Functional Equations and the Harmonic Relations for Multiple Zeta Values

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Abstract

Let $\theta(x)$ denote Jacobi's theta function. We show that the function $F_\xi(x) = (\theta'(0)\theta(x+\xi))/(\theta(x)\theta(\xi))$ satisfies functional equations, which is a generalization of the harmonic relations for multiple zeta values.

1 Introduction

For any multi-index $s = (s_1, \ldots, s_m)$ $(s_i \in \mathbb{N},\ s_m \geq 2)$, the multiple zeta value $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{0<k_1<\ldots<k_m} \frac{1}{k_1^{s_1} \cdots k_m^{s_m}}.$$

We call the positive integer m the depth of $\zeta(s)$.

The harmonic relations (stuffle relations) for multiple zeta values arise from the product of two series ([1],[2],[3]). For example,

$$\zeta(u)\zeta(v) = \{\zeta(u,v) + \zeta(v,u)\} + [\zeta(uv)], \tag{1}$$
$$\zeta(u,v)\zeta(w) = \{\zeta(u,v,w) + \zeta(u,w,v) + \zeta(w,u,v)\} + [\zeta(uw,v) + \zeta(u,vw)],$$

where $u, v,$ and w are integers such that $u, v, w \geq 2$. We can divide the right hand side in each of the above equations into two groups by the depth, which is expressed by using { } and [ ]; The depth of every multiple zeta value in { } is equal to the sum of the depth of the multiple zeta values in the left hand side, and that in [ ] is less than that in { }.

The multiple zeta value $\zeta(s)$ has the following integral representation([4] Sect.10):

$$\begin{aligned}\zeta(s) = &\int_0^1 \frac{dX_{s_1}^{(1)}}{X_{s_1}^{(1)}} \int_0^{X_{s_1}^{(1)}} \frac{dX_{s_1-1}^{(1)}}{X_{s_1-1}^{(1)}} \cdots \int_0^{X_2^{(1)}} dX_1^{(1)} \\ &\cdots \int_0^1 \frac{dX_{s_m}^{(m)}}{X_{s_m}^{(m)}} \int_0^{X_{s_m}^{(m)}} \frac{dX_{s_m-1}^{(m)}}{X_{s_m-1}^{(m)}} \cdots \int_0^{X_2^{(m)}} dX_1^{(m)} \prod_{i=1}^{m} \frac{X_1^{(i)} \cdots X_1^{(m)}}{1 - X_1^{(i)} \cdots X_1^{(m)}}. \quad (2)\end{aligned}$$

Therefore the equations (1) is equivalent to the following equations:

$$f(x)f(y) = \big\{f(x+y)\big(f(x)+f(y)\big)\big\} + \big[f(x+y)\big], \quad (3)$$

$$\begin{aligned}f(x_1)f(x_2)f(y) = &\big\{f(x_1+y)\big(f(x_2+y)f(y) + f(x_2+y)f(x_2) + f(x_1)f(x_2)\big)\big\} \\ &+ \big[f(x_1+y)\big(f(x_2+y) + f(x_2)\big)\big],\end{aligned}$$

where $f(x) = \dfrac{e^x}{1-e^x}$. In fact we obtain the equations (1) by replacing e^x, e^y, e^{x_1}, and e^{x_2} with X, Y, X_1X_2, and X_2 respectively and using the above integral representation.

In this point of view one can regard the usual harmonic relation as the following equation:

$$\Big(\prod_{i=1}^{m} f(x_i)\Big)\Big(\prod_{i=1}^{n} f(y_i)\Big) = \Big\{\sum_p \prod_{i=1}^{m+n} f(x_{p_{i1}} + y_{p_{i2}})\Big\} + \Big[\text{others}\Big]. \quad (4)$$

Here $x_{m+1} = y_{n+1} = 0$ and the sum extends over $p = (p_1, \ldots, p_{m+n+1})$ such that

- $p_1 = (1,1), p_{m+n+1} = (m+1, n+1)$,
- $p_{i+1} - p_i = (1,0)$ or $(0,1)$, $\quad (i = 1, \ldots, m+n)$.

If we denote the degree (depth) of $\prod_{i=1}^{m} f(x_i)$ by m, then the degree (depth) of $\prod_{i=1}^{m+n} f(x_{p_{i1}} + y_{p_{i2}})$ is equal to $m+n$ and the degree (depth) of every term in [others] is less than $m+n$.

By directing our attention to the highest degree (depth), we regared equations of the type $(\prod_{i=1}^{m} \tilde{f}(x_i))(\prod_{i=1}^{n} \tilde{f}(y_i)) = \sum_p \prod_{i=1}^{m+n} \tilde{f}(x_{p_{i1}} + y_{p_{i2}})$ as functional equations with the fundamental structure of the harmonic relations.

