induces the G-relative N-filtration on the hypercohomology, i.e., $NW_k \subset W_{k-2}$ and $N^j : gr^W_{i+k} gr^G_i \xrightarrow{\sim} gr^W_{i-k} gr^G_i$ on $H^n(X_0, A^{\cdot}_{\mathbf{Q}}) = H^n(\overset{\circ}{X}_\infty, \mathbf{Q})$.*

(ii) *F on $\mathcal{V}$ extends to a filtration of $\widetilde{\mathcal{V}}$ in (1.1.8) such that $F^p gr^G_i \widetilde{\mathcal{V}}$ is locally free and $F^p \widetilde{\mathcal{V}}(0) = F^p H^n(X_0, A^{\cdot}_{\mathbf{C}})$ for each i and p.*

Proof. By (1.2.24.i) and [Z,II.(A.1)], we have

$$\begin{aligned} \mathrm{gr}^W_k A^{\cdot}_{\mathbf{C}} &\simeq \bigoplus_{i+j=k} \mathrm{gr}^G_i \mathrm{gr}^L_j A^{\cdot}_{\mathbf{C}} \\ &\overset{Res}{\xrightarrow{\sim}} \bigoplus_{i+j=k} \bigoplus_{q \geqslant \max\{0,-k\}} a_* \Omega^{\cdot}_{\widetilde{\mathcal{Y}}^{(i)} \cap \widetilde{X}_0^{(j+2q+1)}}[-k-2q], \end{aligned}$$

where $\widetilde{\mathcal{Y}}^{(i)} \cap \widetilde{X}_0^{(j')}$ is the normalization of (i-ple in $\mathcal{Y}$, j'-ple in X_0)-locus of $\mathcal{Y} + X_0$ and $a : \widetilde{\mathcal{Y}}^{(i)} \cap \widetilde{X}_0^{(j')} \to X_0$ is the projection (cf. [SZ,(5.22)]). The above isomorphism is compatible with F and we have a similar decomposition for the $\mathbf{Q}$-structure. (1.2.25.i) follows. By (1.2.24.i), [Z,II.(A.1)] and [SZ,(1.5)], we have

$$\mathrm{gr}^W_k A^{\cdot} \simeq \bigoplus_{i+j=k} \mathrm{gr}^G_i \mathrm{gr}^L_j A^{\cdot} \simeq \bigoplus_i \mathrm{gr}^W_k \mathrm{gr}^G_i A^{\cdot}.$$

This is compatible with F. (1.2.25.ii) follows. (1.2.25.iii) also follows by [SZ,(6.8)]. (1.2.25.iv) is already shown in the proof of (1.2.24.iii). This proves the first half of the assertion.

The proof of (i) in the second assertion is analoguous to that of (1.2.24.iii) and we omit it.

As for (ii), set $\Omega(t) := \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y} + X_0)) \otimes \mathcal{O}_{X_t}$ $(t \in \Delta)$. We first note that, for $\theta := f^* d \log t / 2\pi i$,

$$\theta \wedge : (\Omega(0), F) \underset{FQIS}{\xrightarrow{\sim}} (A^{\cdot}_{\mathbf{C}}, F) \qquad \text{(cf. [St,(4.16)])}.$$

This implies $F^p\widetilde{\mathcal{V}}(0) = F^pH^n(X_0, A_{\mathbf{C}}^{\cdot})$. As we have seen in the proof of (1.2.24.iii), F on the E_1 of the spectral sequence of $(R\Gamma R\ell_{t*}\mathbf{Q}_{\overset{\circ}{X}_t}, \tau)$ is strict for d_1. hence gr_F commutes with taking cohomology, and we can compute as

$$\begin{aligned}
&\mathrm{gr}_F^p\mathrm{gr}_i^G\widetilde{\mathcal{V}}(t) = \mathrm{gr}_F^p\mathrm{gr}_i^G H^n(R\Gamma\Omega(t)) \\
&= \mathrm{gr}_F^p E_2^{-i,n+i}(R\Gamma\Omega(t), G) = \mathrm{gr}_F^p E_1^{-i+n,i}(R\Gamma\Omega(t), \mathrm{Dec}\, G) \\
&= \mathrm{gr}_F^p H^n(R\Gamma\mathrm{gr}_{i-n}^{\mathrm{Dec}\, G}\Omega(t)) = H^n(R\Gamma\mathrm{gr}_F^p\mathrm{gr}_{i-n}^{\mathrm{Dec}\, G}\Omega(t)) \\
&= H^n(X_t, \mathrm{gr}_F^p\mathrm{gr}_{i-n}^{\mathrm{Dec}\, G}\Omega(t)).
\end{aligned}$$

From this, we see that $\dim \mathrm{gr}_F^p\mathrm{gr}_i^G\widetilde{\mathcal{V}}(t)$ is upper semi-continuous in $t \in \Delta$. On the other hand, $\dim \mathrm{gr}_i^G\widetilde{\mathcal{V}}(t)$ is constant. Hence $\mathrm{gr}_F^p\mathrm{gr}_i^G\widetilde{\mathcal{V}}$ is locally free by the continuity theorem. ∎

(1.3) In the situation of (1.1), we recall a construction of a CMHC $K^{\cdot}$ whose hypercohomology gives the functorial mixed Hodge structure on the cohomology of $\overset{\circ}{X}_0$ (cf. [D2,III.(8.1.12)]).

Let $K_{\mathbf{Q}}^{\cdot\cdot}$ be a double complex defined by

$$K_{\mathbf{Q}}^{pq} := \begin{cases} I^p(\overset{\circ}{\mathcal{X}}) \otimes_{\mathbf{Q}} a_*\mathbf{Q}_{\widetilde{X}_0^{(q+1)}} & \text{if } p, q \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$d' : K_{\mathbf{Q}}^{pq} \to K_{\mathbf{Q}}^{p+1,q} \qquad \text{is } (-1)^{q+1}d_{I^{\cdot}(\overset{\circ}{\mathcal{X}})}, \text{ and}$$

$$d'' : K_{\mathbf{Q}}^{pq} \to K_{\mathbf{Q}}^{p,q+1} \qquad \text{is the Mayer-Vietoris map } 1 \otimes \left(\sum_i (-1)^i \delta_i^*\right).$$

where $a : \widetilde{X}_0^{(q+1)} \to X_0$ is the projection and $I^{\cdot}(\overset{\circ}{\mathcal{X}})$ is the complex in (1.2.2). The $\mathbf{Q}$-structure is defined as the associated single complex

$$\textbf{(1.3.2)} \qquad K_{\mathbf{Q}}^{\cdot}, \quad d := (-1)^q d' + d'' = -d_{I^{\cdot}(\overset{\circ}{\mathcal{X}})} + 1 \otimes \left(\sum_i (-1)^i \delta_i^*\right) \quad \text{on } K_{\mathbf{Q}}^{pq}$$

Let $K_{\mathbf{C}}^{\cdot\cdot}$ be a double complex defined by

$$K_{\mathbf{C}}^{pq} := \begin{cases} a_* \Omega^p_{\widetilde{X}_0^{(q+1)}}(\log(\mathcal{Y} \cap \widetilde{X}_0^{(q+1)})) & \text{if } p, q \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$d' : K_{\mathbf{C}}^{pq} \to K_{\mathbf{C}}^{p+1,q} \quad \text{is } (-1)^{q+1}(\text{exterior differential}), \text{ and}$$

$$d'' : K_{\mathbf{C}}^{pq} \to K_{\mathbf{C}}^{p,q+1} \quad \text{is the Mayer-Vietoris map } \sum_i (-1)^i \delta_i^*.$$

The $\mathbf{C}$-structure is defined as the associated single complex

(1.3.3) $\quad K_{\mathbf{C}}^{\cdot}, \quad d := (-1)^q d' + d'' = -(\text{exterior differntial}) + \sum_i (-1)^i \delta_i^* \text{ on } K_{\mathbf{C}}^{pq}$

We define filtrations of $K_{\mathbf{Q}}^{\cdot}$ and $K_{\mathbf{C}}^{\cdot}$ by

$$G_i K^{\cdot} := \begin{cases} \tau_i^! K_{\mathbf{Q}}^{\cdot} & \text{over } \mathbf{Q}, \\ W_i(\mathcal{Y}) K_{\mathbf{C}}^{\cdot} & \text{over } \mathbf{C}, \end{cases}$$

(1.3.4)

$$L_j K^{\cdot\cdot} := \bigoplus_{q \geqslant -i} K^{\cdot q} \quad \text{over } \mathbf{Q} \text{ as well as over } \mathbf{C},$$

$$W_k K^{\cdot q} := \begin{cases} \tau_{k+q}^! K_{\mathbf{Q}}^{\cdot q}, \\ W_{k+q}(\mathcal{Y}) K_{\mathbf{C}}^{\cdot q}, \end{cases}$$

$$F^p K_{\mathbf{C}}^{\cdot\cdot} := \bigoplus_{p' \geqslant p} K_{\mathbf{C}}^{p'\cdot}.$$

Lemma (1.3.5). (i) *$(K^{\cdot}, G * L) \to (K^{\cdot}, W)$ is a filtered quasi-isomorphism.*

(ii) *$(K_{\mathbf{Q}}^{\cdot}, G) \otimes \mathbf{C} \underset{FQIS}{\simeq} (K_{\mathbf{C}}^{\cdot}, G)$ and G on $K^{\cdot}$ satisfies*

$$\nu G_i \subset G_i, \qquad gr_i^G K^{\cdot} \underset{QIS}{\cong} a_* K^{\cdot}(\widetilde{\mathcal{Y}}^{(i)} \cap X_0)[-i],$$

hence induces the Gysin filtration on the hypercohomology.

(iii) $(K^{\cdot}_{\mathbf{Q}}, L)\otimes \mathbf{C} \underset{FQIS}{\simeq} (K^{\cdot}_{\mathbf{C}}, L)$ *and* L *on* $K^{\cdot}_{\mathbf{Q}}$ *satisfies*

$$\mathrm{gr}^L_j K^{\cdot}_{\mathbf{Q}} \underset{QIS}{\simeq} a_* \mathbf{Q}_{\overset{\circ}{\widetilde{X}}{}_0^{(-j+1)}}[j],$$

hence induces the Mayer-Vietoris filtration on the hypercohomology.

(iv) $K := ((K^{\cdot}_{\mathbf{Q}}, W), (K^{\cdot}_{\mathbf{C}}, W, F), \alpha)$ *is a CMHC over* $\mathbf{Q}$ *on* X_0*, whose hypercohomology yields the functorial mixed Hodge structure on* $H^{\cdot}(\overset{\circ}{X}_0, \mathbf{Q})$.

(v) *If the spectral sequence of* $R\Gamma K^{\cdot}$ *by the filtration* G *(resp.* L*) degenerates in* $E_2 = E_\infty$*, then* K *with* G *(resp.* L*) is a G-filtered (resp. L-filtered) CMHC over* $\mathbf{Q}$.

(vi) $K^{\cdot}_{\mathbf{C}} = Ker\{\nu : A^{\cdot}_{\mathbf{C}} \to A^{\cdot}_{\mathbf{C}}\}$ *and the filtrations* G, L, W *and* F *on both terms coincide respectively.*

Proof. (i): $(G * L)_k K^{\cdot q} = \sum_{i+j=k}(G_i \cap L_j)K^{\cdot q} = G_{k+q}K^{\cdot q} = W_q K^{\cdot q}$.

The first assertion of (ii) follows immediately by definition. As for the second,

$$\begin{aligned}\mathrm{gr}^G_i K^{\cdot q}_{\mathbf{C}} &= \mathrm{gr}^{W(\mathcal{Y})}_i a_* \Omega^{\cdot}_{\widetilde{X}_0^{(q+1)}}(\log(\mathcal{Y}\cap \widetilde{X}_0^{(q+1)})) \\ &\simeq a_* \Omega^{\cdot}_{\mathcal{Y}^{(i)}\cap \widetilde{X}_0^{(q+1)}}[-i] = a_* K^{\cdot q}_{\mathbf{C}}(\mathcal{Y}^{(i)}\cap \widetilde{X}_0^{(q+1)})[-i].\end{aligned}$$

Similarly, we get the assertion for the $\mathbf{Q}$-structure. The third assertion follows from these.

The first assertion of (iii) follows immediately by definition.

$$\begin{aligned}&\mathrm{gr}^L_j K^{\cdot}_{\mathbf{C}} = K^{\cdot, -j}_{\mathbf{C}}[j] \\ &= a_* \Omega^{\cdot}_{\widetilde{X}_0^{(-j+1)}}(\log(\mathcal{Y}\cap \widetilde{X}_0^{(-j+1)})) \underset{QIS}{\simeq} a_* \mathbf{Q}_{\widetilde{X}_0^{(-j+1)}}[j].\end{aligned}$$

Similarly, we get the assertion for the $\mathbf{Q}$-structure. The third assertion follows from these.

(iv) is found in [D2,III.(8.1.12)]. (v) is easy to verify by using (i) and [SZ,(6.8)]. (vi) is immediate by construction. ∎

We now recall a construction of a **Q**-CMHC $C^{\cdot}$ whose hypercohomology gives the functorial mixed Hodge structure on the cohomology of $(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*)$ (cf. [GNPP, IV.5]).

We are working on a diagram:

(1.3.6)
$$\begin{array}{ccccc} \mathring{\mathcal{X}}^* & \xrightarrow{\mathring{j}} & \mathring{\mathcal{X}} & & \\ \downarrow & & \downarrow{\scriptstyle \ell} & & \\ \mathcal{X}^* & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & X_0 \end{array}$$

As in (1.2.2), the complexes

(1.3.7)
$$\begin{aligned} &I^{\cdot}(\mathring{\mathcal{X}}) := i^{-1}\ell_*\Delta^{\cdot}(\mathring{\mathcal{X}}), \qquad I^{\cdot}(\mathcal{X}^*) := i^{-1}j_*\Delta^{\cdot}(\mathcal{X}^*) \qquad \text{and} \\ &I^{\cdot}(\mathring{\mathcal{X}}^*) := s^{\cdot}(I^{\cdot}(\mathring{\mathcal{X}}) \otimes_{\mathbf{Q}} I^{\cdot}(\mathcal{X}^*)). \end{aligned}$$

are representatives of $i^{-1}R\ell_*\mathbf{Q}_{\mathring{\mathcal{X}}}$, $i^{-1}Rj_*\mathbf{Q}_{\mathcal{X}^*}$ and $i^{-1}R(\ell\mathring{j})_*\mathbf{Q}_{\mathring{\mathcal{X}}^*}$ respectively. The complexes $C^{\cdot}_{\mathbf{Q}}$ and $C^{\cdot}_{\mathbf{C}}$ and their filtrations are defined as

$$C^{\cdot} := \begin{cases} (I^{\cdot}(\mathring{\mathcal{X}}^*)/I^{\cdot}(\mathring{\mathcal{X}}))[1] & \text{over } \mathbf{Q}, \\ (\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))/\Omega^{\cdot}_{\mathcal{X}}(\log\mathcal{Y}))[1] & \text{over } \mathbf{C}, \end{cases}$$

$$G_iC^{\cdot} := \text{image}\begin{cases} (\tau'_i I^{\cdot}(\mathring{\mathcal{X}}^*))[1] \to C^{\cdot}_{\mathbf{Q}}, \\ W_i(\mathcal{Y})\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^{\cdot}_{\mathbf{C}}, \end{cases}$$

(1.3.8)
$$L_jC^{\cdot} := \text{image}\begin{cases} (\tau''_{j+1} I^{\cdot}(\mathring{\mathcal{X}}^*))[1] \to C^{\cdot}_{\mathbf{Q}}, \\ W_{j+1}(X_0)\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^{\cdot}_{\mathbf{C}}, \end{cases}$$

$$W_kC^{\cdot} := \text{image}\begin{cases} (\tau_{k+1} I^{\cdot}(\mathring{\mathcal{X}}^*))[1] \to C^{\cdot}_{\mathbf{Q}}, \\ W_{k+1}(\mathcal{Y}+X_0)\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^{\cdot}_{\mathbf{C}}, \end{cases}$$

$$F^pC^{\cdot}_{\mathbf{C}} := \text{image of } F^{p+1}\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^{\cdot}_{\mathbf{C}}.$$

Lemma (1.3.9). (i) $(C^{\cdot}, G * L) \to (C^{\cdot}, W)$ *is a filtered quasi-isomorphism.*

(ii) $(C^{\cdot}_{\mathbf{Q}}, G) \otimes \mathbf{C} \underset{FQIS}{\simeq} (C^{\cdot}_{\mathbf{C}}, G)$ *and* G *on* $C^{\cdot}$ *satisfies*

$$\nu G_i \subset G_i, \qquad gr_i^G C^{\cdot} \underset{QIS}{\rightleftarrows} C^{\cdot}(\tilde{\mathcal{Y}}^{(i)}, \tilde{\mathcal{Y}}^{(i)} \cap \mathcal{X}^*)[-i],$$

hence induces the Gysin filtration on the hypercohomology.

(iii) $(C^{\cdot}_{\mathbf{Q}}, L) \otimes \mathbf{C} \underset{FQIS}{\simeq} (C^{\cdot}_{\mathbf{C}}, L)$ *and* L *on* $C^{\cdot}_{\mathbf{Q}}$ *satisfies*

$$gr_i^L C^{\cdot}_{\mathbf{Q}} \underset{QIS}{\rightleftarrows} a_* \mathbf{Q}_{\tilde{\mathring{X}}_0^{(j+1)}}[-j],$$

hence induces the Mayer-Vietoris filtration on the hypercohomology.

(iv) $C := ((C^{\cdot}_{\mathbf{Q}}, W), (C^{\cdot}_{\mathbf{C}}, W, F), \alpha)$ *is a CMHC over* $\mathbf{Q}$ *on* X_0, *whose hypercohomology yields the functorial mixed Hodge structure on* $H^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*; \mathbf{Q})$.

(v) *If the spectral sequence of* $R\Gamma C^{\cdot}$ *by the filtration* G *(resp.* L*) degenerates in* $E_2 = E_\infty$, *then* C *with* G *(resp.* L*) is a* G*-filtered (resp.* L*-filtered) CMHC over* $\mathbf{Q}$.

