The Lefschetz Theorem for Foliated Manifolds

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Introduction

These notes are from lectures given at Hokkaido University while I was a Fellow of the Japan Society for the Promotion of Science. It is a pleasure to thank the mathematics department at Hokkaido University for its warm hospitality and the JSPS for its generous support. In particular I want to thank Tatsuo Suwa and Haruo Suzuki for helping with the preparation of these notes and for making my visit to Sapporo so enjoyable.

The aim of the notes is to give a rough outline of the proof of a Lefschetz Theorem for endomorphisms of a differential complex on a compact foliated manifold. The differential operators of the complex are required to differentiate only in leaf directions and the restriction of the complex to any leaf is required to be elliptic. The Theorem we prove is sufficiently general that special cases of it give the Atiyah-Singer G Index Theorem (and so also the Atiyah-Singer Index Theorem), the Atiyah-Bott Lefschetz Theorem and the Connes Index Theorem.

The new results given here are joint work with Connor Lazarov, which was partially supported by a National Science Foundation grant.

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Chapter I. Review of Classical Index and Lefschetz Theorems

§1. The index of an elliptic complex

We denote by M a closed, compact, n dimensional Riemannian manifold.

An elliptic complex (E,d) over M consists of:

a) a finite collection of smooth finite dimensional complex vector bundles $E_0, E_1, \ldots, E_k$.

b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \longrightarrow C^\infty(E_{i+1})$$

where $C^\infty(E_i)$ denotes the smooth sections of E_i .

c) the operators d_i are required to satisfy

$$d_{i+1} \circ d_i = 0 .$$

and a technical condition called ellipticity (see below).

We assume that the d_i are 1^{st} order differential operators. This means that on any coordinate chart $U, x_1, \ldots, x_n$ of M, d_i is given by a matrix of 1^{st} order linear differential operators. To be more specific, suppose

$$E_i|_U \simeq U \times \mathbb{R}^r , \quad E_{i+1}|_U \simeq U \times \mathbb{R}^q .$$

are trivializations of E_i and E_{i+1} over U. With respect to these trivializations, $d_i|_U$ is given by a $q \times r$ matrix $[A_{j\ell}]$ of operators of the form

$$A_{j\ell} = a_0^{j\ell}(x) + a_1^{j\ell}(x)\partial/\partial x_1 + \cdots + a_n^{j\ell}(x)\partial/\partial x_n$$

where each $a_\alpha^{j\ell} \in C^\infty(U)$, the space of smooth complex functions on U. Thus if $s \in C^\infty(E_i)$ and $x \in U$, we may write $s|_U$, using the trivialization above, as

$$s(x) = (s_1(x), \ldots, s_r(x))$$

where each $s_i \in C^\infty(U)$ and we have

$$d_i s(x) = \left(\left(\sum_{\ell=1}^{r} A_{1\ell}s_\ell\right)(x), \ldots, \left(\sum_{\ell=1}^{r} A_{q\ell}s_\ell\right)(x)\right).$$

Ellipticity:

If $x \in M$ and $\xi \in T^*M_x$, the fiber over x of the cotangent bundle T^*M of M, then the symbol of d_i at x, ξ

$$\sigma_{x,\xi}(d_i) : E_{i,x} \longrightarrow E_{i+1,x}$$

is a linear map from the fiber of E_i over x to the fiber of E_{i+1} over x. If d_i is represented in local coordinates by the matrix $[A_{j\ell}(x)]$ as above, then $\sigma_{x,\xi}(d_i)$ is represented by the matrix $[A_{j\ell}(x,\xi)]$ where

$$A_{j\ell}(x,\xi) = a_1^{j\ell}(x)\xi_1 + \cdots + a_n^{j\ell}(x)\xi_n$$

if $\xi = \xi_1 dx_1 + \cdots + \xi_n dx_n$. Note that the term $a_0^{j\ell}(x)$, the zero th order part of $A_{j\ell}(x)$ does not appear in $A_{j\ell}(x,\xi)$.

The complex (E,d) is elliptic provided that for each $x \in M$

and non-zero $\xi \in T^*M_x$, the sequence

$$0 \longrightarrow E_{0,x} \xrightarrow{\sigma_{x,\xi}(d_0)} E_{1,x} \xrightarrow{\sigma_{x,\xi}(d_1)} \cdots \xrightarrow{\sigma_{x,\xi}(d_{k-1})} E_{k,x} \longrightarrow 0$$

is exact.

We may restate this condition as follows: Let $\pi : T^*M \longrightarrow M$ be the projection. Then we have a sequence of fiber bundle maps

$$0 \longrightarrow \pi^*E_0 \xrightarrow{\sigma(d_0)} \pi^*E_1 \xrightarrow{\sigma(d_1)} \cdots \xrightarrow{\sigma(d_{k-1})} \pi^*E_k \longrightarrow 0$$

where for any point $(x,\xi) \in T^*M$,

$$\sigma(d_i) : \pi^*E_{i,(x,\xi)} \quad (= E_{i,x}) \longrightarrow \pi^*E_{i+1,(x,\xi)} \quad (= E_{i+1,x})$$

is just $\sigma_{(x,\xi)}(d_i)$. The complex (E,d) is elliptic provided that this sequence is exact off the zero section. If (E,d) is elliptic, the above sequence defines an element $\sigma(E,d) \in K_c(T^*M)$ called the symbol of (E,d). Here $K_c(T^*M)$ is the K theory of T^*M with compact supports (see [AS]).

Example: The de Rham complex

$T^*_{\mathbb{C}}M$ = complexified cotangent bundle of M

$E_i = \Lambda^i T^*_{\mathbb{C}}M$ the i th exterior power of $T^*_{\mathbb{C}}M$

$C^\infty(E_i)$ = smooth complex i forms on M.

d_i is the usual exterior derivative.

Note: $\sigma_{x,\xi}(d_i) = \Lambda\xi : \Lambda^i T^*_{\mathbb{C}}M_x \longrightarrow \Lambda^{i+1}_{\mathbb{C}} T^*M_x$ where $\Lambda\xi$ is exterior multiplication by ξ .

Some facts about elliptic complexes

1. Define $H^i(E,d) = \ker d_i/\text{image } d_{i-1}$.
Then $\dim H^i(E,d) < \infty$.
This result uses compactness of M strongly.

Define

$$\text{Index}(E,d) = \sum_{i=0}^{k} (-1)^i \dim H^i(E,d).$$

This is a very important invariant. Special cases of (E,d) yield the

i) Euler class $\chi(M)$ of M (de Rham complex).

ii) Signature of M (Signature complex).

iii) $\hat{A}$ genus (Spin complex).

The Atiyah-Singer Index Theorem tells how to compute this invariant from topological information about M and (E,d). In particular the theorem says that Index(E,d) may be computed from the characteristic classes of the tangent bundle TM of M and the characteristic classes of the virtual bundle $\sigma(E,d)$, the symbol of (E,d). See [AS].

2. On each E_i choose an Hermitian inner product denoted $(\ ,\ )_i$. This induces an inner product $\langle\ ,\ \rangle_i$ on $C^\infty(E_i)$ by the formula

$$\langle s_1, s_2\rangle_i = \int_M (s_1(x), s_2(x))_i \, dx.$$

Using $\langle\ ,\ \rangle_i$ we define the adjoints

$$d_i^* : C^\infty(E_i) \longrightarrow C^\infty(E_{i-1})$$

by

$$\langle s_1, d_i^* s_2\rangle_{i-1} = \langle d_{i-1} s_1, s_2\rangle_i$$

where

$$s_1 \in C^\infty(E_{i-1}), \quad s_2 \in C^\infty(E_i) .$$

The Laplacian $\Delta_i : C^\infty(E_i) \longrightarrow C^\infty(E_i)$ is defined by

$$\Delta_i = d_{i-1}d_i^* + d_{i+1}^* d_i .$$

We extend Δ_i to an operator of $L^2(E_i)$, the space of L^2 sections of E_i, as follows. An element $u \in L^2(E_i)$ is in the domain of Δ_i provided that it is the L^2 limit of a sequence $u_n \in C^\infty(E_i)$ such that $\Delta_i u_n$ also converges in $L^2(E_i)$. We then define

$$\Delta_i u = \lim_{n\to\infty} \Delta_i u_n .$$

It is not difficult to show that $\Delta_i u$, if defined, is well defined. [If M were non compact, we would require that each u_n have compact support]. Δ_i is an unbounded operator and its domain is a proper subset of $L^2(E_i)$. Δ_i is a diagonalizable operator. Any eigenvalue λ of Δ_i must be real and non negative since if $\Delta_i s = \lambda s$ for non zero s, we have

$$\begin{aligned}\lambda\cdot\langle s,s\rangle_i &= \langle \Delta_i s,s\rangle_i = \langle (d_{i-1}d_i^* + d_{i+1}^*d_i)s,s\rangle_i \\ &= \langle d_{i-1}d_i^*s,s\rangle_i + \langle d_{i+1}^*d_i s,s\rangle_i \\ &= \langle d_i^*s,d_i^*s\rangle_{i-1} + \langle d_i s,d_i s\rangle_{i+1} \geq 0.\end{aligned}$$

As $\langle s,s\rangle_i > 0$ the result follows. In particular there is a sequence of real numbers

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots , \qquad \lim_{j\to\infty} \lambda_j = \infty$$

such that for each $i = 0, 1, \ldots, k$ there is a sequence of finite dimensional subspaces of $C^\infty(E_i)$, denoted

$$E_i(\lambda_0),\ E_i(\lambda_1),\ E_i(\lambda_2),\ \ldots$$

so that for any $s \in E_i(\lambda_j)$

$$\Delta_i s = \lambda_j s.$$

In addition

$$L^2(E_i) = \bigoplus_{j=0}^{\infty} E_i(\lambda_j).$$

Thus each element in $L^2(E_i)$ can be written as a (possibly infinite) sum of eigenfunctions and we may think of Δ_i as the infinite diagonal matrix

$$\begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & \lambda_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_1 & & \\ & & & & & & \lambda_2 & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_2 \\ & & & & & & & & & \ddots \end{bmatrix}.$$

<u>Other properties of Δ_i</u>

1) $s \in E_i(\lambda_0) = \ker \Delta_i$ if and only if $d_i s = 0$ and $d_i^* s = 0$. The inclusion of $E_i(\lambda_0)$ in $\ker d_i$ induces an isomorphism

$$E_i(\lambda_0) \simeq H^i(E,d).$$

The elements of $E_i(\lambda_0)$ are called harmonic forms. We have

$$\text{Index}(E,d) = \sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_0).$$

2) For each $\lambda_j > 0$, i.e. $j = 1, 2, \ldots$ the sequence

$$0 \longrightarrow E_0(\lambda_j) \xrightarrow{d_0} E_1(\lambda_j) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} E_k(\lambda_j) \longrightarrow 0$$

is exact.

As a corollary, we have immediately

$$\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_j) = 0$$

for all $\lambda_j > 0$.

These results rely on the fact that M is compact. For a general reference for the above facts, see [W].