In this paper we give a function $F_\xi(x;\tau)$ with two parameters ξ, τ which satisfies the functional equations with the fundamental structure of the harmonic relations if we neglect the two parameters. We also obtain the harmonic relations for multiple zeta values by degeneration. This fact guarantees that the functional equations for $F_\xi(x;\tau)$ that is given in this paper are a generalization of the harmonic relations. Furthermore we give their extended functional equations (See Theorem 3.1).

The plan of this paper is the following. In Section 2 we introduce the function $F_\xi(x;\tau)$ which plays a essential role in this paper. In Section 3 we state our main result, i.e., $F_\xi(x;\tau)$ satisfies functional equations with the fundamental structure of the harmonic relations if we neglect the two parameters ξ, τ. We also obtain the harmonic relations by degeneration. Section 4 is devoted to the proof of our main result.

2 The Function $F_\xi(x;\tau)$

Let τ be a complex number with positive imaginary part, i.e., Im $\tau > 0$. In this section we introduce the meromorphic function $F_\xi(x)(= F_\xi(x;\tau))$ on $\mathbb{C}^2$ which plays a essential role in this paper.

Let $\theta(x)$ denote Jacobi's theta function:

$$\theta(x) := \theta(x;\tau) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi\sqrt{-1}(m+\frac{1}{2})^2\tau+2\pi\sqrt{-1}(m+\frac{1}{2})(x+\frac{1}{2})}.$$

The meromorphic function $F_\xi(x)$ is defined by

$$F_\xi(x) = \frac{\theta'(0)\theta(x+\xi)}{\theta(x)\theta(\xi)},$$

where $\theta'(x) = \dfrac{\partial}{\partial x}\theta(x;\tau)$. If $\xi \in \mathbb{C}\setminus\mathbb{Z}+\tau\mathbb{Z}$, then this definition induces that the function $F_\xi(x)$ with respect to x has simple poles on the lattice $\mathbb{Z}+\tau\mathbb{Z}$ and that it's residue at the origin is 1. It also satisfies the following:

PROPOSITION 2.1. *We have*

(i) $F_\xi(x+1) = F_\xi(x), \quad F_\xi(x+\tau) = \mathrm{e}^{-2\pi\sqrt{-1}\xi} F_\xi(x), \quad F_{-\xi}(-x) = -F_\xi(x).$

(ii) $$\sum_{l=1}^{m} F_{-\xi_l}(-x_1-\cdots-x_m)\prod_{\substack{j=1\\(j\neq l)}}^{m} F_{\xi_j-\xi_l}(x_j) + \prod_{j=1}^{m} F_{\xi_j}(x_j) = 0.$$

Proof. The formulas listed in (i) are derived from the formulas below:

$$\theta(x+1) = \theta(x), \quad \theta(x+\tau) = \mathrm{e}^{-\pi\sqrt{-1}\tau-2\pi\sqrt{-1}x}\,\theta(x), \quad \theta(-x) = -\theta(x).$$

In order to prove (ii), let us introduce a function defined by

$$G(\eta) = \Big(\prod_{j=1}^{m} F_{\xi_j+\eta}(x_j)\Big) F_\eta(-x_1 - \cdots - x_m),$$

where $\xi_j,\ \xi_j - \xi_i \notin \mathbb{Z} + \tau\mathbb{Z}, (i, j = 1, \dots, m,\ i \neq j)$. Since the function $G(\eta)$ is doubly periodic, the sum of the residues of $G(\eta)$ at its poles in any period parallelogram is 0. Hence we obtain (ii). □

3 Functional Equations

In this section we state our main result, i.e., $F_\xi(x;\tau)$ satisfies functional equations with the fundamental structure of the harmonic relations if we neglect the two parameters ξ, τ. We also obtain the harmonic relations by degeneration.

We first give some notations. Suppose that $K_1, \dots, K_m$ are non negative integers. Let K and $|K|$ denote $(K_1, \dots, K_m)$ and $K_1 + \cdots + K_m$ respectively. We define the set $P(K)$ such that if $p \in P(K)$ then p satisfies the following conditions:

• $p_1 = (1, \dots, 1), \quad p_{|K|+1} = (K_1 + 1, \dots, K_m + 1)$.