(vi) $C^{\cdot}_{\mathbf{C}} = Coker\{\nu : A^{\cdot}_{\mathbf{C}} \to A^{\cdot}_{\mathbf{C}}\}$ *and the filtrations* G, L, W *and* F *on both terms coincide respectively.*

Proof. (i), (ii) and (iii) are proved analoguously as (1.2.24.i), (1.2.24.ii) and (1.3.5.iii) respectively hence we omit it. (iv) is found in [GNPP,IV.5]. In fact, by the Künneth formula and the residue formula,

$$\mathrm{gr}_k^W C^{\cdot}_{\mathbf{Q}} \underset{QIS}{\simeq} \bigoplus_{i+j=k} a_* \mathbf{Q}_{\tilde{\mathcal{Y}}^{(i)} \cap \tilde{X}_0^{(j)}}[-k].$$

$$\mathrm{gr}_k^W C^{\cdot}_{\mathbf{C}} \underset{QIS}{\simeq} \bigoplus_{i+j=k} a_* \mathbf{C}_{\tilde{\mathcal{Y}}^{(i)} \cap \tilde{X}_0^{(j)}}[-k].$$

These show that $\mathrm{gr}^W C^{\cdot}$ is a CHC hence $C^{\cdot}$ is a CMHC. (v) is easy to verify by using (i) and [SZ,(6.8)]. (vi) is immediate by construction. ∎

(1.4) In the situation of (1.1), we shall construct a mixed version of the Clemens-Schmid sequence after [SZ,II.§7].

Let

$$\textbf{(1.4.1)} \qquad \nu : A^{\cdot} \to A^{\cdot}$$

be the mixed version of the Steenbrink complex and the lifting of the monodromy logarithm in (1.2). From (1.4.1), we have an exact sequence

$$\textbf{(1.4.2)} \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\nu) & \longrightarrow & A^{\cdot} & \xrightarrow{\nu} & A^{\cdot} & \longrightarrow & \mathrm{Coker}(\nu) & \longrightarrow & 0 \\ & & & & & \searrow \quad \nearrow & & & & & \\ & & & & & \mathrm{Im}(\nu) & & & & & \\ & & & & & \nearrow \quad \searrow & & & & & \\ & & & & 0 & & 0 & & & & \end{array}$$

Taking the hypercohomology, we have two long sequences (for n odd or even)

$$\textbf{(1.4.3)} \qquad \to H^n(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \to H^n(\mathring{X}_0) \to H^n(\mathring{X}_\infty) \xrightarrow{N} H^n(\mathring{X}_\infty) \to H^{n+2}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \to$$

over $\mathbf{Q}$ by (1.2.25), (1.3.5.iv) and (1.3.9.iv). This is a mixed version of the Clemens-Schmid sequence.

The following is the Poincaré duality (for the proof, see [Sp]).

Lemma (1.4.4). $H^n(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*; \mathrm{Z}) \simeq H_{2d+2-n}(X_0, Y_0; \mathbf{Z})$ *where* $d+1 = \dim \mathcal{X}$.

Proposition (1.4.5). *In the situation of (1.1), we have, for the cohomology with coefficients in* $\mathbf{Q}$*, the folowing:*

(i) *(1.4.3) is a sequence of mixed Hodge structures over* $\mathbf{Q}$.

(ii) *The filtrations* G *as well as* L *on each term of (1.4.3) are compatible respectively.*

(iii) *If* $\mathcal{Y}$ *is smooth (possibly reducible), then (1.4.3) is a sequence of* G*-filtered mixed Hodge structures over* $\mathbf{Q}$.

(iv) *If* $\mathcal{Y}$ *is smooth (possibly reducible), the Gysin map* $H^0(Y_\infty) \to H^2(X_\infty)$ *is injective and* $H_{2d-1}(X_0) = 0$, *where* $d = \dim X_0$, *then the following parts of (1.4.3) are exact:*

$$H^1(\overset{\circ}{X}_0) \to H^1(\overset{\circ}{X}_\infty) \xrightarrow{N} H^1(\overset{\circ}{X}_\infty) \to H^3(\overset{\circ}{\mathcal{X}}, \overset{\circ}{\mathcal{X}}{}^*).$$
$$H^0(\overset{\circ}{X}_\infty) \to H^2(\overset{\circ}{\mathcal{X}}, \overset{\circ}{\mathcal{X}}{}^*) \to H^2(\overset{\circ}{X}_0) \to H^2(\overset{\circ}{X}_\infty) \xrightarrow{N} H^2(\overset{\circ}{X}_\infty).$$

Proof . (i), (ii) and (iii) follow from (1.2.25), (1.3.5) and (1.3.9).

In order to prove (iv), we first note that taking gr^G on each term of (1.4.2) yield the following commutative diagram consisting of the Clemens-Schmid sequences as horizontal lines and the Thom-Gysin sequences as vertical lines:

(1.4.6)

$$\begin{array}{ccccccccc}
H^{n-1}(\mathcal{Y},\mathcal{Y}^*) & \longrightarrow & H^{n-1}(Y_0) & \longrightarrow & H^{n-1}(Y_\infty) & \longrightarrow & H^{n-1}(Y_\infty) & \longrightarrow & H^{n+1}(\mathcal{Y},\mathcal{Y}^*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^n(\overset{\circ}{\mathcal{X}},\overset{\circ}{\mathcal{X}}{}^*) & \longrightarrow & H^n(\overset{\circ}{X}_0) & \longrightarrow & H^n(\overset{\circ}{X}_\infty) & \longrightarrow & H^n(\overset{\circ}{X}_\infty) & \longrightarrow & H^{n+2}(\overset{\circ}{\mathcal{X}},\overset{\circ}{\mathcal{X}}{}^*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^n(\mathcal{X},\mathcal{X}^*) & \longrightarrow & H^n(X_0) & \longrightarrow & H^n(X_\infty) & \longrightarrow & H^n(X_\infty) & \longrightarrow & H^{n+2}(\mathcal{X},\mathcal{X}^*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^{n-2}(\mathcal{Y},\mathcal{Y}^*) & \longrightarrow & H^{n-2}(Y_0) & \longrightarrow & H^{n-2}(Y_\infty) & \longrightarrow & H^{n-2}(Y_\infty) & \longrightarrow & H^{n}(\mathcal{Y},\mathcal{Y}^*)
\end{array}$$

We shall prove the exactness of the second sequence in (iv) by chasing the diagram (1.4.6). As for the first sequence, the proof is similar and easier and we omit it.

At the first term $H^0(\overset{\circ}{X}_\infty)$, the exactness follows from

$$H^0(X_\infty) \overset{\sim}{\to} H^0(\overset{\circ}{X}_\infty), \qquad H^2(\mathcal{X},\mathcal{X}^*) \overset{\sim}{\to} H^2(\overset{\circ}{\mathcal{X}},\overset{\circ}{\mathcal{X}}^*)$$

by (1.4.4) and from the exactness at $H^0(X_\infty)$ in the usual Clemens-Schmid sequence [Cl]. In the same way, the exactness at $H^2(\overset{\circ}{\mathcal{X}},\overset{\circ}{\mathcal{X}}^*)$ follows from $H^2(\mathcal{X},\mathcal{X}^*) \overset{\sim}{\to} H^2(\overset{\circ}{\mathcal{X}},\overset{\circ}{\mathcal{X}}^*)$, the assumption of the injectivity of $H^0(Y_\infty) \to H^2(X_\infty)$, $H^0(Y_0) \overset{\sim}{\to} H^0(Y_\infty)$ and from the exactness at $H^2(\mathcal{X},\mathcal{X}^*)$ in the usual Clemens-Schmid sequence. Similarly the exactness at $H^2(\overset{\circ}{X}_0)$ follows from the injectivity of $H^1(Y_0) \to H^1(Y_\infty)$ in the usual Clemens-Schmid sequence, $H^0(Y_0) \overset{\sim}{\to} H^0(Y_\infty)$ and from the exactness at $H^2(X_0)$ in the usual Clemens-Schmid sequence. As for the exactness at the first $H^2(\overset{\circ}{X}_\infty)$, notice that $H^2(\mathcal{Y},\mathcal{Y}^*) \to H^4(\mathcal{X},\mathcal{X}^*)$ is injective, because $\{H^3(\mathcal{X},\mathcal{X}^*) \to H^3(\overset{\circ}{\mathcal{X}},\overset{\circ}{\mathcal{X}}^*)\}$ is isomorphic to $\{H_{2d-1}(X_0) \to H_{2d-1}(X_0,Y_0)\}$ by (1.4.4) and the latter is an isomorphism. Now the desired exactness follows similarly from the exactness of the usual Clemens-Schmid sequence, $H^3(\mathcal{X},\mathcal{X}^*) \simeq H_{2d-1}(X_0) = 0$ by (1.4.4) and the assumption, and from the above remark. ∎

It is not yet known in general wether (1.4.3) is exact or not. Proposition (1.4.5.iv) is only a partial result but it is sufficient enough for our later use in the present paper.

Problem (1.4.7). *Prove the exactness of (1.4.3).*

Appendix to §1.

(A.1) As [E2,II.(3.18)] has pointed out, there is a part which is not clear in the proof of [St,(5.9)], i.e., "This implies that $\xi \in P^{q-r}(\widetilde{Y}^{(r+1)},\mathbf{Q})(-r)$." [ibid,p.254

↑11]. We explain the point more precisely. In the notation there, we have

$$\begin{array}{ccc} E_1^{r-1,q-r} & = & H^{q-r}(\widetilde{Y}^{(r)}) \oplus H^{q-r-2}(\widetilde{Y}^{(r+2)}) \oplus H^{q-r-4}(\widetilde{Y}^{(r+4)}) \oplus \dots \\ (-1)^{(r-1)}d_1 \downarrow & & \theta \searrow \quad -\gamma \swarrow \quad \theta \searrow \quad -\gamma \swarrow \dots \\ E_1^{r,q-r} & = & H^{q-r}(\widetilde{Y}^{(r+1)}) \oplus H^{q-r-2}(\widetilde{Y}^{(r+3)}) \oplus \dots \end{array}$$

where the θ and the γ are the Mayer-Vietoris maps and the Gysin maps respectively and we omit the coefficients of the cohomologies as well as the Tate twists. Let

$$\xi = (\xi_i)_{i\geqslant 0} \in Z(E_1^{-r,q+r}) \subset \bigoplus_{i\geqslant 0} H^{q-r-2i}(\widetilde{Y}^{(r+2i)})$$

be a primitive element such that $\widetilde{\nu}^r\xi \in B(E_1^{r,q-r})$ as in the situation in question. Then, by the above diagram, there exists

$$\begin{aligned} &\eta = (\eta_i)_{i\geqslant -1} \in E_1^{r-1,q-r} = \bigoplus_{i\geqslant -1} H^{q-r-2-2i}(\widetilde{Y}^{(r+2+2i)}) \qquad \text{such that} \\ &\widetilde{\nu}^r\xi = (-1)^{r-1}d_1\eta = (\theta\eta_{i-1} - \gamma\eta_i)_{i\geqslant 0}. \end{aligned}$$

In particular, $\xi_0 = \widetilde{\nu}^r\xi_0 = \theta\eta_{-1} - \gamma\eta_0 \in P^{q-r}(\widetilde{Y}^{(r+1)})$, but it is not known wether $\theta\eta_{-1}$ is primitive or not, hence we can not conclude $\theta\eta_{-1} = 0$ (ξ is assumed as $\widetilde{\nu}^r\xi = \theta\eta_{-1}$ there !) by the argument using the polarization on $P^{q-r}(\widetilde{Y}^{(r+1)})$.

However, we can rescue the claim (A.1.1) below (cf. [ibid,p.254,↑0]) along the line of the original proof by using the polarization Q on the whole

$$(E_1^{-r,q+r})_{prim} \overset{\widetilde{\nu}^r}{\underset{\sim}{\to}} (E_1^{r,q-r})_{prim} = \bigoplus_{i\geqslant 0} P^{q-r-2i}(\widetilde{Y}^{(r+1+2i)}).$$

[ibid,(5.9)] now follows from (A.1.1).

Claim (A.1.1). $\xi = 0$.

Proof. We keep the notation in [ibid,§5]. Identifying by $\widetilde{\nu}^r$ above, we have

$$\xi_i = \theta\eta_{i-1} - \gamma\eta_i \qquad (i \geqslant 0).$$

Since θ and γ are adjoint, we can compute as

$$\begin{aligned}
Q(\xi,\xi) &:= \sum_{i\geqslant 0} Q(\xi_i,\xi_i) \\
&= \varepsilon \sum_{i\geqslant 0} \int_{\widetilde{Y}(r+1+2i)} L_0^{n-q} \wedge C\xi_i \wedge \xi_i \\
&= \varepsilon \sum_{i\geqslant 0} \int_{\widetilde{Y}(r+1+2i)} L_0^{n-q} \wedge C(\theta\eta_{i-1} - \gamma\eta_i) \wedge \xi_i \\
&= \varepsilon \sum_{i\geqslant 0} \left(\int_{\widetilde{Y}(r+2i)} L_0^{n-q} \wedge C\eta_{i-1} \wedge \gamma\xi_i - \int_{\widetilde{Y}(r+2+2i)} L_0^{n-q} \wedge C\eta_i \wedge \theta\xi_i \right) \\
&= \varepsilon \sum_{i\geqslant 1} \int_{\widetilde{Y}(r+2i)} L_0^{n-q} \wedge C\eta_{i-1} \wedge (\gamma\xi_i - \theta\xi_{i-1}) = 0,
\end{aligned}$$

where $\varepsilon = (-1)^{(q-r)(q-r-1))/2}$. We used $(\theta\xi_{i-1} - \gamma\xi_i)_{i\geqslant 1} = (-1)^{r+1}d_1\xi = 0$ in the last equality. Hence, by the positive definiteness of Q, we get the assertion. ∎

(A.2) In [Cl], the Clemens-Schmid sequnces are constructed by combining "Wang sequences" and the local cohomology sequences. The mixed versions can be also constructed in this manner.

Lemma (A.2.1). *In the situation and the notation in (1.1)–(1.3), we have a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{\cdot}_{\mathbf{Q}}(\overset{\circ}{X}_\infty)[-1] & \longrightarrow & \rho(A^{\cdot}_{\mathbf{Q}}(\overset{\circ}{X}_\infty),\nu) & \longrightarrow & A^{\cdot}_{\mathbf{Q}}(\overset{\circ}{X}_\infty) & \longrightarrow & 0 \\
 & & QIS\uparrow\wr & & QIS\uparrow\wr & & QIS\uparrow\wr & & \\
0 & \longrightarrow & B^{\cdot}(\overset{\circ}{X}_\infty)[-1] & \longrightarrow & \rho(B^{\cdot}(\overset{\circ}{X}_\infty),\delta) & \longrightarrow & B^{\cdot}(\overset{\circ}{X}_\infty) & \longrightarrow & 0 \\
 & & & & QIS\uparrow\wr & & \| & & \\
 & & 0 & \longrightarrow & \Gamma^{\cdot}(\overset{\circ}{\mathcal{X}}{}^*) & \longrightarrow & B^{\cdot}(\overset{\circ}{X}_\infty) & \overset{\delta}{\longrightarrow} & B^{\cdot}(\overset{\circ}{X}_\infty) \longrightarrow 0
\end{array}$$

whose horizontal lines are exact. The hypercohomology of each horizontal sequence yield a "Wang sequence" in the category of mixed Hodge structures.

The proof is standard and easy by the construction hence we omit it.

In the notation in (1.1.3), (1.2.2) and (1.3.7), we set

$$
\textbf{(A.2.2)}\qquad
\begin{aligned}
&\bar{\Gamma}^{\cdot}(\mathring{\mathcal{X}}^*) := i^{-1}(\ell j)_* \Delta^{\cdot}(\mathring{\mathcal{X}}^*),\\
&\bar{\Gamma}^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) := \mathrm{Ker}\{\Gamma^{\cdot}(\mathring{\mathcal{X}}) \to \bar{\Gamma}^{\cdot}(\mathring{\mathcal{X}}^*)\} \qquad \text{and}\\
&\Gamma^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) := \mathrm{Coker}\{\Gamma^{\cdot}(\mathring{\mathcal{X}}) \to \Gamma^{\cdot}(\mathring{\mathcal{X}}^*)\}[-1].
\end{aligned}
$$

Lemma (A.2.3). *In the situation and the notation in (1.1)–(1.3) and (A.2.2), we have a commutative diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & K^{\cdot}_{\mathbf{Q}} & \longrightarrow & \rho(A^{\cdot}_{\mathbf{Q}}(\mathring{X}_\infty), \nu) & \longrightarrow & C^{\cdot}_{\mathbf{Q}}[-1] & \longrightarrow 0\\
 & & & & QIS\uparrow\wr & & QIS\uparrow\wr & & QIS\uparrow\wr & \\
 & & 0 & \longrightarrow & \Gamma^{\cdot}(\mathring{\mathcal{X}}) & \longrightarrow & \Gamma^{\cdot}(\mathring{\mathcal{X}}^*) & \longrightarrow & \Gamma^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*)[1] & \longrightarrow 0\\
 & & & & \| & & QIS\downarrow\wr & & & \\
0 & \longrightarrow & \bar{\Gamma}^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) & \longrightarrow & \Gamma^{\cdot}(\mathring{\mathcal{X}}) & \longrightarrow & \bar{\Gamma}^{\cdot}(\mathring{\mathcal{X}}^*) & \longrightarrow & 0 &
\end{array}
$$

where the top horizontal sequence is QIS exact and the other horizontal sequences are exact. In particular,

$$
\bar{\Gamma}^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \underset{QIS}{\rightleftarrows} \Gamma^{\cdot}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \underset{QIS}{\rightleftarrows} C^{\cdot}_{\mathbf{Q}}[-2].
$$

The hypercohomology of each horizontal sequence yields a local cohomology sequence in the category of mixed Hodge structures.

Proof. We shall show that the top horizontal sequence is QIS exact in the midle term. The other assertions are standard and easy to see by the construction and we omit their prooves.