Example:

$M = S^1$, (E,d) = the de Rham complex

$$E_0 = \Lambda^0 T^*_{\mathbb{C}} S^1 \qquad C^\infty(E_0) = C^\infty(S^1)$$

$$E_1 = \Lambda^1 T^*_{\mathbb{C}} S^1 \qquad C^\infty(E_1) \simeq C^\infty(S^1)$$

where $C^\infty(S^1)$ denotes smooth $\mathbb{C}$ valued functions on S^1.

$$d: \quad C^\infty(E_0) \longrightarrow C^\infty(E_1)$$

$$df = \frac{\partial f}{\partial \theta}\, d\theta$$

$$d^*: \quad C^\infty(E_1) \longrightarrow C^\infty(E_0)$$

$$d^* g\, d\theta = -\frac{\partial g}{\partial \theta}\,.$$

Thus $\Delta_0 : C^\infty(E_0) \longrightarrow C^\infty(E_0)$ is given by

$$\Delta_0 f = -\partial^2 f/\partial\theta^2$$

and $\Delta_1 : C^\infty(E_1) \longrightarrow C^\infty(E_1)$ is given by

$$\Delta_1 g d\theta = -\frac{\partial^2 g}{\partial\theta^2}\, d\theta .$$

The sequence $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ is given by $0, 1, 4, 9, \dots$

i.e. $\lambda_j = j^2$ and for $j > 0$,

$$E_0(\lambda_j) = \mathbb{C}(\cos j\theta, \sin j\theta)$$

is a 2 dimensional complex vector space and $E_0(\lambda_0) = \mathbb{C}$ the constant functions.

For $\lambda_j > 0$,

$E_1(\lambda_j) = \mathbb{C}((\cos j\theta)d\theta, (\sin j\theta)d\theta)$ and

$E_1(\lambda_0) = \mathbb{C}(d\theta)$.

We now return to the general theory of Δ_i. The fact that Δ_i is diagonal implies that for any function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we may define

$$f(\Delta_i) : L^2(E_i) \longrightarrow L^2(E_i)$$

by : for each $s \in E_i(\lambda_j)$ set $f(\Delta_i)s = f(\lambda_j)s$. i.e. the "matrix" of $f(\Delta_i)$ is

$$\begin{bmatrix} f(0) & & & & & & \\ & \ddots & & & & & \\ & & f(0) & & & & \\ & & & f(\lambda_1) & & & \\ & & & & \ddots & & \\ & & & & & f(\lambda_1) & \\ & & & & & & \ddots \end{bmatrix}.$$

In general the domain of $f(\Delta_i)$ is not all of $L^2(E_i)$. Consider say $f(x) = x$. Then $f(\Delta_i) = \Delta_i$. If f is a bounded Borel function on $[0,\infty)$, the Spectral Mapping Theorem says that $f(\Delta_i)$ is a bounded linear operator on $L^2(E_i)$, in particular the domain $f(\Delta_i) = L^2(E_i)$. Note also that if $f(x)$ goes to zero rapidly enough as $x \to \infty$, then the trace of $f(\Delta_i)$, thought of as the usual trace applied to an infinite matrix (i.e. $\operatorname{tr} f(\Delta_i) = \sum f(\lambda_j) \dim E_i(\lambda_j)$) will be a finite number. In this case, we say $f(\Delta_i)$ is of trace class. See [RS].

We are interested in the family of functions

$$f_t(x) = e^{-tx} \qquad t > 0.$$

<u>Theorem (Seeley, [S])</u>

For $t > 0$, $e^{-t\Delta_i}$ is a smoothing operator on $L^2(E_i)$ and so is of trace class.

Let $\pi_i : M \times M \longrightarrow M$ be projection on the i th factor, $i = 1, 2$. To say $e^{-t\Delta_i}$ is a smoothing operator means that there is a smooth section $k_t^i(x,y)$ of the bundle $\operatorname{Hom}(\pi_2^* E_i, \pi_1^* E_i)$ over $M \times M$, so that for all $s \in L^2(E_i)$,

$$(e^{-t\Delta_i}s)(x) = \int_M k_t^i(x,y)s(y)dy.$$

Note that $k_t^i(x,y)$ is a linear map from $E_{i,y}$ to $E_{i,x}$. $k_t^i(x,y)$ is called the Schwartz kernel of $e^{-t\Delta_i}$.

The trace of $e^{-t\Delta_i}$ can be computed in two ways. Namely as the trace of an infinite matrix i.e.

$$tr_1 e^{-t\Delta_i} = \sum_{j=0}^{\infty} e^{-t\lambda_j} \dim E_i(\lambda_j)$$

and as $tr_2 e^{-t\Delta_i} = \int_M [tr\ k_t^i(x,x)]dx.$

Note that $k_t^i(x,x) : E_{i,x} \longrightarrow E_{i,x}$ so it has a well defined trace.

Proposition: $tr_1 e^{-t\Delta_i} = tr_2 e^{-t\Delta_i}$ (we denote this number by $tr\ e^{-t\Delta_i}$).

Proof. It is easy to see that $k_t^i(x,y)$ must be given as follows: For each λ_j choose on orthonormal basis ϕ_j^v, $v = 1,\dots,$ $\dim E_i(\lambda_j)$ of $E_i(\lambda_j)$. Then

$$k_t^i(x,y) = \sum_{j=0}^{\infty} e^{-t\lambda_j}[\sum_v \phi_j^v(x)\phi_j^v(y)].$$

Here $k_t^i(x,y) : E_{i,y} \longrightarrow E_{i,x}$ acts on $w \in E_{i,y}$ by

$$k_t^i(x,y)w = \sum_{j=0}^{\infty} e^{-t\lambda_j}[\sum_v (\phi_j^v(y),w)_i \cdot \phi_j^v(x)]$$

where $(\, , \,)_i$ is the inner product on $E_{i,y}$. The trace of $k_t^i(x,x)$ is then given by $\sum_{j=0}^{\infty} e^{-t\lambda_j} [\sum_{v} (\phi_j^v(x), \phi_j^v(x))_i]$ and the result follows by integrating over M.

Example: In our example on S^1 we may choose our basis of $L^2(E_0)$ to be $\frac{1}{\sqrt{\pi}} \cos(jx)$, $\frac{1}{\sqrt{\pi}} \sin(jx)$ $j \geq 1$ and the constant function $\frac{1}{\sqrt{2\pi}}$. Then

$$k_t^0(x,y) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{j=1}^{\infty} e^{-j^2 t} [\cos(jx)\cos(jy) + \sin(jx)\sin(jy)]$$

and

$$\int_{S^1} \operatorname{tr} k_t^0(x,x)dx = 1 + \sum_{j=1}^{\infty} 2e^{-j^2 t} = \sum_{j=0}^{\infty} e^{-j^2 t} \dim E_0(\lambda_j).$$

We now return to the general situation and we note that since

$$\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_0) = \operatorname{Index}(E,d),$$

$$\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_j) = 0 \quad \text{for} \quad j > 0,$$

and $e^{-t\lambda_0} = 1$ for all t, we have

Theorem: For all $t > 0$,

$$\begin{aligned} \operatorname{Index}(E,d) &= \sum_{j=0}^{\infty} [\sum_{i=0}^{k} (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j)] \\ &= \sum_{i=0}^{k} [\sum_{j=0}^{\infty} (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j)] \\ &= \sum_{i=0}^{k} (-1)^i \operatorname{tr} e^{-t\Delta_i}. \end{aligned}$$

§ 2. The Lefschetz fixed point formula.

Endomorphisms of elliptic complexes

A collection $T = (T_0, \dots, T_k)$ of $\mathbb{C}$ linear maps T_i: $C^\infty(E_i) \longrightarrow C^\infty(E_i)$ is an endomorphism of the complex (E,d) provided

$$T_{i+1} \circ d_i = d_i \circ T_i$$

for all i.

The T_i then induce linear maps

$$T_i^* : H^i(E,d) \longrightarrow H^i(E,d).$$

Since $H^i(E,d)$ is finite dimensional, we may form $\operatorname{tr} T_i^*$ and we define the Lefschetz number $L(T)$ of the endomorphism T by

$$L(T) = \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i^*).$$

We will be interested in the so called "geometric endomorphisms". To define these, let $f: M \longrightarrow M$ be a smooth map and for $i = 0, \dots, k$, suppose that A_i: $f^*E_i \longrightarrow E_i$ is a smooth bundle map. Then for each $x \in M$, we have a linear map

$$A_{i,x}: \ E_{i,f(x)} \longrightarrow E_{i,x}$$

from the fiber of E_i over $f(x)$, which is the fiber of f^*E_i over x, to $E_{i,x}$ the fiber of E_i over x. For any $s \in C^\infty(E_i)$, we define $T_i s \in C^\infty(E_i)$ by

$$(T_i s)(x) = A_{i,x} \cdot s(f(x)).$$

We assume that the A_i are chosen so that the T_i define an endomorphism of (E,d). We call such an endomorphism the geometric endomorphism determined by f and $A = (A_0, \ldots, A_k)$.

Example:

(E,d) = the de Rham complex of M

f an arbitrary map.

A_i = i th exterior power of the adjoint, df^*, of the differential df of f, extended to $T^*_{\mathbb{C}} M$

$$A_{i,x} = \Lambda^i df^*_x : \Lambda^i T^*_{\mathbb{C}} M_{f(x)} \longrightarrow \Lambda^i T^*_{\mathbb{C}} M_x .$$

Then T_i is the familiar $f^*_i: C^\infty(\Lambda^i T^*_{\mathbb{C}} M) \longrightarrow C^\infty(\Lambda^i T^*_{\mathbb{C}} M)$ and $d_i \circ f^*_i = f^*_{i+1} \circ d_i$. In this case, the Lefschetz number is denoted $L(f)$.

Our aim is to relate the Lefschetz number of a geometric endomorphism to invariants defined on the fixed point set of f. To do so we need f to be non-degenerate along its fixed point set in the sense that at each fixed point p, $df_p: TM_p \longrightarrow TM_p$ has no eigenvectors with eigenvalue $+1$ in directions transverse to the fixed point set. Such fixed points are called non-degenerate. Note: $f = Id_M$ satisfies this condition! For the sake of simplicity we will assume that at each fixed point p, $\det(I - df_p) \neq 0$. The fixed points are then isolated and since M is compact they are finite in number. Denote them by $\{p_1, \ldots, p_q\}$.

<u>Atiyah-Bott Lefschetz Theorem ([AB]):</u> Let f, (E,d) be as above and T a geometric endomorphism defined by f and $A = (A_0, \ldots, A_k)$. Then

$$L(T) = \sum_{j=1}^{q} \frac{\sum_{i=0}^{k} (-1)^i tr(A_{i,p_j})}{|\det(I-df_{p_j})|} .$$

Example: (E,d) the de Rham complex, $T = f^*$. At a non-degenerate fixed point p we have

$$\sum_{i=0}^{k} (-1)^i tr(A_{i,p}) = \sum_{i=0}^{k} (-1)^i tr(\Lambda^i df_p^*) = \det(I-df_p).$$

Thus we have for any map f with non-degenerate fixed points $(p_1, \ldots, p_q)$,

$$L(f) = \sum_{j=1}^{q} \text{sign} \det(I-df_{p_j}).$$

Consider the special case where f is an element of the one parameter group determined by a vector field X with simple zeros (meaning $X : M \longrightarrow TM$ is transverse to the zero section). We assume that the fixed points of f are the same as the zeros of X. As X has simple zeros, the fixed points of f are non-degenerate and at a fixed point p, the degree it has as a zero of X, $\deg_X(p)$ is just

$$\deg_X(p) = \text{sign} \det(I-df_p).$$

Now f is homotopic to the identity map of M so $f^*: H^i(M,\mathbb{C}) \longrightarrow H^i(M,\mathbb{C})$ is the identity map. Thus

$$L(f) = \sum(-1)^i \dim_{\mathbb{C}} H^i(M,\mathbb{C})$$

which is the Euler number $\chi(M)$ of M. Thus we have

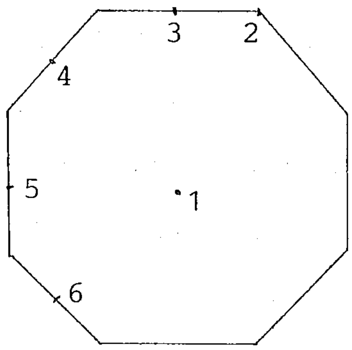
Theorem (Hopf): For a vector field X with simple zeros

$$\chi(M) = \sum_{X(p)=0} \deg_X(p).$$

To obtain the general Hopf theorem (i.e. to drop the requirement that the zeros be simple) we need only observe that any vector field with isolated zeros can be homotoped to one with simple zeros without changing $\sum_{X(p)=0} \deg_X(p)$.