• For every $i = 1, \dots, |K|$, there is a integer l $(1 \leq l \leq m)$ such that

$$p_{i+1} - p_i = \epsilon_l = (0, \dots, \overset{l-th}{1}, \dots, 0).$$

Suppose that $\alpha_i^{(j)}$ is a complex number for positive integers i and j. The $|K|$-tuple vector $\alpha_K(= \alpha)$ is defined by $(\alpha_1^{(1)}, \dots, \alpha_{K_1}^{(1)}, \dots, \alpha_1^{(m)}, \dots, \alpha_{K_m}^{(m)})$. Here we regard $\alpha_i^{(j)}$ in α_K as a blank word if there is integer j such that $K_j = 0$. For example, $\alpha_{(2,0,1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_1^{(3)}), \alpha_{(0,1,2)} = (\alpha_1^{(2)}, \alpha_1^{(3)}, \alpha_2^{(3)})$. For any $p \in P(K)$, we define the complex number $p_i(\alpha)$ as follows:

• If $p_2 - p_1 = \epsilon_l$, then $p_1(\alpha) = \alpha_1^{(l)}$.

• If $p_{i+1} - p_i = \epsilon_{l'}, p_i - p_{i-1} = \epsilon_l, (i = 2, \dots, |K|)$, then $p_i(\alpha) = -\tilde{\alpha}_{p_{i-1,l}}^{(l)} + \tilde{\alpha}_{p_{il'}}^{(l')}$, where $\tilde{\alpha}_i^{(l)} = \alpha_1^{(l)} + \cdots + \alpha_i^{(l)}$.

Our main result is the following:

THEOREM 3.1. *If* $x_{K_j+1}^{(j)} = 0$ $(j = 1, \dots, m)$, *then*

$$\prod_{j=1}^{m}\Big(\prod_{i=1}^{K_j} F_{\alpha_i^{(j)}}(x_i^{(j)})\Big) = \sum_{p \in P(K)} \prod_{i=1}^{|K|} F_{p_i(\alpha)}(x_{p_{i1}}^{(1)} + \cdots + x_{p_{im}}^{(m)}). \qquad (5)$$

Here we regard the empty product $\prod_{i=1}^{0}$ *as one.*

We derive the harmonic relations from the above functional equation. To do this, we give the explicit form of the equation (4). Suppose that K_1, K_2 are positive and that K is equal to (K_1, K_2). For any $k = 1, \ldots, \min\{K_1, K_2\}$, we define the set $P(K;k)$ such that if $p \in P(K;k)$ then p satisfies the following conditions:

- $p_1 = (1,1), \quad p_{|K|-k+1} = (K_1+1, K_2+1)$.
- There are integers $a_1, \ldots, a_k$ $(1 \le a_1 < \cdots < a_k \le |K|-k)$ such that $p_{a_i+1} - p_{a_i} = (1,1)$, $(i = 1, \ldots, k)$.
- For any $i \in \{1, \ldots, |K|-k\} \setminus \{a_1, \ldots, a_k\}$, there is a integer l $(=1,2)$ such that $p_{i+1} - p_i = \epsilon_l$.

Then the explicit form of the equation (4) is the following:

$$\Big(\prod_{i=1}^{K_1} \frac{X_i \cdots X_{K_1}}{1 - X_i \cdots X_{K_1}}\Big)\Big(\prod_{j=1}^{K_2} \frac{Y_j \cdots Y_{K_2}}{1 - Y_j \cdots Y_{K_2}}\Big)$$
$$= \sum_{k=0}^{\min\{K_1,K_2\}} \sum_{p \in P(K;k)} \prod_{i=1}^{|K|-k} \frac{X_{p_{i1}} \cdots X_{K_1} Y_{p_{i2}} \cdots Y_{K_2}}{1 - X_{p_{i1}} \cdots X_{K_1} Y_{p_{i2}} \cdots Y_{K_2}}, \qquad (4')$$

where $P(K;0) = P(K)$. In fact we obtain the above equation by comparing the coefficient of $X_1^{i_1} \cdots X_{K_1}^{i_{K_1}} Y_1^{j_1} \cdots Y_{K_2}^{j_{K_2}}$ $(1 \le i_1 < \ldots < i_{K_1}, 1 \le j_1 < \ldots < j_{K_2})$ in (4).

COROLLARY 3.1. *The functional equations (5) induce the equation (4'), namely the harmonic relations.*

Proof. By [5] Sect.3,

$$F_x(\xi;\tau) = 2\pi\sqrt{-1}\Big[\sum_{j=1}^{\infty} \frac{q^j}{\mathrm{e}(x) - q^j}\,\mathrm{e}(-j\xi) - \sum_{j=1}^{\infty} \frac{q^j}{\mathrm{e}(-x) - q^j}\,\mathrm{e}(j\xi)$$
$$+ \frac{\mathrm{e}(x)}{\mathrm{e}(x) - 1} + \frac{\mathrm{e}(\xi)}{\mathrm{e}(\xi) - 1} - 1\Big], \qquad (|\mathrm{Im}\ x|, |\mathrm{Im}\ \xi| < \mathrm{Im}\ \tau).$$

