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CONTENTS

BOUNDEDNESS OF FOURIER MULTIPLIER OPERATOR DEFINED BY ELLIPTIC TYPE FUNCTION

YONGGEUN CHO
HARMONIC ANALYSIS AND ITS APPLICATIONS AT SAPPORO
AUG 22, 2005

1. Introduction

In this note, we consider a Fourier multiplier operator T_δ defined by

$$\widehat{T_\delta f}(\xi) = \phi\left(\frac{\xi_d - \psi(\xi')}{\delta}\right)\chi(\xi')\widehat{f}(\xi), \tag{1.1}$$

where

$$\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, \quad 0 < \delta \ll 1$$

χ : smooth supported in a small neighborhood of the origin in $\mathbb{R}^{d-1}$

$\phi \in C_0^\infty(-2,2)$ or $\mathcal{S}(\mathbb{R}^1) \cap \{\phi : supp\,\widehat{\phi} \subset \{|t| \sim 1\}\}$.

And ψ is an elliptic type function which is defined as follows (see [13]):

$$\psi : Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1} \to \mathbb{R}$$

is a smooth function with $\psi(0) = \nabla\psi(0) = 0$, all of its eigenvalues of the Hessian $\partial^2_{\xi'}\psi \in [1-\varepsilon, 1+\varepsilon]$(hence its graph is positively curved). The typical example of elliptic type function is $\psi(\xi') = \frac{1}{2}|\xi'|^2$ or $1 - \sqrt{1-|\xi'|^2}$. Any convex surface with non-vanishing Gaussian curvature can be decomposed by finite number of elliptic type functions.

Now let us define the kernel of the operator T_δ be K_δ. Using the stationary and non-stationary phase estimate, one can easily show that if $\phi \in C_0^\infty(-2,2)$, then $|K_\delta(x)| \lesssim \delta^{\frac{d+1}{2}}$ for $|x| \lesssim \delta^{-1}$ and $|K_\delta(x)| \lesssim \delta^M(1+|x|)^{-M}$ for $|x| \gg \delta^{-1}$ and large positive number M, and if $\widehat{\phi} \in C_0^\infty(|t| \sim 1)$, then K_δ is supported in a band $\{|x_d| \sim \delta^{-1}\}$, $|K_\delta(x)| \lesssim \delta^{\frac{d+1}{2}}$ for $|x'| \le c|x_d|$ and $|K_\delta(x)| \lesssim \delta^M(1+|x|)^{-M}$ for $|x'| > c|x_d|$. We call these point-wise estimates *good localization property* of kernel K_δ.

Our aim is to find an optimal pair (p,q) and constant $C(\delta)$ satisfying

$$\|T_\delta f\|_{L^q} \le C(\delta)\|f\|_{L^p}.$$

This note was based on the works done jointly with Y. Kim, S. Lee and Y. Shim [1, 2].

Since by the localization property the kernel K_δ is bounded by $\delta^{\frac{d+1}{2}}$, $(p,q) = (1,\infty)$ and $C(\delta) = C\delta^{\frac{d+1}{2}}$ for some constant C independent of δ. L^2 restriction result shows that $(p,q) = \left(2, \frac{2d+2}{d-1}\right)$ and $C(\delta) = C\delta^{\frac{1}{2}}$. Interpolating these estimates and extending the range of exponent to the endpoint of restriction conjecture or the endpoint of Bochner-Riesz conjecture, we conclude that (p,q) should satisfy $\frac{2d}{d-1} < q \le \infty$ and $\frac{d+1}{q} = (d-1)(1-\frac{1}{p})$, and $C(\delta) = C\delta^{-\frac{d-1}{2}+\frac{d}{p}}$. By a kind of Knapp counter example, we can deduce that

$$\delta^{-\frac{d-1}{2}+\frac{d}{p}} \text{ is optimal along } \frac{d+1}{q} = (d-1)(1-\frac{1}{p}).$$

In the next sections, we will explain that the conjecture is true for $q > \frac{2d+4}{d}$ and introduce some applications of the boundedness of T_δ.

2. MAIN RESULT

Our main achievement is the following.

Theorem 2.1. *Let $0 < \delta \ll 1$ and T_δ be defined by* (1.1). *Then, for p,q satisfying $\frac{2d+4}{d} < q < \frac{2d+2}{d-1}$ and $\frac{d+1}{q} = (d-1)(1-\frac{1}{p})$,*

$$\|T_\delta f\|_q \le C\delta^{-\frac{d-1}{2}+\frac{d}{p}}\|f\|_p.$$

To prove this theorem, we need a series of lemmas. For $0 < \delta \ll 1$, we define multiplier operators T_i for $i = 1,2$ by

$$\widehat{T_i f}(\xi) = \chi_i(\xi')\phi\left(\frac{\xi_n - \psi(\xi')}{\delta}\right)\widehat{f}(\xi)$$

where χ_1, χ_2 are smooth functions supported on Q. Then by the sharp bilinear restriction estimate of Tao [12] we have

Lemma 2.2. *Let $0 < \delta \ll 1$. If $dist(supp\chi_1, supp\chi_2) \sim 1$, then for $\frac{d+2}{d} < p \le 2$,*

$$\|T_1 f T_2 g\|_p \le C\delta\|f\|_2\|g\|_2 \tag{2.1}$$

where the constant C is stable under small (smooth) perturbation ψ.

Here the stability of constant means that

$$\|\partial^\alpha(\psi - \widetilde{\psi})\|_{L^\infty(Q)} \ll 1 \to C \text{ is independent of } \widetilde{\psi}.$$

Now by using the good localization property of kernel K_i of T_i and the argument of Fefferman and Stein ([3], [16, p.422-423]), we can obtain a refinement of Lemma 2.2.

Lemma 2.3. *Let $0<\delta\ll 1$. Then if $dist(supp\chi_1, supp\chi_2)\sim 1$, for p,q satisfying $\frac{2d+4}{d}<q\le 4$ and $2\le p\le q$, then*

$$\|T_1fT_2g\|_{q/2}\le C\delta^{1-d+\frac{2d}{p}}\|f\|_p\|g\|_p$$

where the constant C is stable under small (smooth) perturbation of ψ.

We use a Whitney type decomposition technique introduced in [13], which is useful in exploiting bilinear estimate. For each $j\ge 1$ we dyadically decompose $Q\subset\mathbb{R}^{d-1}$ into $\sim 2^{(d-1)j}$ dyadic cubes Q_k^j of side length $2^{-(j+1)}$. We say $Q_k^j\sim Q_{k'}^j$ to mean that Q_k^j, $Q_{k'}^j$ are not adjacent but have adjacent parent cubes of diameter 2^{-j}. So if $Q_k^j\sim Q_{k'}^j$, $dist(Q_k^j,Q_{k'}^j)\sim 2^{-j}$. By a Whitney decomposition of $Q\times Q$ away from the diagonal D of $Q\times Q$ (e.g. [10], p.16), ignoring some harmless measure zero set, we have

$$Q\times Q\setminus D=\bigcup_{j\ge 1}\bigcup_{Q_k^j\sim Q_{k'}^j}Q_k^j\times Q_{k'}^j. \tag{2.2}$$

Let f_k^j be defined by

$$\widehat{f_k^j}(\xi)=\chi_{Q_k^j}(\xi')\widehat{f}(\xi). \tag{2.3}$$

Since $\sum_{j\ge1}\sum_{Q_k^j\sim Q_{k'}^j}\chi_{Q_k^j}\chi_{Q_{k'}^j}=1$ almost everywhere in $Q\times Q$ from (2.2) and χ_0 is supported in a small neighborhood of the origin, we see

$$T_\delta f(x)\cdot T_\delta f(x)=\sum_{j\ge1}\sum_{Q_k^j\sim Q_{k'}^j}T_\delta f_k^j(x)\cdot T_\delta f_{k'}^j(x). \tag{2.4}$$

Fixing j, we define a bilinear operator by

$$B_j(f,g)(x)=\sum_{Q_k^j\sim Q_{k'}^j}T_\delta f_k^j(x)\cdot T_\delta g_{k'}^j(x).$$

Then, from (2.4) it is easy to see that

$$(T_\delta f(x))^2=\sum_{j\ge1}B_j(f,f). \tag{2.5}$$

Lemma 2.4. *If $2^{2j}\delta<1$, then for p,q satisfying $\frac{2d+4}{d}<q\le 4$, $2\le p<q$, there is a constant C, independent of j,δ, such that*

$$\|B_j(f,g)\|_{q/2}\le C2^{2j(\frac{d+1}{q}-(d-1)(1-\frac{1}{p}))}\delta^{1-d+\frac{2d}{p}}\|f\|_p\|g\|_p \tag{2.6}$$

and if $2^{2j}\delta\ge 1$, then there is a constant C, independent of j,δ, such that for p,q satisfying $\frac{2d}{d-1}\le q\le 4$, $2\le p<q$ and $\frac{d+1}{q}=(d-1)(1-\frac{1}{p})$,

$$\|B_j(f,g)\|_{q/2}\le C2^{-j\frac{4d(d-1)}{d+1}(\frac{1}{p}-\frac{d-1}{2d})}\delta^{\frac{4d}{d+1}(\frac{1}{p}-\frac{d-1}{2d})}\|f\|_p\|g\|_p. \tag{2.7}$$

To finish the proof of Theorem 2.1, let us introduce useful interpolation lemma.

Lemma 2.5 (Interpolation lemma). *Let $\varepsilon_1,\ \varepsilon_2 > 0$. Suppose that T_j be l-linear operators satisfying that for $1 \le p_1^i, p_2^i, < \infty$ (here the superscript i is not an exponent, but an index), $i = 1,\dots l$ and $1 < q_1, q_2 < \infty$, $\|T_j(f^1,\dots,f^l)\|_{q_1} \le M_1 2^{\varepsilon_1 j}\prod_{i=1}^{l}\|f^i\|_{p_1^i}$ and $\|T_j(f^1,\dots,f^l)\|_{q_2} \le M_2 2^{-\varepsilon_2 j}\prod_{i=1}^{l}\|f^i\|_{p_2^i}$. Then $T = \sum T_j$ satisfies*

$$\|T(f^1,\dots,f^l)\|_{q,\infty} \le CM_1^{\theta}M_2^{1-\theta}\prod_{i=1}^{l}\|f^i\|_{p^i,1} \tag{2.8}$$

where $\theta = \varepsilon_2/(\varepsilon_1+\varepsilon_2)$, $1/q = \theta/q_1 + (1-\theta)/q_2$, $1/p^i = \theta/p_1^i + (1-\theta)/p_2^i$, for $i = 1,\dots,l$.

By using the above two lemmas, we give the proof of Theorem 2.1 by summing up the estimates (2.6), (2.7). Applying (2.8) in Lemma 2.5 to (2.6) with $l = 2$, we can see that if for p,q satisfying $\frac{2d+4}{d} < q \le 4$, $2 \le p \le q$ and $\frac{d+1}{q} = (d-1)(1-\frac{1}{p})$, then

$$\|\textstyle\sum_{2^{-2j}\ge\delta}B_j(f,g)\|_{q/2,\infty} \le C\delta^{1-d+\frac{2d}{p}}\|f\|_{p,1}\|g\|_{p,1}. \tag{2.9}$$

Indeed, observe that in (2.6) the exponent on 2^j is negative if $\frac{d+1}{q} < (d-1)(1-\frac{1}{p})$, and positive if $\frac{d+1}{q} > (d-1)(1-\frac{1}{p})$. Using (2.8) in Lemma 2.5, we see that the restricted weak type estimates hold along the line $\frac{d+1}{q} = (d-1)(1-\frac{1}{p})$. On the other hand, from (2.7) we can see that for p,q satisfying $\frac{2d}{d-1} \le q \le 4$, $2 \le p < q$ and $\frac{d+1}{q} = (d-1)(1-\frac{1}{p})$,

$$\|\textstyle\sum_{2^{-2j}<\delta}B_j(f,g)\|_{q/2} \le C\delta^{1-n+\frac{2n}{p}}\|f\|_p\|g\|_p \tag{2.10}$$

because $p < \frac{2d}{d-1}$. From (2.9) and (2.10), using (2.5), it follows that for p,q satisfying $\frac{2d+4}{d} < q < \frac{2d+2}{d-1}$ and $\frac{d+1}{q} = (d-1)(1-\frac{1}{p})$,

$$\|(T_\delta f)^2\|_{q/2,\infty} \le C\delta^{1-d+\frac{2d}{p}}\|f\|_{p,1}^2.$$

This gives restricted weak type estimates for T_δ. Strong boundedness follows from real interpolation among the resulting estimates.

3. Applications

In this section, we introduce two applications of Theorem 2.1

3.1. **Bochner-Riesz operator.** The first is on the Bochner-Riesz operator. Let us define a modified Bochner-Riesz operator T^{α}, $\alpha \in \mathbb{R}$ by

$$\widehat{T^{\alpha}f}(\xi) = \frac{(\xi_d - \psi(\xi'))_+^{\alpha}}{\Gamma(\alpha+1)}\chi(\xi')\widehat{f}(\xi)$$
$$\psi(\xi') = 1 - \sqrt{1-|\xi'|^2}.$$

Obviously, ψ is an elliptic type function. Here, t_+ is the function such that $t_+ = t$ if $t > 0$ and $t_+ = 0$ if $t \leq 0$, and χ is a smooth function supported in a small neighborhood of the origin.

Now we decompose T^α case by case as follows:

$$\text{when } \alpha > 0, \text{ we use } \phi \in C_0^\infty(|t| \sim 1),$$
$$\text{and when } \alpha < 0, \text{ we use } \widehat{\phi} \in C_0^\infty(|\tau| \sim 1).$$

Then we have the following decomposition

$$\widehat{T^\alpha f}(\xi) \sim \sum_{l:\text{large}} 2^{-\alpha l}\phi(2^l(\xi_d - \psi(\xi')))\chi(\xi')\widehat{f}(\xi) + \text{ smooth error.}$$

Let 2^{-l} be δ. Then from the good localization property of K_δ and the interpolation lemma Lemma 2.5, it follows that for the case that $\alpha > 0$,

$$\|T^\alpha f\|_{L^p} \lesssim \|f\|_{L^p}$$
$$\text{if } \alpha > \max\left(d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right) \text{ and } p > \frac{2d+4}{d} \text{ or } p < \frac{2d+4}{d+4}$$

(see [8] for the detailed proof) and for the case that $\alpha < 0$

$$\|T^\alpha f\|_{L^q} \lesssim \|f\|_{L^p}$$
$$\text{if } \alpha < -\frac{d^2-d-2}{2(d^2+d-2)} \text{ and } (1/p, 1/q) \in \Delta_\alpha(d),$$

where

$$\Delta_\alpha(d) \equiv \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in [0,1] \times [0,1] : \frac{1}{p} - \frac{1}{q} \geq \frac{2\alpha}{d+1}, \frac{1}{p} > \frac{d-1}{2d} + \frac{\alpha}{d}, \frac{1}{q} < \frac{d+1}{2d} - \frac{\alpha}{d}\right\}$$

(see [1] for the detailed proof).

3.2. **Convolution operator with singularity on the cone.** The next application is on a Fourier multiplier operator defined in a small neighborhood of light cone. Let $\delta > 0$ and let S_δ be a multiplier operator given by

$$\widehat{S_\delta f}(\eta, \rho, \tau) = \phi(\rho)\psi\left(\frac{(\tau - |\eta|^2/\rho)}{\delta}\right)b(\eta)\widehat{f}(\eta, \rho, \tau)$$

where $\phi \in C_0^\infty(1/2, 2)$, $\psi \in \mathcal{S}(\mathbb{R})$ and b is a smooth function supported in $C_0^\infty(B(0,1))$, $B(0,1) \subset \mathbb{R}^{d-1}$.

Following the similar strategy of proof for S_δ (instead of using the bilinear estimate for elliptic surface, using the bilinear estimate for cone [11, 15]), we can have

Theorem 3.1. *If* $(d-1)(1-\frac{1}{p}) = (d+1)\frac{1}{q}$ *and* $q > q_0 = \frac{2(d^2+2d-1)}{d^2-1}$*, then there is a constant* C *such that*

$$\|S_\delta f\|_q \le C\delta^{\frac{d+1}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p. \tag{3.1}$$

And when $(p,q) = (p_0,q_0)$, $S_\delta f$ *is of restricted weak type* (p_0,q_0)*. Namely,*

$$\|S_\delta f\|_{q_0,\infty} \le C\delta^{\frac{d+1}{2}(\frac{1}{p_0}-\frac{1}{q_0})}\|f\|_{p_0,1}. \tag{3.2}$$

As a consequence of Theorem 3.1, we first obtain a sharp estimate for a convolution operator with kernel defined by

$$K^z(y,s,t) = \left\{ \begin{array}{ll} (ts-\frac{|y|^2}{4})_+^z/\Gamma(z+1) & \text{if } t>0 \\ 0 & \text{if } t \le 0 \end{array} \right\}, \quad Re(z) > -1.$$

The result is that there is a constant C such that

$$\|K^z * f\|_{\widetilde{B}^s_{q,r}} \le C\|f\|_{\widetilde{B}^s_{p,r}} \text{ for } -\frac{d(d+1)^2}{2(d^2+2d-1)} \le Re(z) < 0 \tag{3.3}$$

and for all $f \in \mathcal{S}(\mathbb{R}^{d+1})$, provided p,q satisfies

$$\frac{1}{p} - \frac{1}{q} = 1 + \frac{2Re(z)}{d+1}, \quad 1 + \frac{Re(z)}{d} < \frac{1}{p} < 1 + \frac{Re(z)(d-1)}{d^2+d}, \tag{3.4}$$

where $\widetilde{B}^s_{p,r}$ is a kind of homogeneous Besov space, which is equipped with the norm:

$$\|f\|_{\widetilde{B}^s_{p,r}} := \left(\sum_k 2^{srk}\|\Delta_k f\|_p^r\right)^{\frac{1}{r}}, \quad 1 \le p,r \le \infty, \quad s \in \mathbb{R},$$

where $\widehat{\Delta_k f}(\eta,\rho,\tau) = \phi(|\rho|/2^k)\widehat{f}(\eta,\rho,\tau)$, $(\eta,\rho,\tau) \in \mathbb{R}^{d-1}\times\mathbb{R}\times\mathbb{R}$.

The kernel K^z was studied by Oberlin [9] and Harmse [5] and the estimate (3.3) is a slight improvement of the previous result that $\|K^z * f\|_{L^q} \le C\|f\|_{L^p}$ for $-\frac{n}{2} \le Re(z) \le 0$ with (p,q) satisfying (3.4).

To prove (3.3) let us set

$$K^z_{l,k}(y,s,t) = 2^{zk+zl}\psi_z(t/2^k)\phi_z((s-|y|^2/t)/2^l)$$

with ψ_z,ϕ_z such that $\psi_z \in C_0^\infty(1/2,2)$, $\widehat{\phi}_z \in C_0^\infty((-2,-1/2)\cup(1/2,2))$ and for all $h \in \mathcal{S}(\mathbb{R}^{d+1})$,

$$\langle K^z, h\rangle = \sum_l\sum_k 2^{z(k+l)}\iiint \psi_z(t/2^k)\phi_z(\frac{s-|y|^2/4t}{2^l})h(y,s,t)\,ds\,dy\,dt. \tag{3.5}$$

Then by a dyadic decomposition, we can write $K^z * f$ as

$$K^z * f = \sum_k\sum_l K^z_{l,k} * f \tag{3.6}$$

provided $-(d+1)/2 < Re(z) < 0$. The estimate (3.3) follows from Theorem 3.1 and the interpolation lemma Lemma 2.5. By another use of Theorem 3.1, we can also have the improved sharp estimate for cone multiplier operator with negative index. For the details, see [2].

Apart from sharp estimates, it is possible to get some better bound for the cone multiplier operator with negative index if one use the local smoothing estimates due to Wolff [14] and Wolff and Laba [6] and the argument in [7] based on some scaling method. Recently, G. Garrigos improved the Wolff's local smoothing estimate [4]. Thus a more improvement is now possible.

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REFERENCES

[1] Y. Cho, Y. Kim, S. Lee and Y. Shim *Sharp L^p-L^q estimates for Bochner-Riesz operators of negative index in $\mathbb{R}^n$, $n \geq 3$,* J. Funct. Anal. (1) 218 (2005), 150-167.

[2] Y. Cho, Y. Kim, S. Lee and Y. Shim, *$L^p - L^q$ estimates for convolutions with distribution kernels having singularities on the light cone,* Hokkaido Univ. Preprint Series in Math. ♯684 (2005).

[3] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9-36.

[4] G. Garrigos, *Bergman projections and Wolff's local smoothing inequalities in light-cones,* Symposium "Harmonic Analysis and its Applications at Sapporo", Hokkaido University, Aug 22, 2005.

[5] J. Harmse, *On Lebesgue space estimates for the wave equation*, Indiana Univ. Math. J. **39** (1990), 229-248.

[6] I. Laba and T. Wolff, *A local smoothing estimate in higher dimensions*, J. d'Analyse Math. **88** (2002), 149-171.

[7] S. Lee, *Some sharp bounds for the cone multiplier of negative order in $\mathbb{R}^3$*, Bull. London Math. Soc. **35** (2003), 373-390.

[8] S. Lee, *Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators*, Duke Math. Journ. **122** (2004), 205-232.

[9] D. Oberlin, *Convolution estimates for some distribution with singularites on the light cone*, Duke Math. Journ. **59** (1989), 747-757.

[10] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

[11] T. Tao, *Endpoint bilinear restriction theorems for the cone and some sharp null from estimates*, Math. Z. **238** (2001), 215-268.

[12] T. Tao, *A sharp bilinear restriction estimate for paraboloids*, GAFA **13** (2003), 1359-1384.

[13] T. Tao, A. Vargas and L. Vega *A bilinear approach to the restriction and Kakeya conjecture*, J. Amer. Math. Soc. **11** (1998), 967-1000.

[14] T. Wolff, *Local smoothing type estimates on L^p for large p*, Geom. Funct. Anal. **10** (2000), 1237-1288.

[15] T. Wolff, *A sharp cone restriction estimate*, Annals of Math. **153** (2001), 661-698.

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On the law of the iterated logarithm for gap series

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In this note, we make a brief survey of studies on the law of the iterated logarithm for the series $\sum_{j=1}^{k} f(n_j x)$, where n_j is a rapidly increasing sequence of integers, and f is a function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)\,dx = 0, \quad \|f\|_2^2 = \int_0^1 |f(x)|^2\,dx < \infty. \tag{1}$$

Kac [13] proved the central limit theorem

$$\left|\left\{x \in [0,1] \;\middle|\; \frac{1}{\sqrt{k}\|f\|} \sum_{j=1}^{k} f(n_j x) \le a\right\}\right| \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2}\,du, \qquad (n \to \infty),$$

by assuming the large gap condition

$$\frac{n_{k+1}}{n_k} \to \infty,$$

and β-Lipschitz continuity of f ($\beta > 0$). Later, Takahashi [17] relaxed the smoothness condition to

$$\|f(\cdot + h) - f(\cdot)\|_2 = O((\log 1/h)^{-1-\delta}),$$

where $\delta > 0$.

As to the law of the iterated logarithm, Takahashi [18] proved

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k \log\log k}} = \sqrt{2}\,\|f\|_2, \tag{2}$$

by assuming the large gap condition and β-Lipschitz continuity ($\beta > 0$). We must mention here that we cannot weaken the gap condition to Hadamard gap condition

$$\frac{n_{k+1}}{n_k} > q > 1.$$

It is known by the example given by Erdős-Fortet: for $f(x) = \cos 2\pi x + \cos 4\pi x$ and $n_k = 2^k + 2$, we have

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k \log\log k}} = 2|\cos 2\pi x|.$$

Even we don't have the law of the iterated logarithm exactly in the same way as the sum of independent random variables, we still can expect the upper bound estimate. The result in this direction was first obtained by Takahashi [17]: if one assume the Hadamard gap condition and β-Lipschitz continuity, one has

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k \log\log k}} < C < \infty \qquad \text{a.e. } x. \tag{3}$$

By applying the methods presented by Takahashi, Philipp [20] proved the following uniform law of the iterated logarithm:

$$\limsup_{k\to\infty} \sup_{\|f\|_{\mathrm{BV}}\le 1} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k\log\log k}} < C < \infty,$$

where the Hadamard gap condition is assumed and the supremum is taken over the functions satisfying (1) with total variation less than or equal to 1.

Kaufman-Philipp [14] proved the following variation of the uniform law of the iterated logarithm: Let $\beta > 1/2$ and n_k satisfies the Hadamard gap condition. Then

$$\limsup_{k\to\infty} \sup_{\|f\|_{\mathrm{Lip}\,\beta}\le 1} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k\log\log k}} < C < \infty,$$

where supremum is taken over the class $\mathrm{Lip}\,\beta$ ($\beta > 1/2$) of functions satisfying (1) with β-Lipschitz continuity.

By the way, in the classical case when f is a single trigonometric function, i.e., in the case of lacunary trigonometric series, the central limit theorem and the law of the iterated logarithm were first proved by assuming the large gap condition, and then extended to the case of Hadamard gap condition, and in the last stage is was proved by assuming the Takahashi gap condition:

$$\frac{n_{k+1}}{n_k} > 1 + ck^{-\alpha} \qquad (c > 0,\ \alpha < 1/2),$$

and was also prove that there is a counterexample for these theorems if we relax this condition to $\alpha = 1/2$. From this point of view, it is very natural to expect to extend the above results to Takahashi gap sequence.

It was accomplished by Dhompongsa [6]: If one assume Takahashi gap condition and $\beta > 1/2$, then

$$\limsup_{k\to\infty} \sup_{\|f\|_{\mathrm{Lip}\,\beta}\le 1} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k\log\log k}} < C < \infty.$$

By inspired by this result, Takahashi tried to give a concrete estimate of C, and proved the following law of the iterated logarithm: If one assume the Takahashi gap condition and $f \in \mathrm{Lip}\,\beta$ for some $\beta > \frac{1}{2}$, then

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k\log\log k}} \le \|f\|_A = \sum_{\nu=-\infty}^{\infty} |\widehat{f}(\nu)|. \tag{4}$$

Although it is not a uniform law of the iterated logarithm, it must be noted that it gives a concrete estimate of the upper bound of the law of the iterated logarithm.

The celebrated Bernstein's theorem claims $\|f\|_A < \infty$ and the Fourier series converges uniformly if f is β-Lipschitz continuous for some $\beta > 1/2$. Thus it is very natural to expect that (4) holds under the condition $\|f\|_A < \infty$. Berkes [2] studied in this direction and succeeded in proving

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k\log\log k}} < \infty, \qquad \text{a.e.}$$

under $\|f\|_A < \infty$. By modifying that method we proved the following results [7, 8]:

1) The Takahashi gap condition and $\|f\|_A < \infty$ imply (4).
2) Let the Fourier coefficients of f is parallel, i.e., $\arg \widehat{f}(\nu)$ does not vary for all $\nu > 0$. Then for all $\varepsilon > 0$, there exists $\{n_k\}$ with Hadamard gaps such that

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^k f(n_j x)}{\sqrt{k \log\log k}} \geq \|f\|_A - \varepsilon.$$

3) Let the Fourier coefficients of f is parallel. Then for all $q_k \downarrow 1$, there exists $\{n_k\}$ such that

$$\frac{n_{k+1}}{n_k} > q_k \quad \text{and} \quad \limsup_{k\to\infty} \frac{\sum_{j=1}^k f(n_j x)}{\sqrt{k \log\log k}} = \|f\|_A.$$

By 1), we see that (4) holds under minimal condition $\|f\|_A < \infty$, and even we can say that only (1) implies (4) since (4) is trivial in case $\|f\|_A = \infty$. By 2), we see that this estimate is best possible if we consider the function with parallel Fourier coefficients and the sequence with Hadamard gaps. And by 3), we see the following: if we assume any gap condition weaker than Hadamard's, for any function with parallel Fourier coefficients, there exists a sequence satisfying the gap condition which attains the upper bound.

As to the uniform law of the iterated logarithm, we have the concrete estimate in the following way ([9]): Let $\mathrm{Lip}_{L^2}\,\beta$ be the class of function satisfying (1) and

$$\|f(\,\cdot\, + h) - f(\,\cdot\,)\|_2 = O(h^\beta).$$

In the following results, we assume $X \subset \mathrm{Lip}_{L^2}\,\beta$ for some $\beta > \frac{1}{2}$.