Let $(x,y) \in A^n \oplus A^{n-1}$ such that $d(x,y) = (d_A x, \nu x - d_A y) = 0$ and that there exists $\bar{\eta} \in C^{n-1}$ with $\bar{y} = -d_C \bar{\eta}$ in $C^{\cdot}$. Then, since $\overline{y + d_A \eta} = 0$, there exists $\xi \in A^{n-1}$ satisfying $y + d_A \eta = \nu \xi$. Hence

$$(x,y) - d(\xi,\eta) = (x - d_A\xi, y + d_A\eta - \nu\xi) = (x - d_A\xi, 0) \quad \text{and}$$

$$\nu(x - d_A\xi) = \nu x - d_A \nu \xi = \nu x - d_A(y + d_A\eta) = \nu x - d_A y = 0.$$

Therefore $x - d_A\xi \in K^n$. ∎

Combining the mixed versions of the "Wang sequences" in (A.2.1) and the local cohommology sequnces in (A.2.3) as in [Cl], we get mixed versions of the Clemens-Schmid sequences.

§2. General degenerations of Todorov surfaces.

In this section, we recall and modify the results in [M] (cf. also [T2]) for our later use.

(2.1) We recall first some facts in the code theory. Let $\mathbf{F}_2 := \mathbf{Z}/2\mathbf{Z}$. A *binary linear code* $(V \subset \mathbf{F}_2^I)$ on a finite set I is a vector subspace V of the $\mathbf{F}_2$-vector space $\mathbf{F}_2^I$ of all maps from I to $\mathbf{F}_2$. The *distance* of $\varphi \in \mathbf{F}_2^I$ is $\sharp\{i \in I \mid \varphi(i) = 1\}$. A binary code $(V \subset \mathbf{F}_2^I)$ is *equidistant* if all non-zero elements of V have the same distance; this common distance is called the *distance of the code.* Let $(V \subset \mathbf{F}_2^I)$ be a binary linear code. The *linear subcode* associated to a subset $J \subset I$ is defined as $(\{\varphi \in V \mid \varphi(i) = 0 \text{ if } i \notin J\} \subset \mathbf{F}_2^I)$.

In the case that the set I itself has a structure of $\mathbf{F}_2$-vector space of dimension 4, we define a binary linear code

$$\mathcal{D} := (\{\text{affine linear function on } I\} \subset \mathbf{F}_2^I).$$

Assigning a pair of integers $(k,\alpha)(\mathcal{C}) := (\sharp J, \dim V)$ to a linear subcode $\mathcal{C} = (V \subset \mathbf{F}_2^J)$ of $\mathcal{D}$, we get

Lemma (2.1.1). *There is an order preserving bijection*

$$\{\textit{linear subcode of } \mathcal{D}\}/(\textit{isom. as abstract codes})$$
$$\updownarrow$$
$$\{(k,\alpha)\in \mathbf{Z}^2 \mid 0\leqslant \alpha \leqslant 5, 2^4-2^{4-\alpha}\leqslant k\leqslant \alpha+11\},$$

where we endow these sets orders defined respectively by

$$\mathcal{C}'\leqslant \mathcal{C} \Longleftrightarrow \mathcal{C}' \textit{ is isomorphic to a linear subcode of } \mathcal{C}$$
$$(k',\alpha')\leqslant (k,\alpha) \Longleftrightarrow \alpha'\leqslant \alpha \textit{ and } \alpha-\alpha'\leqslant k-k'.$$

The proof is found in [M,(1.2)]. The assertion about the orders are implicit there, but a careful reading of that proof leads to this assertion.

(2.2) We recall here the definition of Todorov surfaces and K3 surfaces of Todorov type and their relationships.

Definition (2.2.1). *A canonical surface $\bar{X}$ is called a Todorov surface if $\chi(\mathcal{O}_{\bar{X}})=2$ and $\bar{X}$ has an involution σ such that $\bar{X}/\sigma$ is a K3 surface only with rational double points. A pair of integers $(\ell,\alpha):=(c_1^2(X),\log_2\sharp(\textit{2-torsion of Pic}(X)))$ is called the type of $\bar{X}$, where X is the smooth minimal model of $\bar{X}$.*

[M,§5] shows that the values of (ℓ,α) are as in the table (2.3.3) below.

Let (Y,E) be a pair of a smooth minimal K3 surface Y and a disjoint union $E=\sum_{i\in I}E_i$ of (-2)-curves on Y. By using the cup product pairing on $H^2(Y,\mathbf{Z})$ and the reduction modulo 2, we have a homomorphism of modules:

$$\delta:\left(\text{primitive span of}\sum_i \mathbf{Z}[E_i]\,\text{in}\,H^2(Y,\mathbf{Z})\right)\to \mathrm{Hom}\left(\sum_i \mathbf{Z}[E_i],\,\mathbf{F}_2\right)\simeq \mathbf{F}_2^I.$$

$(\mathrm{Im}\,\delta\subset \mathbf{F}_2^I)$ is called the *binary linear code of* (Y,E).

Definition (2.2.2). *Let (ℓ, α) be one of the 11 values for Todorov surfaces in the table (2.3.3) below. A K3 surface of Todorov type (ℓ, α) is a triple $(\bar{Y}, L, E)$ consisting of a K3 surface $\bar{Y}$ only with rational double points, an ample line bundle L on $\bar{Y}$ and a disjoint union $E = \sum_{i\in I} E_i$ of (-2)-curves contained in the exceptional locus of the minimal resolution $\mu : Y \to \bar{Y}$, such that $\mu^*L \otimes \mathcal{O}_Y(E)$ is 2-divisible in Pic(Y) and that $L \cdot L = 2\ell$ and $\dim Im\,\delta = \alpha$ for the associated code. E is called the distinguished (-2)-curves.*

Let $\bar{X}$ be a Todorov surface of type (ℓ, α) and consider the following diagram:

$$\text{(2.2.3)} \qquad \begin{array}{ccccc} \bar{C} & \longrightarrow & \bar{X} & \longleftarrow & \widehat{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \hat{\pi}} \\ \bar{B} & \longrightarrow & \bar{Y} & \xleftarrow{\ \mu\ } & Y \end{array}$$

where $\bar{Y} := \bar{X}/\sigma$, $\bar{C}$ is the canonical curve of $\bar{X}$, $\bar{B} := \pi(\bar{C})$, μ is the minimal resolution, and $\widehat{X} := \bar{X} \times_{\bar{Y}} Y$.

Lemma (2.2.4). *In the above notation, let $\mu^*\bar{B} + E$ be the branch locus of the double cover $\hat{\pi}$. Then there is a bijection:*

$$\{\bar{X} \mid \textit{Todorov surface of type}\,(\ell,\alpha)\}/\textit{isom.}$$

$$\updownarrow$$

$$\{(\bar{Y}, \bar{B}, E) \mid (\bar{Y}, \mathcal{O}_{\bar{Y}}(\bar{B}), E) \textit{ is a K3 surfaces of Todorov type } (\ell,\alpha) \textit{ and } \bar{B} \textit{ satisfies Condition (2.2.5) below}\}/\textit{isom.}$$

Condition (2.2.5). *On the smooth minimal model Y, $B := \mu^*\bar{B}$ is reduced and has at most simple singularities and $B \cap E = \emptyset$.*

The proof of (2.2.4) is found in [M,§4,§5]. We call a data $(\bar{Y}, \bar{B}, E)$ in (2.2.4) a *Todorov triple.*

For a K3 surface $(\bar{Y}, L, E)$ of Todorov type (ℓ, α), it is known that $\sharp I = \ell + 8$, where $E = \sum_{i\in I} E_i$ (see [M,(5.2.ii)]).

(2.3) Finaly, we summarize the main result in [M] about the moduli spaces of Todorov surfaces together with an observation of their general degenerations.

Definition (2.3.1). *A numerical K3 surface is a smooth minimal surface with $p_g = 1$, $q = 0$ and $c_1^2 = 0$ (cf. [U5]).*

Proposition (2.3.2). *The values of type (ℓ, α) of Todorov surfaces are as in the table (2.3.3) below. For each of these values of (ℓ, α), there exists the moduli space of Todorov surfaces of type (ℓ, α) which is irreducible. The general degenerations of Todorov surfaces are those of type (I_1) in Table 0 on the last page, and except the case $(2,1) \to (0,1)$, they go down one step in the direction $\downarrow$ or $\to$ freely under the controle of the associated binary linear code. In case of $(2,1) \to (0,1)$, they go down two steps.*

(2.3.3)

$$(\ell, \alpha) = \begin{array}{lllclc} (8,5) & & & & & \\ (7,4) & & & & & \\ (6,3) & & & & & \\ (5,2) & (4,2) & & & & \\ (4,1) & (3,1) & (2,1) & \vdots & & (0,1) \\ (3,0) & (2,0) & (1,0) & \vdots & (0,0) & (-1,0) \end{array}$$

The left hand side of the vertical dots in the table (2.3.3) correspond to Todorov surfaces.

(0,1) corresponds to numerical K3 surfaces with two double fibers.

(0,0) corresponds to numerical K3 surfaces with one double fiber.

(-1,0) corresponds to K3 surfaces blown up one point.

Proof. The first half of the proposition is proved in [M] by using the code-theory, a suitable version of Nikulin's embedding theorem, and the Torelli theorem and the surjectivity of the period map for K3 surfaces of Todorov type. We prove here the assertion about the degenerations which is implicit there.

There are sixteen (-2)-curves $E = \sum_i E_i$ on a smooth minimal Kummer surface $Y = \mathrm{Km}(A)$ which correspond to the 2-torsion points of the abelian surface A. They form a 4-dimensional $\mathbf{F}_2$-vector space and it is known that the binary linear code of (Y, E) is $\mathcal{D}$ in (1.1). This is the key point of the relationship of the abstract code theory and the geometry from which it is deduced that the binary code associated to any K3 surface of Todorov type is isomorphic to a linear subcode of $\mathcal{D}$ (see [M,(2.1)]).

Let $(\bar{Y}, L, E)$ be a general K3 surface of Todorov type (ℓ, α), i.e., the smooth minimal model Y of $\bar{Y}$ has the Picard number $k+1 = \ell + 9$. Let $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ be the associated binary linear subcode. In case $(\ell, \alpha) \neq (8,5), (2,1)$ or $(1,0)$, L is very ample on $\bar{Y}$ and $\bar{Y}$ has only $k = \ell + 8$ ordinary double points which correspond to E [M,(7.7)]. By (2.1.1), if $(\ell-1, \alpha')$, $\alpha' = \alpha$ or $\alpha - 1$, appears in the table (2.3.3), there is a distinguished (-2)-curve, say E_1, such that the linear subcode of $\mathcal{C}$ associated to the subset $I - \{1\} \subset I$ has invariants $(\ell+8-1, \alpha')$. Take a general member $\bar{B}_1 \in |L|$ and a general member $\bar{B}_0 \in |L|$ subjected that B_0 passes through the ordinary double point on $\bar{Y}$ corresponding to E_1. Let Δ be a small disc in the parameter space of the pencil generated by $\bar{B}_0$ and $\bar{B}_1$ whose center $0 \in \Delta$ corresponds to $\bar{B}_0$. Denote by $\bar{\mathcal{B}} \subset \bar{Y} \times \Delta$ the total space of the family $\{\bar{B}_t\}_{t \in \Delta}$ and by $\mathcal{B} \subset Y \times \Delta$ the proper transform of $\bar{\mathcal{B}}$, which is the total space of the family $\{B_t\}_{t \in \Delta}$ on Y. We can perform a semi-stable reduction of the family of pairs of the double cover of Y branched along $B_t + E$ $(t \in \Delta)$ and their ramification loci in the following way: (i) Set $\mathcal{E}_i := E_i \times \Delta$ $(i \in I)$. Let $\alpha : \mathcal{Y} \to Y \times \Delta$ be the blowing-up along $\mathcal{B} \cap \mathcal{E}_1$. Denote by $W_{\mathcal{Y}}$ the exceptional divisor. (ii) Take the double cover $\beta : \widehat{\mathcal{X}} \to \mathcal{Y}$ branched along the proper transform $\alpha^{-1}(\mathcal{B} + \sum \mathcal{E}_i)$. (iii) Since the $\alpha^{-1}\mathcal{E}_i$ are the total space of families of (-1)-curves on the fibers $\widehat{\mathcal{X}} \to \Delta$, we can contract them to obtain $\gamma : \widehat{\mathcal{X}} \to \mathcal{X}$. Set $\mathcal{B}_{\mathcal{X}} := \gamma(\alpha\beta)^{-1}\mathcal{B}$ and $W_{\mathcal{X}} := \gamma\beta^{-1}W_{\mathcal{Y}}$. Figure 1 at the end of this paper is the central fiber on each step. We thus obtain a family of pairs

(2.3.4) $$f : (\mathcal{X}, \mathcal{B}_{\mathcal{X}}) \to \Delta.$$

It is easy to see that this is a semi-stable degeneration of pairs of smooth minimal models of Todorov surfaces of type (ℓ, α) and their smooth canonical curves whose central fiber X_0 is as the stage (a) in Figure 1 consisting of $\mathbf{P}^2$ and a smooth minimal model of a Todorov surface of type $(\ell - 1, \alpha')$ (for details, cf. [U6,(1.3)]).

In case $(\ell, \alpha) = (8, 5)$, $\bar{Y}$ is a Kummer surface which can be represented as quartic surface with 16 ordinary double points in $\mathbf{P}^3$ by $|2\Theta|$, where Θ is the theta divisor of the associated abelian surface, $L = \mathcal{O}_{\bar{Y}}(2)$ and E corresponds to the above 16 ordinary double points. Hence we can go on in the same way as before.

In case $(\ell, \alpha) = (2, 1)$, it can be seen that the linear system $|L|$ is hyperelliptic and gives a finite double cover $\bar{Y} \to \bar{Z} \subset \mathbf{P}^3$ over a quadric cone $\bar{Z}$ whose branch locus is a union of two smooth quardric sections Q_i $(i = 1, 2)$ meeting transversally (cf. [CD], [M,(5.4)]). The $8 + 2$ ordinary double points on $\bar{Y}$ come from $Q_1 \cap Q_2$ and from the vertex of $\bar{Z}$ counted once and twice respectively. Hence we can find a desired degenerate branch locus $\bar{B}_0 \in |L|$ as a pull-back of a suitable hyperplane section $\bar{H}_0$ of $\bar{Z} \subset \mathbf{P}^3$. The remaining steps of the construction are the same as before and we get a family of pairs like (2.3.4). We remark here that the central fiber X_0 of the resulting semi-stable degeneration of pairs consists of two $\mathbf{P}^2$ and the main component whose type drops as $(2, 1) \to (0, 1)$ in the table (2.3.3) if and only if the hyperplane section $\bar{H}_0$ contains the vertex of $\bar{Z}$.

In case $(\ell, \alpha) = (1, 0)$, the linear system $|L|$ is hyperelliptic and gives a finite double cover $\bar{Y} \to \mathbf{P}^2$ branched along a union of two smooth cubics C_i $(i = 1, 2)$

meeting transversally (cf. [Ca], [M,(5.4)]), and we can go on as in the previous case (for details, see [U5], [U6]). ∎

§3. Moduli and mixed period map.

In this section, we shall formulate a mixed period map for smooth pairs of Todorov surfaces and their canonical curves. For that purpose, (2.2.4) allows us to use Todorov triples instead of Todorov surfaces.

(3.1) Let $(\bar{Y}_r, L_r, E_r)$ be a reference K3 surface of Todorov type (ℓ, α), $\bar{B}_r \in |L_r|$ a reference smooth curve, and $(\bar{Y}_r, \bar{B}_r, E_r)$ a reference Todorov triple (see (2.2)). Let $\mu : Y_r \to \bar{Y}_r$ be the minimal resolution and $B_r := \mu^* \bar{B}_r$. We denote by

(3.1.1) $$[\Lambda] = \Lambda(\bar{Y}_r, \bar{B}_r, E_r)$$

the Thom-Gysin exact sequence

$$\begin{array}{ccccccc} H^2(Y_r, \mathbf{Z}) & \to & H^2(\mathring{Y}_r, \mathbf{Z}) & \to & H^1(B_r, \mathbf{Z}) & \to & 0 \\ \| & & \| & & \| & & \\ \Lambda_Y & & \widetilde{\Lambda} & & \Lambda_3 & & \end{array}$$

together with the cup product pairings on Λ_Y and on Λ_3 and with the fundamental classes

$$b := [B_r],\ e_i := [E_{r,i}] \in \Lambda_Y \qquad (i \in I),$$

where $E_r = \sum_{i \in I} E_{r,i}$, $\mathring{Y}_r := Y_r - (B_r + E_r)$ and $\{e_i \mid i \in I\}$ is considered as an unordered set. We also denote by

$$[\Lambda_Y] = \Lambda(\bar{Y}_r, L_r, E_r)$$

the partial data consisting of Λ_Y, the cup product pairing on it and the fundamental classes $b, \{e_i \mid i \in I\}$, and by

$$[\Lambda_3] = \Lambda(\bar{B}_r)$$

the data Λ_3 equiped with the cup product pairing on it.