Therefore the following formulas hold:

$$\lim_{\xi \to \sqrt{-1}\infty} \lim_{\tau \to \sqrt{-1}\infty} \frac{\sqrt{-1}}{2\pi} F_\xi(x) = f(2\pi\sqrt{-1}x) + 1,$$
$$\lim_{\xi \to -\sqrt{-1}\infty} \lim_{\tau \to \sqrt{-1}\infty} \frac{\sqrt{-1}}{2\pi} F_\xi(x) = f(2\pi\sqrt{-1}x), \qquad (6)$$

where $f(x) = \dfrac{\mathrm{e}^x}{1-\mathrm{e}^x}$. If we lay τ to $\sqrt{-1}\infty$ and lay $\alpha_1^{(1)}, \ldots, \alpha_{K_1}^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_{K_2}^{(2)}$ to $-\sqrt{-1}\infty$ respectively in the equation (5) where $m = 2$, then it follows that

$$\Big(\prod_{i=1}^{K_1} f(x_i^{(1)})\Big)\Big(\prod_{i=1}^{K_2} f(x_i^{(2)})\Big) = \Big\{\sum_{p\in P(K)} \prod_{i=1}^{K_1+K_2} f(x_{p_{i1}}^{(1)} + x_{p_{i2}}^{(2)})\Big\} + \Big\langle \text{others} \Big\rangle. \quad (7)$$

Here $<$ others $>$ comes from the formulas (6). That is, for every term g in $<$ others $>$, there are $p \in P(K)$ and intergers $l, i_1, \ldots, i_l$ $(1 \le i_1 < \cdots < i_l \le |K|)$ such that

$$g = \prod_{j=1}^{l} f(x_{p_{i_j1}}^{(1)} + x_{p_{i_j2}}^{(2)}).$$

Now we replace $\mathrm{e}^{x_i^{(1)}}$ and $\mathrm{e}^{x_i^{(2)}}$ with $X_i \cdots X_{K_1}$ and $Y_i \cdots Y_{K_2}$ respectively. By comparing the coefficient of $X_1^{i_1} \cdots X_{K_1}^{i_{K_1}} Y_1^{j_1} \cdots Y_{K_2}^{j_{K_2}}$ $(1 \le i_1 < \ldots < i_{K_1},\ 1 \le j_1 < \ldots < j_{K_2})$ in the equation (7), one get the equation (4'). By using the integral representation (2), we also obtain the harmonic relations. $\square$

EXAMPLE 3.1. The following equations correspond to the equations (3):

$$\begin{aligned}
&F_\xi(x)F_\eta(y) = F_\eta(x+y)F_{-\eta+\xi}(x) + F_\xi(x+y)F_{-\xi+\eta}(y),\\
&F_{\xi_1}(x_1)F_{\xi_2}(x_2)F_\eta(y)\\
=&F_{\xi_1}(x_1+y)F_{\xi_2}(x_2+y)F_{-\xi_1-\xi_2+\eta}(y)\\
+&F_{\xi_1}(x_1+y)F_{-\xi_1+\eta}(x_2+y)F_{-\eta+\xi_1+\xi_2}(x_2) + F_\eta(x_1+y)F_{-\eta+\xi_1}(x_1)F_{\xi_2}(x_2).
\end{aligned}$$

4 Proof

We will prove our main result (Theorem 3.1). We first give three lemmas.

LEMMA 4.1. *Suppose that K_m is positive. For any $p \in P(K)$, let the function $f_p(x_{K_m}^{(m)})$ be $\prod_{i=1}^{|K|} F_{p_i(\alpha)}(x_{p_{i1}}^{(1)} + \cdots + x_{p_{im}}^{(m)})$. Then $f_p(x_{K_m}^{(m)})$ has quasi periodicity 1 and $\mathrm{e}^{-2\pi\sqrt{-1}\alpha_{K_m}^{(m)}}$ as $x_{K_m}^{(m)}$ goes to $x_{K_m}^{(m)}+1$ and $x_{K_m}^{(m)}+\tau$ respectively.*

Proof. It is obvious that $f_p(x_{K_m}^{(m)}+1) = f_p(x_{K_m}^{(m)})$. We show that $f_p(x_{K_m}^{(m)}+\tau) = \mathrm{e}^{-2\pi\sqrt{-1}\alpha_{K_m}^{(m)}} f_p(x_{K_m}^{(m)})$. Suppose that $K_m > 1$. Let the set I be $\{i \mid p_{i+1} - p_i =$