4) If one assume the Takahashi gap condition, then one has

$$\limsup_{k\to\infty} \sup_{f\in X} \frac{\sum_{j=1}^k f(n_j x)}{\sqrt{k \log\log k}} \leq \sup_{f\in X} \|f\|_A.$$

5) If one assume the large gap condition, then one has

$$\limsup_{k\to\infty} \sup_{f\in X} \frac{\sum_{j=1}^k f(n_j x)}{\sqrt{k \log\log k}} = \sqrt{2} \sup_{f\in X} \|f\|_2.$$

6) Let us assume that X is round in the following sense: for all $f \in X$, one can find $g \in X$ with parallel Fourier coefficients. Then for all $\varepsilon > 0$, there exists $\{n_k\}$ with Hadamard gaps such that

$$\limsup_{k\to\infty} \sup_{f\in X} \frac{\sum_{j=1}^k f(n_j x)}{\sqrt{k \log\log k}} \geq \sup_{f\in X} \|f\|_A - \varepsilon$$

7) Let us assume again that X is round. Then for all $q_k \downarrow 1$, there exists $\{n_k\}$ such that

$$\frac{n_{k+1}}{n_k} > q_k \quad \text{and} \quad \limsup_{k\to\infty} \sup_{f\in X} \frac{\sum_{j=1}^k f(n_j x)}{\sqrt{k \log\log k}} = \sup_{f\in X} \|f\|_A.$$

For β-Lipschitz class with $\beta > 1/2$, every results concerning the law of the iterated logarithm for Hadamard gap sequence were extended to the case of Takahashi gap sequence. Berkes [2] gave a counterexample function f of 1/2-Lipschitz class which does not obey the upper bound estimate for the law of the iterated logarithm for Takahashi gap sequence. Since the class of functions of bounded variation is the subclass of the L^2-1/2-Lipschitz class, it was not clear whether the results of Philipp can be extended in the case of the Takahashi gap sequence. As to this problem, Berkes–Philipp [3, 4] proved the following and showed that it cannot be extended for any gap condition weaker than the Hadamard's: For $f(x) = \langle x \rangle - 1/2$, and for all $q_k \downarrow 1$, there exists $\{n_k\}$ such that

$$\frac{n_{k+1}}{n_k} > q_k \quad \text{and} \quad \limsup_{k\to\infty} \frac{\sum_{j=1}^{k} f(n_j x)}{\sqrt{k \log\log k}} = \infty. \tag{5}$$

It is very natural to ask whether this phenomenon occur for all function with discontinuity. As to this problem, we can prove the following ([10]):

8) Assume that $f \in \mathrm{Lip}_{L^2} \frac{1}{2}$ has $x_0 \in \mathbf{Q}$ such that limits $f(x_0 - 0)$ and $f(x_0 + 0)$ exists and are distinct. Then, for all $q_k \downarrow 1$, there exists $\{n_k\}$ such that (5) holds.
9) Assume that $\|f\|_A = \infty$, and that $\{\Re \widehat{f}(\nu)\}_{\nu>0}$ or $\{\Im \widehat{f}(\nu)\}_{\nu>0}$ is positive, negative or alternating. Then for all $q_k \downarrow 1$, there exists $\{n_k\}$ such that (5) holds.

The result 9) has important examples as $\log|2\sin \pi x|$, $\log|2\cos \pi x| \in \mathrm{Lip}_{L^2} 1/2$. These function plays key role when we investigate limiting behavior of the product of the lacunary trigonometric functions as $\left|\prod_{k=1}^{N} 2\sin \pi n_k x\right|$ and $\left|\prod_{k=1}^{N} 2\cos \pi n_k x\right|$. These functions are not Lipschitz continuous nor bounded, and we cannot apply the classical results by Takahashi. Although Berkes [1] proved (2) for large gap sequence and (3) for Hadamard gap sequence when f is bounded L^2-β-Lipschitz continuous, we cannot apply this results because our functions are not bounded. And these assumptions are much stronger than the these appearing in the central limit theorem by Takahashi. We tried to relax these conditions and obtained the following ([11]):

10) We have (2) for large gap sequence and (3) for Hadamard gap sequence under

$$\|f(\,\cdot\, + h) - f(\,\cdot\,)\|_2 = O((\log \frac{1}{h})^{-3} (\log\log \frac{1}{h})^{-1}).$$

We can also see that the almost sure invariance principle type results under L^2-β-Lipschitz continuity ([12]).

By applying this result, we can conclude that, the large gap condition implies

$$\limsup_{k\to\infty} \left| \prod_{j=1}^{k} 2\cos(\pi n_j x) \right|^{1/\sqrt{k\log\log k}} = e^{\pi/\sqrt{6}},$$

and the Hadamard gap condition implies

$$\limsup_{k\to\infty} \left| \prod_{j=1}^{k} 2\cos(\pi n_j x) \right|^{1/\sqrt{k\log\log k}} < C < \infty.$$

On the other hand, by applying 9), we can conclude that for all $q_k \downarrow 1$, there exists $\{n_k\}$ such that

$$\frac{n_{k+1}}{n_k} > q_k \quad \text{and} \quad \limsup_{k\to\infty} \left| \prod_{j=1}^{k} 2\cos(\pi n_j x) \right|^{1/\sqrt{k\log\log k}} = \infty.$$

References

[1] I. Berkes, On the asymptotic behavior of $\sum f(n_k x)$. I. Main theorems, II. Applications, Z. Wahr. verv. Geb. **34** (1976) 319–345, 347–365

[2] I. Berkes, On the convergence of $\sum f(nx)$ and the Lip 1/2 class Trans. Amer. Math. Soc. **349** (1997) 4143–4158

[3] I. Berkes & W. Philipp, The size of trigonometric and Walsh series and uniform distribution mod 1 J. London Math. Soc. (2) **50** (1994) 454-464

[4] I. Berkes & W. Philipp, Trigonometric series and uniform distribution mod 1 Studia Sci. Math. Hungar. **31** (1996) 15–25

[5] I. Berkes & W. Philipp, A limit theorem for lacunary series $\sum f(n_k x)$ Studia Sci. Math. Hungar. **34** (1998) 1–13

[6] S. Dhompongsa, Uniform laws of the iterated logarithm for Lipschitz classes of functions, Acta Sci. Math. **50** (1986) 105-124.

[7] K. Fukuyama, An asymptotic property of gap series, Acta Math. Hungar., **97** (2002) 209–216.

[8] K. Fukuyama, An asymptotic property of gap series III, Acta Math. Hungar., **102** (2004) 97–106.

[9] K. Fukuyama, A concrete upper bound in the uniform law of the iterated logarithm, Studia Sci. Math. Hungar., **41** (2004) 339-346.

[10] K. Fukuyama, Gap series and function of bounded variation Acta Math. Hungar., (to appear).

[11] K. Fukuyama, The law of the iterated logarithm for gap series of function with weak regularity (preprint)

[12] K. Fukuyama, On the asymptotic property of the product of gap sequence (preprint)

[13] M. Kac, Probability methods in some problems of analysis and number theory, Bull. Amer. Math. Soc. **55** (1949) 641–665.

[14] R. Kaufman and W. Philipp, A uniform law of the iterated logarithm for classes of functions, Ann. Probab., **5** (1978) 930–952.

[15] E. Péter, A probability limit theorem for $\{f(nx)\}$, Acta Math. Hungar., **87** (2000), 23–31.

[16] S. Takahashi, A gap sequence with gaps bigger than the Hadamards Tôhoku Math. J. (2) **13** (1961) 105–111

[17] S. Takahashi, An asymptotic property of a gap sequence, Proc. Japan Acad. **38** (1962), 101-104.

[18] S. Takahashi, The law of the iterated logarithm for a gap sequence with infinite gaps, Tôhoku Math. J., **15** (1963) 281–288.

[19] S. Takahashi, An asymptotic behavior of $\{f(n_k t)\}$, Sci. Rep. Kanazawa Univ., **33** (1988) 27-36.

[20] W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1, Acta Arithmetica, **26** (1975) 241-251.

A NOTE ON PLATE DECOMPOSITIONS OF CONE MULTIPLIERS

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ABSTRACT. We observe that the range of p for Wolff's inequality on plate decompositions of cone multipliers can be improved by using bilinear restriction techniques. This in turn is known to improve the range for sharp L^p results on cone multipliers, local smoothing for the wave equation, convolutions with radial kernels, Bergman projections in tubes over cones, averages over finite type curves in $\mathbb{R}^3$ and associated maximal functions. We also give some improved estimates on square functions associated to cone multipliers.

1. INTRODUCTION

Let $\Gamma = \{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d \;:\; \tau = |\xi|\}$ denote the forward light-cone in $\mathbb{R}^{d+1}$, $d \geq 2$. For fixed $c > 0$ and small $\delta > 0$, we consider δ-neighborhoods of the truncated cone

$$\Gamma_\delta(c) = \{(\tau,\xi) \in \mathbb{R}^{d+1} \;:\; 1 \leq \tau \leq 2 \quad \text{and} \quad \big|\tau - |\xi|\big| \leq c\delta\},$$

with the usual decomposition into plates subordinated to a $\sqrt{\delta}$-*separated* sequence in the sphere $\{\omega_k\} \subset S^{d-1}$:

$$\Pi_k^{(\delta)} = \Big\{(\tau,\xi) \in \Gamma_\delta(c) \;:\; \big|\xi/|\xi| - \omega_k\big| \leq c'\sqrt{\delta}\Big\}; \qquad \text{dist}(\omega_k,\omega_{k'}) \geq \sqrt{\delta} \quad \text{if } k \neq k'. \tag{1.1}$$

Let

$$\alpha(p) := d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, \tag{1.2}$$

the standard Bochner-Riesz critical index in d dimensions. Then Wolff's inequality asserts that for all $\varepsilon > 0$

$$\Big\|\sum_k f_k\Big\|_p \leq C_\varepsilon\, \delta^{-\alpha(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p} \tag{1.3}$$

provided that

$$\text{supp}\ \widehat{f}_k \subset \Pi_k^{(\delta)}. \tag{1.4}$$

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The power $\alpha(p)$ is optimal for each p (except perhaps for $\varepsilon > 0$), and the inequality is conjectured to hold for all $p > 2 + \frac{4}{d-1}$. In his fundamental work [22], Wolff developed a method to prove such inequalities for large values of p, and obtained a positive answer for $d = 2$ and $p > 74$. Subsequently the method has been extended in the paper by Łaba and Wolff [11] to higher dimensions. It is shown there that (1.3) holds for $p > 2 + \frac{32}{3d-7}$ when $d \geq 3$ and $p > 2 + \frac{8}{d-3}$ when $d \geq 4$. In this paper we modify the weakest part of their proof to obtain a better range of exponents in all dimensions (see Table 1 below). The improvement relies on certain square-function bounds which follow from Wolff's bilinear Fourier extension theorem, [23].

Dimension	[22], [11]	Improvements	Conjecture
$d = 2$	$p > 74$	$p > p_2 := 63 + 1/3$	$p > 6$
$d = 3$	$p > 18$	$p > p_3 := 15$	$p > 4$
$d = 4$	$p > 8.4$	$p > p_4 := 7.28$	$p > \frac{10}{3}$
$d \geq 5$	$p > 2 + \frac{8}{d-3}$	$p > p_d := 2 + \frac{8}{d-3}(1 - \frac{1}{d+1})$	$p > 2 + \frac{4}{d-1}$

TABLE 1. Range of exponents for the validity of (1.3) for light-cones.

Theorem 1.1. *Let $d \geq 2$ and p_d as in Table 1. Then, under the assumption* (1.4) *the inequality (1.3) holds for all $\varepsilon > 0$ and all $p \geq p_d$.*

A similar result for decompositions of spheres in $\mathbb{R}^d$ can be formulated as follows. We now let

$$\mathcal{S}_\delta(c) = \{\xi \in \mathbb{R}^d : \big||\xi| - 1\big| \leq c\delta\},$$

and consider the decomposition into rectangular "caps" subordinated to a $\sqrt{\delta}$-separated sequence $\{\omega_k\} \subset S^{d-1}$,

$$C_k^{(\delta)} = \Big\{\xi \in \mathcal{S}_\delta(c) : \big|\xi/|\xi| - \omega_k\big| \leq c'\sqrt{\delta}\Big\}.$$

Theorem 1.2. *The analog of Wolff's inequality for the sphere,*

$$\Big\|\sum_k f_k\Big\|_p \leq C_\varepsilon\, \delta^{-\alpha(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}, \quad supp\ \widehat{f_k} \subset C_k^{(\delta)}, \tag{1.5}$$

holds for $p \geq 2 + \frac{4}{d-1}(2 - \frac{1}{d})$ and all $\varepsilon > 0$.

Again (1.5) is conjectured to hold for the optimal range $p > 2 + 4/(d-1)$. It has been known to hold for $p > 2 + 8/(d-1)$; this follows from a modification of the argument in [11], see also [10]. Note that in two dimensions the range is improved from previously $p > 10$ to $p > 8$.

Remark 1.3. Theorem 1.2 may be extended to convex surfaces with nonvanishing Gaussian curvature and similarly Theorem 1.3 may be extended to cones with $d - 1$ positive principal curvatures. This can be achieved by using scaling and induction on scales arguments such as in §2 of [16].

We proceed to list some of the known implications of Theorem 1.1.

Corollary 1.4. *Let $d \geq 2$ and p_d as in Table 1. Then*

(*i*) *For all $p > p_d$, $\alpha > \frac{d-1}{2} - \frac{d}{p}$, we have*

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|^p_{L^p(\mathbb{R}^d)}dt\Big)^{1/p} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^d)}. \tag{1.6}$$

(*ii*) *For all $p \in (p_d, \infty)$, $\alpha > \frac{d-1}{2} - \frac{d}{p}$ the Fourier multiplier*

$$m_\alpha(\tau, \xi) = (1 - |\xi|^2/\tau^2)^\alpha_+ \tag{1.7}$$

defines a bounded operator in $L^p(\mathbb{R}^{d+1})$.

(*iii*) *Let $\varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ so that φ is radial and not identically zero, and let $\varepsilon > 0$. Let $K_t = \mathcal{F}^{-1}[\varphi \widehat{K}(t\cdot)]$. Then for all Schwartz functions f and $1 < p < \frac{p_d}{p_d - 1}$*

$$\|K * f\|_p \leq C_\varepsilon \sup_{t>0} \|K_t\|_{p+\varepsilon}\|f\|_p.$$

(*iv*) *Let $\chi \in C_0^\infty(\mathbb{R})$ and $s \mapsto \gamma(s) \in \mathbb{R}^3$ a smooth curve satisfying $\sum_{j=1}^n |\langle\theta, \gamma^{(j)}(s)\rangle| \neq 0$ for every unit vector θ and every $s \in supp\ \chi$. For $t > 0$ define the convolution operator A_t by*

$$A_t f(x) = \int f(x - t\gamma(s))\chi(s)ds.$$

Suppose that $\max\{n, 32+2/3\} < p < \infty$. *Then A_t maps $L^p(\mathbb{R}^3)$ into the L^p-Sobolev space $L^p_{1/p}(\mathbb{R}^3)$. Moreover the maximal function $Mf = \sup_t |A_t f|$ defines a bounded operator on $L^p(\mathbb{R}^3)$.*

Parts (i), (ii), (iii) are standard consequences of Theorem 1.1; see [22] for (i) and the local version of (ii). The global version follows by results on dyadic decompositions of multipliers and L^p Calderón-Zygmund theory (see [6] or [17]). The proof of Theorem 1.6 in [15] together with these arguments can be used to deduce (iii) from Theorem 1.1. For (iv) see [16].

Besides the connection to cone multipliers a major motivation for this paper is the relevance of inequalities for plate decompositions for the boundedness properties of the Bergman projection in tube domains over full light cones, see [1], [2]. Denote by $\Delta(Y) = y_0^2 - |y'|^2$ the Lorentz form and consider the forward light cone on which Δ is positive;

$$\Lambda^{d+1} = \{Y = (y_0, y') \in \mathbb{R} \times \mathbb{R}^d : y_0^2 - |y'|^2 > 0, y_0 > 0\}.$$

Let $\mathcal{T}^{d+1} \subset \mathbb{C}^{d+1}$ be the tube domain over Λ^{d+1}, i.e.

$$\mathcal{T}^{d+1} = \mathbb{R}^{d+1} + i\Lambda^{d+1}.$$

Let $w_\gamma(Y) = \Delta(Y)^\gamma$ and consider the weighted space $L^p(\mathcal{T}^{d+1}, w_\gamma)$ with norm

$$\|F\|_{p,\gamma} = \Big(\iint_{\mathcal{T}^{d+1}} |F(X + iY)|^p \Delta^\gamma(Y)\, dY dX\Big)^{1/p}.$$

Let $\mathcal{P}_\gamma$ be the orthogonal projection mapping the weighted space $L^2(\mathcal{T}^{d+1}, w_\gamma)$ to its subspace $\mathcal{A}^p_\gamma$ consisting of the holomorphic functions. Only the case $\gamma > -1$ is interesting since $\mathcal{A}^p_\gamma = \{0\}$ for

$\gamma \le -1$. We are interested in the L^p boundedness properties of $\mathcal{P}_\gamma$. For $\gamma > -1$ the operator $\mathcal{P}_\gamma$ can only be bounded on $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the range

$$1 + \frac{d-1}{2(\gamma+d+1)} < p < 1 + \frac{2(\gamma+d+1)}{d-1}, \tag{1.8}$$

see e.g. Theorem 4.3 in [3], and (1.8) is indeed the conjectured range for L^p boundedness.

Corollary 1.5. *Let $d \ge 2$ and p_d as in Table 1. Then for all $\gamma \ge \frac{d-1}{2}(p_d - \frac{2(d+1)}{d-1})$, the Bergman projection $\mathcal{P}_\gamma$ is a bounded operator in $L^p(\mathcal{T}^{d+1}, w_\alpha)$ in the sharp range (1.8).*

Beyond Corollary 1.5 both Theorem 1.1 and Theorem 4.3 below have implications for the range of boundedness of the Bergman projector $\mathcal{P}_\gamma$ in natural weighted mixed norm spaces. We refer to the derivation of Corollary 1.5 and further discussion to [2] (in particular Proposition 5.5 and Corollaries 5.12 and 5.17).

Our approach to Theorem 1.1 is based on bilinear methods, for which we consider a closely related inequality:

$$\Big\| \sum_k f_k \Big\|_p \le C_\alpha \, \delta^{-\alpha} \Big(\sum_k \|f_k\|_p^2 \Big)^{1/2}. \tag{1.9}$$

One can conjecture the validity of (1.9) for all $\alpha > 0$ and all $2 < p < 2 + \frac{4}{d-1}$, but for the moment no positive result for any such p seems to be known. The limiting point $p = \frac{2(d+1)}{d-1}$ should be the hardest case, since by interpolation and Hölder's inequality it implies both (1.9) and (1.3) in all the conjectured ranges. This kind of inequalities arises naturally in the study of weighted mixed norm inequalities for the Bergman projection operator $\mathcal{P}_\gamma$, see [2].

We shall deduce Theorem 1.1 by using a stronger version of (1.9) for $p = 2(d+3)/(d+1)$, but with a power of $1/\delta$ which is (probably) not optimal. Namely under the assumption (1.4) we have

$$\Big\| \sum_k f_k \Big\|_{\frac{2(d+3)}{d+1}} \le C_\varepsilon \, \delta^{-\frac{d-1}{4(d+3)} - \varepsilon} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_{\frac{2(d+3)}{d+1}}, \tag{1.10}$$

for all $\varepsilon > 0$. We prove this inequality in §2 using the bilinear approach of [20, §5] and the optimal bilinear cone restriction inequality of T. Wolff [23], see Proposition 2.3 below. By Minkowski's inequality and interpolation (1.10) trivially implies non optimal estimates for the inequality (1.9) for all $p \in (2, \infty)$ (see Corollary 2.4 below). In §3 we use these to refine a part of Wolff's proof of (1.3) and obtain the new sharp estimates for large p announced in Table 1. In §4 we improve on some of the square-function results in low dimensions; these yield in particular the estimate

$$\|\mathcal{F}^{-1}[m_\alpha \widehat{f}]\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^4(\mathbb{R}^3)}, \quad \alpha > \frac{5p_2 - 20}{44p_2 - 164}, \tag{1.11}$$

for the cone multiplier in $\mathbb{R}^{2+1}$, improving slightly on the Tao-Vargas result [20].

Further results. Various further improvements on the range of Theorem 1.1 have been recently obtained by the authors, and also by Wilhelm Schlag (personal communication). We plan to take up these matters in a joint paper [9].

Notation. We shall use the notation $A \lesssim B$ if there is a constant (which may depend on d) so that $A \le CB$. We use $A \lessapprox B$ if for every $\varepsilon \in (0,1)$ there is a constant C_ε so that $A \le C_\varepsilon \delta^{-\varepsilon} B$.

2. The bilinear estimate

Following the approach by Tao and Vargas, we first establish an equivalence between linear and bilinear versions of (1.10), which is a higher dimensional analog of Lemma 5.2 in [20].

Lemma 2.1. *Let $d \geq 2$, and suppose that for some $p \in [2, \infty)$ and $\alpha > \max\{0, (d-1)(1/4 - 1/p)\}$*

$$\Big\| \Big(\sum_{\omega_k \in \Omega} f_k \Big) \Big(\sum_{\omega_{k'} \in \Omega'} f_{k'} \Big) \Big\|_{p/2} \leq C\, \delta^{-2\alpha} \Big\| \Big(\sum_{\omega_k \in \Omega} |f_k|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_{\omega_{k'} \in \Omega'} |f_{k'}|^2 \Big)^{1/2} \Big\|_p, \tag{2.1}$$

holds for all $f_k \in \mathcal{S}(\mathbb{R}^{d+1})$ with supp $\hat{f}_k \subset \Pi_k^{(\delta)}$, all pairs of 1-separated subsets $\Omega, \Omega' \subset S^{d-1}$ and all $\delta \ll 1$. Then, it must also hold

$$\Big\| \sum_k f_k \Big\|_p \leq C'\, \delta^{-\alpha} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p. \tag{2.2}$$

Observe that $(d-1)(1/4 - 1/p) \leq \alpha(p)/2$ is equivalent with $d \geq 2$. Thus the restriction on α for $p > 4$ is never severe.

Proof of Lemma 2.1. Let $\Phi : Q \equiv [0,1]^{d-1} \to S^{d-1}$ be a smooth parametrization of (a compact subset of) the sphere and let $\mathcal{D}$ denote the set of all dyadic intervals $I \subset Q$ with $|I| \geq \delta^{\frac{d-1}{2}}$. As in [21, p. 971], we may consider a Whitney decomposition $Q \times Q = \biguplus_{I \sim J} I \times J$, where $I \sim J$ means:

(i) $I, J \in \mathcal{D}$ and $|I| = |J|$;

(ii) If $|I| > \delta^{\frac{d-1}{2}}$, then I and J are not adjacent but their parents are.

(iii) If $|I| = \delta^{\frac{d-1}{2}}$, then I, J have adjacent or equal parents.

For simplicity, we assume (by splitting the sphere into finitely many pieces) that all $\omega_k \in \Phi(Q)$. We also denote $\mathcal{D}_j = \{I \in \mathcal{D} \;:\; |I| = 2^{-j(d-1)}\}$. Then

$$\Big(\sum_k f_k \Big)^2 = \sum_{\omega_k, \omega_{k'} \in \Phi(Q)} f_k\, f_{k'} = \sum_{\sqrt{\delta} \leq 2^{-j} \leq 1} \sum_{\substack{I, J \in \mathcal{D}_j \\ I \sim J}} \Big(\sum_{\omega_k \in \Phi(I)} f_k \Big) \Big(\sum_{\omega_{k'} \in \Phi(J)} f_{k'} \Big).$$

To establish (2.2) we take $L^{p/2}$-norms in the above expression and use Minkowski's inequality in j, so that we reduce the problem to show, for each j

$$\Big\| \sum_{\substack{I, J \in \mathcal{D}_j \\ I \sim J}} \Big(\sum_{\omega_k \in I} f_k \Big) \Big(\sum_{\omega_{k'} \in J} f_{k'} \Big) \Big\|_{p/2} \lesssim (2^{2j}\delta)^{-\alpha} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p^2. \tag{2.3}$$

This is trivial when $2^{-j} \approx \sqrt{\delta}$ since by assumption the number of ω_k's in each I is approximately constant. We consider the general case $\sqrt{\delta} < 2^{-j} \leq 1$. By construction we must have

$$\sum_{I \in \mathcal{D}_j} \sum_{J \sim I} \chi_{I+J} \lesssim 1. \tag{2.4}$$

Indeed, if c_I denotes the center of I, then

$$I + J \subset \big(c_I + B_{c2^{-j}} \big) + \big(c_J + B_{c2^{-j}} \big) \subset 2c_I + B_{c'2^{-j}}.$$

Since for each I there are at most $O(1)$ cubes J with $J \sim I$, and since the centers c_I are 2^{-j} separated, (2.4) follows easily.

From (2.4) it follows that the functions $F_{I,J} = \big(\sum_{\omega_k\in\Phi(I)} f_k\big)\big(\sum_{\omega_{k'}\in\Phi(J)} f_{k'}\big)$ have pairwise (almost) disjoint spectra when $I \sim J \in \mathcal{D}_j$. We may conclude by orthogonality and standard interpolation arguments

$$\Big\| \sum_{I\sim J\in\mathcal{D}_j} F_{I,J} \Big\|_{p/2} \lesssim \max\{1, 2^{j(d-1)(1-4/p)}\}\Big(\sum_{I\sim J\in\mathcal{D}_j} \|F_{I,J}\|_{p/2}^{p/2}\Big)^{2/p}. \tag{2.5}$$

(the case $p/2 = 2$ follows by orthogonality and the cases $p/2 = 1$ and $p/2 = \infty$ are trivial; see e.g. Lemma 7.1 in [20]). Next, we wish to use the bilinear assumption (2.1) to estimate $\|F_{I,J}\|_{p/2}$. This can only be used directly when $2^j \approx 1$, since $\operatorname{dist}(I, J) \sim 1$. For other j's we must use Lorentz transformations to rescale the problem. To do this, let $\{\eta_1, \dots, \eta_d\}$ be an orthonormal basis of $\mathbb{R}^d$ with η_1 being the center of $\Phi(I)$. Then we define $L \in SO(1, d)$ acting on a basis of $\mathbb{R}^{d+1}$ by

$$L(1,\eta_1) = (1,\eta_1), \quad L(-1,\eta_1) = \tfrac{\sigma}{\delta}\,(-1,\eta_1) \quad \text{and} \quad L(0,\eta_\ell) = \sqrt{\tfrac{\sigma}{\delta}}\,(0,\eta_\ell),\ \ell = 2, ..., d,$$

where we choose $\sigma = 2^{2j}\delta$ (so that $\delta < \sigma < 1$). The functions $f_k \circ L$ have now spectrum in (perhaps a multiple) of the plates $\Pi_k^{(\sigma)}$ corresponding to the $\sqrt{\sigma}$-separated centers $\{L(1, \omega_k)\}$. Moreover, by the choice of σ, the plates corresponding to $\omega_k \in \Phi(I)$ and $\omega_{k'} \in \Phi(J)$ are 1-separated, and therefore after a change of variables we can apply (2.1) at scale σ to obtain

$$\begin{aligned}\|F_{I,J}\|_{p/2} &= \Big\| \Big(\sum_{\omega_k\in\Phi(I)} f_k\Big)\Big(\sum_{\omega_{k'}\in\Phi(J)} f_{k'}\Big) \Big\|_{p/2} \\ &\lesssim (2^{2j}\delta)^{-2\alpha} \Big\| \Big(\sum_{\omega_k\in\Phi(I)} |f_k|^2\Big)^{1/2} \Big\|_p \Big\| \Big(\sum_{\omega_{k'}\in\Phi(J)} |f_{k'}|^2\Big)^{1/2} \Big\|_p,\end{aligned}$$

and then also

$$\begin{aligned}\Big(\sum_{I\sim J\in\mathcal{D}_j} \|F_{I,J}\|_{p/2}^{p/2}\Big)^{2/p} &\lesssim (2^{2j}\delta)^{-2\alpha} \Big[\sum_{I\sim J\in\mathcal{D}_j} \Big\|\Big(\sum_{\omega_k\in\Phi(I)} |f_k|^2\Big)^{1/2}\Big\|_p^{\frac{p}{2}} \Big\|\Big(\sum_{\omega_{k'}\in\Phi(J)} |f_{k'}|^2\Big)^{1/2}\Big\|_p^{\frac{p}{2}} \Big]^{\frac{2}{p}} \\ &\lesssim (2^{2j}\delta)^{-2\alpha} \Big[\int \Big(\sum_I \sum_{\omega_k\in\Phi(I)} |f_k|^2\Big)^{p/2}\Big]^{2/p} \le (2^{2j}\delta)^{-2\alpha} \Big\| \Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_p^2,\end{aligned}$$

where we have used the inequalities $2ab \le a^2 + b^2$ in the second inequality and the imbedding $\ell^1 \hookrightarrow \ell^{\frac{p}{2}}$ in the fourth. Combining this with (2.5) we obtain

$$\Big\| \sum_{I\sim J\in\mathcal{D}_j} F_{I,J} \Big\|_{p/2} \lesssim (2^{2j}\delta)^{-2\alpha} \max\{1, 2^{j(d-1)(1-4/p)}\} \Big\| \Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_p^2, \tag{2.6}$$

and by our assumption on α we may sum in j. This proves (2.3) and establishes the lemma. □

We turn to the proof of (a generalization of) the square function estimate (1.10). We shall use the following statement of Wolff's Fourier extension theorem.