We now give an example where f is not in the flow of any vector field.

Let M be the surface of genus 2 and realize M as an octagon with opposite edges identified. Let f be rotation of M by π. Then f has 6 fixed points, namely

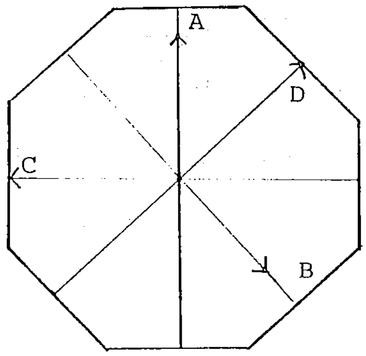


where f is rotation about the point 1.

It is not difficult to calculate that at each fixed point p, $df_p = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, i.e. df_p = rotation by π, so sign $\det(I-df_p) = 1$ and

$$\sum_p \text{sign} \det(I-df_p) = 6.$$

$H^0(M,\mathbb{C}) = H^2(M,\mathbb{C}) = \mathbb{C}$ and f is orientation preserving, so both f_0^* and f_2^* are the identity (because they come from invertible maps on $H^0(M,\mathbb{Z}) = H^2(M,\mathbb{Z}) = \mathbb{Z}$). We may think of $H^1(M;\mathbb{C}) \simeq \mathbb{C}^4$ as being generated by the oriented loops A, B, C, where



Clearly $f_1^* A = -A$ and similarly for B, C, D. Thus with respect to the basis A, B, C, D, $f_1^* = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$

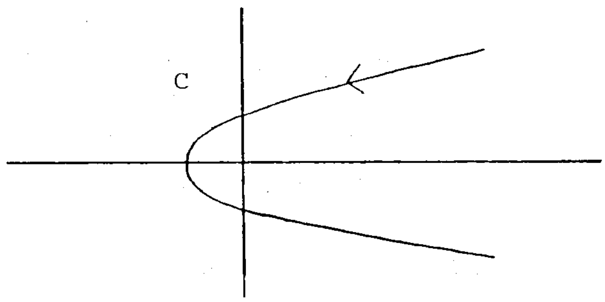
so $\sum_{i=0}^{2} (-1)^i \operatorname{tr} f_i^* = 1 - (-4) + 1 = 6.$

As $L(f) = 6$ and $\chi(M) = -2$, f can not be homotopic to any map in the flow of a vector field as $L(f)$ is a homotopy invariant.

§ 3. Outline of the proof of the Lefschetz theorem

We outline a proof which does not rely in an essential way on the compactness of M. This will allow us to generalize our results to complexes and endomorphisms defined along the leaves of a foliation of a compact manifold even though the leaves may be non compact. A general reference for the material in this section is [RS].

We begin by redefining $e^{-t\Delta_i}$. Let C be the curve in the complex plane



and set

$$e^{-t\Delta_i} = \frac{1}{2\pi i}\int_C \frac{e^{-t\lambda}}{(\lambda I-\Delta_i)}\, d\lambda ,$$

i.e.

$$(e^{-t\Delta_i}s)(x) = \frac{1}{2\pi i}\int_C e^{-t\lambda}[(\lambda I-\Delta_i)^{-1}s](x)d\lambda$$

for $s \in L^2(E_i)$. Now the spectrum of Δ_i, Spec Δ_i, consists of those λ for which $\lambda I - \Delta_i$: domain $\Delta_i \longrightarrow L^2(E_i)$ is <u>not</u> a bijection onto $L^2(E_i)$ with bounded inverse. On any complete manifold, compact or not, Spec Δ_i is a subset of the non negative reals. Thus for all $\lambda \in C$, $(\lambda I-\Delta_i)^{-1}$ is a bounded operator on $L^2(E_i)$, so $e^{-t\Delta_i}$ is defined.

Note that when M is compact, this agrees with our previous definition.

For if M is compact, Spec $\Delta_i = \{0 = \lambda_0 < \lambda_1 < \dots\}$. Let $s \in E_i(\lambda_j)$. Then

$$(e^{-t\Delta_i}s)(x) = \frac{1}{2\pi i}\int_C e^{-t\lambda}[(\lambda I - \Delta_i)^{-1}s](x)d\lambda$$

$$= \frac{1}{2\pi i}\int_C e^{-t\lambda}[(\lambda - \lambda_j)^{-1}\cdot s](x)d\lambda$$

(since $(\lambda I - \Delta_i)s = (\lambda - \lambda_j)s$)

$$= s(x)\cdot\frac{1}{2\pi i}\int_C \frac{e^{-t\lambda}}{\lambda-\lambda_j}\,d\lambda$$

$$= s(x)\cdot e^{-t\lambda_j} \qquad \text{by Cauchy's Theorem.}$$

Some facts about $e^{-t\Delta_i}$

1. $e^{-t\Delta_i}$ is a smoothing operator with Schwartz kernel $k_t^i(x,y)$ a smooth section of $\mathrm{Hom}(\pi_2^*E_i, \pi_1^*E_i)$ (ref [S]).

2. The Spectral Mapping Theorem tells us that $\lim_{t\to\infty} e^{-t\Delta_i} = \pi_{\ker\Delta_i}$ in the strong operator topology. Here $\pi_{\ker\Delta_i}$ is projection onto the kernel of Δ_i. The Schwartz kernel of $\pi_{\ker\Delta_i}$ is always a C^∞ section of $\mathrm{Hom}(\pi_2^*E_i, \pi_1^*E_i)$ for M complete.

3. If M is compact, $\lim_{t\to\infty} \mathrm{tr}\, e^{-t\Delta_i} = \mathrm{tr}\, \pi_{\ker\Delta_i}$.

Recall that an operator A on a Hilbert space H is defined to be positive (written $A \geq 0$) provided that for all $s \in H$,

$$\langle As, s\rangle \geq 0$$

where $\langle\ ,\ \rangle$ is the inner product on H.

Proof of 3. By Spectral mapping theorem,

$$e^{-t\Delta_i} \geq 0, \quad e^{-t_1\Delta_i} \geq e^{-t_2\Delta_i} \quad \text{for } t_1 < t_2 \text{ and}$$

$$\lim e^{-t\Delta_i} = \pi_{\ker\Delta_i}.$$

From [S], we know $e^{-t\Delta_i}$ is smoothing and, since M is compact, of trace class. Because $0 \leq \pi_{\ker\Delta_i} \leq e^{-t\Delta_i}$, $\pi_{\ker\Delta_i}$ is also of trace class. Now trace has the property that if $A_n \leq A_{n+1}$ and A_n trace class then

$$\lim \operatorname{tr} A_n = \operatorname{tr}(\lim A_n).$$

Set $A_n = e^{-\Delta_i} - e^{-n\Delta_i}$ $n = 2, 3, \ldots,$ and apply the above to get

$$\operatorname{tr} e^{-\Delta_i} - \lim \operatorname{tr} e^{-n\Delta_i} = \operatorname{tr} e^{-\Delta_i} - \operatorname{tr} \pi_{\ker\Delta_i}.$$

so $\lim. \operatorname{tr} e^{-n\Delta_i} = \operatorname{tr} \pi_{\ker\Delta_i}.$

Counterexamples to: $\lim\limits_{n\to\infty} A_n = B \Rightarrow \lim\limits_{n\to\infty} \operatorname{tr} A_n = \operatorname{tr} B.$

Let $\mathbb{R}^\infty$ be the Hilbert space of square summable infinite sequences. Let $A_n(x_1, x_2, \ldots) = (0, \ldots, 0, x_n, 0, \ldots)$. Then $A_n \longrightarrow 0$ in the strong operator topology, but $\operatorname{tr} A_n = 1$ for all n.

Even if $A_n \longrightarrow B$ in the norm topology, it is still not necessarily true that $\operatorname{tr} A_n \longrightarrow \operatorname{tr} B$ as the following example shows. Set

$$A_n(x_1, x_2, x_3, \dots) = (\frac{1}{n} x_1, \dots, \frac{1}{n} x_n, 0, \dots)$$

Then $A_n \longrightarrow 0$ in the norm topology, but $\operatorname{tr} A_n = 1$ for all n.

4. Let T_i be as in the Theorem. Then $T_i e^{-t\Delta_i}$ is a smoothing operator with kernel

$$k_t^{T_i}(x,y) = A_{i,x}\, k_t^i(f(x),y).$$

If M is not compact, we need a restriction on f to insure that $k_t^{T_i}(x,y)$ maps $L^2(E_i^L)$ to $L^2(E_i^L)$. We require f to be a diffeomorphism of M of bounded dilation, i.e. there are constants $0 < C_1 < C_2 < \infty$ so that $C_1 \leq |\det df_x| \leq C_2$ for all $x \in M$, and that M have bounded geometry in the sense of Roe [R], (as $|\det df_x|$ depends on the metric on M). $k_t^{T_i}(x,y)$ is always smooth in x and y.

5. Using 3. above we have

$$\lim_{t\to\infty} \operatorname{tr} T_i e^{-t\Delta_i} = \operatorname{tr} T_i \cdot \pi_{\ker\Delta_i}$$

$$= \operatorname{tr}(\pi_{\ker\Delta_i} \cdot T_i \cdot \pi_{\ker\Delta_i}) = \operatorname{tr}(T_i^*).$$

Proof of 5. T_i is a bounded operator. Assume $T_i \geq 0$. Then

$$\operatorname{tr} T_i e^{-t\Delta_i} - \operatorname{tr} T_i \pi_{\ker\Delta_i} = \operatorname{tr}(T_i(e^{-t\Delta_i} - \pi_{\ker\Delta_i}))$$

$$= \operatorname{tr}(T_i^{1/2}(e^{-t\Delta_i} - \pi_{\ker\Delta_i})T_i^{1/2})$$

$T_i^{1/2}$ is self adjoint as we are working on complex Hibert spaces.

Since $e^{-t\Delta_i} - \pi_{\ker\Delta_i}$ 0, $T_i^{1/2}(e^{-t\Delta_i} - \pi_{\ker\Delta_i})T_i^{1/2} \searrow 0$.

So $\operatorname{tr}(T_i^{1/2}(e^{-t\Delta_i} - \pi_{\ker\Delta_i})T_i^{1/2}) \longrightarrow \operatorname{tr} 0 = 0$ as $t \to \infty$

(as in the proof of 3. above).

Thus

$$\operatorname{tr} T_i e^{-t\Delta_i} \longrightarrow \operatorname{tr} T_i \pi_{\ker\Delta_i}.$$

Now for arbitrary T_i we observe that any bounded operator may be written as

$$T_i = g_1 - g_2 + \sqrt{-1}\,(g_3 - g_4) \quad \text{where} \quad g_1, g_2, g_3, g_4 \geq 0.$$

Recall

$$\operatorname{tr} T_i e^{-t\Delta_i} = \int_M \operatorname{tr} k_t^{T_i}(x,x)dx$$

$$= \int_M \operatorname{tr} A_{i,x} k_t^i(f(x),x)dx.$$

6. As $t \to 0$, if $x \neq y$, then $k_t^i(x,y) \longrightarrow 0$ to infinite order and this convergence is uniform in distance(x,y) provided we have global bounds on the coefficients of the Δ_i, T, f and the metrics, and their derivatives to a finite order. See [G]. This

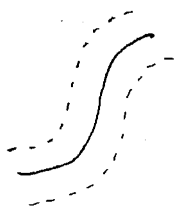
is intentionally vague. What we require is that the metrics and operators be bounded in the sense of Roe[R]. If M is compact this follows. If M is a leaf of a foliation of a compact manifold N and the metrics and operators come from global objects on N, it also follows.