(3.2) Let (Λ, G, F_r) be the reference mixed Hodge structure defined by the complex structure on $\overset{\circ}{Y}_r$, where

$$\Lambda := \widetilde{\Lambda}/\text{torsion},$$

(3.2.1) $$G := (G_1 = 0 \subset G_2 = \operatorname{Im}\{\Lambda_Y \to \Lambda\} \subset G_3 = \Lambda) \quad \text{weight filtration},$$

$$F_r := (F_r^0 = \Lambda \otimes \mathbf{C} \supset F_r^1 \supset F_r^2 \supset F_r^3 = 0) \quad \text{Hodge filtration}.$$

Set

(3.2.2) $$\Lambda_2 := \{\lambda \in \Lambda_Y \mid \lambda \cdot b = \lambda \cdot e_i = 0 \quad (i \in I)\}.$$

Then

$$\operatorname{gr}_2^G \Lambda = G_2\Lambda \hookrightarrow \Lambda_2 \quad \text{with finite cokernel},$$

$$\operatorname{gr}_3^G \Lambda \xrightarrow{\sim} \Lambda_3.$$

Denote

(3.2.3) $$f^p := \dim F_r^p, \qquad f_i^p := \dim \operatorname{gr}_i^G F_r^p.$$

Since Y_r is a smooth minimal K3 surface and B_r is isomorphic to the canonical curve C_r of the Todorov surface of type (ℓ, α) corresponding to $(\bar{Y}_r, \bar{B}_r, E_r)$, we can compute as

(3.2.4)
$$f_2^2 = 1, \qquad f_2^1 = \operatorname{rank} \Lambda_2 - 1 = 12 - \ell,$$

$$f_3^2 = \operatorname{genus} B_r = \operatorname{genus} C_r = (2(C_r)^2 + 2)/2 = \ell + 1,$$

$$f_3^1 = 2f_3^2 = 2(\ell + 1),$$

$$f^2 = f_2^2 + f_3^2 = \ell + 2, \qquad f^1 = f_2^1 + f_3^1 = \ell + 14.$$

Let

(3.2.5)
$$\mathcal{F}_i := \operatorname{Flag}(\Lambda_i \otimes \mathbf{C}; f_i^1, f_i^2),$$

$$\mathcal{F} := \{F \in \operatorname{Flag}(\Lambda \otimes \mathbf{C}; f^1, f^2) \mid \operatorname{gr}_i^G F \in \mathcal{F}_i \text{ for all } i\}.$$

and let

$$\text{(3.2.6)} \qquad \mathrm{gr} : \mathcal{F} \to \mathcal{F}_2 \times \mathcal{F}_3, \qquad F \mapsto (\mathrm{gr}_2^G F,\ \mathrm{gr}_3^G F).$$

The classifying spaces D_i and D of Hodge filtrations on Λ_i and on Λ are defined respectively by

(3.2.7)

D_2 : the one of the two connected components of

$$\{F \in \mathcal{F}_2 \mid F^1 \cdot (F^2 + \bar{F}^2) = 0,\ \omega \cdot \bar{\omega} > 0 \ \ (0 \neq \omega \in F^2)\}$$

which contains the reference Hodge filtration $\mathrm{gr}_2^G F_r$,

$$D_3 := \{F \in \mathcal{F}_3 \mid F^2 \cdot F^2 = 0,\ \sqrt{-1}\omega \cdot \bar{\omega} > 0 \ \ (0 \neq \omega \in F^2)\}$$

$$D := \mathrm{gr}^{-1}(D_2 \times D_3) \subset \mathcal{F},$$

(cf. [Sa,Appendix.§6,II.§7], [U4], [SSU,I.2]).

(3.3) Let $(\bar{Y}, L, E)$ be any K3 surface of Todorov type (ℓ, α) and $\bar{B} \in |L|$ any smooth curve.

Definition (3.3.1). *A $[\Lambda, D]$-marking of a Todorov triple $(\bar{Y}, \bar{B}, E)$ is an isomorphism of data*

$$\eta = (\eta_Y, \tilde{\eta}, \eta_3) : \Lambda(\bar{Y}, \bar{B}, E) \xrightarrow{\sim} [\Lambda]$$

sending the Hodge filtration on $H^2(\overset{\circ}{Y}, \mathbf{C})$ into D.

A $[\Lambda_Y, D_2]$-marking of a K3 surface $(\bar{Y}, L, E)$ of Todorov type is an isomorphism of data

$$\eta_Y : \Lambda(\bar{Y}, L, E) \xrightarrow{\sim} [\Lambda_Y]$$

sending the Hodge filtration on $H^2(Y, \mathbf{C})$ into D_2.

A $[\Lambda_3]$*-marking of a curve* $(\bar{B})$ *is an isometry*

$$\eta_3 : \Lambda(\bar{B}) \xrightarrow{\sim} [\Lambda_3].$$

Notice that a $[\Lambda_Y, D_2]$-marking introduced above coincides with a "special marking" in [M,§7].

We denote by $\mathrm{Aut}[\Lambda, D]$, $\mathrm{Aut}[\Lambda_Y, D_2]$ and $\mathrm{Aut}[\Lambda_3]$ the groups of automorphisms of the data $[\Lambda]$, $[\Lambda_Y]$ and $[\Lambda_3]$ respectively which preserve, in the first two cases, the component D and D_2 respectively.

Lemma (3.3.2). *The natural map*

$$Aut[\Lambda, D] \to Aut[\Lambda_Y, D_2] \times Aut[\Lambda_3]$$

is surjective.

Proof. Since Λ_3 is $\mathbf{Z}$-free, there exists a $\mathbf{Z}$-submodule $\Lambda_3' \subset \Lambda$ such that $\Lambda = \mathrm{Im}\{\Lambda_Y \to \Lambda\} \oplus \Lambda_3'$. Notice also that an automorphism of the data $[\Lambda]$ preserves D if and only if its restriction on the data $[\Lambda_Y]$ preserves D_2. The lemma follows from these observations ∎

For a K3 surface $(\bar{Y}, L, E)$ of Todorov type, let $\mu : Y \to \bar{Y}$ be the minimal resolution. Let $W(\bar{Y})$ be the group of isometries of the latice $H^2(Y, \mathbf{Z})$ generated by the reflections $x \mapsto x + (x \cdot d)d$ $(x \in H^2(Y, \mathbf{Z}))$ where d runs over the fundamental classes of all the exceptional (-2)-curves of μ. We denote by $W(\bar{Y}, E)$ the subgroup of $W(\bar{Y})$ consisting of those elements which preserve the unordered set $\{[E_i] \mid i \in I\}$ of the fundamental classes of the distinguished (-2)-curves $E = \sum_{i \in I} E_i$. Notice that $w \in W(\bar{Y}, E)$ acts on the set of $[\Lambda_Y, D_2]$-markings by $\varphi_Y \mapsto \varphi_Y w^{-1}$. We call an element of the set

$$\{[\Lambda_Y, D_2]\text{-marking of } (\bar{Y}, L, E)\}/W(\bar{Y}, E)$$

a *marking* of the K3 surface $(\bar{Y}, L, E)$ of Todorov type or of a Todorov triple $(\bar{Y}, \bar{B}, E)$ $(\bar{B} \in |L|)$.

(3.4) [M,(7.5)] constructs the coarse moduli space of Todorov surfaces in the following way.

By the Torelli theorem and the surjectivity of the period map for K3 surfaces of Todorov type (ℓ, α), the local universal families are glued together to make up a universal family

$$g : (\bar{\mathcal{Y}}, \mathcal{L}, \mathcal{E}, \bar{\varphi}_{\mathcal{Y}}) \to D_2$$

of marked K3 surfaces of Todorov type (ℓ, α). Let $\mathcal{V} = \mathcal{V}_{(\ell,\alpha)}$ be the Zariski open subset of the projective bundle $\mathbf{P}(g_*\mathcal{L})$ over D_2 consisting of marked Todorov triples $(\bar{Y}, \bar{B}, E, \bar{\varphi}_Y)$, i.e., $\bar{B} \in |L|$ satisfies Condition (2.2.5). Let

$$f : (\bar{\mathcal{Y}}, \bar{\mathcal{B}}, \mathcal{E}, \bar{\varphi}_{\mathcal{Y}}) \to \mathcal{V}$$

be the universal family of the marked Todorov triples of type (ℓ, α). Then the action of $\gamma \in \mathrm{Aut}[\Lambda_Y, D_2]$ on D_2 lifts onto $\mathbf{P}(g_*\mathcal{L})$ by the Torelli theorem for K3 surfaces of Todorov type. In fact, if $\gamma(\bar{Y}, L, E, \bar{\varphi}_Y) = (\bar{Y}', L', E', \bar{\varphi}'_Y)$, there exists uniquely $w \in W(\bar{Y}, E)$ and an isomorphism $\tilde{\gamma} : (\bar{Y}, L, E) \cong (\bar{Y}', L', E')$ such that $(\tilde{\gamma}^{-1})^* = (\varphi'_Y)^{-1}\gamma\varphi_Y w : \Lambda(\bar{Y}, L, E) \cong \Lambda(\bar{Y}', L', E')$. Now define the action of $\gamma \in \mathrm{Aut}[\Lambda_Y, D_2]$ on $\mathbf{P}(g_*\mathcal{L})$ by

$$\gamma(\bar{Y}, \bar{B}, E, \bar{\varphi}_Y) = (\bar{Y}', \tilde{\gamma}\bar{B}, E', \bar{\varphi}'_Y).$$

This action on $\mathbf{P}(g_*\mathcal{L})$ is properly discontinuous since so is that on D_2. The quotients $\mathcal{V}/\mathrm{Aut}[\Lambda_Y, D_2]$ and $D_2/\mathrm{Aut}[\Lambda_Y, D_2]$ are the required coarse moduli spaces of Todorov surfces of type (ℓ, α) and of K3 surfaces of Todorov type (ℓ, α) respectively.

(3.5) We recall here a formulation of a mixed period map for Todorov surfaces with smooth canonical curves.

Let

$$\mathcal{U} = \mathcal{U}_{(\ell,\alpha)} \subset \mathcal{V}_{(\ell,\alpha)} \subset \mathbf{P}(g_*\mathcal{L}) \tag{3.5.1}$$

be the Zariski open subset consisting of those marked Todorov triples $(\bar{Y}, \bar{B}, E, \bar{\varphi}_Y)$ which satisfy the following Condition (3.5.2).

Condition (3.5.2). *On the minimal resolution $\mu : Y \to \bar{Y}$, $B := \mu^*\bar{B}$ is smooth and $B \cap E = \emptyset$.*

We define a mixed period map

$$\begin{aligned} &\Phi : \mathcal{U}/\mathrm{Aut}[\Lambda_Y, D_2] \to D/\mathrm{Aut}[\Lambda, D], \\ &\Phi(\bar{Y}, \bar{B}, E) := \varphi(\text{Hodge filtration on } H^2(\overset{\circ}{Y}, \mathbf{C})), \end{aligned} \tag{3.5.3}$$

where φ is any $[\Lambda, D]$-marking and $\overset{\circ}{Y} = Y - (B + E)$. We see that $\mathrm{Aut}[\Lambda, D]$ acts on D properly discontinuously (cf. [U4,II]) and that, with the aid of the universal family over $\mathcal{U}$, Φ is holomorphic.

§4. Extension of mixed period map

In this section, we shall prove that the local monodromy on $H^2(\overset{\circ}{Y}_\infty, \mathbf{Z})$, arround a "tame" degeneration of Todorov triples in (2.3.2) splits and we shall extend the mixed period map Φ in (3.5.3) to $\bar{\Phi}$ over these degenerations. We shall also show how the data $\bar{\Phi}(0)$ induces the mixed Hodge structure on $H^2(V - (B + E + D)_V, \mathbf{Z})$, where V is the main component of the central fiber $X_0 = V + W$ of a "tame" degeneration and $D = V \cap W$ the double locus. We continue to work on the stage (b) of Figure 1 at the end of this paper. We use the notation in §2.

(4.1) We recall first a general result on the splitting of a nilpotent endomorphism on a vector space over a field. Let

V : a finite dimensional vector space over a field,

G : an increasing filtration of V,

N : a nilpotent endomorphism of V which is compatible with G

The following lemma is found in [SZ,(2.11),(2.16)].

Lemma (4.1.1). *In the above notation, if length $G \leqslant 2$, i.e., for some i, $G_i = 0$ and $G_{i+2} = V$, then the following are equivalent to each other:*

(i) *$G * L$ yields the G-relative N-filtration, where L is the N-filtraion.*

(ii) *The G-relative N-filtration exists.*

(iii) *G is strict for N^j for all non-negative integers j.*

(iv) *G has an N-stable splitting.*

We can show implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). The asssumption length $G \leqslant 2$ is necessary for the step (ii) $\Rightarrow$ (iii). For details, see the above reference.

(4.1.1) is a remarkable fact but it is not sufficient enough for our use. We need an investigation of a local monodromy over $\mathbf{Z}$.

(4.2) Let

(4.2.1) $$\bar{\mathcal{U}} := \bar{\mathcal{U}}_{(\ell,\alpha)} \subset \mathbf{P}(g_*\mathcal{L})$$

be a partial compactification of $\mathcal{U}$ in (3.5.1) added those triples $(\bar{Y}, \bar{B}, E)$ which satisfy the following Condition (4.2.2).

Condition (4.2.2). *The Picard number of the minimal resolution Y of $\bar{Y}$ is $\ell+9$ and $\bar{B}$ is an irreducible, reduced curve with one node. We devide the cases:*

(i) *$\bar{B}$ passes through one of the double points on $\bar{Y}$.*

(ii) *$\bar{B}$ is apart from all the double points on $\bar{Y}$*

Then, by the same argument as (3.4), the quotient

(4.2.3) $$\mathcal{U}/\mathrm{Aut}[\Lambda_Y, D_2]$$

is the coarse moduli space of the triples in question. The central fiber $Y_0 = V \cup W$ of a semi-stable reduction of the degenration of types (4.2.2.i) and (4.2.2.ii) are given in Fugures 1 and 2 at the end of this paper respectively.

(4.3) Let

(4.3.1) $$f : (\mathcal{Y}, \mathcal{B} + \mathcal{E}) \to \Delta$$

be a semi-stable degeneration of type (4.2.2.i) on the stage (b) in Figure 1. By (A.2), we can consider the Thom-Gysin-Clemens-Schmid diagram (1.4.6) over $\mathbf{Z}$ with exact columns. In order to adjust that diagram for our use, we set

(4.3.2)
$$\begin{aligned}
&\widetilde{G}_3(\mathring{Y}_\infty) := H^2(\mathring{Y}_\infty, \mathbf{Z}),\\
&\widetilde{G}_2(\mathring{Y}_\infty) := \mathrm{Im}\{H^2(Y_\infty, \mathbf{Z}) \to \widetilde{G}_3(\mathring{Y}_\infty)\},\\
&\widetilde{G}_3(\mathring{Y}_0) := \mathrm{Coker}\{H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \mathbf{Z}) \to H^2(\mathring{Y}_0, \mathbf{Z})\},\\
&\widetilde{G}_2(\mathring{Y}_0) := \mathrm{Im}\{H^2(Y_0, \mathbf{Z}) \to \widetilde{G}_3(\mathring{Y}_0)\},\\
&\widetilde{G}_3(\mathring{V}) := \mathrm{Coker}\{H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \mathbf{Z}) \to H^2(\mathring{Y}_0, \mathbf{Z}) \to H^2(\mathring{V}, \mathbf{Z})\},\\
&\widetilde{G}_2(\mathring{V}) := \mathrm{Im}\{H^2(V, \mathbf{Z}) \to \widetilde{G}_3(\mathring{V})\},\\
&\widetilde{G}_3(\mathring{W}) := H^2(\mathring{W}, \mathbf{Z}),\\
&\widetilde{G}_2(\mathring{W}) := \mathrm{Im}\{H^2(W, \mathbf{Z}) \to \widetilde{G}_3(\mathring{W})\} = \widetilde{G}_3(\mathring{W}),
\end{aligned}$$

where we use the notation as (1.1.2) applied for (4.3.1) as well as the notation on the stage (b) in Figure 1. Since $H^1(B_0, \mathbf{Z})$ (resp. $H^1(B_\infty, \mathbf{Z})$, $H^1(B_V, \mathbf{Z})$) is $\mathbf{Z}$-free, the Thom-Gysin exact sequence implies that the torsion of $\widetilde{G}_i(\overset{\circ}{Y}_0)$ (resp. $\widetilde{G}_i(\overset{\circ}{Y}_\infty)$, $\widetilde{G}_i(\overset{\circ}{V})$) for $i = 2$, 3 coincide. We denote

$$\begin{aligned} G_i(\overset{\circ}{Y}_\infty) &:= \widetilde{G}_i(\overset{\circ}{Y}_\infty)/(\text{torsion}) \\ G_i(\overset{\circ}{Y}_0) &:= \widetilde{G}_i(\overset{\circ}{Y}_0)/(\text{torsion}) \\ G_i(\overset{\circ}{V}) &:= \widetilde{G}_i(\overset{\circ}{V})/(\text{torsion}) \end{aligned} \tag{4.3.3}$$

In the notation (4.3.3), the Thom-Gysin-Clemens-Schmid diagram becomes

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & H^1(B_0, \mathbf{Z}) & \to & H^1(B_\infty, \mathbf{Z}) & \xrightarrow{N_B} & H^1(B_\infty, \mathbf{Z}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & G_3(\overset{\circ}{Y}_0) & \to & G_3(\overset{\circ}{Y}_\infty) & \xrightarrow{N} & G_3(\overset{\circ}{Y}_\infty) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & G_2(\overset{\circ}{Y}_0) & \to & G_2(\overset{\circ}{Y}_\infty) & \xrightarrow{0} & G_2(\overset{\circ}{Y}_\infty) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array} \tag{4.3.4}$$

We notice that all the columns of the diagram (4.3.4) are exact by (A.2) and the construction. The top row is the case of curves and the exactness is well-known. The bottom row is exact by construction. Hence we see, by chasing the diagram, that the midle row is also exact.

Lemma (4.3.5). *For type (4.2.2.i), there exists a* $\mathbf{Z}$*-basis* $\{e_1, \ldots, e_{m+2g}\}$ *of* $G_3(\overset{\circ}{Y}_\infty)$ *satisfying the following conditions.*

(i) $\{e_1, \ldots, e_m\}$ *is a* $\mathbf{Z}$*-basis of* $G_2(\overset{\circ}{Y}_\infty)$ *and* $\{e_{m+1}, \ldots, e_{m+2g}\}$ *is a lifting of a symplectic* $\mathbf{Z}$*-basis of* $H^1(B_\infty, \mathbf{Z})$.

$$\text{(ii)} \qquad N(e_i) = \begin{cases} -2e_{m+1} & \text{if } i = m+g+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\{e_1, \ldots, e_{m+2g}\}$ be a $\mathbf{Z}$-basis of $G_3(\overset{\circ}{Y}_\infty)$ satisfying the condition (i). Then, by the Picard-Lefschetz formula on $H^1(B_\infty, \mathbf{Z})$ (cf. [SGA,XV.3.4]) and the (4.1.1.iii),

$$N(e_i) = \begin{cases} -2e_{m+1} + x & \text{if } i = m+g+1, \\ 0 & \text{otherwise,} \end{cases}$$

for some $x \in G_2(\overset{\circ}{Y}_\infty)$. Hence it is enough to show

Claim. *Im N in $G_3(\overset{\circ}{Y}_\infty)$ is not primitive.*

By this claim, x is 2-divisible and, replacing e_{m+1} by $e_{m+1} + x/2 \in G_3(\overset{\circ}{Y}_\infty)$, we get the desired basis.