Wolff's bilinear estimate. [23, p. 680]. *Let $p \ge \frac{d+3}{d+1}$, $\varepsilon > 0$ and let E, E' be 1-separated subsets of $\Gamma_{1/N}$. Then, for all smooth f and g supported in E and E', and all N-cubes Q, we have*

$$\|\widehat{f}\widehat{g}\|_{L^p(Q)} \le C_\varepsilon\, N^{-1+\varepsilon}\, \|f\|_2\, \|g\|_2. \tag{2.7}$$

Denote by $\mathcal{Q} \equiv \mathcal{Q}(\delta^{-1/2})$ a tiling of $\mathbb{R}^{d+1}$ with cubes Q of disjoint interior and sidelength $\delta^{-1/2}$, with centers c_Q in $\delta^{-\frac{1}{2}}\mathbb{Z}^{d+1}$.

Proposition 2.2. *Let $d \geq 2$, and suppose that supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$, supp $\widehat{g_k} \subset \Pi_k^{(\delta)}$ and let $\Omega, \Omega' \subset S^{d-1}$ be 1-separated subsets. Suppose $\frac{2(d+3)}{d+1} \leq q \leq p \leq \infty$ and let*

$$\mu(p) = \frac{d}{4} - \frac{d+1}{2p}. \tag{2.8}$$

Then, for all $\varepsilon > 0$

$$\Big(\sum_{Q\in\mathcal{Q}(\delta^{-1/2})} \Big\| \big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|_{L^{q/2}(Q)}^{p/2}\Big)^{2/p} \tag{2.9}$$
$$\lesssim \delta^{-2\mu(p)-\varepsilon}\Big\|\Big(\sum_{\omega_k\in\Omega}|f_k|^2\Big)^{1/2}\Big\|_p\Big\|\Big(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\Big)^{1/2}\Big\|_p.$$

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$ be so that supp $\widehat{\psi} \subset B_{1/10}$ and $\psi(x) > 1$ if $|x_i| \leq 2$, $i = 1, \dots, d+1$; then $\sum_{n\in\mathbb{Z}^{d+1}} \psi(\cdot + n)^2 \approx 1$. Let $\psi_Q = \psi(\sqrt{\delta}(\cdot - c_Q))$, so that $\sum_Q \psi_Q^2 \approx 1$. We write

$$F^Q = \big(\sum_{\omega_k\in\Omega} f_k\big)\,\psi_Q \quad\text{and}\quad G^Q = \big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\,\psi_Q,$$

so that the supports of $\widehat{F^Q}$ and $\widehat{G^Q}$ are 1-separated sets in $\Gamma_{\sqrt{\delta}}$. Thus, we can use Wolff's estimate (2.7) with $N = \delta^{-1/2}$ to obtain

$$\Big\|\Big(\sum_{\omega_k\in\Omega} f_k\Big)\Big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\Big)\Big\|_{L^{q/2}(Q)} \lesssim \|F^Q G^Q\|_{L^{q/2}(Q)} \lessapprox \delta^{1/2}\,\|\widehat{F^Q}\|_2\,\|\widehat{G^Q}\|_2. \tag{2.10}$$

Now, by almost orthogonality we can write

$$\|\widehat{F^Q}\|_2^2 \approx \sum_k \|\widehat{f_k} * \widehat{\psi_Q}\|_2^2 = \Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\psi_Q\Big\|_2^2,$$

and similarly for G^Q. We write $S_\Omega = (\sum_{\omega_k\in\Omega}|f_k|^2)^{1/2}$, $\widetilde{S}_{\Omega'} = (\sum_{\omega_k\in\Omega'}|g_k|^2)^{1/2}$, raise (2.10) to the power $p/2$ and sum in Q. Thus

$$\Big(\sum_Q\Big\|\big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|_{L^{q/2}(Q)}^{p/2}\Big)^{2/p} \lessapprox \sqrt{\delta}\Big(\sum_Q \|S_\Omega\psi_Q\|_2^{p/2}\|S_{\Omega'}\psi_Q\|_2^{p/2}\Big)^{2/p}$$

and by the Cauchy-Schwarz and Hölder inequalities the right hand side is

$$\begin{aligned}
&\lesssim \sqrt{\delta}\Big(\sum_Q \|S_\Omega\psi_Q\|_2^p\Big)^{1/p}\Big(\sum_Q \|S_{\Omega'}\psi_Q\|_2^p\Big)^{1/p} \\
&\lesssim \sqrt{\delta}\Big(\sum_Q \|S_\Omega\psi_Q\|_p^p|Q|^{-1+p/2}\Big)^{1/p}\Big(\sum_Q \|S_{\Omega'}\psi_Q\|_p^p|Q|^{-1+p/2}\Big)^{1/p} \\
&\lesssim \delta^{\frac{1}{2}-(d+1)(\frac{1}{2}-\frac{1}{p})}\,\|S_\Omega\|_p\,\|S_{\Omega'}\|_p
\end{aligned}$$

which yields the assertion. □

We combine Proposition 2.2 for $q = p$ and Lemma 2.1 to obtain

Proposition 2.3. *Let $d \ge 2$, let $\mu(p)$ be as in (2.8) and suppose that $p > \frac{2(d+3)}{d+1}$ and that (1.4) holds. Then, for all $\varepsilon > 0$*

$$\Big\|\sum_k f_k\Big\|_p \le C_\varepsilon\, \delta^{-\mu(p)-\varepsilon} \Big\| \big(\sum_k |f_k|^2\big)^{1/2} \Big\|_p \tag{2.11}$$

We may apply Minkowski's inequality on the right hand side of (2.11) and obtain (1.9) for the limiting case $p = 2(d+3)/(d+1)$. It turns out this is all what is needed to obtain the claimed improvements in Theorem 1.1. This inequality can also be interpolated with the trivial estimates for L^2 and L^∞ to give:

Corollary 2.4. *The inequality (1.9) holds for all $\alpha > \frac{d-1}{4}(\frac{1}{2} - \frac{1}{p})$ when $2 \le p \le 2(d+3)/(d+1)$, and for all $\alpha > \frac{d-1}{4}(1 - \frac{2(d+2)}{p(d+1)}) + \varepsilon$ when $2(d+3)/(d+1) \le p \le \infty$.*

3. Improvement of Wolff's estimate

We turn to Theorem 1.1. The proof in [22, 11] for inequality (1.3) is based on a subtle localization procedure, induction on scales and certain combinatorial arguments. Here we only cite the main modifications leading to the claimed improvements. For simplicity, when δ is fixed (and small) we use the notation $A \lessapprox B$ to indicate the inequality $A \le C_\varepsilon\, \delta^{-\varepsilon} B$ for all $\varepsilon > 0$. Recall that the number of plates $\Pi_k^{(\delta)}$ covering Γ_δ is approximately $\delta^{-\frac{d-1}{2}}$. Also, throughout this section we fix $q(d) = 2(d+3)/(d+1)$.

Due to various reductions (see [11, §3]), it is enough to show that, for all f_k with supp $\hat{f}_k \subset \Pi_k^{(\delta)}$ and $\|f_k\|_\infty \le 1$, and for all $\lambda > 0$ we have

$$\big|\{|\textstyle\sum_k f_k| > \lambda\}\big| \lessapprox \lambda^{-p}\, \delta^{d - \frac{(d-1)p}{2}}\, \|f\|_2^2 \tag{3.1}$$

where $f = \sum_k f_k$. In [22, 11] it is observed that, by Chebyshev's inequality, this property trivially holds for small enough λ; namely for all $\lambda \le \delta^{-\frac{d-1}{2} + \frac{1}{p-2}}$. We use (1.10) to enlarge this range of λ.

Lemma 3.1. *Let $q = q(d) = 2(d+3)/(d+1)$. Then, inequality (3.1) holds for all*

$$\lambda \le \delta^{-\frac{d-1}{2} + \frac{q}{4(p-q)}}. \tag{3.2}$$

Proof. Let $\beta = \frac{d-1}{4(d+3)}$. By Chebyshev's inequality and (1.10), we have

$$\big|\{|f| > \lambda\}\big| \le \lambda^{-q}\, \|f\|_q^q \lessapprox \delta^{-q\beta}\, \lambda^{-q} \big(\textstyle\sum_k \|f_k\|_q^2\big)^{q/2}$$

and estimate

$$\big(\textstyle\sum_k \|f_k\|_q^2\big)^{q/2} \lesssim \delta^{-\frac{d-1}{2}\frac{q/2}{(q/2)'}} \textstyle\sum_k \|f_k\|_q^q \lesssim \delta^{-\frac{d-1}{2}(\frac{q}{2}-1)} \sum_k \|f_k\|_2^2 \,\sup_k \|f_k\|_\infty^{q-2}.$$

Since by assumption $\|f_k\|_\infty \le 1$ and by almost orthogonality $\sum_k \|f_k\|_2^2 \approx \|f\|_2^2$, it suffices to show that in the desired range of λ we have $\delta^{-q\beta - \frac{d-1}{2}(\frac{q}{2}-1)} \lambda^{-q} \le \delta^{-\frac{(d-1)p}{2} - d} \lambda^{-p}$ which is equivalent to (3.2). □

At this point one can proceed exactly as in the proof of Proposition 3.2 of [11] (or p. 1277 in [22], when $d = 2$). The desired gain comes from using $\lambda \ge \delta^{-\frac{d-1}{2} + \frac{q(d)}{4(p-q(d))}}$ (rather than $\lambda \le \delta^{-\frac{d-1}{2} + \frac{1}{p-2}}$) in step (54) of [11] (or (68) of [22]).

For completeness, we shall briefly sketch this procedure here, referring always to the notation in [11]. Localizing with $\sqrt{N}$-cubes Δ as in Lemma 6.1 of [11], one can find a collection of functions $\{f_\Delta\}$ with spectrum in $\Gamma_{\sqrt{\delta}}$ and a number

$$\lambda_* \in (\lambda\delta^{\frac{d-1}{4}+\varepsilon}, c\delta^{-\frac{d-1}{4}}) \tag{3.3}$$

so that

$$\big|\{|f| > \lambda\}\big| \lessapprox \sum_\Delta \big|\{|f_\Delta| > \lambda_*\}\big|$$

and

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \lessapprox \lambda_*^2\lambda^{-2}\,\delta^{-\frac{3d-1}{4}}. \tag{3.4}$$

Here $\mathcal{P}(f_\Delta)$ refers to the set of plates in the wave-packet decomposition of f_Δ. When the cardinality of this set is "small", a further localization argument and induction on scales allows to conclude the theorem (see Lemmas 6.2 and 6.3 in [11]).

In [11, 22], the size of $\operatorname{card}(\mathcal{P}(f_\Delta))$ which ensures the validity of these arguments is controlled in three different ways, each depending on a different combinatorial estimate

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \le c_\varepsilon\delta^\varepsilon\lambda_*^2, \tag{3.5}$$

or

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \le c_\varepsilon\delta^{\frac{3d-3}{8}+\varepsilon}\,\lambda_*^4, \tag{3.6}$$

or, in three dimensions (i.e. $d=2$) only,

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \le c_\varepsilon\delta^{\frac{11}{8}+\varepsilon}\,\lambda_*^9. \tag{3.7}$$

the last estimate being by far the most difficult (see Lemmas 5.2 and 5.3 in [11] and Lemma 3.2 in [22]).

Given the lower bound for λ_* in (3.3) and

$$\lambda \ge \delta^{-\frac{d-1}{2}+\frac{q}{4(p-q)}} \tag{3.8}$$

and given (3.4) it remains to verify the estimates (3.5) in the claimed range $p > p_d$, $d \ge 5$, (3.6) for $p > p_d$, $d = 3,4$ and (3.7) for $p > p_2$.

This is straightforward. By (3.4) and (3.8) we have

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \lessapprox \delta^{-\varepsilon}\lambda_*^2\delta^{d-1-\frac{q(d)}{2(p-q(d))}}\,\delta^{-\frac{3d-1}{4}}$$

which gives in the case $d \ge 4$ the assertion (3.5) if $d-1-\frac{q(d)}{2(p-q(d))}-\frac{3d-1}{4} > 0$ or, after a short computation $p > q(1+\frac{2}{d-3}) = 2+\frac{8}{d-3}\frac{d}{d+1}$. This is the asserted range if $d \ge 5$.

Next we examine the validity of the inequality (3.6) under condition (3.8). We now have

$$\operatorname{card}\big(\mathcal{P}(f_\Delta)\big) \;\le C_\varepsilon\frac{\lambda_*^4\delta^{-\frac{3d-1}{4}}-\varepsilon}{\lambda_*^2\lambda^2} \;\le\; \frac{\lambda_*^4\delta^{-\frac{3d-1}{4}}-\varepsilon}{\lambda^4\,\delta^{\frac{d-1}{2}+2\varepsilon}} \;\le\; \frac{\delta^{-\frac{5d-3}{4}-3\varepsilon}}{\delta^{-2(d-1)+\frac{q(d)}{p-q(d)}}}\,\lambda_*^4.$$

This quantity is $\lesssim \delta^\varepsilon\delta^{\frac{3d-3}{8}}\,\lambda_*^4$ if and only if $\frac{5d-3}{4}-2(d-1)+\frac{q(d)}{p-q(d)}+4\varepsilon < -\frac{3d-3}{8}$, which yields the range gives $p > q(d)(1+\frac{8}{3d-7})$. Notice that this inequality amounts to $p > 7.28$ if $d=4$ and $p > 15$ if $d=3$ which is the assertion in those cases.

Finally we consider the case $d=2$ when $q(2)=10/3$. By (3.4) we need to have $\lambda_*^2\lambda^{-2}\delta^{-5/4-\varepsilon} \le c_\varepsilon \delta^{11/8}\lambda_*^9$, i.e. $\lambda^{-2}\delta^{-21/8-\varepsilon} \le c_\varepsilon \lambda_*^7$ provided that $\lambda_* > \lambda\delta^{1/4+\varepsilon}$. Thus taking the smallest possible λ^* yields $\delta^{-35/8-10\varepsilon} \le \lambda^9$ and this has to hold for all λ satisfying (3.8), i.e. $\lambda \ge \delta^{-\frac{1}{2}+\frac{q(2)}{4(p-q(2))}}$. Taking the minimal λ this is achieved if $35/8 - 10\varepsilon < 9/2 - 9q/(4p-4q)$ with $q = q(2) = 10/3$. Solving in p and letting $\varepsilon \to 0$ yields the range $p > 19q(2) = 63 + 1/3$. □

On the proof of Theorem 1.5. The proof is similar to the proof of Theorem 1.1. Instead of (1.10) we use a square function inequality for the sphere

$$\Big\|\sum_k f_k\Big\|_q \le C_\varepsilon\, \delta^{-\alpha(q)/2-\varepsilon} \Big\| \big(\sum_k |f_k|^2\big)^{\frac12} \Big\|_q, \quad \text{supp } \widehat{f_k} \subset C_k^{(\delta)}, \tag{3.9}$$

with $\alpha(q) = d(1/2-1/q) - 1/2$, and $q = 2(d+2)/d$. In two dimensions this is an old observation by C. Fefferman ([8]), and holds for $q=4$ with $\varepsilon = 0$. In general the proof of (3.9) is rather analogous to the proof of Proposition 2.3; one uses Tao's bilinear Fourier extension inequality [19] (see also [12] for related results). Unlike (2.11) in the conic case the inequality (3.9) is essentially optimal for the given range $q \ge 2(d+2)/d$. We omit further details. □

4. More on square functions

We shall now discuss some improvements of the square function estimate in Proposition 2.3 in low dimensions; thus we seek for estimates of the form

$$\Big\|\sum_k f_k\Big\|_p \le C_\varepsilon\, \delta^{-\beta-\varepsilon} \Big\| \Big(\sum_k |f_k|^2\Big)^{1/2} \Big\|_p, \quad \text{if supp } \widehat{f_k} \subset \Pi_k^{(\delta)}. \tag{4.1}$$

for some $\beta < \min\{\mu(p), (d-1)/4\}$.

We shall assume throughout this chapter the Wolff hypothesis

Hypothesis $\mathcal{W}(w,d)$. For all $\delta \in (0,1)$ and all families $\{h_k\}$ of functions satisfying supp $\widehat{h}_k \subset \Pi_k^{(\delta)}$,

$$\Big\|\sum_k h_k\Big\|_w \le C_\varepsilon \delta^{-\alpha(w)-\varepsilon} \Big(\sum_k \|h_k\|_w^w\Big)^{1/w}, \tag{4.2}$$

where $\alpha(w) = d(1/2-1/w) - 1/2$. Cf. Table 1.

Our improvement is limited to the case where the power $\mu(p)$ in (2.8) satisfies

$$\alpha(p) < \mu(p) \tag{4.3}$$

which holds if and only if $\mu(p) < 1/2 - 1/p$, or, equivalently $p < \frac{2(d-1)}{d-2}$. By the additional restriction $p > \frac{2(d+3)}{d+1}$ we shall get an improvement only in the cases

$$\begin{cases} d=2, & 10/3 < p < \infty, \\ d=3, & 3 < p < 4, \\ d=4, & 14/5 < p < 3. \end{cases} \tag{4.4}$$

If $d=2$ an immediate improvement over the exponent $\mu(p)$ $(=\frac12 - \frac{3}{2p})$ follows from (4.3) in the range $w < p < \infty$ as $L^p(\ell^2)$ is contained in $\ell^p(L^p)$ and thus we obtain (4.1) for $\beta \ge \alpha(p) = \frac12 - \frac2p$ if

$p > w$. In what follows we consider the case $p < w$. While square-function estimates such as (4.1) cannot a priori be interpolated, one still gets the following formally interpolated result.

Lemma 4.1. *Let $d = 2$, suppose that hypothesis $\mathcal{W}(w, 2)$ holds. Then for all family of functions $\{f_k\}$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$ the estimate* (4.1) *holds for* $10/3 \le p \le w$ *with*

$$\beta = \beta_*(p, w) = \frac{3w - 13}{6w - 20} - \frac{9w - 40}{(6w - 20)p} \tag{4.5}$$

In particular note that $\beta_*(4, w) = \frac{3w-12}{24w-80}$ so that $\beta_*(4, 6) = 3/32$. For $w = p_2 = 190/3$ we get only $\beta_*(p_2) = 89/720$ which is worse than $5/44$ exponent which is already known from [20], [23].

Proof of Lemma 4.1. Let φ_k be a bump function adapted to the plate $\Pi_k^{(\delta)}$ which equals 1 on the plate. Define the operator P_k by $\widehat{P_k f} = \varphi_k \widehat{f}$. Each P_k is bounded on $L^p(\mathbb{R}^3)$, $1 \le p \le \infty$, with uniform bounds.

Thus by $\mathcal{W}(2, w)$ and the embedding $L^p(\ell^2) \subset \ell^p(L^p)$ we have the inequality

$$\begin{aligned}\Big\| \sum_k P_k g_k \Big\|_w &\le C_\varepsilon \delta^{-(\alpha(w)+\varepsilon)} \Big(\sum_k \|P_k g_k\|_w^w \Big)^{1/w} \\ &\le C_\varepsilon \delta^{-(\alpha(w)+\varepsilon)} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_w .\end{aligned} \tag{4.6}$$

We also observe that for $2 \le p \le 4$

$$\Big\| \Big(\sum_k |P_k g_k|^2 \Big)^{1/2} \Big\|_p \le C(1 + \log \delta^{-1})^{1/2 - 1/p} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_p \tag{4.7}$$

Indeed the left hand side is estimated by

$$\sup_{\omega \in L^{(p/2)'}} \Big(\sum_k \int |P_k g_k|^2 \omega dx \Big)^{1/2} \lesssim \sup_{\omega \in L^{(p/2)'}} \Big(\sum_k \int |g_k|^2 M_\delta \omega dx \Big)^{1/2}$$

where M_δ is a Besicovich-type maximal operator associated to the light cone which is bounded on L^2 with norm $O(\sqrt{\log(2 + \delta^{-1})})$ if $\delta < 1/2$, see [7], [14]. Thus Hölder's inequality implies (4.7).

Now we can combine Proposition 2.3 and (4.7) to obtain

$$\Big\| \sum_k P_k g_k \Big\|_{10/3} \le C_\varepsilon \delta^{-\frac{1}{20} - \varepsilon} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_{10/3} \tag{4.8}$$

and after a little arithmetic the claimed bound follows by interpolation between (4.6) and (4.8). ☐

For large values of w one can improve on the result of Lemma 4.1. Our approach will be similar to the one by Tao and Vargas [20] in $2 + 1$ dimensions. By using $\mathcal{W}(w, 2)$ in that approach one can slightly improve on the previously known exponents.

Theorem 4.2. *Let $2 \le d \le 4$, and assume $\frac{2(d+3)}{d+1} < p < \min\{\frac{2(d-1)}{d-2}, w\}$. Suppose that hypothesis $\mathcal{W}(w, d)$ holds. Then for p, d as in* (4.4) *and family of functions $\{f_k\}$ with supp $\widehat{f_k} \subset \Pi_k^{(\delta)}$ we get the estimate* (4.1) *for*

$$\beta = \mu(p) - \mathfrak{c}(p) \Big(\frac{1}{p} - \frac{d - 2}{2(d - 1)} \Big) \tag{4.9}$$

where

$$\mathfrak{c}(p) = \frac{d-1}{2}\Big(\frac{\frac{d+1}{2(d+3)} - \frac{1}{p}}{\frac{d+1}{2(d+3)} + \frac{1}{p} - \frac{2(p-1)}{(w-1)p}}\Big). \tag{4.10}$$

The proof will be given in §5.

In $2+1$ dimensions Theorem 4.2 yields inequality (4.1) for the range $10/3 \le p \le w$ with β equal to

$$\beta_{**}(p,w) = \frac{1}{2p}\cdot\frac{(3p^2-2p-20)w - 23p^2+82p-40}{(10+3p)w-23p+10}; \tag{4.11}$$

in particular we have $\beta_{**}(4,w) = \frac{5w-20}{44w-164}$. We compare with (4.5). Notice that $3/32 = \beta_*(4,6) < \beta_{**}(4,6) = 1/10$. A straightforward computation shows the inequality $\beta_{**}(p,w) < \beta_*(p,w)$ holds if and only if $(9p-30)w^2 + (-9p^2-39p+230)w + 23p(3p-10) > 0$ and after factoring we see that for $10/3 < p < w$ we have $\beta_{**}(p,w) < \beta_*(p,w)$ if and only if $(p-\frac{10}{3})(w-\frac{23}{3})(w-p) > 0$. Thus for any $p \in (10/3, w)$ we have

$$\beta_{**}(p,w) < \beta_*(p,w) \quad \iff w > \frac{23}{3} \tag{4.12}$$

so that the L^p result in Theorem 4.2 is better than the result of Lemma 4.1 in the range $w > 23/3$.

As a corollary we obtain

Corollary 4.3. *Let $d=2$ and suppose that $\mathcal{W}(w,2)$ holds for some $w>6$. Let $10/3 < p \le 4$ and let $\alpha > \min\{\beta_*(p,w), \beta_{**}(p,w)\}$ (i.e. $\alpha > \beta_{**}(p,w)$ if $w > 23/3$).*
Then

(i) the smoothing inequality (1.6) *holds true and*

(ii) the Fourier multiplier m_α in (1.7) *defines a bounded operator on $L^p(\mathbb{R}^3)$.*

We also observe that by interpolation we obtain the analogous boundedness results for the range $4 \le p \le w$ under the assumption that $\alpha > \frac12 - \frac2p + \frac{4(w-p)}{p(w-4)}\min\{\beta_*(w,p), \beta_{**}(4,w)\}$.

If we use the result of Theorem 1.1 in $2+1$ dimensions (*i.e.* hypothesis $\mathcal{W}(w,2)$ with $w = p_2 = 190/3$) we obtain this result for $\alpha > \beta_{**}(p, \frac{190}{3}) = \frac{501p^2-134p-3920}{2p(501p+1930)}$. which equals 445/3934 if $p=4$. This represents a slight improvement over the Tao-Vargas result [20] which yields the L^4 boundedness for $\alpha > \frac{5}{44} = 0.11\overline{363}$; note that $\frac{445}{3934} \approx 0.11311642...$. We also see from Lemma 4.1 that the validity of (1.3) for the optimal (conjectured) range $p \ge 6$ implies the L^4 boundedness for $\alpha > 0.09375$; however it has been conjectured that it should hold for all $\alpha > 0$.

Proof of Corollary 4.3. Using the $L^{(p/2)'}(\mathbb{R}^3)$ bounds of the Besicovich maximal operators associated to cones the square function is estimated as in [13], [14] to yield the assertion of Theorem 4.3 for the range $10/3 \le p \le 4$. □

5. Proof of Theorem 4.2

We shall use the following consequence of $\mathcal{W}(w,d)$.

Lemma 5.1. *Suppose that $\mathcal{W}(w,d)$ holds. Then for $r = p'(w-1)$ and every $\varepsilon > 0$*

$$\Big\| \sum_k h_k \Big\|_r \le C_\varepsilon \delta^{-\frac{w'}{p'}\alpha(w)-\varepsilon} \Big(\sum_k \|h_k\|_r^p \Big)^{1/p} \tag{5.1}$$

Proof. Let φ_k be a bump function adapted to the plate $\Pi_k^{(\delta)}$ which equals 1 on the plate. Then $\varphi_k \in \mathcal{F}L^1$ with uniform bounds. Define P_k by $\widehat{P_k f} = \varphi_k \widehat{f}$. Then by hypothesis $\mathcal{W}(w,d)$

$$\Big\| \sum_k P_k h_k \Big\|_w \le C_\varepsilon \delta^{-\alpha(w)-\varepsilon} \Big(\sum_k \|P_k h_k\|_w^w \Big)^{1/w} \le C'_\varepsilon \delta^{-\alpha(w)-\varepsilon} \Big(\sum_k \|h_k\|_w^w \Big)^{1/w}$$

Interpolation with the trivial $\|\sum_k P_k h_k\|_\infty \lesssim \sum_k \|h_k\|_\infty$ yields the assertion. □

To establish Theorem 4.2 we shall work with the following
Hypothesis $\mathcal{H}(\gamma,p)$. For all $\delta < 1$, $\varepsilon > 0$

$$\Big\| \sum_k h_k \Big\|_p \le C_\varepsilon \delta^{-\gamma-\varepsilon} \Big\| \Big(\sum_k |h_k|^2 \Big)^{1/2} \Big\|_p, \tag{5.2}$$

provided that supp $\widehat{h_k} \subset \Pi_k^{(\delta)}$.

By Proposition 2.3 we know already that for $p > 2(d+3)/(d+1)$ this inequality holds true with the exponent $\gamma = \mu(p) = d/4 - (d+1)/2p$ and we seek an improvement in the ranges (4.4).

We use Lemma 5.1 to prove the following proposition which amounts to an improved version of Proposition 5.4 in [20] (who considered the case $w = \infty$ in the $2+1$ dimensional situation). As in §2 we work with a covering $\mathcal{Q}(\delta^{-1/2})$ of $\sqrt{1/\delta}$ cubes.

Proposition 5.2. *Suppose that $2(d+3)/(d+1) < p < \min\{4, w\}$ and suppose that hypotheses $\mathcal{H}(\gamma,p)$ and $\mathcal{W}(w,d)$ hold. Let $r = p'(w-1)$. Then*

$$\Big(\sum_{Q \in \mathcal{Q}(\delta^{-1/2})} \Big\| \sum_k g_k \Big\|_{L^r(Q)}^p \Big)^{1/p} \le C_\varepsilon \delta^{-\frac{\gamma+\alpha(p)}{2}-\varepsilon} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_p. \tag{5.3}$$

provided that supp $\widehat{g_k} \in \Pi_k^{(\delta)}$.