Thus if $f(x) \neq x$, we have

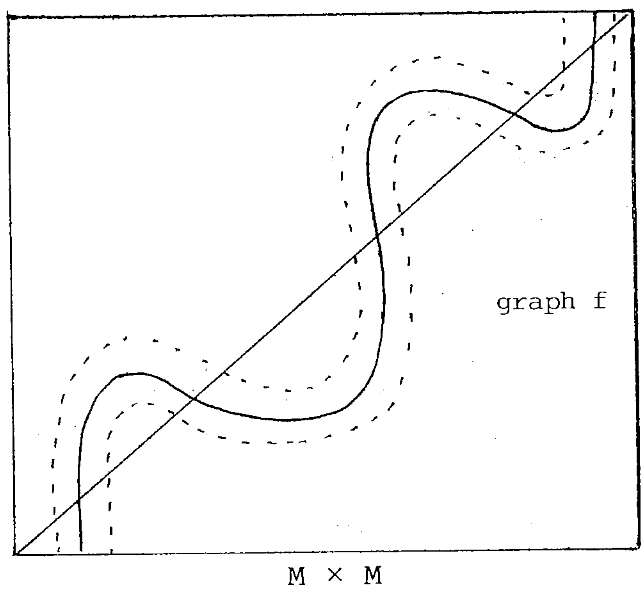
$$\lim_{t\to 0} \operatorname{tr} k_t^{T_i}(x,x) = \lim_{t\to 0} A_{i,x} k_t^i(f(x),x) = 0.$$

Given $\varepsilon > 0$, this convergence is uniform for all x with $\operatorname{distance}(x,f(x)) > \varepsilon$.

PICTURE



ε support of $k_t^{T_i}(x,y)$



Thus $\lim_{t\to 0} \operatorname{tr} k_t^{T_i}(x,x)$ can be computed by integrating only over a neighborhood of the fixed point set of f. This integration can be done using only <u>local</u> information about (E,d), f and T_i. At a fixed point p, this integral equals

$$\frac{\operatorname{tr} A_{i,p}}{|\det(I - df_p)|}.$$

See [AB], [G].

Because of 5 and 6 above, to complete the proof we need only show:

<u>Theorem</u>: $\sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i \circ e^{-t\Delta_i})$ is independent of t.

<u>Proof</u>: Set

$$\Phi(\Delta_i) = e^{-t_1\Delta_i} - e^{-t_2\Delta_i} = \Delta_i \psi_1(\Delta_i)\psi_2(\Delta_i), \text{ where}$$

$$\psi_1(x) = \frac{e^{-\frac{t_1}{2}x} - e^{-\frac{t_2}{2}x}}{x}, \qquad \psi_2(x) = e^{-\frac{t_1}{2}x} + e^{-\frac{t_2}{2}x}.$$

Then $\psi_1(\Delta_i)$, $\psi_2(\Delta_i)$ are smoothing operators as are $d_{i-1}T_{i-1}d_i^*\psi_1(\Delta_i)$, and $\psi_2(\Delta_i)d_{i-1}$. Also note that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ if A and B are smoothing as both are of trace class. Now

$$\sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i e^{-t_1\Delta_i}) - \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i e^{-t_2\Delta_i})$$

$$= \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i \Phi(\Delta_i))$$

$$= \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i \Delta_i \psi_1(\Delta_i)\psi_2(\Delta_i))$$

$$= \sum_{i=1}^{k} (-1)^i \operatorname{tr}(T_i d_{i-1} d_i^* \psi_1(\Delta_i)\psi_2(\Delta_i))$$

$$+ \sum_{i=0}^{k-1} (-1)^i \operatorname{tr}(T_i d_{i+1}^* d_i \psi_1(\Delta_i)\psi_2(\Delta_i).$$

We now show that the first sum is the negative of the second.

$$\sum_{i=1}^{k}(-1)^i tr(T_i d_{i-1} d_i^* \psi_1(\Delta_i)\psi_2(\Delta_i))$$

$$= \sum_{i=1}^{k}(-1)^i tr(d_{i-1} T_{i-1} d_i^* \psi_1(\Delta_i)\psi_2(\Delta_i))$$

$$= \sum_{i=1}^{k}(-1)^i tr(\psi_2(\Delta_i) d_{i-1} T_{i-1} d_i^* \psi_1(\Delta_i))$$

$$= \sum_{i=1}^{k}(-1)^i tr(T_{i-1} d_i^* \psi_1(\Delta_i)\psi_2(\Delta_i) d_{i-1})$$

$$= \sum_{i=1}^{k}(-1)^i tr(T_{i-1} d_i^* d_{i-1} \psi_1(\Delta_{i-1})\psi_2(\Delta_{i-1}))$$

$$= \sum_{i=0}^{k-1}(-1)^{i+1} tr(T_i d_{i+1}^* d_i \psi_1(\Delta_i)\psi_2(\Delta_i))$$

and done.

The second to the last equality follows from

Lemma: Suppose f is differentiable on $\mathbb{C}$. Then $f(\Delta_i)d_{i-1} = d_{i-1}f(\Delta_{i-1})$.

Proof: $d_{i-1}\Delta_{i-1} = \Delta_i d_{i-1}$ implies easily that

$$(\lambda I_i - \Delta_i)^{-1} d_{i-1} = d_{i-1}(\lambda I_{i-1} - \Delta_{i-1})^{-1}$$

where I_i is the identity on $L^2(E_i)$.

Now

$$f(\Delta_i)d_{i-1} = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda I_i - \Delta_i)^{-1} d_{i-1} d\lambda$$

$$= \frac{1}{2\pi i}\int_C f(\lambda) d_{i-1}(\lambda I_{i-1} - \Delta_{i-1})^{-1} d\lambda$$

$$= (d_{i-1})\frac{1}{2\pi i}\int_C f(\lambda)(\lambda I_{i-1}-\Delta_{i-1})^{-1}d\lambda$$

$$= d_{i-1}f(\Delta_{i-1}).$$

Chapter II. The Lefschetz Theorem for Foliated Manifolds

§1. Statement of the theorem

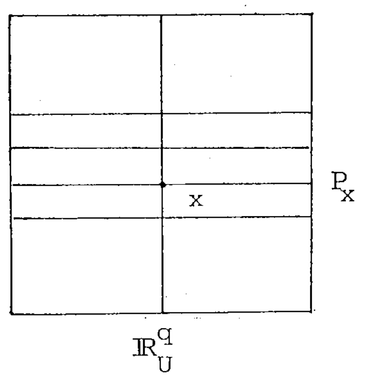
Let M be a compact m dimensional manifold and F a dimension n foliation on M. Then F is an n dimensional subbundle of TM such that for any two sections $X,Y \in C^{\infty}(F)$, $[X,Y] \in C^{\infty}(F)$. The Frobenius Theorem says that for each $x \in M$, there is a neighborhood U of x and a diffeomorphism

$$\phi: \mathbb{R}^n \times \mathbb{R}^q \longrightarrow U \qquad n + q = m$$

so that for all $z \in \mathbb{R}^n \times \mathbb{R}^q$,

$$d\phi(T\mathbb{R}^n_z) = F_{\phi(z)}.$$

Such a (U,ϕ) is called a foliation chart. Given $x \in \mathbb{R}^q$, the submanifold $\phi(\mathbb{R}^n \times \{x\})$ is called a plaque, and is denoted P^U_x. The submanifold $\phi(\{0\} \times \mathbb{R}^q)$ is denoted $\mathbb{R}^q_U$ and is called the transverse submanifold of (U,ϕ). The local picture on M is thus



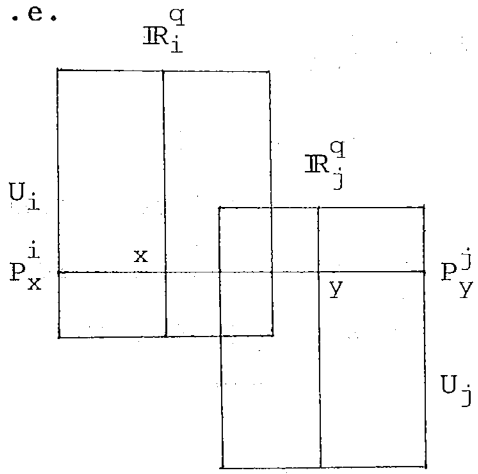
A leaf L of F is a maximal integral (i.e. $TL_x = F_x$ for all $x \in L$) submanifold. Thus $\dim L = n$. The Frobenius Theorem implies that through each point x in M, there passes a unique leaf denoted L_x.

Choose a smooth metric on M. This induces a smooth metric on each L, and L is complete with respect to this metric. Two different metrics on M induce quasi-isometric metrics on the leaf L.

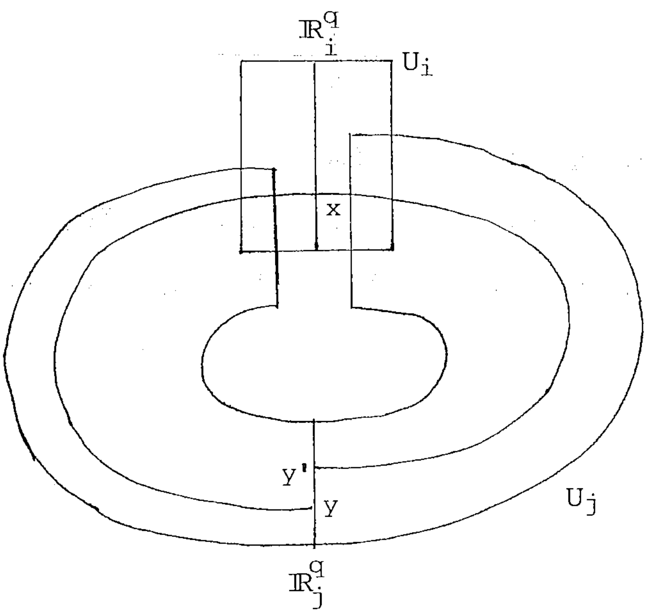
Let $\{(U_i,\phi_i)\}$ be a finite cover of M by foliation charts. If $U_i \cap U_j \neq \phi$ we define a local diffeomorphism f_{ij} from $\mathbb{R}^q_{U_i}$ (hereafter denoted $\mathbb{R}^q_i$) to $\mathbb{R}^q_j$. f_{ij} is defined as follows:

$$f_{ij}(x) = y,$$

if and only if P^i_x, the plaque of x in U_i, has non trivial intersection with P^j_y. i.e.



We may assume that the (U_i,ϕ_i) are chosen so that f_{ij} is always well defined. In particular we avoid choices such as



Transverse measures

A transverse measure ν assigns to any q dimensional submanifold N which is transverse to F a Borel measure denoted ν_N. We say ν is an invariant transverse measure if for all covers by foliation charts $\{(U_i, \phi_i)\}$ we have

$$f_{ij}(\nu_{\mathbb{R}_i^q}) = \nu_{\mathbb{R}_j^q}.$$

Given an invariant transverse measure ν and a function f on M we define

$$\int_M f d\nu$$

as follows:

Let $\{(U_i, \phi_i)\}$ be a finite cover of M by foliation charts. Choose a partition of unity $\{\psi_i\}$ subordinate to the cover. Denote $\nu_{\mathbb{R}_i^q}$ by ν_i and for any plaque P_x^i, denote the volume form obtained from the metric on P_x^i by $dvol_i(x)$. Then set

$$\int_M f d\nu = \sum_i \int_{\mathbb{R}_i^q} [\int_{P_x^i} \psi_i \cdot f dvol_i(x)] d\nu_i .$$

i.e. first integrate $\psi_i f$ over each plaque in U_i to get a function on $\mathbb{R}_i^q$, then integrate this function over $\mathbb{R}_i^q$ with respect to the measure ν_i. It is not difficult to show that $\int_M f d\nu$ is independent of the choice of cover and partition of unity.