$\epsilon_m\}$. Since the number of the elements in I is equal to K_m, there is two positive integers j and j' such that $j = \max\{i \in I\}$ and $j' = \max\{i \in I \setminus \{j\}\}$. Thus it follows that $p_{j+1,m} = K_m + 1,\ p_{j,m} = p_{j'+1,m} = K_m$, and $p_{j',m} = K_m - 1$. By the proposition 2.1 (i),

$$\begin{aligned} f_p(x^m_{K_m} + \tau) = &\Bigg(\prod_{\substack{i=1 \\ (i \le j',\ j+1 \le i)}}^{|K|} F_{p_i(\alpha)}(x^{(1)}_{p_{i1}} + \cdots + x^{(m)}_{p_{im}}) \Bigg) \\ &\times \Bigg(\prod_{i=j'+1}^{j} F_{p_i(\alpha)}(x^{(1)}_{p_{i1}} + \cdots + x^{(m-1)}_{p_{i,m-1}} + x^{(m)}_{K_m} + \tau) \Bigg) \\ = &\,\mathrm{e}^{-\sum_{i=j'+1}^{j} p_i(\alpha)} f_p(x^m_{K_m}). \end{aligned}$$

This completes the proof in the case $K_m \geq 2$ since $\sum_{i=j'+1}^{j} p_i(\alpha) = -\tilde{\alpha}^{(m)}_{p_{j'm}} + \tilde{\alpha}^{(m)}_{p_{jm}} = \alpha^{(m)}_{K_m}$. One can prove the case $K_m = 1$ in a similar way. □

Let $N_1, \dots, N_m$ be positive integers such that $N_i \leq K_i + 1$ $(i = 1, \dots, m)$. For the two lemmas below, we suppose that m is larger than one, that $N_m = K_m$, and that $(N_1, \dots, N_{m-1}) \neq (K_1 + 1, \dots, K_{m-1} + 1)$.

LEMMA 4.2. *If $x^{(m)}_{K_m}$ is equal to $-x^{(1)}_{N_1} - \cdots - x^{(m-1)}_{N_{m-1}}$, then the following identity holds:*

$$0 = \sum_{l \in A_N} \Bigg\{ \prod_{\substack{j \in A_N \\ (j \neq l)}} F_{-\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(j)}_{N_j}}(x^{(j)}_{N_j}) \Bigg(\prod_{i=N_j+1}^{K_j} F_{\alpha^{(j)}_i}(x^{(j)}_i) \Bigg) \Bigg\} \Bigg\{ \prod_{i=N_l+1}^{K_l} F_{\alpha^{(l)}_i}(x^{(l)}_i) \Bigg\},$$

where $A_N = \{l \mid 1 \leq l \leq m,\ N_l \neq K_l + 1\}$.

Proof. It is sufficient to prove the lemma when $A_N = \{1, \dots, m\}$ since the other cases can be proved in a similar way. One has

$$\begin{aligned} &\sum_{l=1}^{m} \Bigg\{ \prod_{\substack{j=1 \\ (j \neq l)}}^{m} F_{-\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(j)}_{N_j}}(x^{(j)}_{N_j}) \Bigg(\prod_{i=N_j+1}^{K_j} F_{\alpha^{(j)}_i}(x^{(j)}_i) \Bigg) \Bigg\} \Bigg\{ \prod_{i=N_l+1}^{K_l} F_{\alpha^{(l)}_i}(x^{(l)}_i), \Bigg\} \\ =& \Bigg\{ \prod_{j=1}^{m-1} \Bigg(\prod_{i=N_j+1}^{K_j} F_{\alpha^{(j)}_i}(x^{(j)}_i) \Bigg) \Bigg\} \\ &\times \underbrace{\Bigg[\sum_{l=1}^{m-1} \Bigg(\prod_{\substack{j=1 \\ (j \neq l)}}^{m-1} F_{-\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(j)}_{N_j}}(x^{(j)}_{N_j}) \Bigg) F_{-\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(m)}_{N_m}}(x^{(m)}_{K_m}) + \prod_{j=1}^{m-1} F_{-\tilde{\alpha}^{(m)}_{N_m} + \tilde{\alpha}^{(j)}_{N_j}}(x^{(j)}_{N_j}) \Bigg]}_{(\diamondsuit)}. \end{aligned}$$

Since $x^{(m)}_{K_m}$ is equal to $-x^{(1)}_{N_1} - \cdots - x^{(m-1)}_{N_{m-1}}$ and the equation in Proposition 2.1 (ii) holds, it follows that $(\diamondsuit) = 0$. This completes the proof. $\Box$