Proof. We group the indices k (and thereore the corresponding plates $\Pi_k^{(\delta)}$ into $O(\delta^{-(d-1)/4})$ disjoint families S_l so that $\operatorname{dist}(\omega_k, \omega_{k'}) \lesssim \delta^{1/4}$ for $k, k' \in S_l$. Define

$$G_l = \sum_{k \in S_l} g_k.$$

As in the proof of Proposition 2.2 we also work with the functions ψ_Q adapted to the cubes $Q \in \mathcal{Q}(\delta^{-1/2})$. By the support propery of $\widehat{\psi_Q}$ the Fourier transform of $\psi_Q G_l$ is supported in a $C\sqrt{\delta}$ plate and these plates forman essentially disjoint plate family. Therefore

$$\begin{aligned} \Big\| \sum_l G_l \Big\|_{L^r(Q)} &\lesssim \Big\| \psi_Q \sum_l G_l \Big\|_r \\ &\lessapprox \delta^{-\frac{w'}{p'}\frac{\alpha(w)}{2}} \Big(\sum_l \|\psi_Q G_l\|_r^p \Big)^{1/p}, \end{aligned} \tag{5.4}$$

by Lemma 5.1 with δ replaced by $\sqrt{\delta}$. By the support property of $\widehat{\psi_Q G_l}$ and Young's inequality

$$\|\psi_Q G_l\|_r \lesssim \delta^{\frac{d-1}{4}(\frac{1}{p}-\frac{1}{r})} \|\psi_Q G_l\|_p \tag{5.5}$$

and therefore

$$\Big(\sum_Q \Big\|\sum_l G_l\Big\|^p_{L^r(Q)}\Big)^{1/p} \lessapprox \delta^{-\frac{w'}{p'}\frac{\alpha(w)}{2}+\frac{d-1}{4}(\frac{1}{p}-\frac{1}{r})}\Big(\sum_Q \|\psi_Q G_l\|^p_p\Big)^{1/p}.$$

A little algebra shows

$$-\frac{w'}{p'}\frac{\alpha(w)}{2}+\frac{d-1}{4}(\frac{1}{p}-\frac{1}{r}) = -\frac{\alpha(p)}{2}$$

with $r = p'(w-1)$. Using some straightforward estimation using the decay of the ψ_Q we also get

$$\Big(\sum_Q \Big\|\sum_l G_l\Big\|^p_{L^r(Q)}\Big)^{1/p} \lessapprox \delta^{-\alpha(p)/2}\Big(\sum_l \|G_l\|^p_p\Big)^{1/p} \tag{5.6}$$

As $\widehat{G}_l$ is supported in a $C\sqrt{\delta}$ plate we may use rescaling arguments as in the proof of Lemma 2.1 to deduce from the hypothesis $\mathcal{H}(\gamma, p)$ applied with parameter $\sqrt{\delta}$ that

$$\|G_l\|_p \lessapprox \delta^{-\gamma/2}\Big\|\Big(\sum_{k\in S_l}|f_k|^2\Big)^{1/2}\Big\|_p$$

and hence

$$\begin{aligned}\Big(\sum_Q \Big\|\sum_l G_l\Big\|^p_{L^r(Q)}\Big)^{1/p} &\le C_\varepsilon \delta^{-\frac{\alpha(p)+\gamma}{2}-\varepsilon}\Big(\sum_l \Big\|\Big(\sum_{k\in S_l}|f_k|^2\Big)^{1/2}\Big\|^p_p\Big)^{1/p}\\ &\le C_\varepsilon \delta^{-\frac{\alpha(p)+\gamma}{2}-\varepsilon}\Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_p\end{aligned}$$

which is the assertion. □

Proof of Theorem 4.2, cont. We first note that Hypothesis $\mathcal{H}(p, \mu(p))$ holds by Proposition 2.3.

Assuming that $\mathcal{H}(p, \gamma)$ holds for some $\gamma \le \mu(p)$ the following estimate for bilinear expressions is an immediate consequence of Proposition 5.2.

$$\Big(\sum_{Q\in\mathcal{Q}}\Big\|\big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|^{p/2}_{L^{r/2}(Q)}\Big)^{2/p} \lessapprox \delta^{-\alpha(p)-\gamma}\Big\|\big(\sum_{\omega_k\in\Omega}|f_k|^2\big)^{\frac12}\Big\|_p\Big\|\big(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\big)^{\frac12}\Big\|_p. \tag{5.7}$$

We now assume that Ω and Ω' are separated as in Proposition 2.2 and interpolate the inequalities (5.7) and (2.9) with $q = 2(d+3)/(d+1)$. As a result we obtain

$$\Big(\sum_{Q\in\mathcal{Q}}\Big\|\big(\sum_{\omega_k\in\Omega} f_k\big)\big(\sum_{\omega_{k'}\in\Omega'} g_{k'}\big)\Big\|^{p/2}_{L^{p/2}(Q)}\Big)^{2/p} \lessapprox \delta^{-2\Gamma(p,\gamma)}\Big\|\big(\sum_{\omega_k\in\Omega}|f_k|^2\big)^{\frac12}\Big\|_p\Big\|\big(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\big)^{\frac12}\Big\|_p.$$

where

$$\Gamma(p,\gamma) = (1-\vartheta)\mu(p) + \vartheta\frac{\alpha(p)+\gamma}{2} \quad \text{with } \vartheta = \Big(\frac1q - \frac1p\Big)\Big/\Big(\frac1q-\frac1r\Big).$$

By Lemma 2.1 we also obtain

$$\Big\|\sum_k f_k\Big\|_p \lessapprox \delta^{-\Gamma(p,\gamma)}\Big\|\big(\sum_{\omega_k\in\Omega}|f_k|^2\big)^{1/2}\Big\|_p. \tag{5.8}$$

Notice that $\alpha(p) < \Gamma(p,\gamma) < \gamma \le \mu(p)$ provided that $\alpha(p) < \gamma \le \mu(p)$. Define a sequence γ_n by setting $\gamma_0 = \mu(p)$ and $\gamma_{n+1} = \Gamma(p,\gamma_n)$ for $n \ge 0$. Then γ_n is decreasing and bounded below and converges to

$$\gamma_* = \frac{1}{1-\vartheta/2}\big((1-\vartheta)\mu(p) + \vartheta\alpha(p)\big) = \mu(p) - \frac{\vartheta}{2-\vartheta}\big(\mu(p) - \alpha(p)\big).$$

We compute that $\vartheta/(2-\vartheta) = (1/q - 1/p)/(1/q + 1/p - 2/r)$ and $\alpha(p) - \mu(p) = \mu(p) - 1/2 + 1/p = (d-2)/4 - (d-1)/2p$ and see that γ_* is equal to the right hand side of (4.9). Thus (5.8) and an iteration yields the assertion of the theorem. □

References

[1] D. Békollé, A. Bonami, M. Peloso and F. Ricci. Boundedness of weighted Bergman projections on tube domains over light cones. Math. Z. **237** (2001), 31-59.

[2] D. Békollé, A. Bonami, G. Garrigós and F. Ricci. Littlewood-Paley decompositions related to symmetric cones and Bergman projections in tube domains. Proc. London Math. Soc. (3) **89** (2004), 317-360.

[3] D. Békollé, A. Bonami, G. Garrigós, C. Nana, M. Peloso and F. Ricci. Lecture notes on Bergman projectors in tube domains over cones: An analytic and geometric viewpoint. To appear in IMHOTEP.

[4] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis, Geometric and Functional analysis, 147-187.

[5] J. Bourgain. Estimates for cone multipliers, Geometric Aspects of Functional Analysis, Operator theory, Advances and Applications, vol. 77, ed. by J. Lindenstrauss and V. Milman, Birkhäuser Verlag, 1995.

[6] A. Carbery. Variants of the Calderón-Zygmund theory for L^p-spaces. Rev. Mat. Iberoamericana 2 (1986), no. 4, 381–396.

[7] A. Córdoba. Geometric Fourier analysis. Ann. Inst. Fourier 32 (1982), 215–226.

[8] C. Fefferman. A note on spherical summation multipliers. Israel J. Math. 15 (1973), 44–52.

[9] G. Garrigós, W. Schlag and A. Seeger, in preparation.

[10] I. Łaba and M. Pramanik. Wolff's inequality for hypersurfaces. Preprint.

[11] I. Łaba and T. Wolff. A local smoothing estimate in higher dimensions. J. Anal. Math. **88** (2002), 149-171.

[12] S. Lee. Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators. Duke Math. J. 122 (2004), no. 1, 205–232.

[13] G. Mockenhaupt. A note on the cone multiplier. Proc. Amer. Math. Soc. **117** (1993), 145–152.

[14] G. Mockenhaupt, A. Seeger and C.D. Sogge. Wave front sets, local smoothing and Bourgain's circular maximal theorem. Annals of Math. **136**, (1992), 207–218.

[15] D. Müller and A. Seeger. Regularity properties of wave propagation on conic manifolds and applications to spectral multipliers. Adv. Math. 161 (2001), no. 1, 41–130.

[16] M. Pramanik and A. Seeger. L^p regularity of averages over curves and bounds for associated maximal operators. Preprint.

[17] A. Seeger. Some inequalities for singular convolution operators in L^p-spaces. Trans. Amer. Math. Soc. 308 (1988), no. 1, 259–272.

[18] T. Tao. Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates. Math. Z. **238** (2001), 215-268.

[19] T. Tao. A sharp bilinear restrictions estimate for paraboloids. Geom. Funct. Anal. 13 (2003), no. 6, 1359–1384.

[20] T. Tao and A. Vargas. A bilinear approach to cone multipliers II. Applications. Geom. Funct. Anal. **10** (2000), 216-258.

[21] T. Tao, A. Vargas and L. Vega. A bilinear approach to the restriction and Kakeya conjectures. J. Amer. Math. Soc. 11 (1998), no. 4, 967–1000.

[22] T. Wolff. Local smoothing type estimates on L^p for large p. Geom. Funct. Anal. **10** (2000), 1237-1288.

[23] T. WOLFF. A sharp bilinear cone restriction estimate. Ann. of Math. (2) **153** (2001), no. 3, 661–698.

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TWO COUNTEREXAMPLES IN THE THEORY OF SINGULAR INTEGRALS

LOUKAS GRAFAKOS

ABSTRACT. In these lectures we discuss examples that are relevant to two questions in the theory of singular integrals. The first question is the L^p boundedness of the maximal operator formed by dilates of Mikhlin-Hörmander multipliers, while the second concerns the L^p boundedness of a well-known object, the classical L^2-bounded Calderón-Zygmund homogeneous singular integral associated with an integrable function on the sphere that is very rough.

1. INTRODUCTION

We denote the Fourier transform of a complex-valued function $f(t)$ on $\mathbf{R}^d$ by

$$\widehat{f}(\tau) = \int_{\mathbf{R}^d} f(t) e^{-2\pi i \tau \cdot t} dt$$

and its inverse Fourier transform by $f^\vee(\tau) = \widehat{f}(-\tau)$. Many linear operators can be expressed in terms of their action on the Fourier transform of the input function. In particular, convolution operators are identified by operators given by multiplication on the Fourier transform, i.e. operators of the form $T_m(f) = (\widehat{f}m)^\vee$. Here we will always be interested in L^2-bounded convolution operators for which the corresponding multiplying functions m (called the Fourier multipliers) must be essentially bounded functions. The Fourier multiplier associated in this way with an operator bounded on $L^p(\mathbf{R}^d)$ is called an L^p Fourier multiplier. The space of all L^p Fourier multipliers on $\mathbf{R}^d$ will be denoted by $M_p(\mathbf{R}^d)$. This is a Banach space (in fact algebra) with norm $\|m\|_{M_p} = \|T_m\|_{L^p \to L^p}$.

The classical Mikhlin multiplier theorem [13] states that if a function $m(\xi)$ on $\mathbf{R}^d$ satisfies

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{1.1}$$

for all multiindices α with $|\alpha| \le [\frac{d}{2}] + 1$, then it must be an L^p Fourier multiplier for all $1 < p < \infty$. This theorem was extended by Hörmander [12] to functions m satisfying the weaker condition

$$\sup_{k \in \mathbf{Z}} \|\varphi(\xi) m(2^k \xi)\|_{L^2_\beta(d\xi)} < \infty \tag{1.2}$$

for some $\beta > d/2$. Here φ is a smooth nonzero bump supported in the annulus $1 < |\xi| < 2$ not vanishing on a smaller annulus and L^2_β is the Sobolev space of functions with "β

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derivatives" in L^2. The space L^2_β is one of the Sobolev spaces L^p_γ with norm

$$\|f\|_{L^p_\gamma} = \big\|\big(\widehat{f}(\xi)(1+|\xi|^2)^{\gamma/2}\big)^\vee\big\|_{L^p(d\xi)},$$

where $1 \le p < \infty$ and $\gamma \in \mathbf{R}$. We remark that in Hörmader's version of this multiplier theorem the Sobolev space L^2_β in (1.2) can be replaced by L^r_γ, where $\gamma > d/r$ and $1 \le r \le 2$; however the least restrictive condition is when $r = 2$.

By duality an L^p Fourier multiplier must always be an $L^{p'}$ Fourier multiplier (where $p' = p/(p-1)$) and hence by interpolation it must be an L^q Fourier multiplier for all q between p and p'. Finding examples of functions that are L^q Fourier multipliers for some $q > 2$ but not L^s Fourier multipliers for some $s > q$ may not be an easy task. A question of this sort will be addressed in section 5.

In the next section we will discuss a problem concerning the L^p boundedness of the supremum of a family of Mikhlin-Hörmander Fourier multipliers.

2. Maximal Mikhlin-Hörmander Fourier multipliers

Suppose that we are given a bounded function on $\mathbf{R}^d$ that satisfies condition (1.1) (or even (1.2)). The question that we would like to address is whether the maximal operator

$$\mathcal{M}_m(f)(x) = \sup_{t>0} \big|(\widehat{f}(\xi)m(t\xi))^\vee(x)\big|$$

is bounded from $L^p(\mathbf{R}^d)$ into itself.

This question is motivated by the almost everywhere convergence questions

$$(\widehat{f}(\xi)m(t\xi))^\vee(x) \to m(0)\, f(x) \qquad \text{for almost all } x \text{ as } \quad t \to 0$$

$$(\widehat{f}(\xi)m(t\xi))^\vee(x) \to m(\infty)\, f(x) \qquad \text{for almost all } x \text{ as } \quad t \to \infty,$$

provided, of course, the quantities $m(0)$ and $m(\infty)$ exist.

We recall that in the usual proof of the Mikhlin-Hörmander multiplier theorem one obtains a weak type $(1,1)$ estimate using the trivial L^2 estimate and a smoothing condition on the kernel. Then the boundedness for the remaining p's follows by interpolation and duality.

By changing our point of view, we may consider $\mathcal{M}_m$ as a linear map from

$$L^p(\mathbf{R}^d) \to L^p(\mathbf{R}^d, L^\infty(\mathbf{R}^+) \tag{2.3}$$

and we may ask whether the classical scalar argument argument based on the weak type $(1,1)$ estimate holds in this setting. In the context of the vector-valued setting described in (2.3) the corresponding multiplier satisfies Mikhlin's condition but for the weak type $(1,1)$ argument to go through one needs to know an initial estimate at a single exponent. In the scalar case, one uses Plancherel's theorem to obtain the L^2 estimate for free but in the vector-valued case the $L^2(\mathbf{R}^d, L^\infty(\mathbf{R}^+))$ estimate cannot be obtained using Plancherel's theorem, in fact as we will shortly see, it may fail.

The underlying problem here is that the Banach space L^∞ is not a UMD space and for this reason many analogues of some of the scalar results in the theory of singular integrals do not hold in the Banach-valued setting.

Theorem 1. *(M. Christ, L. Grafakos, P. Honzík, and A. Seeger* [4]*) There exists a bounded function m such that for all multiindices α there are constants C_α such that*

$$\sup_\xi \sup_k |\partial^\alpha_\xi(\varphi(\xi)m(2^k\xi))| \le C_\alpha,$$

hence m is an L^p Fourier multiplier for all $1 < p < \infty$, but $\mathcal{M}_m$ is unbounded on $L^p(\mathbf{R}^d)$.

We discuss some of the ideas of the proof of this result.

Proof. Let $S = \{1, -1, i, -i\}$. Enumerate the set of all sequences of length N formed by elements of S as follows: $S^N = \{s_1, s_2, \dots, s_{4^N}\}$. Let Φ be a smooth function supported in $\frac{6}{8} \le |\xi| \le \frac{10}{8}$ satisfying $\Phi = 1$ on $\frac{7}{8} \le |\xi| \le \frac{9}{8}$. Define for $N \ge 10$

$$m_N(\xi) = \sum_{\ell=1}^{4^N} \sum_{\nu=1}^{N} s_\ell(\nu)\Phi(2^{-N\ell}2^{-\nu}\xi)$$

and also

$$m(\xi) = \sum_{N=1}^{\infty} m_N(2^{-N8^N}\xi).$$

It is straightforward that for all multiindices α there are constants C_α so that

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

and it is also easy to check that for any $k_0 \in \mathbf{Z}$ we have

$$|\partial^\alpha (m(2^{k_0}\xi)\varphi(\xi))| \le C_\alpha .$$

Pick ψ with Fourier transform supported in $B(0, 1/8)$ with $\|\psi\|_{L^p} = 1$. Let

$$g_N(x) = \sum_{j=1}^{N} e^{2\pi i 2^j x_1}\psi(x)$$

and note that

$$\widehat{g_N}(\xi) = \sum_{j=1}^{N} \widehat{\psi}(\xi - 2^j(1, 0, 0, \dots, 0)).$$

Also let

$$f_{N,p}(x) = N^{-\frac{1}{2}}(2^{N8^N})^{\frac{d}{p}} g_N(2^{N8^N}x)$$

and notice that in view of the Littlewood-Paley Theorem (see [8]) we have that

$$\|g_N\|_{L^p} \approx N^{1/2}$$

while

$$\|f_{N,p}\|_{L^p} \approx C_p .$$

The main ingredient we need is the following lower estimate whose proof we postpone momentarily:

$$\Big\| \sup_{1 \le k \le N4^N} |(m_N(2^k\xi)\widehat{g_N}(\xi))^\vee \Big\|_{L^p} \ge c N . \tag{2.4}$$

This implies that

$$\|\mathcal{M}_m(f_{N,p})\|_{L^p} \ge c\sqrt{N} = c\sqrt{N}\, \|f_{N,p}\|_{L^p} .$$

The reason for this is that $m(\xi) = \sum_{n=1}^{\infty} m_n(2^{-n8^n}\xi)$ and

$$m_n(2^{-n8^n}2^k\xi)\widehat{f_{N,p}}(\xi) = m_n(2^{-n8^n}2^k\xi)\widehat{g_N}(2^{-N8^N}\xi) = 0$$

for all $1 \le k \le N4^N$ unless $n \ne N$.

It remains to prove (2.4). We observe that

$$\sup_{c \in \{1, -, 1, i, -i\}} \mathrm{Re}(c\, z) \ge |z|/\sqrt{2} .$$

Thus for all $x \in \mathbf{R}^d$ and all $j \in \{1, 2, \ldots, N\}$ there is a $c_j(x) \in \{1, -, 1, i, -i\}$ such that

$$\mathrm{Re}\big[c_j(x)\, e^{2\pi i 2^j x_1}\psi(x)\big] \geq |\psi(x)|/\sqrt{2}\,.$$

Therefore there is a $\kappa_x \in \{1, 2, \ldots, 4^N\}$ such that

$$s_{\kappa_x} = (c_1(x), c_2(x), \ldots, c_N(x))\,.$$

We then have

$$\begin{aligned}&\sup_{1\leq k\leq N4^N} |(m_N(2^k\xi)\widehat{g_N}(\xi))^\vee(x)|\\ &\geq \mathrm{Re}\bigg[\int_{\mathbf{R}^d}\sum_{\ell=1}^{4^N}\sum_{\nu=1}^{N} s_\ell(\nu)\Phi(2^{-N\ell-\nu}2^{N\kappa_x}\xi)\sum_{j=1}^{N}\widehat{\psi}(\xi - 2^j e_1)e^{2\pi i x\cdot\xi}d\xi\bigg]\,,\end{aligned}$$

as easily follows by taking $k = N\kappa_x$.

Our choice of exponents makes the previous expression inside the the square brackets zero unless $\ell = \kappa_x$ and $j = \nu$. Also $\Phi = 1$ on support$(\widehat{\psi})$ and hence this expression is at least

$$\sum_{j=1}^{N}\mathrm{Re}\big[s_{k_x(j)}(\widehat{\psi}(\xi - 2^j))^\vee(x)\big] \geq N|\psi(x)|/\sqrt{2}$$

which proves (2.4).

□

3. A POSITIVE RESULT RELATED TO THE PREVIOUS COUNTEREXAMPLE

We recall the main observation in the previous section which can be rephrased as follows:

$$\Big\|\sup_{1\leq k\leq N4^N} |(m_N(2^k\xi)\widehat{g_N}(\xi))^\vee|\,\Big\|_{L^p} \geq c\sqrt{N}\,\|g_N\|_{L^p}\,. \tag{3.5}$$

Replacing $N4^N$ by $\mathcal{N}$ we see that the supremum of a family of $\mathcal{N}$ Mikhlin-Hörmander multipliers has operator norm on L^p at least as big as a constant multiple of $(\log\mathcal{N})^{\frac{1}{2}}$.

The question we would like to address is whether this lower estimate is sharp. We precisely formulate our question.

Question: *Suppose that m_j, $1 \leq j \leq N$, are Mikhlin multipliers satisfying*

$$|\partial^\alpha m_j(\xi)| \leq C_\alpha|\xi|^{-|\alpha|}\,,$$

uniformly in j for all $|\alpha| \leq [\frac{d}{2}]+1$. What is the growth as $N \to \infty$ of the smallest constant $A(N)$ such that

$$\Big\|\sup_{1\leq j\leq N}|(m_j\widehat{f})^\vee|\Big\|_{L^p(\mathbf{R}^d)} \leq A(N)\|f\|_{L^p(\mathbf{R}^d)}$$

holds for all f?

The counterexample in the previous section shows that for $N \geq 10$ we have

$$A(N) \geq c\sqrt{\log N}$$

and we would like to know if the converse inequality also holds for some other constant c'. The following theorem answers this question.

Theorem 2. *(L. Grafakos, P. Honzik, and A. Seeger [9]) Let* $1 < r < 2$ *and suppose*

$$\sup_{1\le j\le N} |\partial^\alpha m_j(\xi)|\,|\xi|^{|\alpha|} \le B$$

for all $|\alpha| \le [\frac{d}{2}] + 1$. *Then for any* $1 < p < \infty$ *there is a constant* $C_{d,p}$ *such that for all* $N \ge 10$ *we have*

$$\Big\| \sup_{1\le j\le N} \big|(m_j \widehat{f})^\vee\big| \Big\|_{L^p(\mathbf{R}^d)} \le C_{d,p} B \sqrt{\log N}\, \|f\|_{L^p(\mathbf{R}^d)} .$$

Therefore for $N \ge 10$ we have that

$$A(N) \le c' \sqrt{\log N}$$

and this shows that $A(N)$ grows indeed like the square root of the logarithm of N as $N \to \infty$.

We will outline a proof of this theorem in the next section, but before we do so, it will be illuminating to discuss a model case that contains the core idea and forms the basic outline of the proof in the general case. The model case comes from the theory of Rademacher multipliers. Let us recall the Rademacher functions defined on the interval $[0,1]$ as follows:

$$\begin{aligned} r_0(t) &= 1 \\ r_1(t) &= \chi_{[0,1/2]} - \chi_{[1/2,1]} \\ r_2(t) &= \chi_{[0,1/4]} - \chi_{[1/4,1/2]} + \chi_{[1/2,3/4]} - \chi_{[3/4,1]} \end{aligned}$$

etc. The inspiration comes by studying the growth in N of the L^p norms of simple-looking maximal functions of the form

$$\sup_{1\le k\le N} \Big| \sum_j a_j^k r_j \Big| ,$$

where a_j^k is a fixed matrix and r_j is the j-th Rademacher function. Let us denote the sequence $(a_j^k)_j$ by a^k.

It turns out that

$$\Big\| \sup_{1\le i\le N} \Big| \sum_j a_j^k r_j \Big| \Big\|_{L^2([0,1])} \le C(N) \sup_{1\le k\le N} \|a^k\|_{\ell^2([0,1])} \tag{3.6}$$

where $C(N)$ grows like $\sqrt{\log N}$ as $N \to \infty$.

To see this we set $F_k = \sum_j a_j^k r_j$. One has the following exponential decay of sums of Rademacher functions (see [15], [8])

$$\big|\{s \in [0,1] :\ |F_k(s)| > \lambda\}\big| \le 2\, e^{-\lambda^2/4\|a^k\|_{\ell^2}^2} . \tag{3.7}$$

Then for $N \ge 10$ we have

$$\begin{aligned} \Big\| \sup_{1\le k\le N} |F_k| \Big\|_{L^2([0,1])}^2 &= \int_0^\infty \lambda \big|\{s \in [0,1] :\ \sup_k |F_k(s)| > \lambda\}\big|\, d\lambda \\ &= \int_0^{u_N} \dots\, d\lambda + \int_{u_N}^\infty \dots\, d\lambda \\ &\le \int_0^{u_N} \lambda\, d\lambda + \int_{u_N}^\infty \sum_{k=1}^N \lambda \big|\{s \in [0,1] :\ |F_k(s)| > \lambda\}\big|\, d\lambda . \end{aligned}$$

We now use (3.7) and calculate the integrals in question. We obtain

$$\begin{aligned}\Big\| \sup_{1\le k\le N} |F_k| \Big\|^2_{L^2([0,1])} &\le \frac{1}{2}u_N^2 + \sum_{k=1}^N \int_{u_N}^\infty 2\,\lambda\, e^{-\lambda^2/4\|a^k\|^2_{\ell^2}} d\lambda \\ &\le \frac{1}{2}u_N^2 + 2N\, e^{-u_N^2/4} \sup_{1\le k\le N} \|a^k\|^2_{\ell^2} \\ &\le c\log N \sup_{1\le k\le N} \|a^k\|^2_{\ell^2}\,,\end{aligned}$$

using the optimal choice of

$$u_N = \sqrt{4\log N} \sup_{1\le k\le N} \|a^k\|_{\ell^2}\,.$$

This proves (3.6).

4. THE GENERAL CASE

We now adapt the idea of the previous section to prove Theorem 2.

Proof. Let D_k be the dyadic cubes in $\mathbf{R}^d$ of sidelength 2^{-k}. We recall the dyadic averaging operator E_k, the martingale difference operator E_k, and the martingale square function $S(f)$ associated with the family of dyadic cubes:

$$\begin{aligned}E_k(f) &= \sum_{Q\in D_k} \chi_Q \frac{1}{|Q|}\int_Q f(t)\,dt\,,\\ D_k(f) &= E_{k+1}(f) - E_k(f)\,,\\ S(f) &= \Big(\sum_k |D_k(f)|^2\Big)^{1/2}\,,\end{aligned}$$

The key element in the proof is the Chang-Wilson-Wolff inequality [3] :

$$\big|\{x\in\mathbf{R}^d : \sup_{k\ge 0}|E_k(g)-E_0(g)| > 2\lambda\,,\ S(g)<\varepsilon\lambda\}\big| \le C_d\, e^{-\frac{c_d}{\varepsilon^2}} \big|\{x\in\mathbf{R}^d : \sup_k |E_k(g)|>\lambda\}\big|$$

which is valid for all functions g, all $\lambda>0$, $\varepsilon\in(0,1)$, and for some fixed constants C_d, c_d (both depending on d).

Recall that we denote by T_m the operator $f\to(\widehat{f}m)^\vee$. Start with

$$\Big\|\sup_{1\le k\le N}|T_{m_k}(f)|\Big\|_{L^p} = \Big(p\,4^p\int_0^\infty \lambda^{p-1}|\{\sup_k |T_{m_k}(f)|>4\lambda\}|\,d\lambda\Big)^{\frac{1}{p}}$$

and control the measure of the set that appears in the previous line by the sum of three terms:

$$|\{\sup_k |T_{m_k}(f)|>4\lambda\}| \quad\le\quad I_\lambda + II_\lambda + III_\lambda\,,$$

where

$$\begin{aligned}I_\lambda &= |\{\sup_k |T_{m_k}(f) - E_0(T_{m_k}(f))| > 2\lambda\ ,\ G_p(f) < \varepsilon_N\lambda/(AB)\}|\,,\\ II_\lambda &= |\{G_p(f) > \varepsilon_N\lambda/(AB)\}|\,,\\ III_\lambda &= |\{\sup_k |E_0(T_{m_k}(f))| > 2\lambda\}|\,.\end{aligned}$$

Here A is a constant and $f \to G_p(f)$ is an L^p bounded maximal operator which controls the square function applied to each T_{m_k}, precisely it satisfies:

$$S(T_{m_k}(f)) \leq A\left(\sup_{\xi} \sup_{|\alpha|\leq[\frac{d}{2}]+1} |\xi|^{|\alpha|}|\partial_\xi^\alpha m_k(\xi)|\right)G_p(f) \leq A\,B\,G_p(f)$$

for all Schwartz functions f. (For a precise definition of G_p, see [9].)

To estimate

$$\left(p\,4^p\int_0^\infty \lambda^{p-1}\,I_\lambda\,d\lambda\right)^{\frac{1}{p}} \tag{4.8}$$

we use the Chang Wilson Wolff theorem to write

$$\begin{aligned}
I_\lambda &\leq |\{\sup_k |T_{m_k}(f) - E_0(T_{m_k}(f))| > 2\lambda\ ,\ G_p(f) < \varepsilon_N\lambda/(AB)\}| \\
&\leq \sum_{k=1}^N |\{|T_{m_k}(f) - E_0(T_{m_k}(f))| > 2\lambda\ ,\ G_p(f) < \varepsilon_N\lambda/(AB)\}| \\
&\leq \sum_{k=1}^N |\{|T_{m_k}(f) - E_0(T_{m_k}(f))| > 2\lambda\ ,\ S(T_{m_k}(f)) < \varepsilon_N\lambda\}| \\
&\leq \sum_{k=1}^N C_d\,e^{-c_d/\varepsilon_N^2}|\{\sup_l |E_l(T_{m_k}(f))| > \lambda\}|\,.
\end{aligned}$$

Insert this estimate in (4.8) to obtain

$$\left(p\,4^p\int_0^\infty \lambda^{p-1}\,I_\lambda\,d\lambda\right)^{\frac{1}{p}} \leq C_d\left(\sum_{k=1}^N e^{-c_d/\varepsilon_N^2}\|\sup_l |E_l(T_{m_k}(f))|\|_{L^p}^p\right)^{\frac{1}{p}} \leq c'B\left[Ne^{-c_d/\varepsilon_N^2}\right]^{\frac{1}{p}} \approx B$$

provided we choose $\varepsilon_N = c''/\sqrt{\log N}$. Here we used that the maximal operator

$$g \to \sup_l |E_l(g)|$$

is controlled by the Hardy-Littlewood maximal function and is therefore L^p bounded for all $1 < p < \infty$, while all T_{m_k} are L^p bounded with norm at most a multiple of B.