Differential complexes on M elliptic along F

A differential complex on M along F consists of :

a) a finite collection of smooth finite dimensional complex vector bundles $E_0, \ldots, E_k$.

b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \longrightarrow C^\infty(E_{i+1})$$

with $d_{i+1} \circ d_i = 0$.

c) each d_i differentiates only in leaf directions.

For the sake of simplicity we assume that each d_i is first order. Then c) means the following. Let (U,ϕ) be a foliation chart with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_q)$, coming from $\mathbb{R}^n \times \mathbb{R}^q$. As U is contractible, $E_i|_U$ and $E_{i+1}|_U$ are trivial, and $d_i|_U$ is given by a $\dim E_{i+1} \times \dim E_i$ matrix of first order linear operators, denoted $[A_{j\ell}]$. To say that d_i differentiates only in leaf directions means that for any $(x,y) \in U$,

$$A_{j\ell}(x,y) = a_0^{j\ell}(x,y) + a_1^{j\ell}(x,y)\partial/\partial x_1 + \cdots + a_n^{j\ell}(x,y)\partial/\partial x_n.$$

We require the $a_k^{j\ell}$ to be smooth complex valued functions on U.

<u>Example</u>: $E_i = \Lambda^i F^* \otimes \mathbb{C}$, thus $E_i|_L$ is the ith exterior power of the complexified cotangent bundle of L for each leaf L.

d_i = exterior derivative along the leaves of F.

This is called the de Rham complex of F.

We now restrict our attention to a single leaf L of F. Note that L need not be compact, although it must be complete.

Denote $E_i|_L$ by E_i^L and by $C_0^\infty(E_i^L)$ the space of smooth sections of E_i^L with compact support. The operator d_i induces one, denoted also by d_i,

$$d_i : C_0^\infty(E_i^L) \longrightarrow C_0^\infty(E_{i+1}^L)$$

and on each leaf L we have the complex

$$0 \longrightarrow C_0^\infty(E_0^L) \xrightarrow{d_0} C_0^\infty(E_1^L) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C_0^\infty(E_k^L) \longrightarrow 0.$$

We say that the complex (E,d) is elliptic along F provided that for each leaf L, the above complex is elliptic. We assume that (E,d) is elliptic along F.

L^2 cohomology of (E,d)

Choose a smooth Hermitian metric on each bundle E_i over M. These induce a metric on each E_i^L and these metrics are also unique up to quasi-isometry. Using the metrics we construct $d_i^* : C_0^\infty(E_{i+1}^L) \longrightarrow C_0^\infty(E_i^L)$ just as we did before. We then construct $\Delta_i^L : C_0^\infty(E_i^L) \longrightarrow C_0^\infty(E_i^L)$ and we extend Δ_i to

$$\Delta_i^L : L^2(E_i^L) \longrightarrow L^2(E_i^L)$$

just as before.

Definition: The ith L^2 cohomology of (E,d) along the leaf L, denoted $H_L^i(E,d)$ is

$$H_L^i(E,d) = \ker \Delta_i^L.$$

The ith L^2 cohomology of (E,d) is denoted $H^i(E,d)$ and it assigns to each leaf L the ith cohomology of (E,d) along L, $H_L^i(E,d)$.

required to be Borel measurable in L. [We are again intentionally vague about this notion. In practice it means that the family (S_L) can be exhibited].

The random operators are added and composed in the obvious way. The natural norm is $\|S\|_\infty = \text{Ess sup}\|S_L\|_{L^2}$ defined to be the smallest $\lambda \geq 0$ such that $\|S_L\|_{L^2} \leq \lambda$ holds almost everywhere. By almost everywhere we mean almost everywhere on the space $R = \cup \mathbb{R}^q_i$, where the $\mathbb{R}^q_i$ come from a finite cover by foliation charts, and the measure on R is the one induced by ν. The random operators on E_i form a von Neumann algebra denoted $W_\nu(F,E_i)$. The measure ν determines a semi finite normal trace on $W_\nu(F,E_i)$. If $S = (S_L) \in W_\nu(F,E_i)$ is an element such that each S_L is given by a smooth kernel $k_L(x,y)$, then

$$\text{tr}_\nu(S) = \int_M \text{tr}\ k_L(x,x)d\nu.$$

For more details on the above constructions we refer to [C], [M-S].

Geometric endomorphisms

Let $f\colon M \longrightarrow M$ be a smooth map and assume that for each leaf L of F, $f(L) \subset L$. For each i, let

$$A_i\colon\ f^*E_i \longrightarrow E_i$$

be a smooth bundle map. We assume that $T_i\colon C^\infty(E_i) \longrightarrow C^\infty(E_i)$ where $(T_is)(x) = A_{i,x}s(f(x))$ satisfy

$$T_id_{i-1} = d_{i-1}T_{i-1}\ .$$

The T_i then induce maps

$$T_i^L : C_0^\infty(E_i^L) \longrightarrow C_0^\infty(E_i^L)$$

satisfying

$$T_i^L d_{i-1} = d_{i-1} T_{i-1}^L .$$

We call such a family $T = (T_0, \ldots, T_k)$ a geometric endomorphism of (E,d) defined by f and $A = (A_0, \ldots, A_k)$. We want the T_i^L to extend to bounded linear maps

$$T_i^L : L^2(E_i^L) \longrightarrow L^2(E_i^L).$$

For this to be true it is necessary to make some restriction on f. The most convenient restriction is to require that $f : M \longrightarrow M$ be a diffeomorphism. This insures that

$$f^* : L^2(E_i^L) \longrightarrow L^2(E_i^L), \quad (f^*s)(x) = s(f(x))$$

is a bounded linear map. As $A_{i,x}$ is globally bounded on M, we then have that $T_i^L : L^2(E_i^L) \longrightarrow L^2(E_i^L)$ is a bounded (independently of L) linear map for all L, so $T_i = (T_i^L)$ is a bounded element of $W_\nu(F,E_i)$.

We shall also need some restricitions on the fixed point set, fix f, of f.

We require:

1) for each L, $\operatorname{fix} f \cap L$ is a union of submanifolds, $\operatorname{fix} f \cap L = \bigcup N_j^L$.

2) for each $x \in \text{fix } f \cap L$, df_x has no eigen vector (in TL_x) with eigen value $+1$ in directions transverse (in L!) to N_j^L $(x \in N_j^L)$.

3) Given $\varepsilon > 0$, denote by $n_\varepsilon(N_j^L)$ the set $\{x \in L \mid \text{distance}_L(x, N_j^L) < \varepsilon\}$.
We assume that there is an $\varepsilon_0 > 0$ such that for all L, j, $n_{\varepsilon_0}(N_j^L)$ is an embedded normal disc bundle in the leaf L and that the $n_{\varepsilon_0}(N_j^L)$ are disjoint. This implies that for each L, the collection of submanifolds $\{N_j^L\}$ is countable. This condition <u>does not</u> follow from 1) and 2).

Note that $f = Id$ satisfies 1), 2), 3).

We now give a counterexample to show that we must assume : $\exists\, \varepsilon > 0$ such that $n_\varepsilon(N_j^L)$ are disjoint.
Counterexample: $M = T^2$ (represented as $\{(x,y) \in \mathbb{R}^2 \mid |x| \leq 2, |y| \leq 2\}$ with opposite sides identified. Let F be the foliation spanned by $\partial/\partial x$. Let $f(x,y)$ be a C^∞ function on T^2 so that

i) $f(x,y) = 1$ if $|x| > 3/2$ or $|y| > 3/2$

ii) $f(x,y) = x^2 - y^2$ if $|x| < 1$ and $|y| < 1$

iii) for fixed $y_0 \neq 0$, $x \longrightarrow f(x,y_0)$ is transverse to 0. i.e. the graph of $f(x,y_0)$ is transverse to the x axis.

Let $g(y)$ be a smooth function on T^2 so that

i) $g(y) = 1$ if $|y| > 3/2$

ii) $g(y) = e^{-1/y^2}$ if $|y| < 1$

iii) $g(y) = 0 \iff y = 0$.

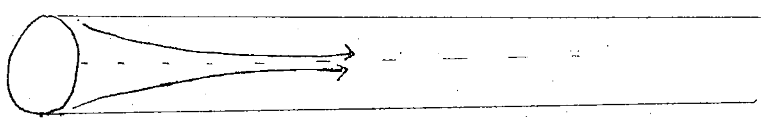
Let X be the vector field on T^2 given by

$$X(x,y) = g(y)f(x,y)\partial/\partial x.$$

Denote by $\phi_1(x,y)$ the time 1 flow of X. ϕ_1 defines a foliation map. On any leaf $L_y = \{(x,y) | -2 \leq x \leq 2\}$, if $y \neq 0$, ϕ_1 has isolated non degenerate fixed points. $\phi_1|_{L_0} = \mathrm{Id}$. Near $(0,0)$ the fixed point set of $\phi_1|_{L_y}$, $y \neq 0$ is $(\pm y, y)$. Thus given $\varepsilon > 0$,

$$n_\varepsilon((-\varepsilon/4, \varepsilon/4)) \cap n_\varepsilon((\varepsilon/4, \varepsilon/4)) \neq \phi.$$

Note: By combining the above example with the suspension of a diffeomorphism of S^1 which is contracting about 0 ($S^1 = [-2,2]/$) we can construct an example of a 2 dim foliation on T^3 which has some leaves of the form $S^1 \times \mathbb{R}$ and with fixed point set of f on $S^1 \times \mathbb{R}$ of the form



i.e. disjoint N_1^L, N_2^L which are asymptotic to each other.

Lefschetz Number of a Geometric Endomorphism

Recall that for each leaf L, π_i^L is the projection of $L^2(E_i^L)$ onto $\ker \Delta_i^L$.

Now set $T^*_{i,L} = \pi^L_i \circ T^L_i \circ \pi^L_i$ and let $T^*_i \in W_\nu(F,E_i)$ be the element $T^*_i = (T^*_{i,L})$. Then T^*_i is an element of tr_ν class and we define the ν Lefschetz number of the geometric endomorphism $T = (T_0, \ldots, T_k)$ to be

$$L_\nu(T) = \sum_{i=0}^{k} (-1)^i tr_\nu(T^*_i).$$

Fixed Point Indices

Let $f : M \longrightarrow M$ be as above with fixed point set $\text{fix } f = \bigcup_{L,j} N^L_j$. Suppose that for each L and j that we are given a function a^L_j defined on N^L_j. We define

$$\int_N a d\nu$$

as follows:

Let (U_i, ϕ_i) be a finite cover of M by foliation charts and $\{\psi_i\}$ a partition of unity subordinate to the cover. Then

$$\int_N a d\nu = \sum_i \int_{\mathbb{R}^q_i} [\sum_{N^L_j \cap P_x \neq \phi} \int_{N^L_j \cap P_x} \psi_i a^L_j dvol(N^L_j)] d\nu_i.$$

Here $dvol(N^L_j)$ is the volume form on N^L_j induced by the metric on M. Note that for any given plaque P_x, only a finite number of N^L_j satisfy $N^L_j \cap P_x \neq \phi$.