In order to give the third lemma, we prepare some notations. Let l be a integer ($l = 1, \ldots, m$) such that $N_l \neq K_l + 1$, and let the multi-index N' be $(N_1 - 1, \ldots, N_{m-1} - 1, K_m - 1)$. For any $p' \in P(N')$, let the subset $P^{(l;p')}$ in $P(K)$ be $\{p \in P(K) \mid p_i = p'_i \ (i = 1, \ldots, |N'| + 1),\ p_{|N'|+2} - p_{|N'|+1} = \epsilon_l\}$. The map $\Phi^{(l)}$ from $P^{(l;p')}$ to $P(K - N' - \epsilon_l)$ is defined by

$$(\Phi^{(l)}(p))_i = p_{|N'|+i+1} - N' - \epsilon_l \quad (p \in P^{(l;p')},\ i = 1, \ldots, |K| - |N'|).$$

Finally the $(|K| - |N'| - 1)$-tuple vector β_l is defined by

$$\begin{aligned}
\beta_l &= (\underline{\beta^{(1)}_{l,1}, \ldots, \beta^{(1)}_{l,K_1-N_1+1}}, \ldots, \underline{\beta^{(l)}_{l,1}, \ldots, \beta^{(l)}_{l,K_1-N_1}}, \ldots, \underline{\beta^{(m)}_{l,1}, \ldots, \beta^{(m)}_{l,K_m-N_m+1}}) \\
&= (\underline{-\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(1)}_{N_1}, \alpha^{(1)}_{N_1+1}, \ldots, \alpha^{(1)}_{K_1}}, \ldots \\
&\qquad \ldots, \underline{\alpha^{(l)}_{N_l+1}, \ldots, \alpha^{(l)}_{K_l}}, \ldots, \underline{-\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(m)}_{N_m}, \alpha^{(m)}_{N_m+1}, \ldots, \alpha^{(m)}_{K_m}}). \qquad (8)
\end{aligned}$$

LEMMA 4.3. *For every $p \in P^{(l;p')}$, we have*

$$p_i(\alpha) = (\Phi^{(l)}(p))_{i-|N'|-1}(\beta), \quad (i = |N'| + 2, \ldots, |K|). \qquad (9)$$

Proof. Let i be a integer such that $|N'| + 2 \leq i \leq |K|$. There are integers a and b such that $p_{i+1} - p_i = \epsilon_a$ and $p_i - p_{i-1} = \epsilon_b$. Then we have $(\Phi^{(l)}(p))_{i-|N'|} - (\Phi^{(l)}(p))_{i-|N'|-1} = \epsilon_a$, $(\Phi^{(l)}(p))_{i-|N'|-1} - (\Phi^{(l)}(p))_{i-|N'|-2} = \epsilon_b$. We first consider the case $i = |N'| + 2$. Since $(\Phi^{(l)}(p))_2 - (\Phi^{(l)}(p))_1 = \epsilon_a$, one obtains $(\Phi^{(l)}(p))_1(\beta) = -\tilde{\alpha}^{(l)}_{N_l} + \tilde{\alpha}^{(a)}_{N_a+\delta_{la}}$, where δ_{la} is the Kronecker's delta. On the other hand, because p is a element in $P^{(l)}$, one has $p_{|N'|+2} - p_{|N'|+1} = \epsilon_l$ and $p_{|N'|+1} = (N_1, \ldots, N_m)$. So it follows that $p_{|N'|+2}(\alpha) = -\tilde{\alpha}^{(l)}_{p_{|N'|+1,l}} + \tilde{\alpha}^{(a)}_{p_{|N'|+2,a}} =$

$(\Phi^{(l)}(p))_1(\beta)$. We next consider the case $i > |N'| + 2$. By direct calculation,

$$\begin{aligned} p_i(\alpha) &= -\tilde{\alpha}^{(b)}_{p_{i-1,b}} + \tilde{\alpha}^{(a)}_{p_{i,a}} \\ &= \begin{cases} \beta^{(l)}_{p_{i,l}-N_l} & (a = b = l) \\ -\tilde{\beta}^{(b)}_{p_{i-1,b}-N_b+1} + \tilde{\beta}^{(l)}_{p_{i,l}-N_l} & (a = l, b \neq l) \\ -\tilde{\beta}^{(l)}_{p_{i-1,l}-N_l} + \tilde{\beta}^{(a)}_{p_{i,a}-N_a+1} & (a \neq l, b = l) \\ -\tilde{\beta}^{(b)}_{p_{i-1,b}-N_b+1} + \tilde{\beta}^{(a)}_{p_{i,a}-N_a+1} & (a \neq l, b \neq l) \end{cases} \\ &= -\tilde{\beta}^{(b)}_{p_{i-1,b}-(N_b-1)-\delta_{lb}} + \tilde{\beta}^{(a)}_{p_{i,a}-(N_a-1)-\delta_{la}} \\ &= -\tilde{\beta}^{(b)}_{(\Phi^{(l)}(p))_{i-|N'|-2,b}} + \tilde{\beta}^{(a)}_{(\Phi^{(l)}(p))_{i-|N'|-1,a}} \\ &= (\Phi^{(l)}(p))_{i-|N'|-1}(\beta), \end{aligned}$$

where $\tilde{\beta}_j = \beta_1 + \cdots + \beta_j$. This completes the proof. □

We are in a position to prove our main theorem.