Next we turn our attention to the corresponding integral for term II_λ

$$\left(p\,4^p\int_0^\infty \lambda^{p-1}\,II_\lambda d\lambda\right)^{\frac{1}{p}}.$$

Using that

$$II_\lambda = |\{G_p(f) > (\varepsilon_N/AB)\lambda\}|,$$

where G_p is L^p bounded, we deduce that

$$\left(p\,4^p\int_0^\infty \lambda^{p-1}\,II_\lambda\,d\lambda\right)^{\frac{1}{p}} \leq C\frac{B\,A_r}{\varepsilon_N}\|G_p(f)\|_{L^p} \leq C_p\frac{B}{\varepsilon_N}\|f\|_{L^p}\,.$$

This last expression is equal to

$$C_r'B\sqrt{\log N}\,\|f\|_{L^p}$$

since ε_N was chosen to be $c''/\sqrt{\log N}$.

Finally we need to control

$$\left(p\,4^p\int_0^\infty \lambda^{p-1}\,III_\lambda d\lambda\right)^{\frac{1}{p}}. \tag{4.9}$$

It turns out that for any $1 < r < p$ one has an estimate (see [9])

$$|E_0(T_{m_k}(f))| \leq C_r\, B\, 2^{-\frac{N}{r}}(MMM(|f|^r))^{\frac{1}{r}} \tag{4.10}$$

whenever $m_k(\xi) = 0$ on $|\xi| \leq 2^N$ (M is the Hardy-Littlewood maximal operator).. This assumption can be made on each multiplier m_k, $k = 1, \dots, N$ as follows: working with f such that $\widehat{f}$ is compactly supported, we may assume that the multipliers m_k are supported in a finite union of dyadic annuli, which, by changing scales, may assume that do not intersect the ball $|\xi| \leq 2^N$.

Insert estimate (4.10) in (4.9) to obtain

$$\Big(p\, 4^p \int_0^\infty \lambda^{p-1}\, III_\lambda\, d\lambda\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^N \|E_0(T_{m_k}(f))\|_{L^p}^p\Big)^{\frac{1}{p}} \leq B\, N^{\frac{1}{p}}\, 2^{-\frac{N}{r}}\|f\|_{L^p}$$

which is trivially controlled by $B\sqrt{\log N}\,\|f\|_{L^p}$. □

5. A PROBLEM INVOLVING HOMOGENEOUS SINGULAR INTEGRALS

Mihklin-Hörmander multipliers correspond to kernels $K(y)$ on $\mathbf{R}^d$ that are singular at the origin, satisfy an estimate $|K(y)| \leq C|y|^{-d}$ for some $C < \infty$ and all $y \neq 0$, and possess a certain amount of smoothness. This smoothness suffices to guarantee the boundedness on all $L^p(\mathbf{R}^d)$ ($1 < p < \infty$) for the corresponding Fourier multiplier operator (given by convolution with K) as well as its weak type $(1,1)$ property.

In this section, we study a problem concerning Fourier multipliers given by convolution with kernels that are homogeneous of degree $-d$ on $\mathbf{R}^d$. Such kernels are determined by their restriction on the unit sphere $\mathbf{S}^{d-1}$. Let K be such a kernel and let Ω be its restriction on $\mathbf{S}^{d-1}$. One may check that the function $\Omega(y/|y|)|y|^{-d}$, $y \neq 0$ coincides with a principal value distribution on $\mathbf{R}^d$ if and only if Ω has mean value zero on the sphere. Only in this case one can make sense of convolution with K.

Let therefore Ω be an integrable function on $\mathbf{S}^{d-1}$ with mean value zero. We will be considering Calderón-Zygmund singular integrals of the form

$$T_\Omega(f)(x) = f * \text{p.v.}\frac{\Omega(x/|x|)}{|x|^d} = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} f(x-y)\frac{\Omega(y/|y|)}{|y|^d}\,dy\,, \tag{5.11}$$

where f is a Schwartz function on $\mathbf{R}^d$. This type of singular integrals were introduced by Calderón and Zygmund in [1].

If Ω is odd then the method of rotations (see [2]) gives

$$T_\Omega(f)(x) = \frac{\pi}{2}\int_{\mathbf{S}^{d-1}} H_\theta(f)(x)\,\Omega(\theta)\,d\theta\,, \tag{5.12}$$

where H_θ is the *directional Hilbert transform*

$$H_\theta(f)(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|t|>\varepsilon} f(x-t\theta)\frac{dt}{t}\,.$$

A simple argument using change of variables yields that the operator H_θ is bounded on L^p exactly when $H_{(1,0,\dots,0)}$ is; the latter is the Hilbert transform in the first variable and the identity operator in the remaining variables and hence it is trivially bounded on $L^p(\mathbf{R}^d)$ (and is of weak type $(1,1)$.)

Thus the boundedness of T_Ω on $L^p(\mathbf{R}^d)$ for Ω odd is an easy consequence of (5.12) and of the boundedness of H_θ on $L^p(\mathbf{R}^d)$ (which is uniform in θ). We point out that, as of this writing, the weak type $(1,1)$ boundedness of T_Ω for Ω odd, remains an open question.

The problem of the L^p boundedness of T_Ω is therefore interesting for Ω even. We begin our discussion by recalling the results of Calderón and Zygmund [2] who showed that if Ω lies in the space $L\log L(\mathbf{S}^{d-1})$, then T_Ω is bounded on $L^p(\mathbf{R}^d)$ for all $1 < p < \infty$. The more delicate issue of the weak type $(1,1)$ property of T_Ω was shown much later by Christ and Rubio de Francia [5] (for $d \le 7$, published only the case $d = 2$) and Seeger [14] for all d.

We note that M. Weiss and A. Zygmund [16] have constructed examples of even functions Ω in $L^1(\mathbf{S}^{d-1})$ such that T_Ω is unbounded on L^2 (even when restricted to continuous functions) and therefore on all other L^p. For an operator to be bounded on $L^2(\mathbf{R}^d)$ a certain condition on Ω is required. A calculation using the Fourier transform gives that the multiplier corresponding to the kernel $\Omega(y/|y|)|y|^{-d}$ is the function

$$\xi \to \int_{\mathbf{S}^{d-1}} \Omega(\theta) \log \frac{1}{|\xi \cdot \theta|} \, d\theta \, .$$

Therefore we have the equivalence

$$\operatorname*{essup}_{|\xi|=1} \left| \int_{\mathbf{S}^{d-1}} \Omega(\theta) \log \frac{1}{|\xi \cdot \theta|} \, d\theta \right| < +\infty \iff T_\Omega : L^2 \to L^2$$

and hence condition

$$\operatorname*{essup}_{|\xi|=1} \int_{\mathbf{S}^{d-1}} |\,\Omega(\theta)| \log \frac{1}{|\xi \cdot \theta|} \, d\theta < +\infty \tag{5.13}$$

implies the L^2 boundedness of T_Ω.

Since condition (5.13) arises naturally, it is reasonable to ask whether it implies the boundedness of T_Ω on L^p for some (or all) $p \neq 2$, The underlying question here is whether the p-independence boundedness property in Calderón-Zygmund theory holds for rough kernels.

This question was answered in the negative by P. Honzík and D. Ryabogin, in collaboration with the author, who constructed an example of an even function Ω on $\mathbf{S}^{d-1}$ such that the corresponding operator T_Ω is bounded on L^p exactly when $p = 2$.

In fact these authors have obtained the following sharper result:

Theorem 3. *(L. Grafakos, P. Honzík, D. Ryabogin* [10]*): Let $0 \le \alpha < \frac{1}{2}$. Then there exists $\Omega \in L^1(\mathbf{S}^{d-1})$ with mean value zero such that*

$$\operatorname*{essup}_{|\xi|=1} \int_{\mathbf{S}^{d-1}} |\,\Omega(\theta)| \log^{1+\alpha} \frac{1}{|\xi \cdot \theta|} \, d\theta < +\infty$$

but T_Ω is unbounded on $L^p(\mathbf{R}^d)$ for all

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \alpha \, .$$

Taking $\alpha = 0$ yields the previous case.

6. The second counterexample

In this section we discuss the counterexample of Theorem 3. In the proof we restrict our attention to the case $d = 2$ and we note that higher dimensional examples can be constructed using the two-dimensional example.

Let $M_p(\mathbf{Z})$ be the space of multipliers on $L^p([0,1])$, i.e. the space of all bounded sequences $(b_m)_{m\in\mathbf{Z}}$ such that the linear operator

$$h(x) \to \sum_{m\in\mathbf{Z}} b_m\left(\int_0^1 h(t)e^{-2\pi imt}dt\right)e^{2\pi imx} \tag{6.14}$$

maps 1-periodic functions h in $L^p([0,1])$ to functions in $L^p([0,1])$. The $M_p(\mathbf{Z})$ norm of the sequence $(b_m)_m$ is then the norm of the operator in (6.14) on $L^p([0,1])$.

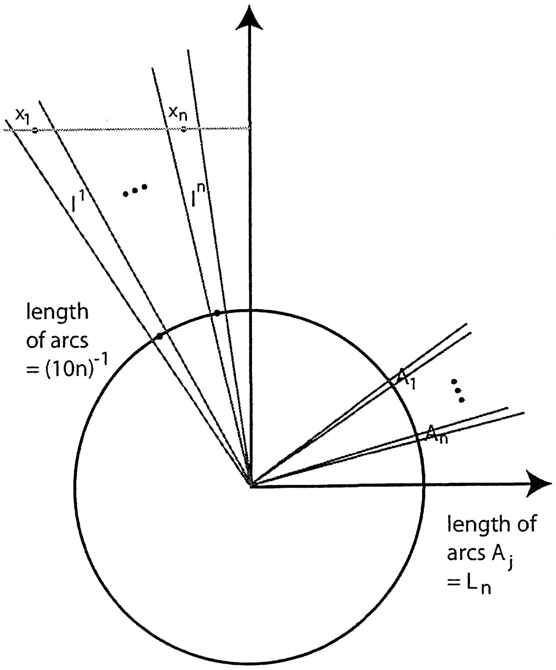

FIGURE 1. The points $x_1,\dots,x_n$ lie on a straight line perpendicular to the vertical coordinate axis. The arcs A_j lie in the first quadrant of the unit circle and have length L_n, a quantity to be determined. The cones I^j lie in the second quadrant and they meet the unit circle on arcs of length $(10n)^{-1}$ centered at the points $x_j/|x_j|$. The centers of the arcs A_j and the points $x_j/|x_j|$ form an angle of size $\pi/2$.

The basis $\{e^{2\pi ikx}\}_{k=-\infty}^{\infty}$ of $L^p([0,1])$, $p\neq 2$, is not unconditional. This means that for all $n=1,2,\dots$ there exist complex sequences a_k^n and $|\varepsilon_k^n|\le 1$ such that

$$\Big\|\sum_{k=1}^{n}\varepsilon_k^n a_k^n e^{2\pi ikx}\Big\|_{L^p[0,1]} \ge c_p\, n^{|\frac12-\frac1p|}\Big\|\sum_{k=1}^{n} a_k^n e^{2\pi ikx}\Big\|_{L^p[0,1]}, \tag{6.15}$$

for some constant c_p. To see this we consider the sequence of $a_k^n=1$ for all k for which the L^p norm if calculated explicitly and gives $\approx n^{1-1/p}$ and a random sequence of ± 1's, which by the Khintchine's inequality gives the constant $\sqrt{n}$.

Rephrased in the language of multipliers, estimate (6.15) is saying that for some constant c_p' we have

$$\|(\dots,0,\dots,0,\varepsilon_1^n,\varepsilon_2^n,\dots,\varepsilon_n^n,0,\dots)\|_{M_p(\mathbf{Z})} \ge c_p' n^{|\frac12-\frac1p|}.$$

We may choose the sequence $\{\varepsilon_k^n\}_{k=1}^n$ to be "maximal" in the sense that its M_p norm is the supremum of the M_p norms of all other sequences of ℓ^∞ norm 1 that satisfy (6.15) for the choice of a_k^n.

We now define

$$m(\omega)(\xi) = \int_{\mathbf{S}^{d-1}} \omega(\theta) \log \frac{1}{|\xi \cdot \theta|} \, d\theta$$

and we also define a similar quantity

$$m_\alpha(\omega)(\xi) = \int_{\mathbf{S}^{d-1}} |\omega(\theta)| \log^{1+\alpha} \frac{1}{|\xi \cdot \theta|} \, d\theta \,,$$

while for each integer n we define a even function

$$\Omega_n = \sum_{k=1}^n \varepsilon_k^n \underbrace{C(n) \sum_{j=0}^3 (-1)^j \chi_{A_k \text{ rotated by } \frac{j\pi}{2}}}_{\omega_k^n} \,,$$

where A_k are the arcs of Figure 1. Here $C(n)$ is a constant chosen so that

$$m_\alpha(\omega_k^n)(x_k/|x_k|) = 1/2$$

for all k. Finally we denote by $D(n)$ the constant

$$D(n) = m(\omega_k^n)(x_k/|x_k|) \,.$$

It is not difficult to check that

$$\begin{aligned} C(n) &\approx L_n^{-1} |\log L_n|^{-1-\alpha} \\ D(n) &\approx |\log L_n|^{-\alpha}. \end{aligned}$$

while for all $x \notin \bigcup_{j=0}^3 (I^k \text{ rotated by } \frac{j\pi}{2}) \cap \mathbf{S}^1$ we have

$$\begin{aligned} |m(\omega_k^n)(x)| &\lesssim (\log n) |\log L_n|^{-1-\alpha} \\ m_\alpha(\omega_k^n)(x) &\lesssim (\log n)^{1+\alpha} |\log L_n|^{-1-\alpha} \,. \end{aligned}$$

It follows from these estimates that

$$\|\Omega_n\|_{L^1(\mathbf{S}^1)} \lesssim n \, (\log n) \, |\log L_n|^{-1} \,. \tag{6.16}$$

On the other hand we have

$$m(\Omega_n)(x_k) = D(n) \, \varepsilon_k^n + \sum_{1 \le i \ne k \le n} \varepsilon_i^n m(\omega_i^{\varepsilon_n})(x_k) = D(n) \varepsilon_k^n + o_k^n,$$

where o_k^n is an error term which satisfies

$$|o_k^n| \le D(n)/4$$

provided

$$n^{4n} \lesssim L_n^{-1} \,.$$

We now take a look at certain multiplier norms. Using a deLeeuw type argument [6] we can restrict the $M_p(\mathbf{R}^2)$ norm of a multiplier to its values at the points $(x_k)_k$ which lie on a line parallel to the horizontal coordinate axis and we can thus estimate from below the

$M_p(\mathbf{R}^2)$ norm of the multiplier by the $M_p(\mathbf{Z})$ norm of the sequence of the first coordinates of its values at the points $(x_k)_k$. This way we obtain

$$\begin{aligned}\|m(\Omega_n)\|_{M_p(\mathbf{R}^2)} &\geq c_p\|(..,0,m(\Omega_n)(x_1),...,m(\Omega_n)(x_n),0,...)\|_{M_p(\mathbf{Z})}\\ &\geq c_p\,D(n)\Big[\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\|_{M_p(\mathbf{Z})} -\\ &\qquad \frac{1}{D(n)}\|(...,0,o_1^n,...,o_n^n,0,...)\|_{M_p(\mathbf{Z})}\Big]\\ &\geq \frac{1}{2}\,c_p\,D(n)\,\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\|_{M^p(\mathbf{Z})}\,,\end{aligned}$$

since the inequality

$$\frac{1}{D(n)}\Big\|(...,0,o_1^n,...,o_n^n,0,...)\Big\|_{M_p(\mathbf{Z})} > \frac{1}{2}\,\Big\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\Big\|_{M_p(\mathbf{Z})}$$

would contradict the "maximal" choice of $(\varepsilon_k^n)_{k=1}^n$.

We now recall that

$$\Big\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\Big\|_{M_p(\mathbf{Z})} \geq c_p'\,n^{|\frac{1}{2}-\frac{1}{p}|}\,,$$

which implies that

$$\|m(\Omega_n)\|_{M_p(\mathbf{R}^2)} \geq c'\,D(n)\,n^{|\frac{1}{2}-\frac{1}{p}|} \approx |\log L_n|^{-\alpha}n^{|\frac{1}{2}-\frac{1}{p}|}\,.$$

We finally choose the L_n's. We had the restriction

$$n^{4n} \leq L_n^{-1}$$

while the need to make the expression on the right in equation (6.16) equal to a constant forces us to choose

$$|\log L_n| \approx n\log n\,.$$

With this choice of L_n and all the facts we have accumulated so far we have

$$\begin{aligned}\|T_{\Omega_n}\|_{L^p\to L^p} &= \|m(\Omega_n)\|_{M_p(\mathbf{R}^2)}\\ &\geq c'|\log L_n|^{-\alpha}n^{|\frac{1}{2}-\frac{1}{p}|}\\ &\approx (\log n)^{-\alpha}n^{|\frac{1}{2}-\frac{1}{p}|-\alpha}\,.\end{aligned}$$

We have now constructed a sequence of even integrable functions Ω_n with L^1 norm at most a constant such that

$$\|T_{\Omega_n}\|_{L^p\to L^p} \to \infty$$

when $|\frac{1}{2}-\frac{1}{p}| > \alpha$.

To complete the proof we need some functional analysis. Let $\mathcal{B}_\alpha$ the Banach space of all even integrable functions Ω on $\mathbf{S}^1$ with mean value zero with norm

$$\|\Omega\|_{\mathcal{B}_\alpha} \equiv \|\Omega\|_{L^1(\mathbf{S}^1)} + \|m_\alpha(\Omega)\|_{L^\infty(\mathbf{S}^1)} < \infty.$$

Consider the family of linear maps

$$\Omega \to T_\Omega(f) \quad : \quad \mathcal{B}_\alpha \to L^p(\mathbf{R}^2)$$

indexed by the set

$$U = \{f \in L^p(\mathbf{R}^2) : \ \|f\|_{L^p} = 1\}.$$

If no claimed Ω existed, then for all $\Omega \in \mathcal{B}_\alpha$ we would have

$$\sup_{f\in U}\|T_\Omega(f)\|_{L^p} \leq C(\Omega) < \infty\,.$$

The uniform boundedness principle implies the existence of a constant $K < \infty$ such that

$$\|T_\Omega\|_{L^p \to L^p} = \sup_{f \in U} \|T_\Omega(f)\|_{L^p} \le K \, \|\Omega\|_{\mathcal{B}_\alpha}$$

for all $\Omega \in \mathcal{B}_\alpha$. But this contradicts the construction of Ω_n's for $|\frac{1}{2} - \frac{1}{p}| > \alpha$.

7. CONDITIONS THAT DISTINGUISH BETWEEN p'S

Theorem 3 suggests that one should look for conditions on Ω that distinguish boundedness on $L^p(\mathbf{R}^d)$ for different values of p's.

A natural condition that one should introduce in the study of this problem is the following:

$$CL(\alpha) \qquad\qquad \operatorname*{essup}_{|\xi|=1} \int_{\mathbf{S}^{d-1}} |\,\Omega(\theta)|\, \log^{1+\alpha} \frac{1}{|\xi \cdot \theta|}\, d\theta < +\infty$$

A result of Stefanov and the author [11] says that $CL(\alpha)$ implies the L^p boundedness of T_Ω whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\alpha}{2(1+\alpha)}\,.$$

This was improved by Fan, Guo, Pan [7] when $\alpha > 1$ but Theorem 3 is only concerned with the case $\alpha \le 1/2$. As for $\alpha \le 1/2$ we have $\alpha > \alpha/(2(1+\alpha))$, it remains an open question to find out what happens in between. We pose therefore the following question:

(a) Assume that $CL(\alpha)$ holds for some $\alpha < 1/2$. Does it follow that

$$T_\Omega : L^p \to L^p$$

whenever

$$\alpha \ge \left|\frac{1}{p} - \frac{1}{2}\right| \ge \frac{\alpha}{2(1+\alpha)} \qquad ?$$

For $\alpha \ge 1/2$ the counterexample does not work and one may guess that in this case T_Ω is bounded on L^p for the whole range

(b) Assume that $CL(\alpha)$ holds for some $\alpha \ge 1/2$. Does it follow that

$$T_\Omega : L^p \to L^p$$

for all $1 < p < \infty$?

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REFERENCES

[1] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[2] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[3] S. Y. A. Chang, M. Wilson, and T. Wolff. *Some weighted norm inequalities concerning the Schrödinger operator,* Comment. Math. Helv. **60** (1985), 217–246.

[4] M. Christ, L. Grafakos, P. Honzík, and A. Seeger, *Maximal functions associated with multipliers of Mikhlin-Hörmander type,* Math. Zeit. **249** (2005), 223–240.

[5] M. Christ and J.-L. Rubio de Francia, *Weak type* $(1,1)$ *bounds for rough operators II,* Invent. Math. **93** (1988), 225–237.

[6] K. deLeeuw, *On* L^p *multipliers*, Ann. Math, **81** (1964), 364–379.

[7] D. Fan, K. Guo, and Y. Pan, *A note of a rough singular integral operator*, Math. Ineq. and Appl., **2** (1999), 73–81.

[8] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education, Upper Saddle River, NJ 2004.

[9] L. Grafakos, P. Honzik, and A. Seeger, *On Maximal functions for Mikhlin-Hörmander multipliers,* Advances in Math., to appear.

[10] L. Grafakos, P. Honzik, and D. Ryabogin, *On the p-independence boundedness property of Calderón-Zygmund theory,* J. für die Reine und Angew. Math., to appear.

[11] L. Grafakos and A. Stefanov, *L^p bounds for singular integrals and maximal singular integrals*, Indiana Univ. Math. J., **47** (1998), 455–469.

[12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces,* Acta Math. **104** (1960), 93–139.

[13] S. G. Mikhlin, *On the multipliers of Fourier integrals,* (Russian) Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 701–703.

[14] A. Seeger, *Singular integral operators with rough convolution kernels*, Jour. Amer. Math. Soc., **9** (1996), 95–105.

[15] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1971.

[16] M. Weiss and A. Zygmund, *An example in the theory of singular integrals,* Studia Math. **26** (1965), 101–111.

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FALCONER'S DISTANCE SET CONJECTURE FOR POLYGONAL NORMS

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ABSTRACT. A conjecture of Falconer [F86] asserts that if E is a planar set with Hausdorff dimension strictly greater than 1, then its Euclidean distance set $\Delta(E)$ has positive one-dimensional Lebesgue measure. We review recent work on the analogous question with the Euclidean distance replaced by non-Euclidean norms $\|\cdot\|_X$ in which the unit ball is a polygon, and construct explicit examples of sets with large Hausdorff dimension whose distance set has Lebesgue measure 0.

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§0. INTRODUCTION

A conjecture of Falconer [F86] asserts that if a set $E \subset \mathbb{R}^2$ has Hausdorff dimension strictly greater than 1, then its Euclidean distance set

$$\Delta(E) = \Delta_{l_2^2}(E) = \left\{ \|x - x'\|_{l_2^2} : x, x' \in E \right\}$$

has positive one-dimensional Lebesgue measure. The current best result in this direction is due to Wolff [W99], who proved that the conclusion is true if E has Hausdorff dimension greater than 4/3. Erdogan [Er03], [E04] extended this result to higher dimensions, proving that the same conclusion holds for subsets of $\mathbb{R}^d$ with Hausdorff dimension greater than $\frac{d}{2} + \frac{1}{3}$. This improves on the earlier results of Falconer [F86], Mattila [M87], and Bourgain [B94].

An analogous question may be posed for more general n-dimensional normed spaces. Let X be the n-dimensional vector space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$. We define the X-distance set of a set $E \subset X$:

$$\Delta_X(E) = \{\|x - x'\|_X : x, x' \in E\},$$

and ask how the size of $\Delta_X(E)$ depends on the dimension of E as well as on the properties of the norm $\|\cdot\|_X$. Simple examples show that Falconer's conjecture as stated above, but with $\Delta(E)$ replaced by $\Delta_X(E)$, cannot hold for all normed spaces X. For instance, let X be the 2-dimensional plane with the norm

$$\|x\|_{l_\infty^2} = \max(|x_1|, |x_2|)$$

and let $E = F \times F$, where F is a subset of $[0, 1]$ with Hausdorff dimension 1 such that $F - F := \{x - x' : x, x' \in F\}$ has measure 0. (It is an easy exercise to modify the Cantor set construction to produce such a set.) Then E has Hausdorff dimension 2, but its l_∞^2-distance set $F - F$ has measure 0.

Here and below, we use $\dim(E)$ to denote the Hausdorff dimension of E, $|F|_d$ to denote the d-dimensional Lebesgue measure of F, and $|A|$ to denote the cardinality of a finite set A.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Definition 0.1. *Let X be a 2-dimensional normed space, and let $0 < \alpha < 2$. We will say that the α-Falconer conjecture holds in X if for any set $E \subset X$ with dim$(E) > \alpha$ we have $|\Delta_X(E)|_1 > 0$.*

The above considerations indicate that the range of α for which the α-Falconer conjecture holds in X will depend on the properties of the norm on X, and in particular that the curvature of the distance function should play a role. Indeed, let

$$BX = \{x \in X : \|x\|_X \leq 1\}$$

be the unit ball in X. In the example with the product of Cantor sets, the unit ball was a square (no curvature), and the α-Falconer conjecture fails for all $\alpha < 2$. On the other hand, we do have an α-Falconer conjecture with $\alpha > 4/3$ (and expect it to hold with $\alpha > 1$) in a 2-dimensional plane is equipped with the Euclidean norm, where the unit ball is strictly convex and its boundary ∂BX has nonvanishing curvature. This motivates several natural questions: For what range of α does the α-Falconer conjecture hold in X if ∂BX has everywhere nonvanishing curvature? What if only know that ∂BX is strictly convex, but make no curvature assumptions? Does the α-Falconer conjecture with $\alpha < 2$ fail for all polygons and for all $\alpha < 2$?

With regard to the first two questions, the following partial results are known.

Theorem 1. *(Iosevich-Łaba* [IL04]*) The 3/2-Falconer conjecture holds in any 2-dimensional vector space X over $\mathbb{R}$ such that BX is strictly convex and ∂BX has the property that the diameter of the chord*

$$\{x \in BX : x \cdot v \geq \max_{y \in BX}(y \cdot v) - \epsilon\},$$

where v is a unit vector and $\epsilon > 0$, is bounded by $C\sqrt{\epsilon}$ uniformly for all v and ϵ.

Erdogan [Er03], [Er04] observes that if we make the stronger assumption that BX is strictly convex and that ∂BX is smooth and has nonvanishing Gaussian curvature, then his arguments for the Euclidean case extend to X, with only minor changes. Thus, with these assumptions, the 4/3-Falconer conjecture holds in X, and moreover this result extends to higher dimensions.

Theorem 2. *(Erdogan* [Er03], [Er04]*) If X is an n-dimensional vector space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$ such that the unit ball BX has a smooth boundary with nonvanishing Gaussian curvature, and if $E \subset X$ has dim$X > \frac{n}{2} + \frac{1}{3}$, then $\Delta_X(E)$ has positive measure.*

The methods of [W99], [Er03], [Er04], [IL04] are Fourier-analytic. The general strategy, due to Falconer [F86] and Mattila [M87], employs decay estimates on the Fourier transform of measures supported on ∂BX. In [Er03], [Er04], Mattila's approach is combined with a weighted modification of the bilinear restriction estimate of Tao [T03]. [IL04] uses recent stationary-phase type estimates available for non-smooth surfaces, see eg. [BRT98].

We do not know what the optimal range of α should be for the strictly convex case. However, there are no known counterexamples to the 1-Falconer conjecture in normed spaces with BX strictly convex.

On the other hand, the polygonal case has been resolved entirely by Falconer [Fa04], along with its higher-dimensional analogue:

Theorem 3. *If X is an n-dimensional vector space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$ such that the unit ball BX is a polytope with finitely many faces, then there is a compact set $E \subset X$ with $\dim E = n$ and $|\Delta_X(E)|_1 = 0$.*

In particular, if X is a 2-dimensional space and BX is a polygon with finitely many sides, then the α-Falconer conjecture fails for all $\alpha < 2$. The proof is based on consideration of "typical" intersections of homothetic copies of fixed Borel subsets of $\mathbb{R}^n$, and, as such, is not constructive.

The purpose of the remainder of this paper is to give explicit examples of subsets of X with large dimension whose X-distance set has measure zero, for large classes of 2-dimensional spaces X such that BX is a symmetric polygon with finitely many sides. Throughout the sequel, we will always assume that X is 2-dimensional, with BX as above. In general, we do not know how to construct explicit sets E with $\dim E = 2$ and $|\Delta_X(E)|_1 = 0$. However, we have the following construction.