The Lefschetz Theorem: Let M, F, f, T, A and (E,d) be as above. To each $N^L_j \subset \text{fix } f$ we may associate a function a^L_j which depends on f, A, the symbols of the Δ_i, and the metrics and their jets to a finite order only on N^L_j so that

$$L_\nu(T) = \int_N a\,d\nu.$$

<u>Some Examples</u>

1) (E,d) = the de Rham, signature, Dolbeault or spin complex of F.

i) If $f = Id$, $T = Id$, then a_j^L is the usual local integrand formula given by the Atiyah-Singer Index Theorem. We thus recover the Connes Index Theorem for foliated manifolds for these operators. If we take the codimension 0 foliation of M which has one leaf (namely M), we recover the Atiyah-Singer Index Theorem for these operators.

ii) In general, i.e. $f \neq Id$, $T = f^*$, a_j^L is the usual local integrand formula given by the Atiyah-Singer G Index Theorem. In particular, for the de Rham complex

$$a_j^L = G(N_j^L) \text{ sign } \det(I - df_n)$$

where $G(N_j^L)$ is the usual local integrand for the Euler class of N_j^L and df_n is the action of $df|_L$ restricted to the normal bundle of N_j^L in L.

If we take the codimension 0 foliation, we recover the Atiyah-Singer G Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

2) If N_j^L consists of a single point p then

$$a_j^L = \frac{\sum_{i=0}^{k} (-1)^i \operatorname{tr} A_{i,p}}{|\det(I - df_{L,p}|}$$

where $df_{L,p}$ is the linear map on TL_p, given by the restriction of df_p.

§2. Computation of an example of $L_\nu(f)$

We now construct a foliated manifold M and a diffeomorphism f of M preserving the foliation which has non zero Lefschetz numbers for all the classical complexes. The manifold M is a flat T^2 bundle over Σ_4, the surface of genus 4. First we give an algebraic construction of M and f, then we show how to realize them geometrically.

Let $\Gamma \subset SL_2\mathbb{R}$ be a subgroup generated by elements $\alpha_j = \theta^{-j}\alpha\theta^j$, $j = 0, \ldots, 7$ where $\alpha = \begin{bmatrix} d & 0 \\ 0 & d^{-1} \end{bmatrix}$ and θ is rotation by $\pi/16$. For proper choice of α, $\Sigma_4 = \Gamma\backslash SL_2\mathbb{R}/SO_2$. We take for a fundamental domain of Σ_4 a regular 16 gon D centered at zero in the Poincaré disc ($\cong SL_2\mathbb{R}/SO_2$). The action of the generators we have chosen for Γ identifies opposite edges of D by translation along the geodesic through the midpoints of the respective edges. The elements α_j satisfy one relation, namely

$$\alpha_0\alpha_1^{-1}\alpha_2\alpha_3^{-1}\alpha_4\alpha_5^{-1}\alpha_6\alpha_7^{-1}\alpha_0^{-1}\alpha_1\alpha_2^{-1}\alpha_3\alpha_4^{-1}\alpha_5\alpha_6^{-1}\alpha_7 = \mathrm{Id}.$$

We note that the SO_2 bundle $\Gamma\backslash SL_2\mathbb{R}$ over Σ_4 is a non trivial double cover of the orthonormal frame bundle $\Gamma\backslash PSL_2\mathbb{R}$ of Σ_4 and so defines a spin structure on Σ_4.

To determine a flat T^2 bundle over Σ_4, we need only define a homomorphism $h : \pi_1\Sigma_4 \longrightarrow \mathrm{Diff}\, T^2$. The bundle $M = (SL_2\mathbb{R}/SO_2)\times_h T^2$ is obtained from $(SL_2\mathbb{R}/SO_2)\times T^2$ by identifying (x,t) with $(\gamma x, h(\gamma)t)$ for all $\gamma \in \pi_1\Sigma_4$. The natural foliation $\tilde{F}$ on $(SL_2\mathbb{R}/SO_2)\times T^2$, whose leaves are $(SL_2\mathbb{R}/SO_2)\times\{t\}$, then descends to a foliation F on M transverse to the fibers of M.

To this end we denote by A the element of $\mathrm{Diff}\, T^2$ determined by the affine map of $\mathbb{R}^2$, $(x,y) \longrightarrow (-x + e, -y + e)$ and by B the element determined by $(x,y) \longrightarrow (-x,-y)$. Here we set $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then define $h : \pi_1\Sigma_4 \longrightarrow \mathrm{Diff}\, T^2$ by

$$h(\alpha_j) = \begin{cases} A & j = 0,\ 3,\ 4,\ 7 \\ B & j = 1,\ 2,\ 5,\ 6\ . \end{cases}$$

Note that $A^2 = B^2 = \mathrm{Id}$, so h preserves the relation among the α_j and defines a homomorphism. Also note that $[AB]^n = \mathrm{Id} \iff n = 0$ since AB is determined by the affine map $(x,y) \longrightarrow (x+e,\ y+e)$. This implies that all the leaves of F are non-compact.

The diffeomorphisms A and B preserve Lebesgue measure dt on T^2. Thus dt determines an invariant transverse measure ν on F. Note that for any fiber T^2 of M, $\nu(T^2) = 1$.

A point in M will be denoted $[gSO_2, t]$ where $g \in SL_2\mathbb{R}$, and $t \in T^2$. Let $r \in SO_2$ be rotation by $\pi/4$. Define $f : M \longrightarrow M$ by

$$f([gSO_2, t]) = [rgSO_2, t].$$

Lemma: f is well defined and preserves F.

Proof: If f is well defined, it obviously preserves F. To see that f is well defined, note that the action of r on the fundamental domain D is to rotate it about its center by $\pi/2$ (not $\pi/4$). One then easily checks that $r\alpha_j = \alpha_{j+4}r$ or $r\alpha_j = \alpha_{j+4}^{-1}r$ for all j, where the addition of subscripts is mod 8. Now for each α_j we have

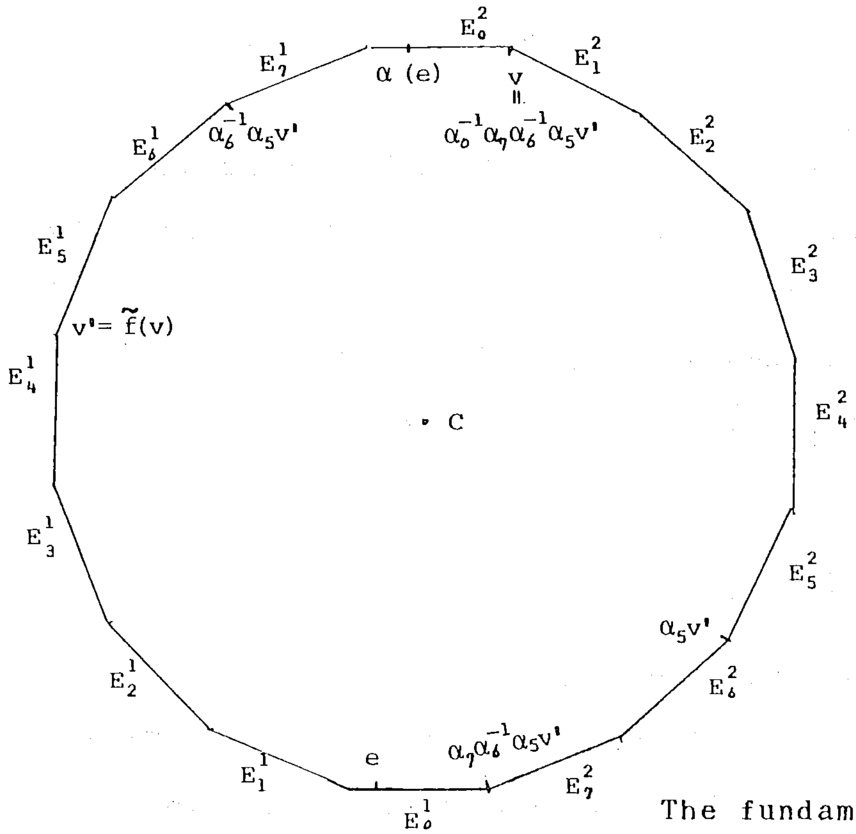
$$f([\alpha_j gSO_2,\ h(\alpha_j)t]) = [r\alpha_j gSO_2,\ h(\alpha_j)t]$$

$$= [\alpha_{j+4}^{\pm 1} rgSO_2,\ h(\alpha_j)t] = [rgSO_2,\ h(\alpha_{j+4}^{\mp 1}\alpha_j)t]$$

$$= [rgSO_2,\ t] = f([gSO_2,\ t])$$

since $h(\alpha_{j+4}^{\mp 1}\alpha_j) = Id$ for all j. As an arbitrary $\gamma \in \Gamma$ can be written as a product of the α_j we have that f is well defined.

In order to determine the fixed point set of f, we now give a geometric construction of M and f. To construct M, we identify points on the boundary of $D \times T^2$ in the follwing way.



The fundamental domain D.

The edge E_j^1 is identified to the edge E_j^2 by the action of the isometry α_j of the Poincaré disc $SL_2\mathbb{R}/SO_2$.
We identify $E_j^1 \times T^2$ to $E_j^2 \times T^2$ by $(e,t) \sim (\alpha_j(e),\ h(\alpha_j)t)$.
Then M equals $D \times T^2/\sim$. $D \times T^2$ is foliated by leaves of the form $D \times \{t\}$ and the above identifications respect this

foliation, so it induces a foliation on M and this foliation is just F. The map $\tilde{f} : D \times T^2 \longrightarrow D \times T^2$ given by rotation by $\pi/2$ on the D factor and the identity on the T factor induces f on M.

We write (d,t) for a point in $D \times T^2$ and $[d,t]$ for the point it determines in M. It is clear that all the points $[c,t]$, $t \in T^2$ are fixed by f and that the action of df on $TL_{[c,t]}$ is rotation by $\pi/2$.

The only other possible fixed points are the points $[v,t]$, $t \in T^2$. These are in fact fixed since

$$\tilde{f}(v,t) = (v',t) = (v,\ h(\alpha_0^{-1}\alpha_7\alpha_6^{-1}\alpha_5)t) = (v,t).$$

It is also easy to see that the action of df on $TL_{[v,t]}$ is rotation by $\pi/2$.

The metric we put on M is the one induced from $D \times T^2$ by the Poincaré metric on D and the natural metric on T^2. The orientation we put on F is the one it receives from the natural orientation on D.

The local fixed point indices and Lefschetz numbers $L_\nu(f)$ for $T = df$ for the classical complexes are given below.

1. de Rham Complex

The local index at an isolated nondegenerate fixed point p is sign of $\det(I-df_p)$ ([AB], II, § 3). As df_p is rotation by $\pi/2$ $\det(I-df_p) = 2$ for all fixed points and we have

$$L_\nu(f) = \int_N 1\, d\nu = \int_{T^2} 1\, dt + \int_{T^2} 1\, dt = 2.$$

Now $L_\nu(f) = \sum_{i=0}^{2} (-1)^i tr_\nu(f_i^*)$ where $f_{i,L}^* : H_L^i(L;\mathbb{R}) \longrightarrow H_L^i(L;\mathbb{R})$ where L is a leaf of F. As L is a non compact complete surface, we have $H_L^0(L;\mathbb{R}) = H_L^2(L;\mathbb{R}) = 0$. Thus $tr_\nu(f_0^*) = tr_\nu(f_2^*) = 0$ and $tr_\nu(f_1^*) = -2$. This implies that for ν almost all L, $H_L^1(L;\mathbb{R}) \neq 0$, i.e. for almost all L, there are non zero harmonic L^2 one forms on L.

2. Signature Complex

For each leaf L, $f|_L$ is an isometry so we may consider the action of f on the signature complex of F.