We now prove the above claim. Since the restriction map $H^2(W, \mathbf{Z}) \to H^2(D, \mathbf{Z})$ is surjective and the fundamental class of B_W is sent to the 2-divisible element $[B_D]$ of $H^2(D, \mathbf{Z})$, where $B_D := B_W \cap D =: \{p, q\}$, the Mayer-Vietoris sequence implies an exact sequence

$$\textbf{(4.3.6)} \qquad 0 \to \widetilde{G}_2(\overset{\circ}{Y}_0) \xrightarrow{r} \widetilde{G}_2(\overset{\circ}{V}) \oplus \widetilde{G}_2(\overset{\circ}{W}) \to H^2(D, \mathbf{Z})/\mathbf{Z}[B_D] \to 0.$$

Since $\widetilde{G}_2(\overset{\circ}{W})$ and $H^2(D, \mathbf{Z})/\mathbf{Z}[B_D]$ are isomorphic through the above map, (4.3.6) splits hence we have, in particular,

$$(\text{torsion of } \operatorname{Im} r) \oplus \widetilde{G}_2(\overset{\circ}{W}) = (\text{torsion of } \widetilde{G}_2(\overset{\circ}{V})) \oplus \widetilde{G}_2(\overset{\circ}{W}).$$

It is easy to compute, by the Thom-Gysin sequence and the Mayer-Vietoris sequence, the following:

$$H^1(\overset{\circ}{V}, \mathbf{Z}) = H^1(\overset{\circ}{W}, \mathbf{Z}) = 0. \qquad H^2(\overset{\circ}{W}, \mathbf{Z}) : \text{2-torsion.}$$

$$H^3(\overset{\circ}{Y}_0, \mathbf{Z}) = 0.$$

By the Clemens-Schmid-Thom-Gysin diagram, we see

$$H^2(\mathcal{Y}, \mathcal{Y}^*; \mathbf{Z}) \rightrightarrows H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*, \mathbf{Z}),$$

so, by chasing the diagram, we have

$$\mathrm{Im}\{H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \mathbf{Z}) \to H^2(\mathring{Y}_0, \mathbf{Z})\} \subset \mathrm{Im}\{H^2(Y_0, \mathbf{Z}) \to H^2(\mathring{Y}_0, \mathbf{Z})\}.$$

Notice also that

$$\mathrm{Im}\{H^0(B_V, \mathbf{Z}) \oplus H^0(B_W, \mathbf{Z}) \xrightarrow{\text{restriction}} H^0(B_D, \mathbf{Z})\} = \mathbf{Z}(p+q),$$

$$\mathrm{Im}\{H^0(\mathring{D}, \mathbf{Z}) \xrightarrow{\text{residue}} H^0(B_D, \mathbf{Z})\} = \mathbf{Z}(p-q).$$

Hence, by the above results, the Mayer-Vietoris-Thom-Gysin diagram is arranged as

(4.3.7)

$$\begin{array}{ccccccccccc}
 & & 0 & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & \frac{\mathbf{Z}p+\mathbf{Z}q}{\mathbf{Z}(p+q)+\mathbf{Z}(p-q)} & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \to & \frac{\mathbf{Z}p+\mathbf{Z}q}{\mathbf{Z}(p+q)} & \to & H^1(B_0, \mathbf{Z}) & \to & H^1(B_V, \mathbf{Z}) & \to & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \to & H^1(\mathring{D}, \mathbf{Z}) & \to & \widetilde{G}_3(\mathring{Y}_0) & \to & \widetilde{G}_3(\mathring{V}) \oplus \widetilde{G}_3(\mathring{W}) & \to & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & 0 & \to & \widetilde{G}_2(\mathring{Y}_0) & \to & \widetilde{G}_2(\mathring{V}) \oplus \widetilde{G}_2(\mathring{W}) & \to & \frac{H^2(D,\mathbf{Z})}{\mathbf{Z}[B_D]} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

We see, from (4.3.7) together with the remarks after (4.3.2) and (4.3.6), that the image $H^1(\mathring{D}, \mathbf{Z})$ in $G_3(\mathring{Y}_0)$ is 2-divisible. Put

$$\widehat{H}^1(\mathring{D}) : \text{primitive span of image of } H^1(\mathring{D}, \mathbf{Z}) \text{ in } G_3(\mathring{Y}_0).$$

Then we have

(4.3.8)
$$\begin{array}{ccccccccc} 0 & \to & \frac{\mathbf{Z}p+\mathbf{Z}q}{\mathbf{Z}(p+q)} & \to & H^1(B_0,\mathbf{Z}) & \to & H^1(B_V,\mathbf{Z}) & \to & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \\ 0 & \to & \widehat{H}^1(\overset{\circ}{D}) & \to & \mathrm{gr}_3^{G(\overset{\circ}{Y}_0)} & \to & \mathrm{gr}_3^{G(\overset{\circ}{V})} & \to & 0 \end{array}$$

From (4.3.4), (1.4.5.ii) and the primitivity of $\widehat{H}^1(\overset{\circ}{D})$, we have a commutative diagram

(4.3.9)
$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Im}\,N & \to & \mathrm{Ker}\,N & \to & \mathrm{gr}_2^L G(\overset{\circ}{Y}_\infty)_3 & \to & 0 \\ & & \downarrow\alpha & & \uparrow\wr & & \downarrow & & \\ 0 & \to & \widehat{H}^1(\overset{\circ}{D}) & \to & G_3(\overset{\circ}{Y}_0) & \to & G_3(\overset{\circ}{V}) & \to & 0 \end{array}$$

We see, from (4.3.4), that

(4.3.10)
$$\mathrm{Ker}\,N/G_2(\overset{\circ}{Y}_\infty) \xrightarrow{\sim} \mathrm{Ker}\,N_B,$$
$$(\mathrm{Im}\,N + G_2(\overset{\circ}{Y}_\infty))/G_2(\overset{\circ}{Y}_\infty) \xrightarrow{\sim} \mathrm{Im}\,N_B$$

by the induced maps. Taking gr_3^G of (4.3.9), we have, by (4.3.10) and (4.3.8), a commutative exact diagram

(4.3.11)
$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Im}\,N_B & \to & \mathrm{Ker}\,N_B & \to & \mathrm{gr}_2^L H^1(B_\infty,\mathbf{Z}) & \to & 0 \\ & & \downarrow\alpha_B & & \uparrow\wr & & \downarrow & & \\ 0 & \to & \frac{\mathbf{Z}p+\mathbf{Z}q}{\mathbf{Z}(p+q)} & \to & H^1(B_0,\mathbf{Z}) & \to & H^1(B_V,\mathbf{Z}) & \to & 0 \end{array}$$

Since $\mathrm{Im}\,N_B$ in $\mathrm{Ker}\,N_B$ is 2-divisible and $H^1(B_V,\mathbf{Z})$ is $\mathbf{Z}$-free, α_B in (4.3.11) is not isomorphic hence not so is α in (4.3.9). This proves our claim. ∎

Remark (4.3.12) The same assertion as Lemma (4.3.5) holds also for the type (4.2.2.ii) on the stage (b) in Figure 2 at the end of this article. The proof is similar, but now terms $\widetilde{G}_i(\overset{\circ}{V}) \oplus \widetilde{G}_i(\overset{\circ}{W})$ $(i=3,2)$ etc. are replaced by

$$\widetilde{G}_3(\overset{\circ}{V} \sqcup \overset{\circ}{W}) := (H^2(\overset{\circ}{V},\mathbf{Z}) \oplus H^2(\overset{\circ}{W},\mathbf{Z}))/\mathbf{Z}([D_{\overset{\circ}{V}}] - [D_{\overset{\circ}{W}}]),$$
$$\widetilde{G}_2(\overset{\circ}{V} \sqcup \overset{\circ}{W}) := \mathrm{Im}\{H^2(V,\mathbf{Z}) \oplus H^2(W,\mathbf{Z}) \to \widetilde{G}_3(\overset{\circ}{V} \sqcup \overset{\circ}{W})\} \quad \text{etc.,}$$

where $[D_{\overset{\circ}{V}}]$ and $[D_{\overset{\circ}{W}}]$ are the fundamental classes of $\overset{\circ}{D} \subset \overset{\circ}{V}$ and $\overset{\circ}{D} \subset \overset{\circ}{W}$ respectively. The splitting of (4.3.6) in the present case is given by the image of $H^2(W, \mathbf{Z})$ in $\widetilde{G}_2(\overset{\circ}{V} \sqcup \overset{\circ}{W})$. We omit the details.

(4.4) Let

$$\{e_1, \ldots, e_{m+2g}\} \tag{4.4.1}$$

be a $\mathbf{Z}$-basis of Λ in (3.2.1) satisfying the condition (4.3.5.i) and let N be an endomorphism of Λ defined by

$$N(e_i) = \begin{cases} -e_{m+1} & \text{if } i = m+g+1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.2}$$

Since N splits, we can construct easily a partial compactification of the classifying space $D/\mathrm{Aut}[\Lambda, D]$ in (3.5.3) added only the boudary component of codimension 1 associated to N by the method of troidal compactifications for locally symmetric Siegel spaces (cf. [AMRT], [CCK]). As a set, this is defined by

$$\begin{aligned} &\overline{D/\mathrm{Aut}[\Lambda, D]} := (D/\mathrm{Aut}[\Lambda, D]) \sqcup (\exp(\mathbf{C}N)D/\exp(\mathbf{C}N))/\mathrm{Norm}_{\mathbf{Z}}(N), \\ &\text{where} \quad \mathrm{Norm}_{\mathbf{Z}}(N) := \{\gamma \in \mathrm{Aut}[\Lambda, D] \mid \gamma^{-1}N\gamma = N\}. \end{aligned} \tag{4.4.3}$$

The analytic structure is defined through the following construction of $\overline{D/\mathrm{Aut}[\Lambda, D]}$.

Let $D_c \simeq \Delta^\lambda \times (U \times \Delta^{\mu-1}) \times \Delta^\nu$ be a small open subset of D, where Δ is the unit disc, U is the upper half plane and the decomposion is the one into $D_2 \times D_3 \times$ (extension data) (see (3.2.7)). Construct

$$\begin{array}{ccc} D_c & \simeq & \Delta^\lambda \times (U \times \Delta^{\mu-1}) \times \Delta^\nu \\ \downarrow \varepsilon & & \\ D_c/\exp(\mathbf{Z}N) & \simeq & \Delta^\lambda \times (\Delta^* \times \Delta^{\mu-1}) \times \Delta^\nu \\ \cap & & \\ \overline{D_c/\exp(\mathbf{Z}N)} & \simeq & \Delta^\lambda \times \Delta^\mu \times \Delta^\nu \\ \downarrow & & \\ \overline{D_c/\exp(\mathbf{Z}N)}/\mathrm{Norm}_{\mathbf{Z}}(N) & & \end{array} \tag{4.4.4}$$

where $\varepsilon := 1 \times (\exp 2\pi i(\ ) \times 1) \times 1$. Patching up by

$$\begin{array}{lccc} & (D_c/\exp(\mathbf{Z}N))/\mathrm{Norm}_{\mathbf{Z}}(N) & \hookrightarrow & D/\mathrm{Aut}[\Lambda, D] \\ (4.4.5) & \cap & & \\ & \overline{D_c/\exp(\mathbf{Z}N)}/\mathrm{Norm}_{\mathbf{Z}}(N), & & \end{array}$$

we obtain $\overline{D/\mathrm{Aut}[\Lambda, D]}$. As in the case of locally symmetric Siegel spaces, this has a structure of V-manifold (= orbifold).

Proposition (4.4.6). *The mixed period map Φ in (3.5.3) extends holomorphically to*

$$\bar{\Phi} : \bar{\mathcal{U}}/Aut[\Lambda_Y, D_2] \to \overline{D/Aut[\Lambda, D]}$$

which sends a boudary point to its nilpotent orbit, where the source is (4.2.3) and the target (4.4.3).

Proof. By construction (4.2.1), the boundary $\bar{\mathcal{U}} - \mathcal{U}$ is a smooth divisor on $\bar{\mathcal{U}}$. Localizing the situation at a boundary point, we may assume

$$\mathcal{U} = \Delta^{11-\ell} \times (\Delta^* \times \Delta^{\ell}) \subset \bar{\mathcal{U}} = \Delta^{11-\ell} \times \Delta^{\ell+1}$$

with local coordinates $t = (t_1, t')$, where $t_1 = 0$ is the boundary and t' the other coordinates. Take a point $\tau \in \mathcal{U}$ and fix an isomorphism of the data in (3.1.1)

$$\pi : \Lambda(\bar{Y}_\infty, \bar{B}_\infty, E_\infty) \xrightarrow{\sim} \Lambda(\bar{Y}_\tau, \bar{B}_\tau, E_\tau).$$

By definition (or by (3.3.2)), for any $\mathbf{Z}$-basis $\{e_1(\infty), \ldots, e_{m+2g}(\infty)\}$ of $G_3(\mathring{Y}_0)$ and the monodromy logarithm N_∞ satisfying the condition (4.3.5) (see also (4.3.12)), there exists a $[\Lambda, D]$-marking in (3.3.1)

$$\eta = (\eta_Y, \tilde{\eta}, \eta_3) : \Lambda(\bar{Y}_\tau, \bar{B}_\tau, E_\tau) \xrightarrow{\sim} [\Lambda] \quad \text{such that}$$

$$\eta\pi e_i(\infty) = e_i \qquad (m + 1 \leqslant i \leqslant m + 2g).$$

Hence we have $N_\infty = (\eta\pi)^{-1}(2N)(\eta\pi)$ for the types (i) and (ii) in (4.2.2) on the stage (b). For each fixed t', let $F(\infty, t')$ be the limit Hodge filtration as $t_1 \to 0$. We define

$$\bar{\Phi}(0,t') := \exp(\mathbf{C}N)(\eta\pi F(\infty,t'))/\exp(\mathbf{C}N) \bmod \mathrm{Norm}_{\mathbf{Z}}(N).$$

In order to see that $\bar{\Phi}$ is holomorphic, we observe its period matrix. We first examine the type (4.2.2.i). By (4.3.5.ii), gr_2^W and the extension data of the period matrix are invariant under the action of the local monodromy $T_\infty := \exp N_\infty$. gr_3^W of the period matrix is the only part which is affected by T_∞. To see this part more precisely, let $\{e_1(t), \ldots, e_{m+2g}(t)\}$ be a horizontal frame of the local system $\{\Lambda(t) := H^2(\overset{\circ}{Y}_t, \mathbf{Z})/(\mathrm{torsion})\}_{t\in\mathcal{U}}$ which coincides with $\eta^{-1}\{e_1, \ldots, e_{m+2g}\}$ at $t = \tau$. $e_{m+g+1}(t)$ is multi-valued. Let $\{\omega_1(t), \ldots, \omega_{g+1}(t)\}_{t\in\mathcal{U}}$ be a frame of the Hodge filter F^2 satisfying $\omega_{1+i}(t) \equiv e_{m+g+i}(t) \bmod \sum_{j=1}^{m+g} \mathbf{C}e_j(t)$ $(1 \leqslant i \leqslant g)$. Then the period matrix for F^2 is of the form

$$\begin{aligned}&(\omega_1(t); \ldots, \omega_{g+1}(t))\\ &= (e_1(t), \ldots; e_{m+1}(t), \ldots; e_{m+g+1}(t), \ldots, e_{m+2g}(t))\begin{pmatrix} A(t) & B(t) \\ 0 & Z(t) \\ 0 & 1_g \end{pmatrix}\end{aligned}$$

The (1,1)-part $z_{11}(t)$ of $Z(t)$ is the only part which is multi-valued. By (4.3.5.ii), we can compute as

$$z_{11}(t) = 2(\log t_1)/2\pi i + s(t),$$

where $s(t)$ extends holomorphically over $\bar{\mathcal{U}}$, which is equivalent to the existence of the limit Hodge filter $F^2(\infty, t')$. Hence, by (4.4.4) and (4.4.5), we have

$$\begin{array}{ccccccc} \Phi & : & \mathcal{U} & \to & D_c/\exp(\mathbf{Z}N) & \to & D/\mathrm{Aut}[\Lambda, D] \\ & & \cap & & \cap & & \cap \\ \bar{\Phi} & : & \bar{\mathcal{U}} & \to & \overline{D_c/\exp(\mathbf{Z}N)} & \to & \overline{D/\mathrm{Aut}[\Lambda, D]} \end{array}$$

where the (1,1)-part of $Z(t)$ on the midle stage becomes

$$\text{(4.4.7)} \qquad \exp(2\pi i z_{11}(t)) = t_1^2 \exp(2\pi i s(t)).$$

This shows that $\bar{\Phi}$ is holomorphic for the type (4.2.2.i).

As for the type (4.2.2.ii), a similar argument works and instead of (4.4.7) we get

$$\text{(4.4.8)} \qquad \exp(2\pi i z_{11}(t)) = t_1 \exp(2\pi i s(t))$$

because of the $(2:1)$ base extension in the semi-stable reduction of pairs in Figure 2 on the last page. ∎

§5. Inheritance of induction hypothesis and infinitesimal mixed Torelli theorem

(5.1) The following result is useful for our inductive approach of the mixed Torelli problem by using the degeneration of the type (4.2.2.i).

Proposition (5.1.1). *In the notation of (4.3), we see for the family (4.3.1) the following:*

(i) $G_3(\overset{\circ}{Y}_0) \xrightarrow{\sim} Ker\{N : G_3(\overset{\circ}{Y}_\infty) \to G_3(\overset{\circ}{Y}_\infty)\}$.

(ii) *The Gysin filtrations G are isomorphic under (i).*

(iii) *The Mayer-Vietoris filtration L and the N-filtration L are isomorphic under (i).*

(iv) *Before the shiftings [2], $W_0 \supset (G * L)_0$ with index 2 on $G_3(\overset{\circ}{Y}_0)$ and $W = G * L$ on $G_3(\overset{\circ}{Y}_\infty)$.*

(v) *(i) is an isomorphism of mixed Hodge structures with weight filtrations* $(G*L)[2]$ *on both terms.*

Proof. (i) and (ii) follow immediately from (4.3.4). Since $\widetilde{G}_3(\overset{\circ}{W}) = H^2(\overset{\circ}{W}, \mathbf{Z})$ is a 2-torsion and N satisfies (4.3.5.ii), (4.3.9) implies a commutative exact diagram

$$\textbf{(5.1.2)}\qquad \begin{array}{ccccccccc} 0 & \to & \operatorname{Im} N & \to & \operatorname{Ker} N & \to & \operatorname{gr}_2^L G_3(\overset{\circ}{Y}_\infty) & \to & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \\ 0 & \to & H^1(\overset{\circ}{D}, \mathbf{Z}) & \to & G_3(\overset{\circ}{Y}_0) & \to & G_3(\overset{\circ}{V}) \oplus \widetilde{G}_3(\overset{\circ}{W}) & \to & 0. \end{array}$$

This shows (iii).