Theorem 3A. *Let BX be a symmetric convex polygon with $2K$ sides. Then there is a set $E \subset [0,1]^2$ with Hausdorff dimension $\geq K/(K-1)$ such that $|\Delta_X(E)|_1 = 0$.*

Using recent results on Diophantine approximations, we can improve this for almost all polygons BX. Fixing a coordinate system, we can define for any non-degenerate segment $I \subset X$ its *slope* $Sl(I)$: if the line containing I is given by an equation $u_1x_1 + u_2x_2 + u_0 = 0$, then we set $Sl(I) = -u_1/u_2$. We write $Sl(I) = \infty$ if $u_2 = 0$.

Theorem 3B. *For any integer $K \geq 3$ and for almost all $\gamma_1, \ldots, \gamma_K$ (satisfying the Diophantine condition stated in Section 2), the following is true. If BX is a symmetric convex polygon with $2K$ sides, and the slopes of non-parallel sides are equal to $\gamma_1, \ldots, \gamma_K$, then there is an explicit compact set $E \subset X$ such that the Hausdorff dimension of E is 2 and the Lebesgue measure of $\Delta_X(E)$ is 0.*

Actually, we will prove the stronger result: if $K \geq 3$ and if the slopes of 3 non-parallel sides of BX are fixed, then for almost all choices of slopes of other $K-3$ non-parallel sides we can construct the set E as claimed. More specifically, the construction can be carried out provided that the slopes $\gamma_1, \ldots, \gamma_K$ satisfy a certain Diophantine condition stated in Section 2. (Note that for $K \leq 3$ Theorem 3B follows from Theorem 3C below.).

The sets we construct are Cantor-type sets E defined as intersections of a sequence of sets E_j, each of which is a union of balls of radii decreasing to 0 as $j \to \infty$. The main step in the construction is finding a suitable set A_j of the centers of the balls used at j-th step. On one hand, for the distance set of E to be small we need estimates on the size of certain projections (depending on BX) of the difference set $A_j - A_j$. On the other hand, for the lower bound on the dimension of E we require that A_j be well separated, i.e. we need a suitable bound from below on $|a - a'|$ for all $a, a' \in A_j$, $a \neq a'$. This is done in Section 1 in the setting of Theorem 3A. The proof of Theorem 3B is given in Section 2: there, we use the Diophantine condition just mentioned to improve the separation constants.

If we assume that there is a coordinate system in which the slopes of all sides of K are algebraic, then a stronger result is known [KŁ04]. Note in particular that Theorem 3C applies to all polygons BX with 4 or 6 sides.

Theorem 3C. [KŁ04] *If BX is a polygon with finitely many sides, and if there is a coordinate system in which all sides of BX have algebraic slopes, then there is* *a compact $E \subset X$ such that the Hausdorff dimension of E is 2 and the Lebesgue measure of $\Delta_X(E)$ is 0.*

In fact, [KŁ04] gives a recipe for an explicit construction of the set E claimed in the theorem. First, a suitable discrete set of points is constructed in [KŁ04]; to obtain the Cantor-type set E, one then follows the procedure described in [IŁ04].

§1. PROOF OF THEOREM 3A

We may assume that $K \geq 4$, since otherwise Theorem 3C applies. We use $B(x,r)$ to denote the closed Euclidean ball with center at x and with radius r. We also denote $A - A = \{a - a' : \ a, a' \in A\}$ and $A \cdot v = \{a \cdot v : \ a \in A\}$.

Let $b_1, \ldots, b_K$ be vectors such that

$$BX = \bigcap_{k=1}^{K} \{x : \ |x \cdot b_k| \leq 1\}.$$

Then for any $x \in X$,

$$\|x\|_X = \max_{1 \leq k \leq K} |x \cdot b_k|. \tag{1.3}$$

Let also $a_1, \ldots, a_K$ be unit vectors parallel to the K sides of BX, so that

$$a_j \cdot b_j = 0, \ j = 1, \ldots, K. \tag{1.4}$$

Lemma 1.1. *Assume that $K \geq 4$. Then there are arbitrarily large integers n for which we may choose sets $A = A(n) \subset B(0, 1/2)$ such that $|A| = n$ and*

$$|(A - A) \cdot b_k| \ll n^{1-1/K}, \ k = 1, 2, \ldots, K, \tag{1.1}$$

(in particular, $|\Delta_X(A)| \ll n^{1-1/K}$), and

$$\|x - x'\|_X \gg n^{-1/2}, \ x, x' \in A, \ x \neq x', \tag{1.2}$$

with the implicit constants independent of n.

Proof. Fix a large integer N, and let $u_1, \ldots, u_K$ be numbers in $[1, 2]$, to be determined later. Define

$$S = \Big\{ \sum_{k=1}^{K} \frac{j_k}{N} u_k a_k, \ j_k \in \{1, \ldots, N\} \Big\}.$$

We claim that the set

$$U = \{(u_1, \ldots, u_K) \in \mathbb{R}^K : \ |S| < N^K\}$$

has K-dimensional measure 0. Indeed, if $|S| < N^K$, then we must have

$$\sum_{k=1}^{K} \frac{j_k}{N} u_k a_k = 0$$

for some $j_1, \dots, j_K \in \{1-N, \dots, N-1\}$, not all zero. Fix such $j_1, \dots, j_K$. Then the $2 \times K$ matrix with columns $\frac{j_k}{N} u_k a_k$, $k = 1, \dots, K$, has rank at least 1, hence its nullspace has dimension at most $K-1$. It follows that U is a union of a finite number of hyperplanes of dimension at most $K-1$, therefore has K-dimensional measure 0 as claimed.

We will assume henceforth that $(u_1, \dots, u_K) \notin U$. Then $|S| = N^K$ and $S \subset B(0, 2K)$. Our goal is to obtain (1.1), (1.2) for $n = N^K$ and $A = (4K)^{-1}S$.

We first prove that (1.1) holds, i.e.

$$
(1.5) \qquad |(S-S) \cdot b_k| \ll N^{K-1} \ll n^{1-1/K}, \; k = 1, 2, \dots, K.
$$

Indeed, let $x \in S - S$, then $x = \sum_{k=1}^{K} \frac{j_k}{N} u_k a_k$ for some $j_1, \dots, j_K \in \{1-N, \dots, N-1\}$. Fix $k_0 \in \{1, \dots, k\}$, then

$$
x \cdot b_{k_0} = \sum_{k=1}^{K} \frac{j_k}{N} u_k a_k \cdot b_{k_0} = \sum_{k \neq k_0} \frac{j_k}{N} u_k a_k \cdot b_{k_0},
$$

where we also used (1.4). The last sum can take at most $(2N)^{K-1}$ possible values, which proves (1.5).

It remains to verify that there is a choice of $u_1, \dots, u_K$ for which (1.2) also holds. We will do so by proving that if t is a sufficiently small constant, depending only on K and on the angles between the non-parallel sides of BX, then the set

$$
(1.6) \qquad \{(u_1, \dots, u_K) \in [1,2]^K : \; \|x\|_X \leq tN^{-K/2} \text{ for some } x \in S - S\}
$$

has K-dimensional Lebesgue measure strictly less than 1.

Let $x \in S-S$, then $x = \sum_{k=1}^{K} \frac{j_k}{N} u_k a_k$ for some $j_k \in \{1-N, \dots, N-1\}$. Suppose that $x \neq 0$ and

$$
(1.7) \qquad \|x\|_X \leq tN^{-K/2}.
$$

Assume that $|j_{k_1}| \geq |j_{k_2}| \geq \cdots \geq |j_{k_K}|$, and that $|j_{k_1}| \in [2^s, 2^{s+1})$ for some integer s such that $1 \leq 2^s \leq N$. If we had $|j_{k_2}| < 2^{s-2}/K$, then we would also have

$$
\|x\|_X \geq \|\frac{j_{k_1}}{N} u_{k_1} a_{k_1}\|_X - \sum_{k \neq k_1} \|\frac{j_k}{N} u_k a_k\|_X \geq \frac{2^s}{N} - K \cdot \frac{2 \cdot 2^{s-2}}{KN} = \frac{2^{s-1}}{N} \geq \frac{1}{2N}.
$$

But if $K \geq 4$, then (1.7) implies that $\|x\|_X \leq tN^{-2}$, which contradicts the last inequality if $t \leq 1$ and $N > 2$. It follows that

$$
(1.8) \qquad |j_{k_1}| \geq 2^s, \; |j_{k_2}| \geq 2^{s-2}/K.
$$

Fix j_{k_1}, j_{k_2} as in (1.8). Fix also $y = \sum_{k \neq k_1, k_2} \frac{j_k}{N} u_k a_k$, and consider the set of $(u_{k_1}, u_{k_2}) \in \mathbb{R}^2$ such that (1.7) holds, i.e.

$$
\|\frac{j_{k_1}}{N} u_{k_1} a_{k_1} + \frac{j_{k_2}}{N} u_{k_2} a_{k_2} + y\|_X \leq tN^{-K/2}.
$$

By (1.8), this set has 2-dimensional measure

$$\le c_1(tN^{-K/2})^2 \cdot \frac{N}{2^s} \cdot \frac{NK}{2^{s-2}} = 4c_1K \cdot t^2N^{2-K}/2^{2s}.$$

Here and through the rest of the proof of the lemma, c_1, c_2, c_3 denote constants which may depend on K and on the angles between the non-parallel sides of BX, but are independent of t and N.

Integrating over u_k, $k \neq k_1, k_2$, we see that the set

$$\Big\{(u_1, \dots, u_K) \in [1,2]^K : \ \|\sum_{k=1}^{K} \frac{j_k}{N} u_k a_k\|_X \le tN^{-K/2}\Big\},$$

with fixed $j_1, \dots, j_K$ such that

$$2^s \le \max_{k=1,\dots,K} |j_k| < 2^{s+1}, \tag{1.9}$$

has K-dimensional measure $\le 4c_1K \cdot t^2N^{2-K}/2^{2s}$.

The number of K-tuples $j_1, \dots, j_K$ satisfying (1.9) is $\le (2^{s+2})^K$, hence summing over all such K-tuples we get a set of measure

$$\le c_2t^2N^{2-K}2^{(K-2)s}.$$

Now sum over all s with $2^s \le N$. We find that the measure of the set in (1.6) is

$$\le c_2 \sum_{s:1\le 2^s \le N} t^2N^{2-K}2^{(K-2)s} \le c_3t^2N^{2-K}N^{K-2} = c_3t^2.$$

This is less than 1 if $t < \sqrt{c_3}$, as claimed.

Proof of Theorem 3A. We construct E as follows. Take a small positive number c which will be specified later. Let $A_j = A(n_j)$ be as in Lemma 1.1, where a nondecreasing sequence $\{n_j\}$ and a sequence $\{N_j\}$ are such that

$$N_j = \prod_{\nu=1}^{j} n_\nu, \quad n_j \to \infty \, (j \to \infty), \quad \log n_{j+1}/\log N_j \to 0 \, (j \to \infty). \tag{1.10}$$

(We consider that the empty product for $j = 0$ is equal to 1.) Also, fix $s = (K-1)/K > 1/2$. Let also c be small enough so that for any j the discs $B(x, cn_j^{-s})$, $x \in A_j$, are mutually disjoint and contained in $B(0,1)$; this is possible by (1.2). Denote

$$\delta_j = cn_j^{-s}, \quad \Delta_j = \prod_{\nu=1}^{j} \delta_j = c^jN_j^{-s}.$$

Let $E_1 = \bigcup_{x\in A_1} B(x, \delta_1)$. We then define $E_2, E_3, \dots$ by induction. Namely, suppose that we have constructed E_j which is a union of N_j disjoint closed discs B_i of radius Δ_j each. Then E_{j+1} is obtained from E_j by replacing each B_i by the image of $\bigcup_{x\in A_{j+1}} B(x, \delta_{j+1})$ under the unique affine mapping which takes $B(0,1)$ to B_i and preserves direction of vectors. We then let $E = \bigcap_{j=1}^{\infty} E_j$.

We will first prove that E has Hausdorff dimension at least $1/s$. The calculation follows closely that in [F85], pp. 16–18.

Let $\mathcal{B}_j$ be the family of all discs of radius Δ_j used in the construction of E_j, and let $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{B}_j$, where we set $\mathcal{B}_0 = \{B(0,1)\}$. We then define

$$\mu(F) = \inf \Big\{ \sum_{i=1}^{\infty} N_{j(i)}^{-1} : \ F \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), B(x_i, r_i) \in \mathcal{B}_{j(i)} \Big\}, \tag{1.11}$$

for all $F \subset E$. Clearly, μ is an outer measure on subsets of E. Observe that if $B = B(x, \Delta_j) \in \mathcal{B}_j$, then

$$N_j^{-1} = n_{j+1} \cdot N_{j+1}^{-1} = \sum_{B' \in \mathcal{B}_{j+1} : B' \subset B} (N_{j+1})^{-1}, \tag{1.12}$$

hence the sum in (1.11) does not change if we replace a disc $B \in \mathcal{B}_j$ by all its subdiscs from the next iteration $\mathcal{B}_{j+1}$. In particular, we may assume that all the discs in the covering of F in (1.11) have radius less than δ for any $\delta > 0$.

We first claim that if $B_0 = B_0(x_0, r_0) \in \mathcal{B}_j$ then

$$\mu(E \cap B_0) = N_j^{-1}. \tag{1.13}$$

The inequality $\mu(E \cap B_0) \leq N_j^{-1}$ is obvious, by taking a covering of $E \cap B_0$ by the single ball B_0. Let now $E \cap B_0 \subset \bigcup_i B_i$, where $B_i \in \mathcal{B}$ has radius $r_i = \Delta_{j(i)}$. We need to prove that

$$\sum r_i^{1/s} \geq r_0^{1/s}. \tag{1.14}$$

Since E is compact and B_i are open relative to E, we may assume that the covering is finite. We may also assume that all B_i are disjoint, since otherwise we may simply remove any discs contained in any other disc of the covering. If the covering consists of the single disc B_0, we are done. Otherwise, let B_I be one of the covering discs with smallest r_i, say $B_I \in \mathcal{B}_j$, and let $\tilde{B}_I \in \mathcal{B}_{j-1}$ be such that $B_I \subset \tilde{B}_I$. Then $\tilde{B}_I \subset B_0$, hence all discs in $\mathcal{B}_j$ contained in $\tilde{B}_I$ are also contained in B_0. By the minimality of r_I, these discs belong to the covering $\{B_i\}$. We then replace all these discs by the single disc $\tilde{B}_I$; by (1.12), the sum on the left side of (1.14) does not change. Iterating this procedure, we eventually arrive at a covering consisting only of B_0, which proves (1.14).

Next, we prove that for any $s' > s$

$$\mu(E \cap B) \ll r^{1/s'} \tag{1.15}$$

for any disc $B = B(x, r)$, not necessarily in $\mathcal{B}$, where the constant in $\ll$ may depend on s'. We may assume that $r \leq 1$, since otherwise we have from (1.13) with $B_0 = B(0,1)$

$$\mu(E \cap B) \leq \mu(E) = 1 \leq r^{1/s'},$$

which proves (1.15). Let $j \geq 0$ be such that $r \in (\Delta_{j+1}, \Delta_j]$, and consider all discs in $\mathcal{B}_j$ which intersect $E \cap B$. They are closed, mutually disjoint discs which intersect

B and have radius no less than r; hence there are at most 6 such discs. Applying (1.13) to each of these discs and summing up, we have

$$\mu(E \cap B) \leq 6N_j^{-1}.$$

Moreover,

$$r > \Delta_{j+1} = N_j^{-s} n_{j+1}^{-s} c^{-j-1},$$

and we get (1.15) using (1.10).

Thus, if $s' > s$ and $\{B_i\}_{i=1}^{\infty}$ is a covering of E by discs of radii r_i, then from (1.15) we have

$$\sum_{i=1}^{\infty} r_i^{1/s'} \gg \sum_{i=1}^{\infty} \mu(E \cap B_i) \geq \mu(E).$$

Taking the infimum over all such coverings, we see that

$$H_{1/s'}(E) > 0.$$

Since $s' > s$ is arbitrary, we conclude that the Hausdorff dimension of E is at least $K/(K-1)$.

It remains to prove that $|\Delta_X(E)|_1 = 0$. From (1.1) we have

$$|(A-A) \cdot b_k| \leq Cn^{1-1/K}, \ k = 1, 2, \ldots, K, \tag{1.16}$$

with C independent of n. We choose c small enough so that

$$cC < 1/2. \tag{1.17}$$

Let D_j be the set of the centers of the discs in $\mathcal{B}_j$. We claim that

$$|(D_j - D_j) \cdot b_k| \leq C^j N_j^s, \ k = 1, 2, \ldots, K. \tag{1.18}$$

Indeed, for $j = 1$ this is (1.16). Assuming (1.18) for j, we now prove it for $j+1$. Let $x, x' \in D_{j+1}$. Then $x \in B(y, \Delta_j)$, $x' \in B(y', \Delta_j)$, $y, y' \in D_j$. We write

$$(x - x') \cdot b_k = (y - y') \cdot b_k + ((x-y) - (x'-y')) \cdot b_k. \tag{1.19}$$

The first term on the right is in $(D_j - D_j) \cdot b_k$, hence has at most $C^j N_j^s$ possible values. Also, by construction $x - y, x' - y'$ are in $\Delta_j A_{j+1}$, hence the second term is in $\Delta_j(A_{j+1} - A_{j+1}) \cdot b_k$ and has at most Cn_{j+1}^s possible values, by (1.16). This gives at most $C^{j+1} N_{j+1}^s$ possible values for (1.19), as required.

By (1.18), (1.3) and the triangle inequality, $\Delta_X(E_j)$ can be covered by at most $KC^j N_j^s$ intervals of length $2c_0\Delta_j = 2c_0 c^j N_j^{-s}$, where c_0 is the X-diameter of $B(0,1)$. It follows that

$$|\Delta_X(E_j)|_1 \leq 2Kc_0(cC)^j \leq 2Kc_0(1/2)^j,$$

by (1.17). The last quantity goes to 0 as $j \to \infty$. Since $\Delta_X(E) \subset \Delta_X(E_j)$, this proves our claim that $|\Delta_X(E)|_1 = 0$. The proof of the theorem is complete.

Remark. It is easy to check that the set constructed in the proof of Theorem 3A has the Hausdorff dimension exactly $K/(K-1)$.

§2. PROOF OF THEOREM 3B

The case $K \le 3$ is covered by Theorem 3C. We consider that $K > 3$ and denote $d = K - 3$. Denote

$$\bar{l} = (l_1, \dots, l_d) \in \mathbb{Z}_+^d,$$

$$\mathcal{L}(L) = \{\bar{l} : 0 \le l_k < L \, (k = 1, \dots, d)\}.$$

For a real vector $\bar{\gamma} = (\gamma_1, \dots, \gamma_d)$ we write $\bar{\gamma} \in (KM)$ if for any positive integer L and for any $\varepsilon > 0$

$$\inf \left| \sum_{\bar{l} \in \mathcal{L}(L)} n_{\bar{l}} \gamma_1^{l_1} \dots \gamma_d^{l_d} \right| \left(\max_{\bar{l} \in \mathcal{L}(L)} |n_{\bar{l}}| \right)^{(1+\varepsilon)L^d} > 0,$$

where infimum is taken over all nonzero integral vectors $\{n_{\bar{l}} : \bar{l} \in \mathcal{L}\}$. The following theorem easily follows from the results of Kleinbock and Margulis [KM98].

Theorem A. *For almost all $\bar{\gamma} \in \mathbb{R}^d$ we have $\bar{\gamma} \in (KM)$.*

The results of [KM98] have been refined in [BKM01], [Be02], [BBKM02].
Now we formulate the main result of this section.

Theorem 4. *Let $\bar{\gamma} \in (KM)$, $K = d+3$, and let BX be a symmetric convex polygon with $2K$ sides, and the slopes of non-parallel sides are equal to $\gamma_1, \dots, \gamma_d, 0, 1,$ and ∞, then there is a compact $E \subset X$ such that the Hausdorff dimension of E is 2 and the Lebesgue measure of $\Delta_X(E)$ is 0.*

Formally, Theorem 4 deals with polygons BX of special kind, but it is easy to see that for any polygon we can make slopes of three sides of it equal to $0, 1, \infty$ by a choice of a coordinate system. Indeed, if I_1, I_2, I_3 are 3 non-parallel sides of BX, then, taking the x_1-coordinate axis and the x_2-coordinate axis of a new coordinate system parallel to I_1 and I_3 respectively, we get $Sl(I_1) = 0$, $Sl(I_3) = \infty$; moreover, the slope of I_2 can be made equal to 1 by scaling and, if necessary, reflecting, the x_2-coordinate axis. Thus, combining Theorem A and Theorem 4 we get Theorem 3 (and also its stronger version mentioned in the end of §0).

We use notation introduced in the beginning of §1. To prove Theorem 4, we need a lemma similar to Lemma 1.1.

Lemma 2.1. *Assume that $K, d, \bar{\gamma}, BX$ satisfy the conditions of Theorem 4. Then for any $\varepsilon > 0$ there are arbitrarily large integers n for which we may choose sets $A = A(n) \subset B(0, 1/2)$ such that $|A| = n$ and*

$$|(A - A) \cdot b_k| \ll n^{(1/2)+\varepsilon}, \quad k = 1, 2, \dots, K, \tag{2.1}$$

(in particular, $|\Delta_X(A)| \ll n^{(1/2)+\varepsilon}$), and

$$\|x - x'\|_X \gg n^{-1/2-\varepsilon}, \quad x, x' \in A, \quad x \ne x', \tag{2.2}$$

where the implicit constants may depend on ϵ but are independent of n.

Proof. Fix a positive integer $L > 1/\varepsilon$. Next, fix a large integer N. Define

$$S_0 = \left\{ \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d} : j_{\bar{l}} \in \{1, \dots, N\} \right\}. \tag{2.3}$$

and $S = S_0 \times S_0$, that is

$$S = \{(x_1, x_2) : x_1, x_2 \in S_0\}.$$

For any $x \in S_0$ we have

$$|x| \le \sum_{\bar{l} \in \mathcal{L}(L)} |\gamma_1|^{l_1} \dots |\gamma_d|^{l_d} = \sum_{l=0}^{L-1} |\gamma_1|^l \cdots \sum_{l=0}^{L-1} |\gamma_d|^l \le \gamma^{dL},$$

where

$$\gamma = \max(|\gamma_1|, \dots, |\gamma|_d) + 1.$$

Therefore, $S \subset B(0, 2\gamma^{dL})$. Our goal is to check that $|S| = n$ and to obtain (2.1), (2.2) for $n = N^{2L^d}$ and $A = (4\gamma^{dL})^{-1} S$.

We consider that a_k $(k = 1, \dots, d)$ are parallel to the sides with slopes $\gamma_1, \dots, \gamma_d$ respectively and $a_{d+1}, a_{d+2}, a_{d+3}$ are parallel to the sides with slopes $0, 1, \infty$ respectively. Thus, we can take $b_k = (-\gamma_k, 1)$ for $k = 1, \dots, d$, $b_{d+1} = (0, 1)$, $b_{d+2} = (-1, 1)$, $b_{d+3} = (1, 0)$.

We first prove (2.1) for $k = 1, \dots, d$, i.e.

$$|(S - S) \cdot b_k| \ll n^{(1/2)+\varepsilon}. \tag{2.4}$$

Indeed, for $x \in (S - S) \cdot b_{k_0}$, $k_0 = 1, 2, \dots, d$, we have a representation

$$x \cdot b_{k_0} = -\gamma_k \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j'_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d} + \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j''_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d},$$

where

$$j'_{\bar{l}}, j''_{\bar{l}} \in \{1 - N, \dots, N - 1\} \quad (\bar{l} \in \mathcal{L}(L)).$$

Denote

$$\mathcal{L}(L, k_0) = \{\bar{l} : 0 \le l_k < L\, (k = 1, \dots, d; k \ne k_0),\, 0 \le l_{k_0} \le L\}.$$

Then we have

$$x \cdot b_{k_0} = \sum_{\bar{l} \in \mathcal{L}(L, k_0)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d}$$

with

$$j_{\bar{l}} \in \{2 - 2N, \dots, 2N - 2\} \quad (\bar{l} \in \mathcal{L}(L, k_0)).$$

Hence,

$$|(S - S) \cdot b_{k_0}| \ll (4N)^{L^d + L^{d-1}}.$$

By the choice of L we have $L^d+L^{d-1} < (1+\varepsilon)L^d$, and we get (2.4). for $k = 1, \dots, d$. Next, (2.4) holds for $k = d+1, d+2, d+3$ because for those k and for $x \in (S-S)\cdot b_k$ we have a representation

$$x \cdot b_{k_0} = \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d}$$

with

$$j_{\bar{l}} \in \{2-2N, \dots, 2N-2\} \quad (\bar{l} \in \mathcal{L}(L)).$$

Hence,

$$|(S-S)\cdot b_{k_0}| \le (4N)^{L^d},$$

and we again get (4.2) for sufficiently large N. So, (2.1) is proved.

Now observe that the supposition $\overline{\gamma} \in (KM)$ implies that elements of S_0 with different representations (2.3) are distinct. This gives $|S_0| = N^{L^d}$ and thus $|S| = |S_0|^2 = n$ as required. Moreover, since for any $x, x' \in S_0$ there is a representation

$$x - x' = \sum_{\bar{l} \in \mathcal{L}(L,k_0)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d}$$

with

$$j_{\bar{l}} \in \{1-N, \dots, N-1\} \quad (\bar{l} \in \mathcal{L}(L, k_0)).$$

we conclude from the supposition $\overline{\gamma} \in (KM)$ that for $x \neq x'$

$$|x - x'| \gg (2N)^{-(1+0.1\varepsilon)L^d - 1}. \tag{2.5}$$

By the choice of L, we have $(1+0.1\varepsilon)L^d + 1 \le (1+1.1\varepsilon)L^d$, and from (2.5) we get for sufficiently large N and distinct $y, y' \in A$

$$\|y - y'\|_X \gg (4\gamma^{dL})^{-1}(2N)^{-(1+1.1\varepsilon)L^d} \gg N^{-(1+2\varepsilon)L^d} = n^{-1/2-\varepsilon}.$$

This completes the proof of Lemma 2.1.

Proof of Theorem 4. We construct E as follows. Let $A_j = A(n_j)$ be as in Lemma 2.1 with $\varepsilon = \varepsilon_j$, where a nondecreasing sequence $\{n_j\}$, a sequence $\{N_j\}$, and a sequence $\{\varepsilon_j\}$ are such that

$$N_j = \prod_{\nu=1}^{j} n_\nu, \quad n_j \to \infty\, (j \to \infty), \quad \log n_{j+1}/\log N_j \to 0,\ \varepsilon_j \to 0\, (j \to \infty).$$

(We consider that the empty product for $j = 0$ is equal to 1.) Let also all n_j be large enough so that for any j the discs $B(x, n_j^{-1/2-2\varepsilon_j})$, $x \in A_j$, are mutually disjoint and contained in $B(0,1)$; this is possible by (2.2). Denote

$$\delta_j = n_j^{-1/2-2\varepsilon_j}, \quad \Delta_j = \prod_{\nu=1}^{j} \delta_j.$$

Let $E_1 = \bigcup_{x \in A_1} B(x, \delta_1)$. We then define $E_2, E_3, \dots$ by induction. Namely, suppose that we have constructed E_j which is a union of N_j disjoint closed discs B_i of radius Δ_j each. Then E_{j+1} is obtained from E_j by replacing each B_i by the image of $\bigcup_{x \in A_{j+1}} B(x, \delta_{j+1})$ under the unique affine mapping which takes $B(0,1)$ to B_i and preserves direction of vectors. We then let $E = \bigcap_{j=1}^{\infty} E_j$. The verification of properties $\dim(E) = 2$ and $|\Delta_X(E)| = 0$ is exactly as in the proof of Theorem 3A.

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REFERENCES

[B94] J. Bourgain, Hausdorff dimension and distance sets, Israel J. Math. **87** (1994), 193–201.

[BBKM02] V.V. Beresnevich, V.I. Bernik, D.Y. Kleinbock, and G.A. Margulis, Metric Diophantine approximation: the Khintchine—Groshev theorem for nondegenerate manifolds, Moscow Mathematical Journal **2** (2002), 203–225.

[Be02] V. Beresnevich, A Groshev type theorem for convergence on manifolds, Acta Math. Hung. **94** (2002), 99–130.

[BKM01] V. Bernik, D. Kleinbock, and G. Margulis, Khintchine–type theorems on manifolds: the convergence case for standard and multiplicative version, Int. Math. Research Notices No. 9 (2001), 453–486.

[BRT98] L. Brandolini, M. Rigoli, G. Travaglini, Average decay of Fourier transforms and geometry of convex sets, Rev. Mat. Iberoamericana **14** (1998), 519–560.

[Er03] M. B. Erdogan, On Falconer's distance set conjecture, to appear in Rev. Mat. Iberoamericana.

[Er04] M. B. Erdogan, A bilinear Fourier extension theorem and applications to the distance set problem, Int. Math. Research Notices No. 23 (2005), 1411-1425,

[F85] K.J. Falconer, The geometry of fractal sets, Cambridge University Press (1985).