At each fixed point p, $df_p : TL_p \longrightarrow TL_p$ is an isometry of the oriented 2 dim space TL_p. Thus df_p is given by a rotation of TL_p through a well defined (because of the orientation) angle θ_p. The fixed point index of f at p is then

$$-i \cot(\theta_p/2)$$

(see [AB], II, Theorem 6.27). Thus in our case $\theta_p = \pi/2$ so the fixed point index at each fixed point is $-i$ and the Lefschetz number for f is

$$L_\nu(f) = -2i.$$

3. Dolbeault Complex

The surface Σ_4 is a complex manifold and this complex structure lifts to a complex structure on each leaf L of F.

The map f covers a holomorphic map on Σ_4, so f restricted to any leaf is holomorphic. Denote by $\Lambda^{p,q}$ the bundle on M over F,

$$\Lambda^{p,q} = \Lambda^p T^* F \otimes_{\mathbb{C}} \Lambda^q \bar{T}^* F$$

where T^*F and $\bar{T}^*F$ are respectively the holomorphic and antiholomorphic cotangent bundles of F. A section of $\Lambda^{p,q}$ is then a form of type p,q on each leaf L. Since f is a holomorphic map of each leaf f^* induces an endomorphism of the p Dolbeault complex, $p = 0, 1$

$$0 \longrightarrow C^\infty(\Lambda^{p,0}) \xrightarrow{\bar{\partial}} C^\infty(\Lambda^{p,1}) \longrightarrow 0.$$

We denote the Lefschetz number in this case by $L_\nu(f^p)$. By equation (4.8) of [AB], II, the local index at a nondegenerate isolated fixed point p is given by

$$\frac{\mathrm{tr}_{\mathbb{C}} \Lambda^p(df_p)}{\det_{\mathbb{C}}(I - df_p)} .$$

Here $df_p : TL_p \longrightarrow TL_p$ maps the real tangent space of TL_p to itself. However, TL_p also has a complex structure and df_p preserves that structure. Thus we may think of df_p as a complex linear map of the complex space TL_p. The df_p in the above formula is to be understood in this way.

Now $df_p : TL_p \longrightarrow TL_p$ in our example, considered as a complex linear map, is just multiplication by i. Thus for $p = 0$, the local indices are $\frac{1}{1-i}$ and $L_\nu(f^0) = \frac{2}{1-i} = 1+i$, while for $p = 1$ the local indices are $i/(1-i)$ and $L_\nu(f^1) = \frac{2i}{1-i} = i-1$.

4. Spin Complex

The surface Σ_4 is a spin manifold so each leaf L is a spin manifold. As we have noted above a spin structure on Σ_4 is given by

$$\Gamma\backslash SL_2\mathbb{R} \longrightarrow \Gamma\backslash PSL_2\mathbb{R}$$

where $\Gamma\backslash PSL_2\mathbb{R}$ is orthonormal frame bundle of Σ_4. Thus we may exhibit a spin structure on F by

$$SL_2\mathbb{R}\times_h T^2 \longrightarrow PSL_2\mathbb{R}\times_h T^2$$

where $PSL_2\mathbb{R}\times_h T^2$ is the orthonormal frame bundle of the foliation F. In this representation $f : M \longrightarrow M$ is given by

$$f([gSO_2, t]) = [rgSO_2, t]$$

and $df : PSL_2\mathbb{R}\times_h T^2 \longrightarrow PSL_2\mathbb{R}\times_h T^2$ is given by

$$df([\pm g, t]) = ([\pm rg, t])$$

and we indicate the class of g in $PSL_2\mathbb{R} = SL_2\mathbb{R}/\pm I$ by $\pm g$. It is clear that df has two liftings $\widetilde{df}$ to $SL_2\mathbb{R}\times_h T^2$, namely

$$\widetilde{df}([g, t]) = [rg, t]$$

and

$$\widetilde{df}_\pi([g, t]) = [r_\pi g, t]$$

where r_π is rotation by $\frac{5\pi}{4}$.

The local fixed point index for f for the spin complex at an isolated non degenerate fixed point p is given by ([AB], II, Theorem 8.35)

$$\pm \frac{i}{2} \operatorname{cosec}(\theta_p/2)$$

where θ_p is the angle of rotation (i.e. $\frac{\pi}{2}$) of $\underline{df_p}$ on TL_p. To resolve the ambiguity for the indices we must choose a lifting of df to a spin covering (see [G] Theorem 4.5.2 in this regard). For the lifting $\widetilde{df}$ the local index is $-i/\sqrt{2}$ and the Lefschetz number $L_\nu(f, \widetilde{df})$ is $-2i/\sqrt{2}$. For the lifting $\widetilde{df}_\pi$ the local index is $i/\sqrt{2}$ and the Lefschetz number $L_\nu(f, \widetilde{df}_\pi)$ is $2i/\sqrt{2}$.

§3. Outline of the proof of the Lefschetz theorem

We first collect some facts about the Schwartz kernel of $e^{-t\Delta_i}$.

We assume that at each point $x \in M$, we may choose local coordinates so that with respect to these coordinates, the symbol of Δ_i, $\sigma(\Delta_i)$, which has an expression as

$$\sigma(\Delta_i) = a^2(x,\xi) + a^1(x,\xi) + a^0(x,\xi)$$

where each a^i is a square matrix and is of order i in ξ, satisfies

$$a^2(x,\xi) = \sum_{i,j=1}^{n} g_{ij}\xi_i\xi_j \cdot I$$

where (g_{ij}) is the induced metric on T^*F. [Each entry of a^1 is of the form $\sqrt{-1}\sum_{k=1}^{n} b_k\xi_k$ where $b_k(x)$ is a C^∞ function and each entry of a^0 is a C^∞ function]. The classical operators all satisfy this condition.

Let $k^i_{t,L}(x,y)$ be the Schwartz kernel of $e^{-t\Delta_i}$ on L. Then there is an asymptotic expansion as $t \longrightarrow 0$ of the form

$$k^i_{t,L}(x,y) \sim K^i_{t,L}(x,y) =$$

$$\sum_{k=0}^{r} \sum_{\substack{j=0 \\ j+k \text{ even}}}^{3k} t^{(k-n)/2} \int e^{(i(x-y)\cdot\xi/\sqrt{t})} b^i_{k,j}(x,\xi) \frac{e^{-|\xi|^2}}{[\frac{j+k+2}{2}]!} d\xi$$

(See [G]).

Here $|\xi|^2 = \Sigma g_{ij}\xi_i\xi_j$ and r is sufficiently large. Each

$b^i_{k,j}(x,\xi)$ is homogeneous of degree j in ξ and if we write

$$b^i_{k,j}(x,\xi) = \sum_{|\alpha|=j} b^i_{k,j,\alpha}(x)\xi^\alpha$$

then each $b^i_{k,j,\alpha}(x)$ is given as a canonical polynomial in the a^i and their derivatives to a finite order. As these are all globally bounded on M, this implies that for fixed t, $K^i_{t,L}(x,y)$ is bounded on M independently of x,y and L.

To say $k^i_{t,L}(x,y) \sim K^i_{t,L}(x,y)$ means that given j, there is c_j so that for sufficiently large r,

$$|k^i_{t,L}(x,y) - K^i_{t,L}(x,y)| < c_j t^j.$$

c_j depends continuously on a^2, a^1, a^0 and so since these are bounded on M, we have that the above inequality is independent of x,y and L. Thus for small t, we have that $k^i_{t,L}(x,y)$ is uniformly bounded on M and so $tr_\nu(e^{-t\Delta_i}) < \infty$.

We appeal to [C] for the fact that $k^i_{t,L}(x,y)$ is transversely measurable.

References for the above are [ABP], [G] and [T].

Now as $e^{-(t+s)\Delta_i} = e^{-t\Delta_i}e^{-s\Delta_i}$ we have that for all $t > 0$, $e^{-t\Delta_i}$ is tr_ν class. Alternately we have by the Spectral Mapping Theorem that on each L, $0 \leq e^{-t\Delta_i} \leq e^{-s\Delta_i}$ if $t \geq s$. This implies that for all $x \in L$, $0 \leq tr\, k^i_{t,L}(x,x) \leq tr\, k^i_{s,L}(x,x)$. Thus we have

$$0 \leq \int_M tr\, k^i_{t,L}(x,x)d\nu \leq \int_M tr\, k^i_{s,L}(x,x)d\nu$$

and $e^{-t\Delta_i}$ is tr_ν class for all $t > 0$.

If $|x-y| > \varepsilon$, $K^i_{t,L}(x,y)$ can be integrated by parts to prove

$$|K^i_{t,L}(x,y)| \leq c(\varepsilon,j)t^j \qquad \text{as} \quad t \to 0.$$

$c(\varepsilon,j)$ is independent of x,y and L. As $K^i_{t,L}(x,y)$ vanishes to infinite order off the diagonal, so does $k^i_{t,L}(x,y)$.

__Proposition__: $\sum_{i=0}^{k} (-1)^i tr_\nu(T_i e^{-t\Delta_i})$ is independent of t.

As T_i is globally bounded on M and $e^{-t\Delta_i}$ is tr_ν class, $T_i e^{-t\Delta_i}$ is tr_ν class. The proof of this proposition is essentially the same as in the compact case and is omitted.

__Proposition__: $\lim_{t\to\infty} tr_\nu(T_i e^{-t\Delta_i}) = tr_\nu(\pi_i T_i \pi_i) \quad (= tr_\nu T_i^*)$.

__Proof__: As tr_ν is a normal trace

(i.e. $f_n \nearrow f \Rightarrow tr_\nu(f_n) \longrightarrow tr_\nu(f)$)

we may apply the argument used in the classical case to conclude that

$$\lim_{t\to\infty} tr_\nu(T_i e^{-t\Delta_i}) = tr_\nu(T_i \pi_i).$$

But π_i is tr_ν class (as $0 \leq \pi_i \leq e^{-t\Delta_i}$ for all $t > 0$) and $\pi_i = \pi_i^2$. Thus

$$tr_\nu(T_i \pi_i) = tr_\nu(T_i \pi_i^2) = tr_\nu(\pi_i T_i \pi_i)$$

and done.

Alternate proof: For this we assume d is of Dirac type. All the classical operators are of Dirac type. From [R], II, Lemma 1.2, we have that

i) for $t > t_0 > 0$, there is $c > 0$ so that for all $x,y \in L$, $|k^i_{t,L}(x,y)| \leq c$.

ii) On any compact subset of $L \times L$, $k^i_{t,L}(x,y)$ converges uniformly to $k_L^{\pi_i}(x,y)$, the kernel of the projection onto $\ker \Delta_i$.

Since T_i is uniformly bounded on M, i) and ii) are true with $k^i_{t,L}(x,y)$ replaced by $A_{i,x}k^i_{t,L}(f(x),y)$ (= Schwartz kernel of $T_i e^{-t\Delta_i}$) and $k_L^{\pi_i}(x,y)$ replaced by $A_{i,x}k_L^{\pi_i}(x,y)$, (= Schwartz kernel of $T_i\pi_i$). Thus for any plaque P we have

$$\lim_{t\to\infty} \int_P tr(A_{i,x}k^i_{t,L}(f(x),x))dx$$

$$= \int_P tr(A_{i,x}k_L^{\pi_i}(f(x),x))dx.$$

As $\int_P tr(A_{i,x}k^i_{t,L}(f(x),x))dx$ is bounded by a constant for all plaques P of a fixed finite open cover by foliation charts, we may apply the Bounded Convergence Theorem to the measure space $R = \cup R^q_i$, $\nu = \cup \nu_i$ to obtain the result that $\lim_{t\to\infty} tr_\nu(T_i e^{-t\Delta_i}) = tr_\nu(T_i\pi_i)$. But as before π_i is tr_ν class and $\pi_i^2 = \pi_i$ so $tr_\nu(T_i\pi_i) = tr_\nu(\pi_i T_i \pi_i) = tr_\nu(T_i^*)$.