As for the first part of (iv), we see in the same way as (1.3.2), (1.3.4) and (1.3.5.i) that a complex $K^\cdot_{\mathbf{Z}}$ over $\mathbf{Z}$ and its filtrations G, L and W are defined and that they satisfy $W = G*L$ on $K^\cdot_{\mathbf{Z}}$. The spectral sequence of hypercohomology of $(K^\cdot_{\mathbf{Z}}, W)$ degenerates in $E_2 = E_\infty$ [D2,II]. We compute the E_2^2:

$$E_1^{-k,2+k} = H^2(Y_0, \operatorname{gr}_k^W K^\cdot_{\mathbf{Z}}) = \bigoplus_{i+j=k} H^{2-i+j}(\widetilde{(\mathcal{B}+\mathcal{E})}^{(i)} \cap \widetilde{Y}_0^{(-j+1)}, \mathbf{Z}).$$

$$E_2^{1,1} = E_1^{1,1} = H^1(D, \mathbf{Z}) = 0$$

$$E_2^{-1,3} = E_1^{-1,3} = H^1(B_V, \mathbf{Z})$$

$$E_2^{-2,4} = E_1^{-2,4} = 0.$$

Hence, before shifting [2], we have

$$W_{-1} = 0 \subset W_0 = \operatorname{Ker}\{H^2(\overset{\circ}{Y}_0, \mathbf{Z}) \xrightarrow{\alpha} H^1(B_V, \mathbf{Z})\} \subset W_1 = H^2(\overset{\circ}{Y}_0, \mathbf{Z}),$$

where α is the composite of the Mayer-Vietoris map and the residue map. On the other hand, since

$$G_{-1} = 0 \subset G_0 = \operatorname{Im}\{H^2(Y_0, \mathbf{Z}) \to H^2(\overset{\circ}{Y}_0, \mathbf{Z})\} \subset G_1 = H^2(\overset{\circ}{Y}_0, \mathbf{Z}),$$

$$L_{-2} = 0 \subset L_{-1} = \operatorname{Im}\{H^1(\overset{\circ}{D}, \mathbf{Z}) \to H^2(\overset{\circ}{Y}_0, \mathbf{Z})\} \subset L_0 = H^2(\overset{\circ}{Y}_0, \mathbf{Z}),$$

we have

$$(G*L)_{-1} = 0 \subset (G*L)_0 = G_0 + L_{-1} \subset (G*L)_1 = H^2(\overset{\circ}{Y}_0, \mathbf{Z})$$

before the shiftings. By the part of the original Mayer-Vietoris-Thom-Gysin diagram like (4.3.7), we see

$$(G*L)_0 \subset W_0 \quad \text{with index 2 on } H^2(\overset{\circ}{Y}_0, \mathbf{Z})$$

hence so is that on $G_3(\overset{\circ}{Y}_0)$.

The second part of (iv) follows from (4.3.5) and the argument of the proof of the step (iv) $\Rightarrow$ (i) in (4.1.1) which is valid also over $\mathbf{Z}$ (cf. [SZ,(2.11)]). (v) is a consequence of (i)–(iv) and (1.3.5). ∎

(5.2) We have the following partial result at present for the infinitesimal mixed period map.

Proposition (5.2.1). *The infinitesimal mixed Torelli theorem holds for the extension $\bar{\Phi}$ of the mixed period map in (4.4.6) at interior points $\in \mathcal{U}$ and at boundary points $\in \bar{\mathcal{U}} - \mathcal{U}$ of the type (4.2.2.i) in the tangential directions of the boundary.*

Proof. Let $(Y, B+E)$ be a pair of the smooth minimal model and its branch locus of $(\bar{Y}, \bar{B}+E) \in \mathcal{U}$. By taking the dual, we see that

$$H^1(T_Y(-\log(B+E))) \to \mathrm{Hom}(H^1(\Omega_Y^1(\log(B+E))), H^2(\mathcal{O}_Y))$$

is injective. This proves the first half of our assertion.

For a degeneration of pairs of the type (4.2.2.i), the locally trivial (= equi-singular) small deformations of the pair on the stage (a) in Figure 1 on the last page within the limits of these pairs corresponds exactly to those on the stage (b), i.e., there are no problems of ordinary double points nor the Todorov involution. On the stage (a), they are determined by the deformations of the pair of the main component and the union of its branch locus and the double curve. The latter are determined by the deformations of $(V, (B + E + D)_V)$ on the stage (b). As before,

$$H^1(T_V(-\log(B + E + D)_V)) \to \mathrm{Hom}(H^1(\Omega_V^1(\log(B + E + D)_V)), H^2(\mathcal{O}_V))$$

is injective. This implies the second half of our assertion, because, for a boundary point $t_0 \in \bar{\mathcal{U}}$, $\bar{\Phi}(t_0)$ induces the Hodge filtration on $G_3(\overset{\circ}{V}) \otimes \mathbf{C} = H^2(\overset{\circ}{V}, \mathbf{C})/\mathbf{C}[\overset{\circ}{D}]$ by (5.1.2) and the difference of

$$H^2(\overset{\circ}{V}, \mathbf{C})/\mathbf{C}[\overset{\circ}{D}] \hookrightarrow H^2(\overset{\circ}{V} - \overset{\circ}{D}, \mathbf{C}) = H^2(V - (B + E + D)_V, \mathbf{C})$$

affects their gr_F^p only for $p = 2$. ∎

Problem (5.2.2). *Solve infinitesimal mixed Torelli problem for $\bar{\Phi}$.*

Problem (5.2.3). *Solve local mixed Torelli problem for $\bar{\Phi}$.*

Problem (5.2.4). *Solve generic mixed Torelli problem for Φ.*

Remark (5.3.5). (i) In cases $(\ell, \alpha) = (1, 0)$ and $(2, 1)$, the generic mixed Torelli theorem is verified for geometric monodromy in an elementary way ([L], [SSU, II.2]).

(ii) The number of moduli of Todorov surfaces is 12 for every (ℓ, α). On the other hand, a hypersurface of $(\bar{\mathcal{U}}/\mathrm{Aut}[\Lambda_Y, D_2])_{(\ell-1, \alpha')}$, $\alpha' = \alpha$ or $\alpha - 1$, is glued as the boundary locus of the degenerations of the type (4.2.2.i). Hence the induction step will not proceed naively.

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Table 0

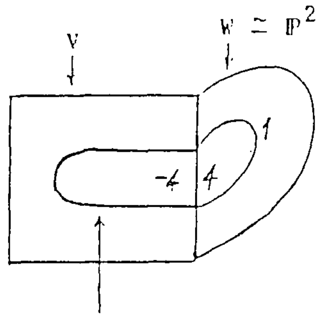
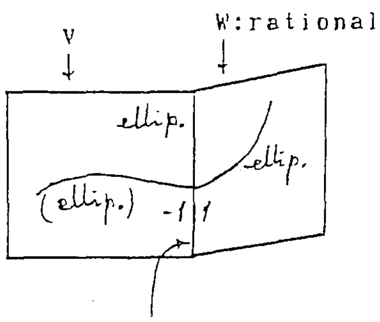
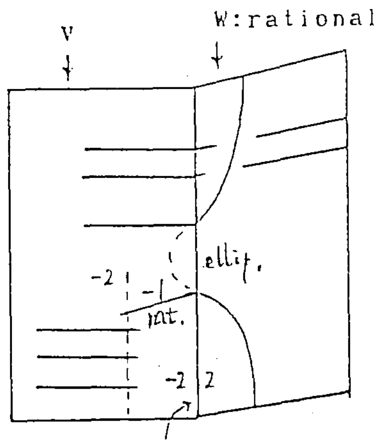
degeneration of branch locus	central fiber of semi-stable degeneration of pairs: (X_0, Y_0), $X_0 = V + W$	change of (p_g, q, c_1^2) of V	local monodromy on $H^2(X_\infty)$
(I_1) passing an isolated branch point A_1	 genus drops by 1 (no base extension)	$(0, 0, -1)$	I
(I_2) passing D_4	 section of fibration on V, $\tilde{E}_8$ on V (base extension of 2:1 once)	$(0, +1, -1)$	I
(II_1) having ordinary quadruple point	 part of singular fiber of fibration by curves of genus 2 on V, $\tilde{E}_8$ on V (base extension of 2:1 once)	$(-1, 0, -1)$	II

Figure 1

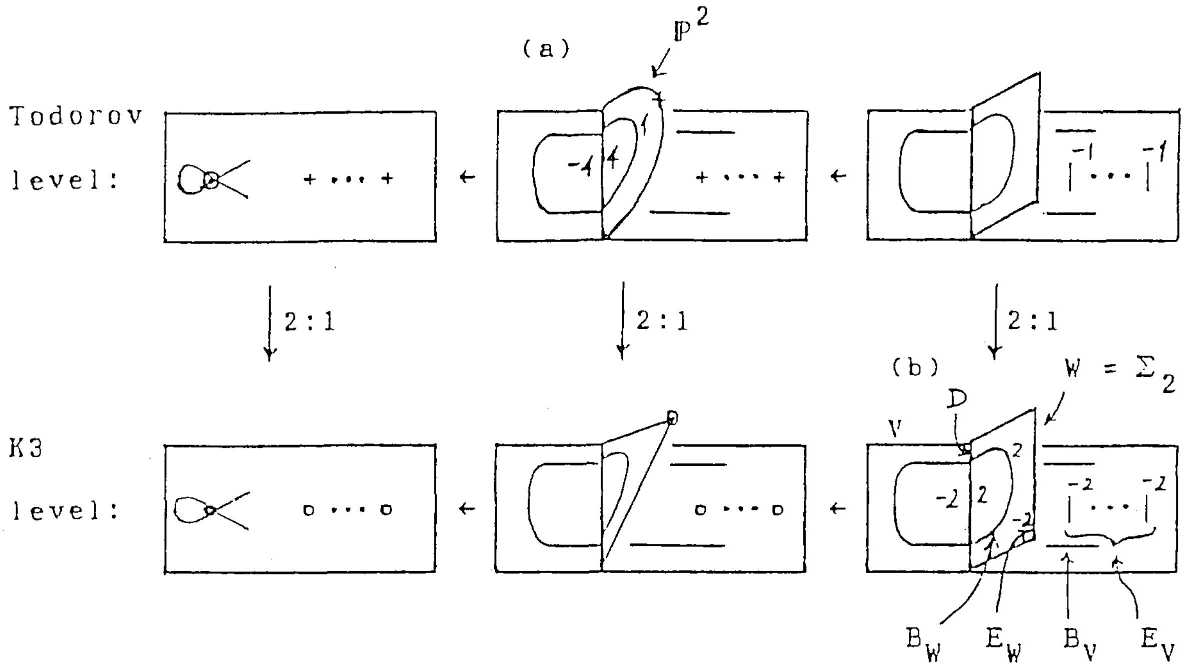
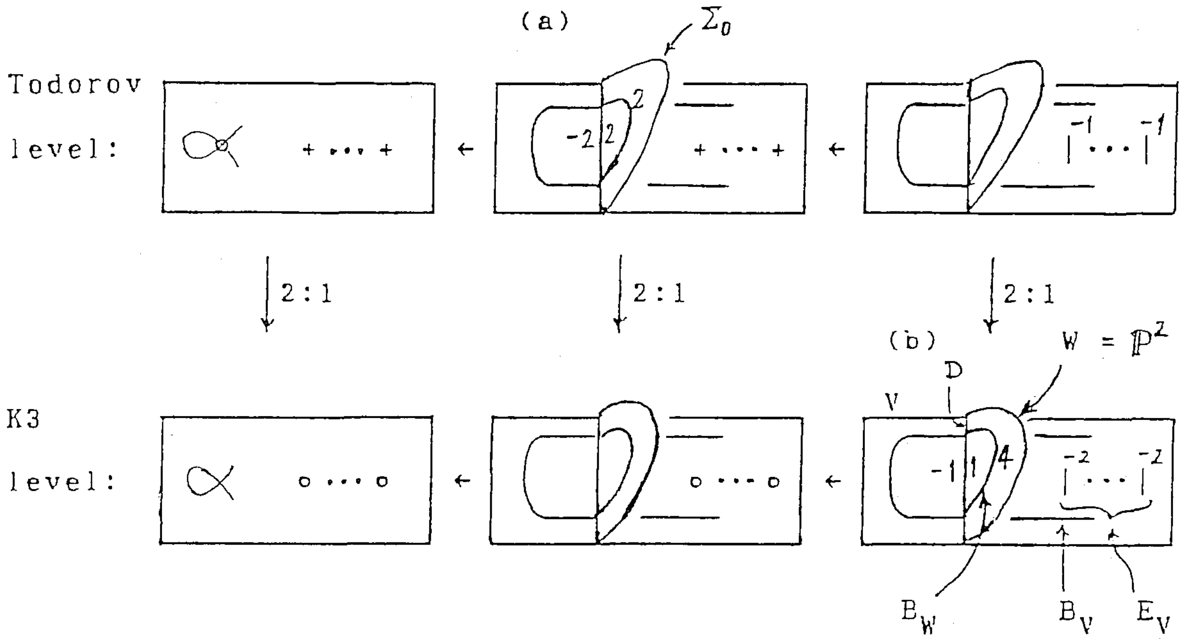


Figure 2



Classification of Logarithmic Enriques Surfaces

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Contents

Introduction

This is an expository survey on the article [14] which will be published elsewhere.

Normal projective surfaces with only quotient singularities appear in studies of threefolds and semi-stable degenerations of surfaces (cf. Kawamata [5], Miyanishi [6], Tsunoda [11]). We have been interested in such singular surfaces with logarithmic Kodaira dimension $-\infty$ (cf. Miyanishi-Tsunoda [8], Zhang [12,13]). In the present paper, we shall study the case of logarithmic Kodaira dimension 0.

Let $\bar{V}$ be a normal projective rational surface with only quotient singularities but with no rational double singular points. Let $K_{\bar{V}}$ be the canonical divisor of $\bar{V}$ as a Weil divisor. We call $\bar{V}$ a logarithmic Enriques surface if $H^1(\bar{V},O_{\bar{V}}) = 0$ and $K_{\bar{V}}$ is a trivial Cartier divisor for some positive integer N. The smallest one of such integers N is called the index of $K_{\bar{V}}$ and denoted by Index($K_{\bar{V}}$) or simply by I. Since $IK_{\bar{V}}$ is trivial, there is a $\mathbb{Z}/I\mathbb{Z}$-covering $\pi: \bar{U} \to \bar{V}$, which is unique up to isomorphisms and étale outside Sing $\bar{V}$. Then $\bar{U}$, called the canonical covering of $\bar{V}$, is a Gorenstein surface, and the minimal resolution of singularities of $\bar{U}$ is an abelian surface or a K3 surface.

Let $f: V \to \bar{V}$ be a minimal resolution of singularities of $\bar{V}$ and set $D := f^{-1}(\mathrm{Sing}\ \bar{V})$. We often confuse $\bar{V}$ deliberately with (V,D) or (V,D,f).

§1 is a preparation and contains a proof of an inequality (cf. Proposition 1.5) which plays an important role in the whole theory; in particular, if $I \geq 3$ then $c := \#(\mathrm{Sing}\ \bar{V}) \leq (D,K_V) \leq c-1-(K_V^2)$, and if $I \geq 4$ then $c < -3(K_V^2)$. In §2, it is proved that if a positive integer p is a factor of I then $\bar{U}/(\mathbb{Z}/p\mathbb{Z})$ is a logarithmic Enriques surface, as well. We also prove that $I \leq 66$; this result is originally due to S. Tsunoda. Moreover,

$I \leq 19$ if I is a prime number. §§ 3 - 5 are devoted to the proofs of the following three theorems:

Theorem 3.1. *Let* $\bar{V}$ *or synonymously* (V,D) *be a logarithmic Enriques surface with* $\mathrm{Index}(K_{\bar{V}}) = 2$. *Then there is a logarithmic Enriques surface* $\bar{W}$ *or* (W,B) *with* $\mathrm{Index}(K_{\bar{W}})=2$ *and* $\#(\mathrm{Sing}\ \bar{W}) = 1$ *such that* V *is obtained from* W *by blowing up all singular points of* B *(i.e., intersection points of irreducible components of* B*) and then blowing down several (-1)-curves on the blown-up surface.*

Moreover, $\#(\mathrm{Sing}\ \bar{U}) = \#(\mathrm{Sing}\ \bar{V}) \leq \#\{$*irreducible component of* $D\}$ ≤ 10. *The case with* $\#(\mathrm{Sing}\ \bar{V}) = 10$ *occurs and, in this case, there is a (-2)-rod of Dynkin type* A_{19} *on* U.

Theorem 4.1. *Let* (V,D), *or synonymously* (V,D), *be a log Enriques surface whose canonical covering* $\tilde{U}$ *is an abelian surface. Then* I $(= \mathrm{Index}(K_{\bar{V}})) = 3$ *or* 5. *More precisely, we have:*

(1) *Suppose* $I = 3$. *Then* $\rho(\bar{U}) = \rho(\bar{V}) = 4$ *and* D *consists of nine isolated (-3)-curves. Hence* $\bar{U}$ *is a singular abelian surface.*

(2) *Suppose* $I = 5$. *Then* $\rho(\bar{U}) = \rho(\bar{V}) = 2$, *and* D *consists of five connected components each of which consists of one (-2)-curve and one (-3)-curve.*

Theorem 5.1. *Let* (V,D) *be a logarithmic Enriques surface such that* I $(= \mathrm{Index}(K_{\bar{V}}))$ *is a prime number and the canonical covering* $\bar{U}$ *is a K3-surface. Then* $I \neq 2, 13$. *Moreover, the singularity type of* $\bar{V}$ *is explicitly given. In particular,* $(D,K_V) = c-1-(K_V^2)$.