[F86] K.J. Falconer, On the Hausdorff dimension of distance sets, Mathematika **32** (1986), 206–212.

[F04] K.J. Falconer, Dimension of intersections and distance sets for poolyhedral norms, preprint, 2004.

[I01] A. Iosevich, Curvature, combinatorics and the Fourier transform, Notices Amer. Math. Soc. **46** (2001), 577–583.

[IŁ03] A. Iosevich and I. Łaba, Distance sets of well-distributed planar point sets, Discrete Comput. Geometry **31** (2004), 243–250.

[IŁ04] A. Iosevich and I. Łaba, K-distance sets, Falconer conjecture and discrete analogs, preprint, 2003.

[KŁ04] S. Konyagin and I. Łaba, Distance sets of well distributed planar sets for polygonal norms, Israel J. Math., to appear.

[KM98] D.Y. Kleinbock and G.A. Margulis, Flows on homogeneous spaces and Diophantine approximation on manifolds, Ann. Math. (2) **148** (1998), 339–360.

[M87] P. Mattila, Spherical averages of Fourier transforms of measures with finite energy; dimension of intersections and distance sets, Mathematica **34** (1987), 207–228.

[T03] T. Tao, A sharp bilinear restriction estimate for paraboloids, Geom. Funct. Anal. **13** (2003), 1359–1384.

[W99] T. Wolff, Decay of circular means of Fourier transforms of measures, Int. Math. Res. Notices **10** (1999), 547–567.

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LECTURES ON NEHARI'S THEOREM ON THE POLYDISK

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ABSTRACT. We outline a proof that the 'little' Hankel operators on the Hardy space $H^2(\mathbb{C}_+^d)$ are bounded iff on the symbol is in Chang—Fefferman product $\mathrm{BMO}(\mathbb{C}_+^d)$. This extends the classical Nehari theorem to a range of product domains. This is a Theorem of the author, Sarah Ferguson [8], and Erin Terwilleger [12].

1. INTRODUCTION

These notes concern the subject of Nehari's theorem, on Hardy space of the disk, and products of the disk. The theorem on the disk is classical; the same question on products of the disk, the polydisk of the title, is a new result of the author, Sarah Ferguson and Erin Terwilleger [8, 12]. The proof in the product setting is much more complicated. It relies upon a delicate induction on parameters, built around the harmonic analysis associated with product theory, as developed by S.-Y. Chang, R. Fefferman and J.-L. Journé [3, 4, 9, 10]. These notes will provide an approach to this result that is more leisurely than the research articles on the subject. We in particular include a great many references, and a description of related results and concepts.

The key concepts of this paper concern the intertwined topics of Hankel operators, Hardy space, Hilbert transforms, commutators, and paraproducts. Let us describe classical Hankel operators.

$L^2(\mathbb{R})$ splits into the sum $H^2_+(\mathbb{C}_+) \oplus H^2_-(\mathbb{C}_+)$ of functions which are *analytic* and *antianlytic* respectively. Let $\mathrm{P}_\pm$ be the corresponding orthogonal projections, which are realized by the

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Fourier projections onto positive and negative axes. Namely,

$$\widehat{f}(\xi) \stackrel{\text{def}}{=} \int f(x)\, \mathrm{e}^{-ix\xi}\; dx\,,$$

$$\mathrm{P}_{\pm} f(x) \stackrel{\text{def}}{=} \int \widehat{f}(\xi)\, \mathrm{e}^{ix\xi}\; d\xi\,.$$

Consider a function b on $L^2(\mathbb{R})$, and the operator M_b of pointwise multiplication by b. That is, $\mathrm{M}_b\, \varphi \stackrel{\text{def}}{=} b \cdot \varphi$. A *Hankel operator with symbol* b is an operator H_b from $H^2_+(\mathbb{C}_+)$ to $H^2_-(\mathbb{C}_+)$ given by $\mathrm{H}_b\, \varphi \stackrel{\text{def}}{=} P_- \mathrm{M}_b\, \overline{\varphi}$. It is clear that this definition only depends on the analytic part of b.

Clearly, if b is bounded then so is the Hankel operator H_b. But, as it turns out this is not necessary. The Nehari theorem says that the necessary and sufficient condition for boundedness is that there is a bounded function β for which $\mathrm{P}_+\, \beta = \mathrm{P}_+\, b$.

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2. Meyer Wavelets

We recall some basic facts about the Meyer wavelets [14]. Throughout this paper, $\mathcal{D}$ denotes the dyadic grid. Thus,

$$\mathcal{D} \stackrel{\text{def}}{=} \{[j2^k, (j+1)2^k) \,:\, j, k \in \mathbb{Z}\}\,. \tag{2.1}$$

Define translation and dilation operators by

$$\mathrm{Tr}_y\, f(x) \stackrel{\text{def}}{=} f(x-y)\,, \qquad y \in \mathbb{R}\,, \tag{2.2}$$

$$\mathrm{Dil}^p_s\, f(x) \stackrel{\text{def}}{=} s^{-1/p} f(x/s)\,, \qquad 0 < s, p < \infty\,, \tag{2.3}$$

$$\mathrm{Dil}^p_I\, f(x) \stackrel{\text{def}}{=} \mathrm{Tr}_{c(I)}\, \mathrm{Dil}^p_{|I|}\, f(x)\,,\ I \text{ is an interval}\,. \tag{2.4}$$

In the second definition, s denotes the *scale* of the dilation, and the normalization is chosen to preserve $L^p(\mathbb{R})$ norm. In the last definition, we extend the definition of dilation to an interval, which incorporates a translation to the center of I, denoted $c(I)$, and a dilation by the length of I.

Y. Meyer [14] found a Schwartz function w, with

$$\widehat{w} \text{ is supported on } 2\pi \leq |\xi| \leq 8\pi, \tag{2.5}$$

and the functions $\{w_I \,:\, I \in \mathcal{D}\}$ form an orthonormal basis for $L^2(\mathbb{R})$. Here, we use the same notation as in the case of the Haar basis, $w_I = \mathrm{Dil}^2_I$.

Let us recall that the Hardy space $\mathrm{Re}(H^1(\mathbb{C}_+))$ and it's dual have convenient expressions in terms of the Meyer wavelet. Specifically, we have

$$\|f\|_{\mathrm{Re}(H^1(\mathbb{C}_+))} \simeq \Big\| \Big[\sum_{I\in\mathcal{D}} \frac{|\langle f, w_I\rangle|^2}{|I|} \mathbf{1}_I \Big]^{1/2} \Big\|_1 ,$$

$$\|f\|_{\mathrm{Re}(H^1(\mathbb{C}_+))^*} = \|f\|_{\mathrm{BMO}(\mathbb{C}_+)} \simeq \sup_{J \text{ is an interval}} \Big[|J|^{-1} \sum_{\substack{I\in\mathcal{D}\\ I\subset J}} |\langle f, w_I\rangle|^2 \Big]^{1/2} .$$

Let us write the Meyer wavelet $w = u + v$, where u is the analytic part of w, and v is the antianalytic part. In particular u is also a Schwartz function.

3. The Nehari Theorem on the Disk

The classical result that we are interested in is:

3.1. **Theorem** (Nehari's Theorem, [17]). *The Hankel operator* H_b *from* $H^2_+(\mathbb{C}_+)$ *to* $H^2_-(\mathbb{C}_+)$ *iff there is a bounded function* β *with* $P_+ b = P_+\beta$. *Moreover,*

$$\|\mathrm{H}_b\| = \inf_{\beta\,:\,\mathrm{P}_-\,\beta = \mathrm{P}_-\,b} \|\beta\|_\infty$$

There are three proofs of this fact in the literature. In the new results, we will need to rely upon methods from two of these proofs. For much more on the proofs of Nehari's theorem, and the rich theory that has flowed from it, see the books by V.V. Peller [19], or N. Nikolskii [18].

Factorization. Given a bounded Hankel operator H_b, we want to show that we can construct a bounded function β so that the analytic part of b and β agree. This proof is the one found by Nehari [17]. We begin with a basic computation of the norm of the Hankel operator H_b:

$$\begin{aligned} \|\mathrm{H}_b\| &= \sup_{\|\varphi\|_{H^2_+(\mathbb{C}_+)}=1} \sup_{\|\psi\|_{H^2_+(\mathbb{C}_+)}=1} \int \mathrm{H}_b\,\psi \cdot \overline{\varphi}\, dx \\ &= \sup_{\|\varphi\|_{H^2_+(\mathbb{C}_+)}=1} \sup_{\|\psi\|_{H^2_+(\mathbb{C}_+)}=1} \langle (\mathrm{P}_+\, b), \psi\cdot\varphi\rangle \end{aligned} \tag{3.2}$$

But, it is a classical fact that every $f \in H^1(\mathbb{C}_+)$ splits into a product of functions in $H^2(\mathbb{C}_+)$. We read from the equality above that the analytic part of b defines a bounded linear functional on $H^1(\mathbb{C}_+)$ a subspace of $L^1(\mathbb{C}_+)$.

The Hahn Banach Theorem applies, giving us an extension of this linear functional to all of L^1, with the same norm. But a linear function on L^1 is a bounded function, hence we have constructed a bounded function β with the same analytic part as b.

Duality. In this proof, the H^1—BMO duality is decisive. The calculation (3.2) shows that $\mathrm{P}_+ b$ is a bounded linear functional on H^1. Therefore, we have

$$\|\mathrm{H}_b\| \simeq \|\mathrm{P}_+ b\|_{\mathrm{BMO}}$$

(This is not equality, since we are not choosing to adopt a canonical norm for BMO.) In addition, we have $\mathrm{BMO} = L^\infty + \mathrm{H}\, L^\infty$, where H is the Hilbert transform. Therefore, we can select $\beta \in L^\infty$ which has the same analytic part as b.

4. Aspects of Product Hardy Theory

We describe the elements of product Hardy space theory, as developed by S.-Y. Chang and R. Fefferman [3, 4, 6, 7] as well as Journé [9, 10]. By this, we mean the Hardy spaces associated with domains like $\mathbb{C}_+ \otimes \mathbb{C}_+$, with boundary $\mathbb{R} \otimes \mathbb{R}$. In particular, the boundary is flat, and while we work with several variables, we are very far from the pseudoconvex case.

We view $\mathbb{R}^d$ as a tensor product of one dimensional spaces. In particular, previously, we used the splitting of $L^2(\mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2_-(\mathbb{C}_+)$. This leads to a decomposition of $L^2(\mathbb{R}^d)$ into 2^d components. To describe them, let us set $\mathrm{P}_{\pm,j}$ to be the one dimensional Fourier projection operator $\mathrm{P}_\pm$ acting on the jth coordinate. For $\sigma \in \{-,+\}^d$, set

$$\mathrm{P}_\sigma = \bigotimes_{j=1}^{d} \mathrm{P}_{\sigma(j),j}$$

Likewise, we set $H^2_\sigma(\mathbb{C}^d_+)$ to be the range of the orthogonal projection P_σ. We then have

$$L^2(\mathbb{R}^d) = \bigoplus_{\sigma \in \{+,-\}^d} H^2_\sigma(\mathbb{C}^d_+)$$

Among these 2^d Hardy spaces, we distinquish $H^2_\oplus(\mathbb{C}^d_+)$ in which $\sigma \equiv +$, and likewise for $H^2_\ominus(\mathbb{C}^d_+)$. The corresponding orthogonal projections are $\mathrm{P}_\oplus$ and $\mathrm{P}_\ominus$.

4.1. *Remark.* The (real) Hardy space $H^1(\mathbb{R}^d)$ typically denotes the class of functions with the norm

$$\|f\|_1 + \sum_{j=1}^{d} \|\mathrm{R}_j\, f\|_1$$

where R_j denote the Reisz transforms. This space is invariant under the one parameter family of isotropic dilations, while $H^1(\mathbb{C}^d_+)$ is invariant under dilations of each coordinate seperately. That is, it is invariant under a d parameter family of dilations. That is why we refer to 'multiparameter' theory, or 'd parameters.'

As before, the real H^1, $\mathrm{Re}\, H^1(\mathbb{C}^d_+)$ has a variety of equivalent norms, in terms of square functions, maximal functions and Hilbert transforms. For our discussion of paraproducts, it is appropriate to make some definitions of translation and dilation operators which extend the definitions in (2.2)—(2.4). (Indeed, here we are adopting broader notation than we really need, in anticipation of a discussion of multiparameter paraproducts.) Define

$$\operatorname{Tr}_y f(x-y) \stackrel{\text{def}}{=} f(x-y), \qquad y \in \mathbb{R}^d, \tag{4.2}$$

$$\operatorname{Dil}^p_{t_1,\dots,t_d} f(x_1,\dots,x_d) \stackrel{\text{def}}{=} (t_1\cdots t_d)^{-1/p} f(x_1/t_1,\dots,x_d/t_d), \qquad t_1,\dots,t_d>0 \tag{4.3}$$

$$\operatorname{Dil}^p_R \stackrel{\text{def}}{=} \operatorname{Tr}_{c(R)} \operatorname{Dil}^p_{|R_1|,\dots,|R_d|}. \tag{4.4}$$

In the last definition $R = R_1 \times \cdots \times R_d$ is a rectangle, and the dilation incorporates the locations and scales associated with R. $c(R)$ is the center of R.

Let $\mathcal{D}^d = \mathcal{D} \times \cdots \times \mathcal{D}$ denote the d fold product of the dyadic intervals. These are the dyadic rectangles in $\mathbb{R}^d$. For a non negative bump function φ^1 with $\int \varphi^1\, dx = 1$, define the (strong) maximal function by

$$\mathrm{M}\cdots\mathrm{M} f(x) = \sup_{R\in\mathcal{D}^d} \operatorname{Dil}^2_R \varphi^1(x)\langle f, \operatorname{Dil}^2_R \varphi^1\rangle$$

We use the superscript on φ^1 to indicate that it has a non zero integral.

4.5. **Theorem.** *All of the norms below are equivalent, and can be used as a definition of real* $\operatorname{Re} H^1(\mathbb{C}^d_+)$.

$$\|\mathrm{M}\cdots\mathrm{M} f\|_1, \qquad \sum_{\sigma\in\{0,1\}^d} \|\mathrm{P}_\sigma f\|_1 \qquad \sum_{j=1}^d \sum_{\mathrm{A}_j\in\{\mathrm{I},\mathrm{H}_j\}} \Big\|\prod_{j=1}^d \mathrm{A}_j f\Big\|_1$$

In the last expression, we are summing over all choices of operators A_j *being either the identity operator, or* H_j, *the Hilbert transform computed in the* j*th direction.*

The dual $\operatorname{Re}\mathrm{H}^1(\mathbb{C}^d_+)^* = \mathrm{BMO}(\mathbb{C}^d_+)$ is the product BMO space. We describe the characterization of this space obtained by S.-Y. Chang and R. Fefferman [4]. We need the product wavelet basis. For a rectangle $R = \prod_{j=1}^d R_{(j)} \in \mathcal{D}^d$ set

$$w_R(x_1,\dots,x_d) = \prod_{j=1}^d w_{R_{(j)}}(x_j) = \operatorname{Dil}^2_R w_{[0,1]^d}(x)$$

We will use the notation u_R for the corresponding tensor products of the analytic Meyer wavlets. The product BMO space has the equivalent norm

$$\|b\|_{\mathrm{BMO}(\mathbb{C}^d_+)} \simeq \sup_{U\subset\mathbb{R}^d} \Big[|U|^{-1} \sum_{R\subset U} |\langle b, w_R\rangle|^2\Big]^{1/2} \tag{4.6}$$

What is essential about this definition is that the supremum is taken over all sets $U \subset \mathbb{R}^d$ of finite measure. That is, the range of test sets that are required in this setting are much more complicated than the set of intervals, or even rectangles.

It is the Theorem of Chang and Fefferman that

4.7. **Theorem.** *We have the equivalence of norms*

$$\|f\|_{(\operatorname{Re} H^1(\mathbb{C}^d_+))^*} \simeq \|f\|_{\mathrm{BMO}(\mathbb{C}^d_+)}$$

That is, $\mathrm{BMO}(\mathbb{C}^d_+)$ *is the dual to* $\operatorname{Re} H^1(\mathbb{C}^d_+)$.

To define *analytic* $\mathrm{BMO}(\mathbb{C}^d_+)$, it suffices to replace the Meyer wavelets above by the analytic Meyer wavelets.

4.1. **Journé's Lemma.** The explicit definition of BMO in (4.6) is quite difficult to work with. In the first place, it is not an intrinsic definition, in that one needs some notion of wavelet to define it. Secondly, the supremum is over a very broad class of objects: All subsets of $\mathbb{R}^d$ of finite measure. There are simplier definitions, (that unfortunately are not intrinsic) that in particular circumstances are sufficient. This is the surprising conclusion of Journé's Lemma.

For our purposes, there are two appropriate definitions. Set $\|f\|_{\mathrm{BMO(rec)}}$ to be the supremum in (4.6), but with the important restriction that the sets U are taken to be rectangles. Historically, this was the first natural guess for the correct definition of $\mathrm{BMO}(\mathbb{C}^d_+)$. But, in a key moment, L. Carleson [2] produced examples of functions which acted as linear functionals on $\mathrm{H}^1(\mathbb{C}^d_+)$ with norm one, yet had arbitrarily small BMO(rec) norm. This example is recounted at the beginning of R. Fefferman's article [7]. Despite this fact, Journé Lemma shows that in certain circumstances, the an 'impovrished' BMO norm can dominate $\mathrm{BMO}(\mathbb{C}^d_+)$ norm.

The second definition is forced upon us by the complexities of three and higher parameters. Say that a collection of rectangles $\mathcal{U} \subset \mathcal{D}^d$ *has* $d-1$ *parameters* iff there is a choice of coordinate j so that for all $R, R' \in \mathcal{U}$ we have $R_j = R'_j$, that is the jth coordinate of the rectangles agree. A collection of rectangles has a *shadow* given by $\mathrm{sh}(\mathcal{U}) \stackrel{\text{def}}{=} \bigcup\{R : R \in \mathcal{U}\}$.

We then define

$$\|f\|_{\mathrm{BMO}_{-1}(\mathbb{C}^d_+)} \stackrel{\text{def}}{=} \sup_{\substack{\mathcal{U} \text{ has } d-1 \\ \text{parameters}}} \Big[|\mathrm{sh}(\mathcal{U})|^{-1} \sum_{R \in \mathcal{U}} |\langle f, w_R\rangle|^2\Big]^{1/2} \tag{4.8}$$

Observe that in $d = 2$ this reduces to the rectangular BMO definition. We use the -1 subscript to indicate that we have 'lost one parameter' in the definition.

There is another ingredient that we need, a 'dilate' of the set U, which should be taken just barely bigger than U itself.

4.9. **Lemma** (Journé's Lemma in $d-1$ parameters). *For all $\eta > 0$, and collections of rectangles $\mathcal{U}$ whose shadow has finite measure, we can construct $V \subset \operatorname{sh}(\mathcal{U})$ and a function $\operatorname{Emb} : \mathcal{U} \longrightarrow [1, \infty)$ so that*

$$\operatorname{Emb}(R) \cdot R \subset V, \qquad R \in \mathcal{U}, \tag{4.10}$$

$$|V| < (1+\eta)|\operatorname{sh}(U)|, \tag{4.11}$$

$$\Big\| \sum_{R \subset U} \operatorname{Emb}(R;U)^{-2d} \langle f, w_R \rangle w_R \Big\|_{\mathrm{BMO}(\mathbb{C}^d_+)} \le K_\eta \|f\|_{\mathrm{BMO}_{-1}(\mathbb{C}^d_+)} \tag{4.12}$$

The last inequality holds for all functions f, with the constant K_η depending only on η.

Notice that the power on the embeddedness term is quite large, twice the number of parameters. Also, concerning the conclusions, if we were to take $\operatorname{Emb}(R) \equiv 1$, then certainly the first conclusion (4.10) would be true. But, the last conclusion would be false for the Carleson examples in particular. This choice is obviously not permitted in general.

The formulations of Journé's Lemma given here are not the typical ones found in Journé's orginal Lemma, or J. Pipher's extension to three dimensional case. These papers give the more geometric formulation of these Lemmas, and J. Pipher's article implicitly contains the geometric formulation needed to prove the Lemma above (provided one is satisfied with the estimate $|V| \lesssim |\operatorname{sh}(\mathcal{U})|$). See Pipher [20]. Lemma 4.9, as formulated above, was found in Lacey and Terwilleger [12]; the two dimensional variant (which is much easier) appeared in Lacey and Ferguson [8]. The paper of Cabrelli, Lacey, Molter and Pipher [1] is a comprehensive survey of issues related to Journé's Lemma. See in particular Sections 2 and 4.

5. Multiparameter Paraproducts

We present the paraproducts in the manner in which they arise in our problem. The reader is referred to Journé [10] where one can find a Theorem which implies the result we need. Recently, multiparameter paraproducts have received new attention, and extensions by Muscalu, Pipher, Tao and Thiele [15, 16]. Our discussion is drawn from Lacey and Metcalfe [11] (also inspired by [15, 16]). In particular, the main theorem of that paper will prove the statement below.

We will have need of paraproducts which are presented in a somewhat different way. We make some definitions. For $\vec{s} \in \mathbb{Z}^d$, let us set

$$\Delta \mathrm{U}_{\vec{j}} = \sum_{\substack{R \in \mathcal{D}^d \\ |R_s| = 2^{j_s}, \ 1 \le s \le d}} u_R \otimes u_R \,.$$

Recall that u is the analytic Meyer wavelet, and u_R is the wavelet associated with R. For a subset of coordinates $J \subset \{1,\dots,d\}$ set

$$\mathrm{U}_{\vec{j},J} \stackrel{\mathrm{def}}{=} \sum_{\substack{\vec{k}\in\mathbb{Z}^d \\ k_s=j_s\,,\ s\in J \\ k_s\ge j_s\,, s\notin J}} \Delta\, \mathrm{U}_{\vec{k}}$$

For those coordinates $s \in J$, we take the wavelet projection onto that scale, while for those coordinates $s \notin J$, we sum over all larger scales.

Write $R' \lesssim_J R$ iff $|R'_s| \le |R_s|$ for $s \notin J$ and $|R'_s| = |R_s|$ for $s \in J$. Thus $R' \lesssim_J R$ iff this pair of rectangles contributes to the sum $\mathrm{U}_{\vec{j},J}$.

5.1. **Theorem.** *For all $J \subset \{1,\dots,d\}$, and $\vec{k} \in \mathbb{Z}^d$ with $\|\vec{k}\|_\infty \le 8$, we have*

$$\Big\|\sum_{\vec{j}\in\mathbb{Z}^d} (\Delta\, \mathrm{U}_{\vec{j},J}\, b)\cdot \overline{\mathrm{U}_{\vec{j}+\vec{k},J}\,\varphi}\Big\|_2 \lesssim \|b\|_{\mathrm{BMO}(\mathbb{C}^d_+)}\|\varphi\|_2$$

Moreover, suppose we have the following seperation condition: Fix an integer $A > 0$. Suppose that

$$\text{if } \langle b, u_{R'}\rangle \ne 0,\ \langle \varphi, u_R\rangle \ne 0 \text{ with } R' \lesssim_J R, \text{ then } AR \cap R' = \emptyset. \tag{5.2}$$

We then have the estimate

$$\Big\|\sum_{\vec{j}\in\mathbb{Z}^d} (\Delta\, \mathrm{U}_{\vec{j},J}\, b)\cdot \overline{\mathrm{U}_{\vec{j}+\vec{k},J}\,\varphi}\Big\|_2 \lesssim A^{-100d}\|b\|_{\mathrm{BMO}(\mathbb{C}^d_+)}\|\varphi\|_2 \tag{5.3}$$

Implied constants are independent of the choice of $\vec{k}$.

6. Nehari Theorem in Several Variables

The Hankel opererators we are are concerned with are maps from $H^2_\oplus(\mathbb{C}^d_+)$ to $H^2_\ominus(\mathbb{C}^d_+)$ given by $\mathrm{H}_b\,\varphi \stackrel{\mathrm{def}}{=} \mathrm{P}_\ominus\,\mathrm{M}_b\,\overline{\varphi}$. This definition only depends upon $\mathrm{P}_\oplus\, b$. The Nehari Theorem in this context is:

6.1. **Theorem.** *We have the equivalence*

$$\|\mathrm{H}_b\| \simeq \|b\|_{\mathrm{BMO}(\mathbb{C}^d_+)} \tag{6.2}$$

where the latter space is S.-Y. Chang and R. Fefferman BMO*, the dual to the Hardy space $H^1(\mathbb{C}^d_+)$. (In particular, this is the analytic version of that space.)*

6.3. *Remark.* These are the 'little' Hankel operators, in that we are taking the 'smallest' reasonable projection above. To define the 'big' Hankel operators, one would replace $\mathrm{P}_\ominus$ above by $\mathrm{I}-\mathrm{P}_\oplus$. We refer the reader to Cotlar and Sadosky [5] for the theory of these Hankel operators.

There is an an equivalent formulation in terms of factorization. Classical factorization of $H^1(\mathbb{C}_+)$ functions does not extend to $H^1(\mathbb{C}_+^d)$. The Nehari theorem is equivalent to *weak factorization*. The formalization of this is done in terms of a tensor products of $H^2(\mathbb{C}_+^d$. We define a projective tensor product norm by

$$\|f\|_{H^2(\mathbb{C}_+^d)\hat{\otimes}H^2(\mathbb{C}_+^d)} \stackrel{\text{def}}{=} \inf\Big\{\sum_j \|\varphi_j\|_{H^2(\mathbb{C}_+^d)}\|\psi\|_{H^2(\mathbb{C}_+^d)} \,:\, f = \sum_j \varphi_j\psi_j\,,\ \varphi_j,\psi_j \in H^2(\mathbb{C}_+^d)\Big\}$$

6.4. **Theorem.** *Either of the equivalences of norms below are a consequence of the other equivalence.*

$$\|\mathrm{H}_b\| \simeq \|\mathrm{P}_{\oplus}\, b\|_{\mathrm{BMO}(\mathbb{C}_+^d)} \tag{6.5}$$

$$\|f\|_{H^1(\mathbb{C}_+^d)} \simeq \|f\|_{H^2(\mathbb{C}_+^d)\hat{\otimes}H^2(\mathbb{C}_+^d)} \tag{6.6}$$

The last equivalence of norms is the weak factorization statement in $H^1(\mathbb{C}_+^d)$.

6.7. **Corollary.** *We have*

$$\|\mathrm{H}_b\| = \inf\big\{\|\beta\|_\infty \,:\, \mathrm{P}_{\oplus}\, b = \mathrm{P}_{\oplus}\, \beta\big\}\,.$$

Theorem 6.4 was known and elementary; once weak factorization (6.6) is known, Corollary 6.7 is easy. Thus, Theorem 6.1 is the main point. The proof we give is an induction on d, using weak factorization in $H^1(\mathbb{C}_+^{d-1})$ in a critical moment.

The inequality $\|\mathrm{H}_b\| \lesssim \|b\|_{\mathrm{BMO}(\mathbb{C}_+^d)}$, turns out to be quite easy, and so the issue is to establish the lower bound on the norm of the Hankel operator.

The central difficulty here lies in the subtle nature of BMO in the higher parameter case. We adopt a 'bootstapping' approach motivated by a critical Lemma of Journé. A crude description of the method is as follows.

It suffices to assume that $b = \mathrm{P}_{\oplus}\, b \in \mathrm{BMO}(\mathbb{C}_+^d)$ is of norm one, and find an absolute lower bound on $\|H_b\|$. We begin by using the induction hypothesis to establish

$$\|H_b\| \gtrsim \|b\|_{\mathrm{BMO}_{-1}(\mathbb{C}_+^d)},$$

where the latter norm is BMO norm 'with one less parameter' defined in (4.8). Thus, we are free to impose the additional hypothesis that $\|b\|_{\mathrm{BMO}_{-1}(\mathbb{C}_+^d)}$ is less than some fixed, absolute constant. Observe that implicitly, this forces b to be the type of functions which Carleson discovered.

Yet, Journé's Lemma gives modest sufficient conditions for this improvished norm to dominate the true BMO norm. The lower bound for the norm of H_b can then be explicitly estimated as a main term, plus several error terms. Each of the error terms is a paraproduct, which can be controlled with Journé's Lemma and the fact that the improvished norm is small.

Proof of Theorem 6.4. We discuss the proof of Theorem 6.4 and Corollary 6.7. Observe that the computation (3.2) is quite general. In the language we have introduced above, it shows immediately that

$$\|\mathrm{H}_b\| \simeq \|\mathrm{P}_\oplus b\|_{H^2(\mathbb{C}^d_+)\widehat{\otimes}H^2(\mathbb{C}^d_+)^*} \tag{6.8}$$

That is, the Hankel norms are equivalent to the *dual norm* of the tensor product norm.

The equivalence of (6.5) and (6.6) is then immediate.