Proof of Theorem 3.1. We use induction on $|K| = k$. The claim is trivial for $k = 1$; Assume it is true for $|K| < k$. It is sufficient to prove the equation (5) when $m \geq 2$ and $K_m \geq 1$. We define the function $f(x^{(m)}_{K_m})$ by $f(x^{(m)}_{K_m}) =$ (L.H.S in (5)) $-$ (R.H.S in (5)). By virtue of Lemma 4.1, it has quasi periodicity 1 and $\mathrm{e}^{-2\pi\sqrt{-1}\alpha^{(m)}_{K_m}}$ as $x^{(m)}_{K_m}$ goes to $x^{(m)}_{K_m} + 1$ and $x^{(m)}_{K_m} + \tau$ respectively. Its possible poles are

$$-x^{(1)}_{N_1} - \cdots - x^{(m-1)}_{N_{m-1}}, \quad (1 \leq {}^{\forall}N_i \leq K_i + 1, i = 1, \ldots, m-1).$$

Suppose that $x^{(1)}_{N_1}, \ldots, x^{(m-1)}_{N_{m-1}}$ are generic. Hence the order of every possible pole is 1 or less. We shall calculate their residues. Let X denote $-x^{(1)}_{K_1+1} - \cdots - x^{(m-1)}_{K_{m-1}+1}$. Since $X = 0$,

$$\operatorname*{Res}_{x^{(m)}_{K_m}=X} f(x^{(m)}_{K_m}) = \prod_{j=1}^{m-1}\left(\prod_{i=1}^{K_j} F_{\alpha'^{(j)}_i}(x^{(j)}_i)\right) \prod_{i=1}^{K_m-1} F_{\alpha'^{(m)}_i}(x^{(m)}_i) - \sum_{p\in P(K')} \prod_{i=1}^{|K'|} F_{p_i(\alpha')}(x^{(1)}_{p_{i1}} + \cdots + x^{(m)}_{p_{im}}).$$

Here K' and α' are defined by

$$K' = \begin{cases} (K_1, \ldots, K_{m-1}), \\ (K_1, \ldots, K_{m-1}, K_m - 1), \end{cases} \qquad \alpha' = \begin{cases} (\alpha_1^{(1)}, \ldots, \alpha_{K_{m-1}}^{(m-1)}), & (K_m = 1), \\ (\alpha_1^{(1)}, \ldots, \alpha_{K_m - 1}^{(m)}), & (K_m \geq 2). \end{cases}$$

It follows from the induction hypothesis that $\operatorname{Res}_{x_{K_m}^{(m)}=0} f(x_{K_m}^{(m)}) = 0$. We next calculate the residues on the other poles. Suppose that $N_1, \ldots, N_{m-1}$ are positive integers such that $N_i \leq K_i + 1$ $(i = 1, \ldots, m-1)$ and $(N_1, \ldots, N_{m-1}) \neq (K_1 + 1, \ldots, K_{m-1} + 1)$. Let N_m be K_m, and let the multi-indices N' and N be $(N_1 - 1, \ldots, N_{m-1} - 1, K_m - 1)$ and $(N_1, \ldots, N_{m-1}, K_m)$ respectively. Let X denote $-x_{N_1}^{(1)} - \cdots - x_{N_{m-1}}^{(m-1)}$. Then

$$\begin{aligned} &\operatorname*{Res}_{x_{K_m}^{(m)}=X} f(x_{K_m}^{(m)}) \\ &= - \operatorname*{Res}_{x_{K_m}^{(m)}=X} \sum_{\substack{p \in P(K) \\ (p_{|N'|+1} = N)}} \prod_{i=1}^{|K|} F_{p_i(\alpha)}(x_{p_i 1}^{(1)} + \cdots + x_{p_i m}^{(m)}) \\ &= - \sum_{p' \in P(N')} \prod_{i=1}^{|N'|} F_{p'_i(\alpha)}(x_{p'_i 1}^{(1)} + \cdots + x_{p'_i m}^{(m)}) \\ &\qquad \times \underbrace{\left[\sum_{l \in A_N} \sum_{p \in P^{(l;p')}} \prod_{i=|N'|+2}^{|K|} F_{p_i(\alpha)}(x_{p_{i1}}^{(1)} + \cdots + x_{p_{im}}^{(m)}) \right]}_{(\clubsuit)}. \end{aligned}$$