In §6, we consider the remaining case where the canonical covering $\bar{U}$ of $\bar{V}$ is singular. Possible types of singularities of $\bar{V}$ and $\bar{U}$ are given when $I := \mathrm{Index}(K_{\bar{V}}) = 3$ or 5. As a corollary, we see that if there is a singularity of Dynkin type E_k ($k = 6$, 7 or 8) on $\bar{U}$ then $I = 5$, 25, 7, 11, 13, 17 or 19. It remains to consider possible combinations of singularities on $\bar{V}$. We obtain the following theorem (cf. Proposition 6.2 and Lemma 6.3):

Theorem 6.1. *Let* (V,D) *be a logarithmic Enriques surface such that* I *is an odd prime number and* $\mathrm{Sing}\,\bar{U} \neq \phi$. *Then* $c := \#(\mathrm{Sing}\,\bar{V}) \le \mathrm{Min}\{16, 23-I\}$, $\#(\mathrm{Sing}\,\bar{U}) \le (24-I)/2$ *and* $-1 \le \rho(\bar{V})-c \le 4$, *where* $\rho(\bar{V})$ *is the Picard number of* $\bar{V}$. *Moreover, if* $c = 16$ *or* $\rho(\bar{V})-c = 4$, *then* $I = 5$ *or* 3, *respectively and* $\mathrm{Sing}\,\bar{V}$ *is precisely described in Proposition* 6.2; *particularly,* $(D, K_V) = c-1-(K_V^2)$.

We found an example of logarithmic Enriques surface (V,D) with $(c,I) = (15,3)$. Moreover, there is a (-2)-fork Γ of Dynkin type D_{19} on the minimal resolution U of the canonical covering $\bar{U}$ of (V,D). By contracting Γ on U we get the canonical covering $\bar{U}'$ of a new log Enriques surface (V',D'). In particular, U is a K3-surface with $\rho(U) = 20$. Such a K3-surface is probably new. Note that $\bar{U}'$ can not be a quartic surface of $\mathbb{P}^3$ (cf. Kato-Naruki [4]).

Terminology. We refer to [8; §§1.1-1.5] or [9; §2] for the definitions of (admissible rational) rods, twigs and forks, and the definition of $B^{\#}$ for a reduced effective divisor B. A $(-n)$-curve on a nonsingular projective surface is a nonsingular rational curve of self-intersection number -n. A (-2)-rod (resp. fork) is a rod (resp. fork) whose irreducible components are all (-2)-curves.

Notation. Let V be a nonsingular projective surface and let D, D_1 and D_2 be divisors on V.

K_V: Canonical divisor of V,

$\kappa(V)$: Kodaira dimension of V,

$\bar{\kappa}(X)$: Logarithmic Kodaira dimension of a non-complete surface X,

$\rho(V)$: Picard number of V,

$h^i(V,D)$: $= \dim H^i(V,D)$,

$\#(D)$: The number of all irreducible components of $\mathrm{Supp}(D)$,

f^*D: Total transform of D,

$f'D$: Proper transform of D,

$D_1 \sim D_2$: D_1 and D_2 are linearly equivalent,

$D_1 \equiv D_2$: D_1 and D_2 are numerically equivalent,

$e(D)$: Euler number of D,

Σ_n: Hirzebruch surface of degree n.

§1. Preliminaries

We work over the complex number field $\mathbb{C}$. Let $\bar{V}$ be a normal projective algebraic surface defined over $\mathbb{C}$ and let $f: V \to \bar{V}$ be a minimal resolution of $\mathrm{Sing}(\bar{V})$. Denote by D the reduced effective divisor whose support is $f^{-1}(\mathrm{Sing}\ \bar{V})$. Then there is a $\mathbb{Q}$-divisor $D^\#$ such that $f^*(K_{\bar{V}}) \equiv K_V+D^\#$ and $0 \le D^\# \le D$.

Definition 1.1. $\bar{V}$ is said to be *a log (= logarithmic) Enriques surface* if the following three conditions are satisfied:

(1) $\bar{V}$ has only quotient singularities and $\mathrm{Sing}(\bar{V}) \neq \phi$.

(2) $NK_{\bar{V}}$ is a trivial Cartier divisor for some positive integer N, which is equivalent to saying that $ND^{\#}$ is an integral divisor.

(3) $q(\bar{V}) := \dim H^1(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$.

(1) implies that $\mathrm{Supp}(D-D^{\#}) = \mathrm{Supp}\, D$ (cf. [8; §1.5 and §2.5]). Let Δ be a connected component of D. Then Δ is an admissible rational rod or an admissible rational fork, which are defined in [9; §2] (cf. Brieskorn [2; Satz 2.11]). $f(\Delta)$ is a rational double singular point if and only if Δ is a (-2)-rod or a (-2)-fork. We can define the direct image f_*F for each divisor F on V as in the case where f is a morphism between nonsingular surfaces. Then the property of linear equivalence "$\sim$" between divisors on V is preserved under f_*. By [8; Lemma 2.4], there exists a positive integer P such that for each Weil divisor $\bar{F}$ on $\bar{V}$, $P\bar{F}$ is linearly equivalent to a Cartier divisor. Let $\bar{F}_1$ and $\bar{F}_2$ be two Weil divisor on $\bar{V}$, we define the intersection number of $\bar{F}_1$ and $\bar{F}_2$ by $(\bar{F}_1, \bar{F}_2) := (1/P^2)(f^*(P\bar{F}_1), f^*(P\bar{F}_2))$.

We often identify $\bar{V}$ with (V,D,f) or (V,D).

Proposition 1.2. *Let* (V,D) *be a log Enriques surface. Then* $\kappa(V) \leq \bar{\kappa}(V-D) = 0$. *Moreover, if* $\kappa(V) = 0$, *then* $\bar{V}$ *has only rational double singular points and either* V *is a* K3*-surface or* V *is an Enriques surface.*

Proof. By virtue of [8; Lemma 1.10], we have $h^0(V, n(D+K_V)) = h^0(V, n(D^{\#}+K_V)) = 1$, for each positive integer n satisfying $n(D^{\#}+K_V) \sim f^*(nK_{\bar{V}}) \sim 0$. Therefore, $\bar{\kappa}(V-D) = 0$.

Suppose that $\kappa(V) = 0$. Then there exists a positive integer N such that $ND^{\#}$ is an integral divisor and NK_V is linearly equivalent to an effective divisor Δ. Since $0 \equiv N(D^{\#}+K_V) \sim ND^{\#}+\Delta$, we have $D^{\#} = \Delta = 0$. $D^{\#} = 0$ means that D consists of (-2)-rods and (-2)-forks (cf. [8; §1.5]). Namely, $\bar{V}$ has only rational double singular points. Note that V is a minimal surface, for $NK_V \sim 0$. By the classification theory of nonsingular surfaces and by the hypotheses that $\kappa(V) = 0$ and $q(V) = q(\bar{V}) = 0$, we see that V is a K3 surface or an Enriques surface. Q.E.D.

Let (V,D) be a log Enriques surface. Denote by $\tilde{D}$ the reduced divisor $\mathrm{Supp}\, D^{\#}$. Then $D-\tilde{D}$ consists of exactly those connected components of D which are contracted to rational double singular points on $\bar{V}$. Therefore, $(V,\tilde{D})$ is also a log Enriques surface.

In view of Proposition 1.2 and the above argument, we assume, untill the end of the present article, the following two conditions:

(1) $\kappa(V) = -\infty$, *hence* V *is a rational surface,*

(2) $\mathrm{Supp}(D^{\#}) = \mathrm{Supp}(D) \neq \phi$.

Definition 1.3. Let $\bar{V}$ be a log Enriques surface. We denote by $\mathrm{Index}(K_{\bar{V}})$ or simply by I, the smallest positive integer such that $IK_{\bar{V}}$ is a Cartier divisor.

Actually, $IK_{\bar{V}} \sim 0$ which is proved in the following lemma.

Lemma 1.4. (1) $(K_V^2) \leq -1$, $I \geq 2$, $IK_{\bar{V}} \sim 0$ *and* $I(D^{\#}+K_V) \sim 0$.
(2) *Let* N *be a positive integer. Then* $h^0(V,-NK_V) \neq 0$ *if and only if* I *is a divisor of* N.

Proof. (1) Since $K_V \equiv -D^{\#}$, Supp $D^{\#}$ = Supp $D \neq \phi$ and D has negative definite intersection matrix, we have $(K_V^2) \leq -1$. If $I = 1$, then $\bar{V}$ is Gorenstein. Hence $K_V = f^*K_{\bar{V}}$ and $D^{\#} = 0$ because $\bar{V}$ has only rational singularities. This contradicts the assumptions that $\mathrm{Sing}(\bar{V}) \neq \phi$ and Supp $D^{\#}$ = Supp D. Hence $I \geq 2$. Note that $I(D^{\#}+K_V) \equiv 0$. Hence $I(D^{\#}+K_V) \sim 0$ and $IK_{\bar{V}} \sim 0$ by the additional assumption that V is rational. In particular, $h^0(V,-IK_V) \neq 0$.

(2) Suppose that $h^0(V,-NK_V)\neq 0$. Then $-NK_V$ is linearly equivalent to an effective divisor Δ. Note that $ND^{\#}-\Delta \sim N(D^{\#}+K_V) \equiv 0$. Since $D^{\#}$ has negative definite intersection matrix, we have $ND^{\#} = \Delta$. Hence $ND^{\#}$ is an integral divisor. So, $NK_{\bar{V}}$ is a Cartier divisor. Then, N is divisible by I by the definition of I. Q.E.D.

The inequality(**) in the following proposition is very helpful in proving Theorems 5.1 and 6.1.

Proposition 1.5. *Let* (V,D) *be a log Enriques surface and let* c *be the number of connected components of* D. *Let* p *and* q *be integers satisfying* $1 \leq q < p \leq I-1$ $(I := \mathrm{Index}(K_{\bar{V}}))$. *Then we have:*

$$(*) \qquad c \le (D,K_V) < \frac{2c(p-q)^2+(p-p^2)(K_V^2)}{(p-q)(p+q-1)}, \quad \textit{and}$$

$$(**) \qquad (D,K_V) \le c-1-(K_V^2) \qquad \textit{if} \quad I \ge 3.$$

If $I \ge 4$ *then* $c < -3(K_V^2)$. *If* $c = 1$ *then* $I = 2$. (*The case* $c = 1$ *has been treated in* [10; Proposition 2.2]).

Proof. Let p, q be the same as in the statement. We claim first that $h^2(V,(p-q)D+pK_V) = h^0(V,-(p-q)D-(p-1)K_V) = 0$. Indeed, suppose that $h^0(V,-(p-q)D-(p-1)K_V) \ne 0$. Then $h^0(V,-(p-1)K_V) \ne 0$. Hence I is a divisor of $(p-1)$ and $I \le p-1$ by Lemma 1.4. This contradicts the assumption $p \le I-1$.

Next, we claim that $h^0(V,(p-q)D+pK_V) = 0$. Suppose, on the contrary, that $h^0(V,(p-q)D+pK_V) \ne 0$. Then $h^0(V,[pD^\#]+pK_V) = h^0(V,pD+pK_V) \ne 0$ (cf. [8; Lemma 1.10]). Here, $[pD^\#]$ is the maximal effective integral divisor such that $pD^\#-[pD^\#]$ is effective. Let Δ be an effective divisor such that $[pD^\#]+pK_V \sim \Delta$. Then $p(D^\#+K_V) \sim \Delta+(pD^\#-[pD^\#])$. Since $D^\#+K_V \equiv 0$, we have $\Delta = 0$ and $pD^\# = [pD^\#]$ which is an integral divisor. Hence I is a factor of p and $I \le p$. This contradicts the assumption $p \le I-1$.

Write $D = \sum_{i=1}^{n} D_i$ where D_i's are irreducible components of D. Note that D consists of rational trees. Hence we have $\sum_{i<j} (D_i,D_j) = n-c$. Therefore, $2p_a(D)-2 = (D,D+K_V) = \sum_i (D_i^2) + \sum_i (D_i,K_V) + 2\sum_{i<j} (D_i,D_j) = \sum_i (2p_a(D_i)-2) + 2(n-c) = -2c$. Hence, $p_a(D) = 1-c$.

Applying the Riemann-Roch theorem, we obtain:

$$0 \geq -h^1(V,(p-q)D+pK_V) = \frac{1}{2}\{[(p-q)D+pK_V][(p-q)D+(p-1)K_V]\}+1.$$

Hence we have:

$$\begin{aligned} 0 &> [(p-q)D+pK_V][(p-q)D+(p-1)K_V] \\ &= (p-q)^2(D^2)+(2p-1)(p-q)(D,K_V)+(p^2-p)(K_V^2) \\ &= (p-q)^2[-2c-(D,K_V)]+(2p-1)(p-q)(D,K_V)+(p^2-p)(K_V^2) \\ &= -2c(p-q)^2+(p-q)(p+q-1)(D,K_V)+(p^2-p)(K_V^2). \end{aligned}$$

Thence follows the second half of the inequality(*). Setting $p = 2$ and $q = 1$, we obtain the inequality(**).

Since $\mathrm{Supp}\ D^{\#} = \mathrm{Supp}\ D$, each connected component Δ_i of D contains an irreducible component D_i with $(D_i^2) \leq -3$. Hence $(\Delta_i, K_V) \geq (D_i, K_V) = -2-(D_i^2) \geq 1$. Therefore, $(D,K_V) \geq c$.

Suppose $I \geq 4$. Setting $p = 3$ and $q = 2$ in the inequality (*), we obtain $c < (2c-6(K_V^2))/4$, i.e., $c < -3(K_V^2)$.

Consider the case $c=1$. Suppose $I \geq 3$. Then $(D,K_V) \leq -(K_V^2)$ by the inequality(**). Hence $(D-D^{\#},K_V) = (D+K_V,K_V) \leq 0$ because $D^{\#}+K_V \equiv 0$. Since $D-D^{\#} \geq 0$, we have $(D-D^{\#},K_V) = 0$. Hence $D-D^{\#}$, whose support coincides with $\mathrm{Supp}\ D$, consists of (-2)-curves. Hence $D^{\#} = 0$, $\mathrm{Supp}\ D = \mathrm{Supp}\ D^{\#} = \phi$ and $\mathrm{Sing}\ \bar{V} = \phi$. This is a contradiction.

Q.E.D.

§2. Canonical coverings of logarithmic Enriques surfaces

Let $\bar{V}$ (or synonymously (V,D,f)) be a log Enriques surface. Denote by V^0 the smooth part $\bar{V} - (\mathrm{Sing}\ \bar{V}) = V - D$. By the relation $\mathcal{O}(ID^{\#}) \cong \mathcal{O}(-K_V)^{\otimes I}$ ($I := \mathrm{Index}(K_{\bar{V}})$) and a nonzero global section of

$O(ID^{\#})$, we can define a $\mathbb{Z}/I\mathbb{Z}$-covering $\hat{\pi}: \hat{U} \to V$ such that $\hat{U}$ is normal and the restriction π^0 of $\hat{\pi}$ to $U^0 := \hat{\pi}^{-1}(V^0)$ is finite and étale. Actually, $\hat{U}$ is connected and $\hat{\pi}^{-1}(D)$ is contractible to quotient singular points on a normal projective surface $\bar{U}$ (cf. [13; Cor. 5.2]). Let $\pi: \bar{U} \to \bar{V}$ be the finite morphism induced by $\hat{\pi}$. Note that π^0 is induced by the relation $I(-K_{V^0}) \sim 0$ and $\bar{U}$ is the normalization of $\bar{V}$ in the function field $\mathbb{C}(U^0)$. Note that $K_{U^0} \sim \pi^{0*}(K_{V^0}+(I-1)(-K_{V^0})) \sim 2\pi^{0*}K_{V^0} \sim 2K_{U^0}$ and $K_{U^0} \sim 0$. Hence $K_{\bar{U}} \sim 0$ and there are only rational double singular points on $\bar{U}$. Let $g: U \to \bar{U}$ be a minimal resolution of singularities of $\bar{U}$. Then $K_U \sim 0$. Hence U is an abelian surface or a K3-surface. Note that $\bar{U} = U$ when U is an abelian surface.

Definition 2.1. The surface $\bar{U}$ (resp. the map $\pi: \bar{U} \to \bar{V}$) defined above is called *the canonical covering* (resp. *the canonical map*) *of* $\bar{V}$.

Assume $I = pq$ with $p < I$ and $q < I$. Set $\bar{U}_1 = \bar{U}/(\mathbb{Z}/p\mathbb{Z})$ where $\mathbb{Z}/p\mathbb{Z}$ is considered as a subgroup of $\mathbb{Z}/I\mathbb{Z}$ which acts on $\bar{U}$. Then $\bar{V} = \bar{U}/(\mathbb{Z}/I\mathbb{Z}) = \bar{U}_1/(\mathbb{Z}/q\mathbb{Z})$ where the action of $\mathbb{Z}/q\mathbb{Z} \cong (\mathbb{Z}/I\mathbb{Z})/(\mathbb{Z}/p\mathbb{Z})$ on $\bar{U}_1$ is induced by the action of $\mathbb{Z}/I\mathbb{Z}$ on $\bar{U}$. The we have the following lemma whose proof is easy and omitted.

Lemma 2.2. *Let* J *be a positive integer. Then* $JK_{\bar{U}_1}$ *is a Cartier divisor if and only if* p *is a divisor of* J. *Moreover,* $\bar{U}_1$ *is a rational log Enriques surface with* $\mathrm{Index}(K_{\bar{U}_1}) = p$. *If* $\bar{U}$ *is nonsingular then* 2 *is not a divisor of* I.

In view of the above lemma, we assume that I $(= \mathrm{Index}(K_{\bar{V}}))$ is a prime number in order to obtain the information about $\bar{U}$, e.g., the singularity type of $\bar{U}$. Possible divisors of I are given in the following lemma. The idea of the proof is found in [10; p.108].

Lemma 2.3. *Let* $\bar{V}$ *be a log Enriques surface. Then* $\varphi(I) \le b_2(U) - \rho(U) \le 21$, *where* $\varphi(I)$ *is the Euler function and* $b_2(U)$ *is the second Betti number. Hence each prime divisor of* I *is not greater than* 19. *Moreover,* $2 \le I \le 66$. *Finally, if* I *is not a prime number then* $2|I$, $3|I$ *or* $5|I$.