To complete the proof of the Lefschetz Theorem, we now compute

$$\lim_{t\to 0} \operatorname{tr}_\nu(T_i e^{-t\Delta_i}).$$

As T_i is uniformly bounded on M, the Schwartz kernel $k_{t,L}^{T_i}(x,y)$ of $T_i e^{-t\Delta_i}$ which is $A_{i,x}k_{t,L}^{i}(f(x),y)$ is asymptotic as $t \to 0$ (uniformly on M!) to

$$A_{i,x}K_{t,L}^{i}(f(x),y) =$$

$$\sum_{k,j} t^{(k-n)/2} \int e^{(i(f(x)-y)\cdot\xi/\sqrt{t}} A_{i,x}b_{k,j}^{i}(f(x),y)\frac{e^{-|\xi|^2}}{[\frac{j+k+2}{2}]!}\, d\xi$$

Set $c_{k,j}^{i} = A_{i,x}b_{k,j}^{i}(f(x),y)$. Then $c_{k,j}^{i}$ is given by a canonical polynomial in A, f, the a^i, the metrics, and their derivatives to a finite order.

Let $B_i(t)$ be the element of $W_\nu(F, E_i)$ whose Schwartz kernel on each leaf L, denoted $K_{t,L}^{T_i}(x,y)$, is

$$K_{t,L}^{T_i}(x,y) = A_{i,x}K_{t,L}^{i}(f(x),y).$$

Set $B(t) = \sum_{i=0}^{k} (-1)^i B_i(t)$ and denote its Schwartz kernel by $K_{t,L}(x,y)$. As $t \to 0$, $\operatorname{tr}_\nu(T_i e^{-t\Delta_i})$ is asymptotic to $\operatorname{tr}_\nu(B_i(t))$ so $\sum_{i=0}^{k} (-1)^i \operatorname{tr}_\nu(T_i e^{-t\Delta_i})$, which is independent of t, is asymptotic to $\sum_{i=0}^{k} (-1)^i \operatorname{tr}_\nu(B_i(t)) = \operatorname{tr}_\nu(B(t))$.

Thus to compute $\Sigma(-1)^i tr_\nu(T_i e^{-t\Delta_i})$ we need only compute $tr_\nu(B(t))$ and since this is independedt of t, only the zero th order term in t can be non zero (and equals $\Sigma(-1)^i tr_\nu(T_i e^{-t\Delta_i})$).

Recall that $\exists\, \varepsilon_0 > 0$ such that the embedded normal bundles $n_{\varepsilon_0}(N_j^L)$ are disjoint. If $\text{distance}_L(f(x),x) > \varepsilon_1 > 0$, then $tr\, K_{t,L}^{T_i}(x,x) \longrightarrow 0$ to infinite order in t uniformly on M. Thus the same is true of $tr\, K_{t,L}(x,x)$, so we may assume $tr(K_{t,L}(x,x)) = 0$ on $M - \bigcup_{L,j} n_\varepsilon(N_j^L)$ where ε [to be specified later] satisfies $0 < \varepsilon < \varepsilon_0$.

We now wish to claim that given $\varepsilon > 0$, there is $\varepsilon_1 > 0$ so that for all $x \in M - \bigcup_{L,j} n_\varepsilon(N_j^L)$, $\text{distance}_L(x,f(x)) > \varepsilon_1$. In general this is false. However if we assume that the fixed point set of f in any coordinate chart U looks like (fix $f \cap$ plaque) $\times\mathbb{R}_U^q$, it is true. We shall assume this.

Denote by $⨍_{n_\varepsilon} tr\, B(t)$, the collection of functions $⨍_{n_\varepsilon(N_j^L)} tr(K_{t,L}(x,x))$, i.e. to each N_j^L we associate the function given by the integral over the fiber of $n_\varepsilon(N_j^L)$ of the function $tr(K_{t,L}(x,x))$. Then $⨍_{n_\varepsilon} tr\, B(t)$ assigns to each N_j^L a function.

<u>Proposition</u>:

$$tr_\nu(B(t)) = \int_N [⨍_{n_\varepsilon} tr\, B(t)]d\nu.$$

<u>Proof</u>: Let $\{U_i\}$ be a finite measurable <u>partition</u> of M, (i.e. $\bigcup_i U_i = M$, $U_i \cap U_j = \phi$, $i \neq j$). Choose the U_i so that

1) there is an open connected set W_i with

$$W_i \subset U_i \subset \bar{W}_i$$

2) $\bar{U}_i \subset V_i$ where V_i is a foliation chart for F.

Given $\varepsilon > 0$, set

$$U_i^{\varepsilon} = \{x \in M \mid \text{distance}_L(x, U_i \cap L) \leq \varepsilon \quad \text{for all leaves } L,\ x \in L\}.$$

(thus U_i^{ε} is a foliation ε neighborhood of U_i).
Choose ε so that $0 < \varepsilon < \varepsilon_0$ and $U_i^{\varepsilon} \subset V_i$ for all i.

Now for each i, choose an open cover $V_i, V_{i,1}, \ldots, V_{i,r_i}$ of M by foliation charts and let $\psi_i, \psi_{i,1}, \ldots, \psi_{i,r_i}$ be a partation of unity subordinate to the cover with

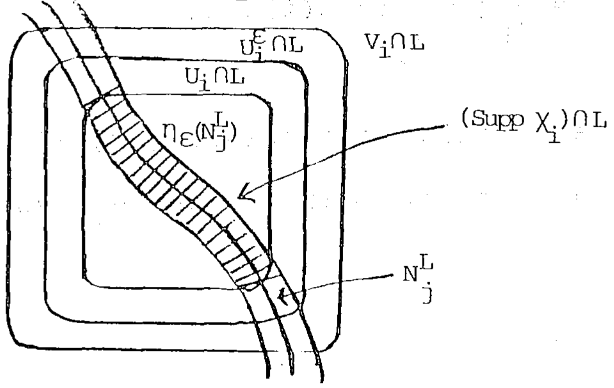
$$\psi_i \big|_{U_i^{\varepsilon}} \equiv 1.$$

Denote by $\pi : n_{\varepsilon}(N_j^L) \longrightarrow N_j^L$ the projection and for each i, let χ_{U_i} be the characteristic function of U_i. Let χ_i be the measurable function on M,

$$\chi_i(x) = \begin{cases} \chi_{U_i}(\pi(x)) & x \in \bigcup_{L,j} n_{\varepsilon}(N_j^L) \\ 0 & \text{otherwis e.} \end{cases}$$

Thus χ_i is the characteristic function of $n_{\varepsilon}\big|_{(\cup N_j^L) \cap U_i}$.

[See picture below, which gives the situation on a single leaf L].



Now $tr_\nu(B(t)) = \sum_i tr_\nu(\chi_i \cdot B(t))$. If we compute $tr_\nu(\chi_i \cdot B(t))$ using the cover V_i, $V_{i,1}$, ... and the partition of unity ψ_i, $\psi_{i,1}$, ... , since $\psi_i \equiv 1$ on the support of $\chi_i \cdot B(t)$, we have

$$tr_\nu(\chi_i \cdot B(t)) = \int_{\mathbb{R}^q_i} [\int_P \chi_i tr(K_{t,L}(x,x))dx]d\nu_i .$$

In each plaque P, the situation is as pictured above (we assume without loss of generality that there is only one N^L_j with $N^L_j \cap P \neq \phi$). Then

$$\int_P \chi_i tr(K_{t,L}(x,x))dx = \int_{n_\varepsilon|_{N^L_j \cap P \cap U_i}} tr(K_{t,L}(x,x))dx.$$

But for any bundle n over base N,

$$\int_n f = \int_N [\fint_n f].$$

Thus $\int_P \chi_i tr(K_{t,L}(x,x))dx = \int_{N^L_j \cap P \cap U_i} [\fint_{n_\varepsilon(N^L_j)} tr(K_{t,L}(x,x))]dn$

$$= \int_{N^L_j \cap P} \chi_i [\fint_{n_\varepsilon(N^L_j)} tr(K_{t,L}(x,x))]dn$$

and we have

$$tr_\nu(\chi_i B(t)) = \int_{\mathbb{R}^q_i} [\int_{(\cup N^L_j)\cap P} \chi_i \{\fint_{n_\varepsilon} tr(K_{t,L}(x,x))\}dn]d\nu_i$$

$$= \int_N [\chi_i \fint_{n_\varepsilon} tr(B(t))]d\nu$$

where the last equality follows immediately provided we compute using V_i, $V_{i,1}$, ... and ψ_i, $\psi_{i,1}$, ...

Summing over i gives the proposition.

Now to compute $tr_\nu(B(t))$ or rather $\int_{n_\varepsilon} tr(B(t))$.

Case 1. $L \subset \text{fix } f$, so $N_0^L = L$. Then

$$\int_{n_\varepsilon(L)} tr\, B(t) = tr\, K_{t,L}(x,x)|_L$$

and since $f(x) = x$ for all $x \in L$, $tr\, K_{t,L}(x,x)|_L$ is given by

$$\sum_{i=0}^{k} (-1)^i \sum_{\ell,j} t^{(\ell-n)/2} \int tr\, c^i_{\ell,j} \cdot \frac{e^{-|\xi|^2}}{[\frac{j+\ell+2}{2}]!}\, d\xi.$$

Since $c^i_{\ell j}$ is homogeneous of order j in ξ and vanishes for $\ell+j$ odd, $c^i_{\ell j}$ is of odd order in ξ if ℓ is odd. Thus the integral is zero if ℓ is odd and so if $n = \dim L$ is odd, there is no zero th order term and we set $a_0^L = 0$ in this case. If n is even, we set

$$a_0^L = \sum_{i=0}^{k} (-1)^i \sum_j \int tr\, c^i_{n,j} \cdot \frac{e^{-|\xi|^2}}{[\frac{j+n+2}{2}]!}\, d\xi$$

and note that a_0^L is given as a polynomial in A, f, the a^i, the metrics, and their derivatives to a finite order.

Case 2. $L \cap \text{fix } f = \bigcup_j N_j^L \neq L$.

<u>Lemma</u>: $\int_{n_\varepsilon(N_j^L)} tr(K_{t,L}(x,x)) \sim \sum_{\substack{\ell,j \\ j+\ell \text{ even}}} t^{(j+\ell-p)/2} d^i_{\ell,j}$, where $\dim N_j^L = p$,

and where $d^i_{\ell,j}$ depends only on A, f the a^i, the metrics, and their derivatives to finite order on N_j^L. This asymptotic expansion is independent of x and L, i.e. given q, and ε, there is

$c(q,\varepsilon)$ so that for all N_j^L,

$$\left| \oint_{n_\varepsilon(N_j^L)} tr(K_{t,L}(x,x)) - \sum t^{(j+\ell-p)/2} d_{\ell,j}^i \right| \leq c(q,\varepsilon)t^q.$$

To see this expand $tr(K_{t,L}(x,x))$ on $n_\varepsilon(N_j^L)$ in a Taylor series (about the zero section N) in t and then integrate over the fiber of $n_\varepsilon(N_j^L)$. See [G].

Thus in our calculation of $\int_N \oint_{n_\varepsilon} tr(B(t))d\nu$, we may replace $\oint_{n_\varepsilon} tr(K_{t,L}(x,x))$ by a where $a = (a_j^L)$ and

i) $a_j^L \equiv 0$ if $\dim N_j^L = p$ is odd (for in this case the asymptotic expansion has no zero th order term).

ii) $a_j^L = \sum_{i=0}^{k} (-1)^i \sum_{j+\ell=p} d_{\ell,j}^i$ if $\dim N = p$ is even.

This completes the proof of the Theorem.

To identify the a_j^L for the classical complexes, we appeal to [G] or [ABP] where the calculation of these is made. As this is a purely local problem, we may use the proofs in the compact case without alternation. Details will appear elsewhere.

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