Here $A_N = \{ l \mid 1 \leq l \leq m, \; N_l \neq K_l + 1 \}$ and $P^{(l;p')} = \{ p \in P(K) \mid p_i = p'_i \; (i = 1, \ldots, |N'| + 1), \; p_{|N'|+2} - p_{|N'|+1} = \epsilon_l \}$. We shall show that $(\clubsuit) = 0$. We fix $p' \in P(N')$ and denote $P^{(l)}$ the set $P^{(l;p')}$. Since the map $\Phi^{(l)}$ in Lemma 4.3 is bijective and the equation (9) holds,

$$\begin{aligned} (\clubsuit) &= \sum_{l \in A_N} \sum_{p \in P^{(l)}} \prod_{i=1}^{|K|-|N'|-1} F_{(\Phi^{(l)}(p))_i}(\beta_l)(x_{p_{i+|N'|+1,1}}^{(1)} + \cdots + x_{p_{i+|N'|+1,m}}^{(m)}) \\ &= \sum_{l \in A_N} \sum_{q \in P(K - N' - \epsilon_l)} \prod_{i=1}^{|K|-|N'|-1} F_{q_i(\beta_l)}(x_{q_{i1}+N_1-1}^{(1)} + \cdots + x_{q_{il}+N_l}^{(l)} + \cdots + x_{q_{im}+N_m-1}^{(m)}) \\ &= \sum_{l \in A_N} \sum_{q \in P(K - N' - \epsilon_l)} \prod_{i=1}^{|K|-|N'|-1} F_{q_i(\beta_l)}(y_{l,q_{i1}}^{(1)} + \cdots + y_{l,q_{im}}^{(m)}). \end{aligned}$$

Here β_l is the $(|K|-|N'|-1)$-tuple vector (8), and for integers j and i $(1 \leq j \leq m,\ 1 \leq i \leq K_j - N_j + 2 - \delta_{lj})$, $y_{l,i}^{(j)}$ is defined by

$$y_{l,i}^{(j)} = \begin{cases} x_{i+N_j-1}^{(j)} & (j \neq l), \\ x_{i+N_l}^{(l)} & (j = l). \end{cases}$$

By the induction hypothesis and Lemma 4.2,

$$(\clubsuit) = \sum_{l \in A_N} \prod_{j=1}^{m} \Bigg(\prod_{i=1}^{K_j - N_j + 1 - \delta_{lj}} F_{\beta_{l,i}^{(j)}}(y_{l,i}^{(j)}) \Bigg)$$

$$= \sum_{l \in A_N} \Bigg\{ \prod_{\substack{j \in A_N \\ (j \neq l)}} F_{-\tilde{\alpha}_{N_l}^{(l)} + \tilde{\alpha}_{N_j}^{(j)}}(x_{N_j}^{(j)}) \Bigg(\prod_{i=N_j+1}^{K_j} F_{\alpha_i^{(j)}}(x_i^{(j)}) \Bigg) \Bigg\} \Bigg\{ \prod_{i=N_l+1}^{K_l} F_{\alpha_i^{(l)}}(x_i^{(l)}) \Bigg\} = 0.$$

Hence $\mathrm{Res}_{x_{K_m}^{(m)}=X} f(x_{K_m}^{(m)}) = 0$. Thus, for generic $\alpha_{K_m}^{(m)}, x_1^{(1)}, \ldots, x_{K_{m-1}}^{(m-1)}$, the function $f(x_{K_m}^{(m)})$ is an entire function with quasi periodicity. It must vanish for generic points. It also vanishes for all points by analyticity. $\square$

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References

[1] M. E. Hoffman, *Multiple hormonic series*, Pacific J. Math **152** (1992), no.2, 275–290.

[2] M. Waldschmidt, *Multiple polylogarithms: an introduction*, Number theory and discrete mathematics (Chandigarh, 2000), Trends Math., Birkhäuser, Basel, 2002, pp. 1–12.

[3] M. Kaneko, *Multiple Zeta Values and Poly-Bernoulli Numbers (in Japanese)*, Tokyo Metropolitan University Seminar Note, 1997.

[4] V. V. Zudilin, *Algebraic relations for multiple zeta values*, Russian Math. Surveys **58:1** (2003), 1–29.

[5] D. Zagier, *Periods of modular forms and Jacobi theta functions*, Invent. Math **104** (1991), 449–465.