Set $V^0 = \bar{V} - \mathrm{Sing}\,\bar{V}$ and $U^0 = \pi^{-1}(V^0)$. Then $\pi\colon U^0 \to V^0$ is étale. Hence $e(U^0) = Ie(V^0)$ and the following lemma holds.

Lemma 2.4. *Let* $\bar{V}$ *be a log Enriques surface. Let* $I := \mathrm{Index}(K_{\bar{V}})$ *and let* c *and* $\tilde{c}$ *be the numbers of all connected components of* $\mathrm{Sing}\,\bar{V}$ *and* $\pi^{-1}(\mathrm{Sing}\,\bar{V})$, *respectively. We use the notations* $\pi\colon \bar{U} \to \bar{V}$ *and* $g\colon U \to \bar{U}$ *as set at the beginning of* §2. *Then we have:*

$$e(U) + \rho(\bar{U}) - \rho(U) - \tilde{c} = I(\rho(\bar{V}) - c + 2),$$

where $e(U)$ *is the Euler number.*

Suppose further that $\tilde{c} = c$ *(this hypothesis is satisfied if* I *is a prime number) and that* U *is a K3-surface. Then we have:*

$$c \le 21 + \rho(\bar{U}) - \rho(U) \le 21 \quad \textit{and} \quad 1 \le \rho(\bar{V}) - c + 2 \le 23/I.$$

Lemma 2.5. *Let* $\bar{V}$ *be a log Enriques surface. Suppose that* $\bar{U}$ *is nonsingular and* I (= $\mathrm{Index}(K_{\bar{V}})$) *is a prime number. Then for each singular point* y *of* $\bar{V}$, $\pi^{-1}(y)$ *consists of a single smooth point, and* $\hat{O}_{\bar{V},y} \cong \mathbb{C}[[X,Y]]/C_{I,q}$ *with a cyclic subgroup* $C_{I,q}$ *of* $GL(2,\mathbb{C})$, *where* $1 \le q \le I-2$ *and* $\mathrm{g.c.d.}(q,I) = 1$. *The action of* $C_{I,q}$ *is given by:* $gX = \xi X$ *and* $gY = \xi^q Y$, *where* g *is a generator of* $C_{I,q}$ *and* ξ *is a primitive* I*-th root of the unity.*

Proof. This follows from the smoothness of $\bar{U}$ and the assumption that y is not a rational double singular point. Q.E.D.

The proof of Theorem 3.1 is omitted.

Proof of Theorem 4.1. By Lemma 2.2, I is not divisible by 2. We have $\varphi(I) \le b_2(\bar{U}) - \rho(\bar{U}) = 6 - \rho(\bar{U}) \le 5$ by Lemma 2.3. Hence $I = 3$ or 5, and we have $\rho(\bar{U}) \le 2$ if $I = 5$ and $\rho(\bar{U}) \le 4$ if $I = 3$. Then Theorem 4.1 can be proved by using Lemmas 2.4 and 2.5.

Employ the notations as set at the beginning of §2. We are going to prove Theorem 5.1. Let $\bar{V}$ be such a log Enriques surface that the canonical covering $\bar{U}$ is a K3-surface and the index I of $K_{\bar{V}}$ is a prime number. Since $\bar{U}$ is nonsingular, we can apply Lemma 2.5. Let $m_1, \cdots, m_a$ be integers such that the following three conditions are satisfied:

(1) $1 = m_1 < m_2 < \cdots < m_a < I-1$,

(2) the singularity $(\mathbb{C}^2/C_{I,m_i}, 0)$ is not isomorphic to the

singularity $(\mathbb{C}^2/C_{I,m_j},0)$ if $i \neq j$,

(3) for each $1 \le k \le I-2$, the singularity $(\mathbb{C}^2/C_{I,k},0)$ is isomorphic to a singularity $(\mathbb{C}^2/C_{I,m_i},0)$ for some m_i with $m_i \le k$.

$(m_1, m_2, \cdots, m_a)$ is uniquely determined and easily found (cf. [2; Satz 2.11]). Let n_i be the number of all singular points of $\bar{V}$ which have the same singularity as $(\mathbb{C}^2/C_{I,m_i},0)$. By our assumption that $\bar{V}$ has no rational double singular points, we have $\Sigma\, n_i = c$ (= #(Sing $\bar{V}$)). A precise description of $(n_1, n_2, \cdots, n_a)$ is given in the following theorem:

Theorem 5.1. *We use the above notations. Let* $\bar{V}$, *or synonymously* (V,D), *be a log Enriques surface. Suppose that the canonical covering* $\bar{U}$ *is a K3-surface and the index* I *of* $K_{\bar{V}}$ *is a prime number. Then* $\rho(\bar{V}) = c-2+(24-c)/I$, *and one of the following cases occurs, where* $\Sigma\, n_i = c$:

(1) $(c,I) = (3,3)$. *Then* $(m_1, \cdots, m_a) = (1)$, $c = n_1 = 3$ *and* $\rho(V) = 11$. *Hence* D *consists of three isolated (-3)-curves.*

(2) $(c,I) = (4,5)$. *Then* $(m_1, \cdots, m_a) = (1,2)$, $(n_1,n_2) = (1,3)$ *and* $\rho(V) = 13$.

(3) $(c,I) = (3,7)$. *Then* $(m_1, \cdots, m_a) = (1,2,3)$, $(n_1,n_2,n_3) = (0,1,2)$ *and* $\rho(V) = 12$.

(4) $(c,I) = (2,11)$. *Then* $(m_1, \cdots, m_a) = (1,2,3,5,7)$, $(n_1, \cdots, n_5) = (0,0,0,1,1)$ *and* $\rho(V) = 11$.

(5) $(c,I) = (13,11)$. *Then* $(m_1, \cdots, m_a) = (1,2,3,5,7)$, $(n_1, \cdots, n_5) = (3,4,0,0,6)$, $(4,1,1,0,7)$, $(4,2,0,1,6)$ *or* $(5,0,0,2,6)$ *and* $\rho(V) = 47$, 48, 49 *or* 51, *respectively.*

(6) $(c, I) = (7, 17)$. *Then* $(m_1, \cdots, m_a) = (1,2,3,4,5,8,10,11)$ *and* $(n_1, \cdots, n_8) = (1,0,1,1,0,0,2,2)$, $(1,0,0,1,1,0,3,1)$, $(0,2,1,0,0,0,3,1)$, $(0,2,0,0,1,0,4,0)$, $(1,1,1,0,0,0,0,4)$, $(1,1,0,0,1,0,1,3)$, $(1,0,1,0,0,1,4,0)$, $(2,0,0,0,0,2,1,2)$, $(1,2,0,0,0,1,0,3)$, $(1,1,0,2,0,0,0,3)$, $(1,1,0,1,0,1,2,1)$, $(1,0,0,3,0,0,2,1)$, $(0,3,0,1,0,0,1,2)$, $(0,3,0,0,0,1,3,0)$ *or* $(0,2,0,2,0,0,3,0)$.

(7) $(c, I) = (5, 19)$. *Then* $(m_1, \cdots, m_a) = (1,2,3,4,6,7,8,9,14)$, $(n_1, \cdots, n_9) = (1,0,0,0,0,1,0,1,2)$, $(1,0,0,0,2,0,0,0,2)$, $(0,1,1,0,0,1,0,0,2)$ *or* $(0,2,0,0,1,0,0,0,2)$ *and* $\rho(V) = 29$, 29, 24 *or* 26, *respectively.*

In particular, $(D, K_V) = c-1-(K_V^2)$.

Conversely, if $\bar{V}$ *is a log Enriques surface of which the singularity type belongs to one of the above cases, then the canonical covering* $\bar{U}$ *is a K3-surface.*

Finally, for each prime number I *with* $3 \leq I \leq 19$ *and* $I \neq 13$, *there is a log Enriques surface* $\bar{V}$ *such that* I *is the index of* $K_{\bar{V}}$ *and the canonical covering* $\bar{U}$ *of* $\bar{V}$ *is a K3-surface*

Proof. At first, we show the converse part. Let $\bar{V}$ be a log Enriques surface of which the singularity type belongs to one of the cases of Theorem 5.1. Every singular point x of $\bar{V}$ has the same singularity as $(\mathbb{C}^2/G_x, 0)$ with a cyclic subgroup G_x of $GL(2,\mathbb{C})$ of order I. Since the canonical covering $\pi: \bar{U} \to \bar{V}$ has degree I and is an étale cyclic covering outside $\mathrm{Sing}\,\bar{V}$, we see that $\bar{U}$ is nonsingular. Then $\bar{U}$ is a K3-surface in view of Theorem 4.1. Now we shall prove a main part of Theorem 5.1.

By Lemma 2.4, we obtain the first assertion and that $c \leq 21$.

In particular, $I|(24-c)$. By Lemma 2.2, we have $I \geq 3$. Hence $3 \leq I \leq 19$ by Lemma 2.3. We treat only the case $I = 3$. The other cases can be treated similarly. We use also Proposition 1.5.

Assume $I = 3$. Then $(m_1, \cdots, m_a) = (1)$ and D consists of c isolated (-3)-curves D_i $(1 \leq i \leq c)$. Note that $D^{\#} = \frac{1}{3}D$ and $(K_V^2) = (D^{\#})^2 = -c/3$. Hence we have $c/3 + 10 = \rho(V) = \rho(\bar{V}) + \#(D) = c - 2 + (24-c)/3 + c$. This implies $c = 3$ and $\rho(V) = 11$.

The final part can be proved by constructing concretly examples. Q.E.D.

Theorem 6.1 is a consequence of the following Proposition 6.2 and Lemma 6.3. We need some preparation.

Let (V,D) be a log Enriques surface such that I is an odd prime number and $\mathrm{Sing}\ \bar{U} = \sum_{i=1}^{6} m_i A_i$ for some integers $m_i \geq 0$ $(1 \leq i \leq 6)$. The second condition means, by definition, that $\mathrm{Sing}\ \bar{U}$ consists of m_i singularities $\{x_{ij}\}$ $(1 \leq j \leq m_i)$ of Dynkin type A_i for each $1 \leq i \leq 6$. Let m_0 be the number of all singularities $\{y_{0j}\}$ of $\bar{V}$ such that $x_{0j} := \pi^{-1}(y_{0j})$ is a smooth point of $\bar{U}$. Then the singularities $y_{ij} := \pi(x_{ij})$ $(0 \leq i \leq 6)$ exhaust $\mathrm{Sing}\ \bar{V}$ and are isomorphic to $(\mathbb{C}^2/C_{I(i+1),k_i}, 0)$ for some $1 \leq k_i \leq I(i+1)-2$ with $\mathrm{g.c.d.}(I(i+1),k_i) = 1$. We have also $\sum_{i=1}^{6} m_i = \#(\mathrm{Sing}\ \bar{U})$ and $\sum_{i=0}^{6} m_i = c$.

In the case $I = 5$, let $n_1, \cdots, n_{10}$ be respectively the numbers of all singularities $\{y_{\alpha j}\}$ of $\bar{V}$ such that $(\alpha, k_\alpha) =$ (0,1), (0,2), (1,1), (1,3), (2,2), (2,11), (3,3), (3,11), (4,4), (4,9). Then $m_i = n_{2i+1} + n_{2i+2}$ $(0 \leq i \leq 4)$.

In the case $I = 7$, let $n_1, \cdots, n_9$ be the numbers of all singularities $\{y_{\alpha j}\}$ of $\bar{V}$ such that $(\alpha, k_\alpha) = (0,1), (0,2), (0,3), (1,1), (1,3), (1,9), (2,2), (2,5), (2,8)$, respectively. Then $m_i = n_{3i+1}+n_{3i+2}+n_{3i+3}$ $(0 \le i \le 2)$.

In general, if $I=3$ then $\mathrm{Sing}\, \bar{U} = \sum_{i \ge 1} m_i A_i + \sum_{j \ge 4} \delta_j D_j$ for some integers $m_i \ge 0$ and $\delta_j \ge 0$. Set $m_0 := c - \#(\mathrm{Sing}\, \bar{U}) = c - \sum_{i \ge 1} m_i - \sum \delta_j$.

The bounds for c and $\rho(\bar{V})-c$ are given below.

Proposition 6.2. *Let* (V,D) *be a log Enriques surface such that* I *is an odd prime number and* $\mathrm{Sing}\, \bar{U} \neq \phi$. *Then we have* $2 \le c \le \mathrm{Min}\,\{16, 23-I\}$ *and* $c-1 \le \rho(\bar{V}) \le c+4$. *More precisely, we have:*

(1) *Suppose* $I = 3$. *Then* $c \le 15$ *and* $\rho(\bar{V}) \le c+4$. *Moreover, if* $c=15$, *then* $\rho(\bar{V}) = 14$, $\rho(V) = 29$, $\mathrm{Sing}\, \bar{U} = 6A_1$ *and* $(m_0, m_1) = (9,6)$. *If* $\rho(\bar{V}) = c+4$, *then* $\sum_{i=0}^{3} m_i + \delta_4 = c$, $\mathrm{Sing}\, \bar{U} = D_4, A_3, A_2$ *or* A_1, $(m_0, \cdots, m_3, \delta_4) = (1,0,0,0,1), (2,0,0,1,0), (3,0,1,0,0)$ *or* $(4,1,0,0,0)$ *and* $\rho(V) = 11, 12, 13$ *or* 14, *respectively.*

(2) *Suppose* $I = 5$. *Then* $c \le 16$ *and* $\rho(\bar{V}) \le c+2$. *Moreover, if* $c = 16$, *then* $\rho(\bar{V}) = 15$, $\rho(V) = 40$, $\mathrm{Sing}\, \bar{U} = 3A_1$, $(m_0, m_1) = (13,3)$ *and* $(n_1, \cdots, n_4) = (4,9,3,0)$. *If* $\rho(\bar{V}) = c+2$, *then* $\sum_{i=0}^{2} m_i = c$, $\mathrm{Sing}\, \bar{U} = A_2$ *or* A_1, $(m_0, m_1, m_2) = (1,0,1)$ *or* $(2,1,0)$, $(n_1, \cdots, n_6) = (0,1,0,0,0,1)$ *or* $(0,2,0,1,0,0)$ *and* $\rho(V) = 11$ *or* 12, *respectively.*

(3) *Suppose* $I = 7$. *Then* $c \le 15$ *and* $\rho(\bar{V}) \le c+1$. *Moreover, if* $c = 15$, *then* $\rho(\bar{V}) = 14$, $\mathrm{Sing}\, \bar{U} = 2A_1$, $(m_0, m_1) = (13,2)$, $(n_1, \cdots, n_6) = (0,11,2,2,0,0), (1,8,4,2,0,0), (2,5,6,2,0,0)$ *or* $(3,2,8,2,0,0)$ *and*

$\rho(V) = 44, 45, 46$ *or* 47, *respectively. If* $\rho(\bar{V}) = c+1$, *then* $c = 2$, $\rho(V)=11$, $\mathrm{Sing}\ \bar{U} = A_1$, $(m_0,m_1) = (1,1)$ *and* $(n_1,\cdots,n_6) = (0,0,1,0,0,1)$.

(4) *Suppose* $I \geq 11$. *Then* $\rho(\bar{V}) = c-1$.

In particular, we have $24-kI \leq c+\rho(U)-\rho(\bar{U}) = 24-I(\rho(\bar{V})-c+2) \leq 24-I$, *where* $k = 6$ (*resp.* 4, 3 or 1) *if* $I = 3$ (*resp.* $I = 5$, $I = 7$ *or* $I \geq 11$) (*cf. Lemma* 2.4). *Moreover,* $(D,K_V) = c-1-(K_V^2)$ *when the upper bound of* c *or* $\rho(\bar{V})-c$ *in* (1), (2) *and* (3) *is attained.*

Proof. Since $I \geq 3$ we have $c \geq 2$ by Proposition 1.5. We use the result $1 \leq \rho(\bar{V})-c+2 = (24+\rho(\bar{U})-\rho(U)-c)/I \leq 21/I \leq 7$ in Lemma 2.4. In particular, we obtain the assertion(4), and $c-1 \leq \rho(\bar{V}) \leq c+5$ and $c = 24+\rho(\bar{U})-\rho(U)-I(\rho(\bar{V})-c+2) \leq 23-I \leq 20$. Moreover, if $\rho(\bar{V}) = c+5$ then $I = 3$ and $24+\rho(\bar{U})-\rho(U)-c = 21$, whence $c = 2$ and $\mathrm{Sing}\ \bar{U} = A_1$. In proving the assertion(1), we will show that this case does not occurs. Therefore, in order to prove Proposition 6.2, we have only to consider the case where $I = 3$, 5 or 7 and show the assertions (1), (2) and (3). (1), (2) and (3) can be proved similarly as in the proof of Theorem 5.1. We omit the proof. Q.E.D.

Lemma 6.3. *Let* $\bar{V}$ *be a log Enriques surface. Then* $\#(\mathrm{Sing}\ \bar{U}) \leq \mathrm{Min}\ \{10, (24-p)/2\}$ *for every prime divisor* p *of* I.

Proof. It suffices to consider the case where $\mathrm{Sing}\ \bar{U} \neq \phi$. In this case, if $g\colon U \to \bar{U}$ is a minimal desingularization then U is a K3-surface. In view of Lemma 2.2, we may assume that $I = p$ which is a prime number. For each $x \in \mathrm{Sing}\ \bar{U}$, we have $\pi(x) \in \mathrm{Sing}\ \bar{V}$ and $\pi^{-1}\pi(x) = x$. Hence, $\#(\mathrm{Sing}\ \bar{U}) \leq c$. Note that $\rho(U)-\rho(\bar{U})$ is the number of all irreducible components of exceptional divisors of g, which is apparently not less than $\#(\mathrm{Sing}\ \bar{U})$. So, we have $\#(\mathrm{Sing}\ \bar{U}) \leq \mathrm{Min}\ \{c,\ \rho(U)-\rho(\bar{U})\} \leq [c+\rho(U)-\rho(\bar{U})]/2 = [24-I(\rho(\bar{V})-c+2)]/2 \leq (24-I)/2$ by Lemma 2.4. This, together with Theorem 3.1, implies Lemma 6.3. Q.E.D.

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