□

Proof of Corollary 6.7. We can assume that the symbol of the Hankel operator H_b is in analytic BMO. Then, (6.8) and (6.6) show that b defines a bounded linear functional on $H^1(\mathbb{C}^d_+) \subset L^1(\mathbb{R}^d)$. Appeal to the Hahn Banach Theorem to extend this linear functional to all of $L^1(\mathbb{R}^d)$, with the same norm. The Corollary follows. □

7. Proof of Theorem 6.1

The upper bound on the norm of a Hankel operator is easy. Observe that, trivially,

$$H^2(\mathbb{C}^d_+)\widehat{\otimes}H^2(\mathbb{C}^d_+) \subset H^1(\mathbb{C}^d_+)\,.$$

For the dual spaces, we have the reverse inclusion. In particular, the $\mathrm{BMO}(\mathbb{C}^d_+)$ norm is larger than the dual tensor product norm. Thus, by (6.2),

$$\begin{aligned}\|\mathrm{H}_b\| &\simeq \|\mathrm{P}_\oplus b\|_{H^2(\mathbb{C}^d_+)\widehat{\otimes}H^2(\mathbb{C}^d_+)^*}\\ &\lesssim \|\mathrm{P}_\oplus b\|_{\mathrm{BMO}(\mathbb{C}^d_+)}\end{aligned}$$

Thus, the primary difficulty is in establishing the lower bound on the norm of the Hankel operator.

7.1. **The Initial Lower Bound.**

The proof is by induction on dimension d, and we take the classical Nehari Theorem as the base case in the induction. Thus, we assume that Theorem 6.1 holds in dimension $d-1$, and prove it in dimension d.

Take b to be in analytic $\mathrm{BMO}(\mathbb{C}^d_+)$, and of norm one. We recall that this means in particular, that we have

$$\sup_{U\subset\mathbb{C}^d_+}\Big[|U|^{-1}\sum_{\substack{R\in\mathcal{D}^d\\ R\subset U}}|\langle b, v_R\rangle|^2\Big] = 1\,, \tag{7.1}$$

where we recall that the supremum is over all subsets U of finite measure, and that the functions v_R are the analytic Meyer wavelets associated to dyadic rectangles in $\mathbb{C}^d_+$.

Let us argue that

$$\|\mathrm{H}_b\| \gtrsim \|b\|_{\mathrm{BMO}_{d-1}(\mathbb{C}^d_+)} \tag{7.2}$$

This last norm is given in (4.8), and in particular, it is a supremum as in (7.1), with an additional restriction on rectangles that contribute to that sum.

Now, this inequality we are to prove, by (6.8), reduces to showing

$$\|\mathrm{P}_\oplus b\|_{(H^2(\mathbb{C}^d_+)\widehat{\otimes}H^2(\mathbb{C}^d_+))^*} \gtrsim \|\mathrm{P}_\oplus b\|_{\mathrm{BMO}_{d-1}(\mathbb{C}^d_+)}\,. \tag{7.3}$$

Now, we can assume that $b = \mathrm{P}_\oplus b$ is a Schwartz function, and that $\|b\|_{\mathrm{BMO}_{d-1}(\mathbb{C}^d_+)} = 1$. Thus, after a permutation of coordinate and a possible dilation, we can take a collection of rectangles $\mathcal{U}$ which achieves the supremum in the $\mathrm{BMO}_{d-1}(\mathbb{C}^d_+)$ norm.

In particular, we can assume that $|\mathrm{sh}(\mathcal{U})| = 1$; there is an interval I of length one so that for all $R \in \mathcal{U}$ we have $R_1 = I$; for $\psi = \sum_{R\in\mathcal{U}} \langle b, v_R\rangle v_R$ we have $\langle b, \psi\rangle = 1$. Then, it suffices to see that $\|\psi\|_{H^2(\mathbb{C}^d_+)\widehat{\otimes}H^2(\mathbb{C}^d_+)} \lesssim 1$.

Write $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ as (x_1, x') with $x' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. Each rectangle $R \in \mathcal{U}$ has the same first coordinate. So the first coordinate in the in product that defines the Meyer analytic wavelet v_R is independent of R. Therefore, we can write $\psi(x) = \psi_1(x_1)\psi'(x')$ where $\psi_1(x_1) \in H^1(\mathbb{R})$ is of norm one. It can writen as $\psi_1 = \alpha_1 \cdot \beta_1$ with α_1 and β_1 of $H^2(\mathbb{R})$ norm one.

ψ' satisfies something similar. Observe that

$$\|\psi'\|_{H^1(\mathbb{C}^{d-1}_+)} \leq |U|^{1/2}\|\psi'\|_2 \leq 1\,.$$

Hence, ψ' is in $H^1(\mathbb{C}^{d-1}_+)$, and is of norm at most one. In fact, it has norm comparable to one, since by construction

$$\|\psi'\|_{\mathrm{BMO}(\mathbb{C}^{d-1}_+)} \leq 1$$

and $\langle \psi', \psi'\rangle = 1$. Thus, by the induction hypothesis, we have

$$\|\psi'\|_{H^1(\mathbb{C}^{d-1}_+)} \simeq \|\psi'\|_{H^2(\mathbb{C}^{d-1}_+)\widehat{\otimes}H^2(\mathbb{C}^{d-1}_+)} \simeq 1\,.$$

Thus, ψ' can be writen as a sum of products of $\alpha'_j \cdot \beta'_j$ with

$$\sum_j \|\alpha'_j\|_{H^2(\mathbb{C}^{d-1}_+)}\|\beta'_j\|_{H^2(\mathbb{C}^{d-1}_+)} \simeq 1\,.$$

But then, it is clear that we can write

$$\psi(x_1, x') = \sum_j \alpha_1(x_1)\alpha'_j(x') \cdot \beta_1(x_1)\beta'_j(x')$$

and so $\|\psi\|_{H^2(\mathbb{C}^d_+)\widehat{\otimes}H^2(\mathbb{C}^d_+)} \lesssim 1$.

7.2. **The BMO($\mathbb{C}^d_+$) lower bound.** Our task is to 'bootstrap' from the weaker inequality (7.2). Namely, for an absolute constant η_{-1} whose value is to be specified, it suffices to consider Hankel symbols b which satisfy $b = \mathrm{P}_\oplus b$; b is Schwartz function; $\|b\|_{\mathrm{BMO}(\mathbb{C}^d_+)} = 1$; and $\|b\|_{\mathrm{BMO}_{-1}(\mathbb{C}^d_+)} < \eta_{-1}$. (The subscript $_{-1}$ mimics our notation for the reduced parameter BMO space.)

We show by direct computation that $\|\mathrm{H}_b\| \gtrsim 1$, namely we will apply the Hankel operator to a particular $H^2(\mathbb{C}^d_+)$ function, and provide a lower bound on the norm of the image.

Here is how we select the function to apply the Hankel to. Select a collection of rectangles $\mathcal{U}$ which acheive the supremum in the definition of BMO($\mathbb{C}^d_+$) norm. Thus,

$$\sum_{R\in\mathcal{U}} |\langle b, v_R\rangle|^2 = |\mathrm{sh}(\mathcal{U})| \,.$$

Moreover, we can, after taking an appropriate dilation, that $|\mathrm{sh}(\mathcal{U})| = 1$, and that if $R \subset \mathrm{sh}(\mathcal{U})$, then $R \in \mathcal{U}$.

The function we apply the Hankel to the wavelet projection of b onto the wavelets associated with $\mathcal{U}$, $\alpha = \sum_{R\in\mathcal{U}} \langle b, v_R\rangle v_R$.

Observe that

$$\begin{aligned} \|\mathrm{H}_b\,\alpha\| &= \|\mathrm{P}_\oplus |\alpha|^2\|_2 \gtrsim \||\alpha|^2\|_2 = \|\alpha\|_4^2 \\ &\simeq \Big\| \Big[\sum_{R\in\mathcal{U}} \frac{|\langle b, v_R\rangle|^2}{|R|} \mathbf{1}_R \Big]^{1/2} \Big\|_4^2 \geq \Big[\sum_{R\in\mathcal{U}} |\langle b, v_R\rangle|^2 \Big]^{1/2} \simeq 1 \,. \end{aligned}$$

Here, we are relying on the Littlewood Paley inequalities, to pass to the wavelet square function; that $\mathcal{U}$ has shadow equal to one in measure, and that L^4 norms dominate L^2 norms on a probability space. Thus, we have $\|\mathrm{H}_\alpha\,\alpha\| \geq \eta_0 > 0$, for absolute η_0.

This is in fact our main estimate. Our task is to show that for η_{-1} sufficiently small, that we have

$$\|\mathrm{H}_{b-\alpha}\,\alpha\| < \tfrac{1}{2}\eta_0 \,. \tag{7.4}$$

This can be done with the aid of Journé's Lemma.

Fix a second small parameter η_{J} whose value will be specified below. (The subscript $_{\mathrm{J}}$ is for 'Journé.') Apply Lemma 4.9. There is a set $V \supset \mathrm{sh}(\mathcal{U})$ and a function $\mathrm{Emb} : \mathcal{U} \longrightarrow [1,\infty)$ for which these conditions hold. $|V| < 1+\eta_{\mathrm{J}}$; $\mathrm{Emb}(R)R \subset V$ for all $R \in \mathcal{U}$; $\|\widetilde{\alpha}\|_{\mathrm{BMO}(\mathbb{C}^d_+)} \leq K_{\eta_{\mathrm{J}}}\eta_{-1}$ where

$$\widetilde{\alpha} \stackrel{\mathrm{def}}{=} \sum_{R\in\mathcal{U}} \mathrm{Emb}(R)^{-2d} \langle b, v_R\rangle v_R \,. \tag{7.5}$$

We now decompose the symbol b. We have already defined α. Set

$$\beta \stackrel{\text{def}}{=} \sum_{\substack{R\subset V\\ R\not\in\mathcal{U}}} \langle b, v_R\rangle v_R \,. \tag{7.6}$$

Thus, these are the rectangles with are 'close' to $\mathcal{U}$, but not in it, as defined by the set V. Define γ by $b = \alpha + \beta + \gamma$. To verify (7.4), it suffices to show that

$$\|\mathrm{H}_\beta\,\alpha\| < K\eta_{\mathrm{J}}^{1/4}\,, \tag{7.7}$$

$$\|\mathrm{H}_\gamma\,\alpha\| < K_{\eta_{\mathrm{J}}}\eta_{-1}\,. \tag{7.8}$$

One then specifies η_{J} so that the top line is no more than $\frac{1}{4}\eta_0$. The constant $K_{\eta_{\mathrm{J}}}$ that appears in the second line is absolute, so we can then fix η_{-1} sufficiently small to prove (7.4).

The inequality for β is easily availible to us, by the particular form of the Journé Lemma we are using. Observe first that

$$1 + \sum_{\substack{R\subset V\\ R\not\in\mathcal{U}}} |\langle b, v_R\rangle|^2 = \sum_{R\subset V} |\langle b, v_R\rangle|^2 \le 1 + \eta_{\mathrm{J}}\,.$$

Therefore, $\|\beta\|_2 \le \sqrt{\eta_{\mathrm{J}}}$. On the other hand, the $\mathrm{BMO}(\mathbb{C}^d_+)$ norm of β is less than or equal to one. Thus, we have $\|\beta\|_4 \lesssim \eta_{\mathrm{J}}^{1/4}$. A Hankel operator is at worst a product, thus

$$\|\mathrm{H}_\beta\,\alpha\| \le \|\beta\|_4\|\alpha\|_4 \le K\eta_{\mathrm{J}}^{1/4}\,.$$

So it remains to verify (7.8).

An Initial Calculation. We make an explicit computation of a Hankel operator. Namely, restricting attention to one dimension, we have

$$\mathrm{H}_{v_I}\,\overline{v_J} = \mathrm{P}_+(v_I\overline{v_J}) = \begin{cases} 0 & 8|J| < |I| \\ \mathrm{P}_+(v_I\overline{v_J}) & |I| \le 8|J| \end{cases} \tag{7.9}$$

This follows from the Fourier localization properties of the Meyer wavelet. If J is much smaller than I, then $v_I\overline{v_J}$ is purely anti analytic, giving us the first case above.

We apply the observation above to the term $\mathrm{H}_\gamma\,\overline{\alpha}$. This leads us to the conclusion that we need to bound

$$\|\mathrm{H}_\gamma\,\alpha\| \le \Big\| \sum_{(R,R')\in\mathcal{A}} \langle b, u_R\rangle\, \overline{\langle \varphi, u_R\rangle}\, u_R\, \overline{u_R} \Big\|_2\,,$$

$$\mathcal{A} \stackrel{\text{def}}{=} \{(R,R') : R\subset U\,,\ R'\not\subset V\,, |R'_s| \le 64|R_s|\,, 1\le s\le d\}\,.$$

It is essential to observe that this sum can be written as a finite sum of the paraproducts in Theorem 5.1, applied to the functions α and γ. This sum varies of choices of $\vec{k}$ with

$|\vec{k}| \le 6$, and arbitrary $J \subset \{1, \dots, d\}$. (The subset J consists of those coordinates s for which $|R_s| \simeq |R'_s|$.)

We use Theorem 5.1 to provide an estimate of the L^2 norm of the sum above an absolute constant times η_{-1}. In particular, we want to use the more technical estimate (5.3) to acheive this end.

We will need to decompose the collection $\mathcal{A}$ into appropriate parts to which this estimate applies. That is the purpose of this definition. For an integer $n \ge 1$, take

$$\alpha_n \stackrel{\text{def}}{=} \sum_{\substack{R \subset U \\ 2^{n-1} \le \operatorname{Enl}(R;U) \le 2^n}} \langle b, u_R \rangle$$

We claim that

$$\|\mathrm{H}_\gamma \, \alpha_n\| \lesssim 2^{-n} \eta_{-1} \,. \tag{7.10}$$

It follows from Lemma 4.9 that we have the estimate

$$\|\alpha_n\|_{\mathrm{BMO}(\mathbb{C}^d_+)} \lesssim 2^{2dn} \eta_{-1} \,, \tag{7.11}$$

indeed, this is the point of this definition. From other parts of the expansion of the Hankel operator, we need to find some decay in n.

Nevertheless, from this estimate and the upper bound on Hankel operator norms, we have the estimate

$$\|\mathrm{H}_\gamma \, \alpha_n\| \lesssim \|b\|_{\mathrm{BMO}(\mathbb{C}^d_+)} \|\alpha_n\|_2 \lesssim 2^{2dn} \eta_{-1} .$$

We use this estimate for $n < 20$, say.

Now, for $R \subset U$ with $2^{n-1} \le \operatorname{Enl}(R;U) \le 2^n$, and rectangle R' with $(R, R') \in \mathcal{A}$, it follows that we must have $2^{n-9} R \cap R' = \emptyset$. That is, (5.2) is satisfied with the value of A in that display being $A \simeq 2^n$ for $n \ge 20$. Thus, we conclude that

$$\|\mathrm{H}_\gamma \, \alpha_n\| \lesssim 2^{-50n} \eta_{-1}, \qquad n \ge 20 \,.$$

This completes our proof of (7.10), and the proof of the lower bound on the norm of Hankel operators.

7.12. *Remark.* The subject of Nehari's theorem is foundational to the subject of operator theory. See for instance the books of Nikolskii [18] and Peller [19]. It might well be that the positive solution to Nehari's theorem in the polydisk will ultimately be seen as the starting point of a broader theory, whose outlines we can currently only dimly see.

References

[1] Carlos Cabrelli, Michael Lacey, Ursula Molter, and Jill C Pipher, *Variations on the Theme of Journe's Lemma*, To appear in Houston J Math., available at arXiv:math.CA/0412174. ↑7

[2] L. Carleson, *A counterexample for measures bounded on H^p spaces for the bidisk*, Mittag-Leffler Rep. No. 7, Inst. Mittag-Leffler (1974). ↑6

[3] Sun-Yung A. Chang and Robert Fefferman, *Some recent developments in Fourier analysis and H^p-theory on product domains*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 1–43.MR 86g:42038 ↑1, 4

[4] ———, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), no. 1, 179–201.MR 82a:32009 ↑1, 4, 5

[5] Mischa Cotlar and Cora Sadosky, *Nehari and Nevanlinna-Pick problems and holomorphic extensions in the polydisk in terms of restricted BMO*, J. Funct. Anal. **124** (1994), no. 1, 205–210.MR1284610 (95f:47047) ↑8

[6] Robert Fefferman, *Harmonic analysis on product spaces*, Ann. of Math. (2) **126** (1987), no. 1, 109–130.MR 90e:42030 ↑4

[7] R. Fefferman, *Bounded mean oscillation on the polydisk*, Ann. of Math. (2) **110** (1979), no. 2, 395–406.MR 81c:32016 ↑4, 6

[8] Sarah H. Ferguson and Michael T. Lacey, *A characterization of product BMO by commutators*, Acta Math. **189** (2002), no. 2, 143–160.1 961 195 ↑1, 7

[9] Jean-Lin Journé, *A covering lemma for product spaces*, Proc. Amer. Math. Soc. **96** (1986), no. 4, 593–598.MR 87g:42028 ↑1, 4

[10] ———, *Calderón-Zygmund operators on product spaces*, Rev. Mat. Iberoamericana **1** (1985), no. 3, 55–91.MR 88d:42028 ↑1, 4, 7

[11] Jason Metcalfe and Michael T. Lacey, *Paraproducts in One and Several Parameters*, available at `arXiv:math.CA/0502334`. ↑7

[12] Michael T. Lacey and Erin Terwilleger, *Little Hankel Operators and Product BMO* (2004). ↑1, 7

[13] Brett Wick, Michael T. Lacey, and Erin Terwilleger, *Remarks on Product VMO*, Proceedings AMS, Posted July 7, 2005, (to appear in print), available at `arXiv:math.CA/0405097`. ↑

[14] Yves Meyer and Ronald Coifman, *Wavelets*, Cambridge Studies in Advanced Mathematics, vol. 48, Cambridge University Press, Cambridge, 1997. Calderón-Zygmund and multilinear operators; Translated from the 1990 and 1991 French originals by David Salinger.MR 98e:42001 ↑2

[15] Camil Muscalu, Jill Pipher, Terrance Tao, and Christoph Thiele, *Bi-parameter paraproducts*, arxiv:math.CA/0310367. ↑7

[16] ———, *Multi-parameter paraproducts*, available at `arXiv:math.CA/0411607`. ↑7

[17] Z. Nehari, *On bounded bilinear forms*, Ann. of Math. (2) **65** (1957), 153–162. ↑3

[18] Nikolai K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002.MR1864396 (2003i:47001a) ↑3, 14

[19] Vladimir V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.MR1949210 (2004e:47040) ↑3, 14

[20] Jill Pipher, *Journé's covering lemma and its extension to higher dimensions*, Duke Math. J. **53** (1986), no. 3, 683–690.MR 88a:42019 ↑7

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The John-Nirenberg type inequality for non-doubling measures

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Abstract

For the Tolsa RBMO, the (weighted) John-Nirenberg type inequality is shown and by applying this some vector-valued inequalities are proved.

1 Introduction

In this paper we discuss the (weighted) John-Nirenberg type inequality for the sharp maximal operator due to X. Tolsa.

By "cube" $Q \subset \mathbf{R}^d$ we mean a compact cube whose edges are parallel to the coordinate axes. Its center will be denoted by z_Q and its side length will be denoted by $\ell(Q)$. By $Q(x,l)$ we will also denote the cube with center at x and sidelength l. For $\rho > 0$, ρQ will denote a cube concentric to Q with its sidelength $\rho \ell(Q)$. Throughout this paper μ will be a (positive) Radon measure on $\mathbf{R}^d$ satisfying the growth condition:

$$\mu(Q(x,l)) \le C_0 \, l^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } l > 0, \tag{1}$$

where C_0 and $n \in (0,d]$ are some fixed numbers. We do not assume that μ is doubling. By $\mathcal{Q}(\mu)$ we will denote the set of all cubes $Q \subset \mathbf{R}^d$ with positive μ-measures.

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It is well known that the doubling property of the underlying measure is a basic condition in the classical Calderón-Zygmund theory of harmonic analysis. Recently, more attention has been paid to non-doubling measures. It has been shown that many results of this theory still hold without assuming the doubling property.

Nazarov, Treil and Volberg developed the theory of the singular integrals for the measures with growth condition to investigate the analytic capacity on the complex plane [5], [6]. Tolsa showed that the analytic capacity is subadditive and that it is bi-Lipschitz invariant [13], [14]. The research, which was started from their pioneer works using the modified maximal operator, has been developed in many ways:

Garcia-Cuerva and Eduardo Gatto defined a potential operator for the measures with growth condition [2]. Tolsa defined for the growth measures RBMO (regular bounded mean oscillation), the Hardy space $H^1(\mu)$ and the Littlewood-Paley decomposition [10], [11]. He also gave his $H^1(\mu)$ space in terms of the grand maximal operator [12]. Chen and Sawyer modified the definition of RBMO to investigate the commutator of the potential operator and RBMO [1]. Deng, Han and Yang defined for the growth measures the Besov space and the Triebel-Lizorkin space [3], [4]. The authors defined for the growth measures the Morrey space and established some inequalities [8], [9]. The aim of this paper is to introduce the (weighted) John-Nirenberg type inequality for the growth measures, which can be applied to obtaining the vector-valued sharp maximal inequality for the Morrey space.

Given two cubes $Q \subset R$, we denote

$$\delta(Q, R) := \int_{\ell(Q)}^{\ell(Q_R)} \frac{\mu(Q(z_Q, l))}{l^n} \frac{dl}{l},$$

where Q_R denotes the smallest cube concentric with Q containing R. We say that Q is a doubling cube if $\mu(2Q) \leq 2^{d+1} \mu(Q)$. By $\mathcal{Q}(\mu, 2)$ we will denote the set of all doubling cubes. Given $Q \in \mathcal{Q}(\mu)$, we set Q^* as the smallest doubling cube R of the form $R = 2^j Q$ with $j = 0, 1, \ldots$ [2]

Our BMO here is RBMO (regular bounded mean oscillation) introduced by Tolsa [10] which are the suitable substitutes for the classical spaces. Denoting the average of f over the cube Q by $m_Q(f) := \dfrac{1}{\mu(Q)} \displaystyle\int_Q f \, d\mu$, we say that $f \in L^1_{loc}(\mu)$ is an element of RBMO if it satisfies

$$\|f\|_* := \sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(x) - m_{Q^*}(f)| \, d\mu(x) + \sup_{\substack{Q \subset R \\ Q, R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{1 + \delta(Q, R)} < \infty.$$

For the many other equivalent norms we refer to [10](Lemma 2.10). The advantage of RBMO is the following John-Nirenberg theorem due to Tolsa.

[2] By the growth condition (1) there are a lot of big doubling cubes. Precisely speaking, given any cube $Q \in \mathcal{Q}(\mu)$, we can find $j \in \mathbf{N}$ with $2^j Q \in \mathcal{Q}(\mu, 2)$. Meanwhile, for μ-a.e. $x \in \mathbf{R}^d$, there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. So we can say that there are a lot of small doubling cubes too (see [10]).

THEOREM 1 *Let $f \in$RBMO and $Q \in \mathcal{Q}(\mu)$.*

(1) *There exist positive constants C and C' independent on f such that*

$$\mu\{x \in Q \,|\, |f(x) - m_{Q^*}(f)| > \lambda\} \le C\,\mu\left(\frac{3}{2}Q\right)\exp\left(-\frac{C'\lambda}{\|f\|_*}\right), \quad \lambda > 0.$$

(2) *Let $q \in [1, \infty)$. Then there exists a constant C independent on f such that*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |f(x) - m_{Q^*}(f)|^q \, d\mu(x)\right)^{\frac{1}{q}} \le C\,\|f\|_*.$$

For $f \in L^1_{loc}(\mu)$, we define two maximal operators also due to Tolsa: The sharp maximal operator $M^\sharp f(x)$ is defined as

$$M^\sharp f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |f(x) - m_{Q^*}(f)| \, d\mu(x) + \sup_{\substack{x \in Q \subset R \\ Q, R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{1 + \delta(Q,R)}$$

and $Nf(x)$ is defined as $Nf(x) := \sup_{x \in Q \in \mathcal{Q}(\mu,2)} m_Q(|f|)$. It is well known that N is a bounded operator on $L^p(\mu)$ with $p > 1$ and by $\|N\|_p$ we will denote the operator norm. Since there are a lot of small doubling cubes, we have also a pointwise estimate: $|f(x)| \le Nf(x)$ for μ-a.e. $x \in \mathbf{R}^d$. In this paper the weight w will be a non-negative function on $\mathbf{R}^d$ satisfying a mild condition:

$$w \in L^{p_0}(\mu) \text{ for some } p_0 > 1 \tag{2}$$

and $w(A)$, $A \subset \mathbf{R}^d$, will denote $\int_A w(x)\, d\mu(x)$.

We shall prove the following theorem.

THEOREM 2 *Suppose that w satisfies (2). For every $f \in L^1_{loc}(\mu)$, $Q_0 \in \mathcal{Q}(\mu)$, $q \in [1,\infty)$ and $\alpha \in (0,1)$, there exists a constant C independent on f such that*

$$\left(\int_{Q_0} |f(x) - m_{(Q_0)^*}(f)|^q \, w(x)^\alpha \, d\mu(x)\right)^{\frac{1}{q}} \le C\left(\int_{\frac{3}{2}Q_0} (M^\sharp f(x))^q \, W(x)^\alpha \, d\mu(x)\right)^{\frac{1}{q}}.$$

Here, denoting N^j as the j-th composition of the operator N, we put

$$W(x) := \sum_{j=1}^{\infty} (2\beta)^{1-j} N^j w(x), \quad \beta \ge \|N\|_{p_0}. \tag{3}$$

2 Proof of Theorem 2

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_0 and C_1, do not change in different occurrences.

The cubes with generation In the sequel we follow [12] with minor modifications.

LEMMA **3** *The following properties hold :*

(1) *For* $\rho > 1$ *and* $Q \in \mathcal{Q}(\mu)$, *we have* $\delta(Q, \rho Q) \leq C_0 \log \rho$.

(2) *Let* $Q \in \mathcal{Q}(\mu)$ *and suppose that* $2^k Q$, $k = 1, 2, \ldots, k_0$, *are no doubling cubes. Then* $\delta(Q, 2^{k_0} Q) \leq C_0\, 2^{n+1} \log 2$.

(3) *Let* $Q \in \mathcal{Q}(\mu)$ *and* $\alpha > 0$. *Suppose that, for some* $c > 0$,

$$\alpha \leq \mu(2^k Q) \leq c\,\alpha, \quad k = 0, 1, \ldots, k_0.$$

Then $\delta(Q, 2^{k_0} Q) \leq 2^n \log 2\, c\, C_0\, c_n$, *where* $c_n := \sum_{k=0}^{\infty} 2^{-nk}$.

Proof. (1) follows easily from the growth condition. We prove (2) first. The dyadic argument yields that $\delta(Q, 2^{k_0} Q) = \int_{\ell(Q)}^{\ell(2^{k_0} Q)} \frac{\mu(Q(z_Q, l))}{l^n} \frac{dl}{l} \leq 2^n \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^k Q)}{\ell(2^k Q)^n}$. By the growth condition (1) we have $d - n \geq 0$ and the assumption and the definition of the doubling cubes imply $2^{d+1} \mu(2^{k-1} Q) \leq \mu(2^k Q)$. These observations yield

$$\delta(Q, 2^{k_0} Q) \leq 2^n \log 2 \frac{\mu(2^{k_0} Q)}{\ell(2^{k_0} Q)^n} \sum_{k=1}^{k_0} (2^{n-d-1})^{k_0 - k} \leq C_0\, 2^{n+1} \log 2.$$

Next we prove (3). It follows by the dyadic argument and the assumption that

$$\delta(Q, 2^{k_0} Q) \leq 2^n \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^k Q)}{\ell(2^k Q)^n} \leq 2^n \log 2\, c \frac{\alpha}{\ell(Q)^n} \sum_{k=0}^{k_0} 2^{-nk} \leq 2^n \log 2\, c\, C_0\, c_n.$$

∎

Given two cubes $Q \subset R$, we denote

$$\tilde{\delta}(Q, R) := \int_{\ell(Q)}^{\ell(Q^R)} \frac{\mu(Q(z_Q, l))}{l^n} \frac{dl}{l},$$

where Q^R denotes the largest cube concentric with Q contained in R. We will treat points $x \in \operatorname{supp}(\mu)$ as if they were cubes (with $\ell(x) = 0$). So, for $x \in \operatorname{supp}(\mu)$ and some cube $Q \ni x$, the notations $\tilde{\delta}(x, Q)$ and x^Q make sense.

Let $C_1 = C_0\, 2^{n+1} \log 2$. Fix $Q_0 \in \mathcal{Q}(\mu)$ and let $Q_1 = \frac{3}{2} Q_0$.

LEMMA **4** *If* $\alpha > 3C_1$, *then for each* $x \in Q_0 \cap \operatorname{supp}(\mu)$ *with* $\tilde{\delta}(x, Q_1) > \alpha$ *there exists some doubling cube* $Q \subset Q_1$ *centered at* x *satisfying*

$$|\tilde{\delta}(Q, Q_1) - \alpha| \leq 2C_1.$$

Proof. Let R be the biggest cube of the form $2^{-k}x^{Q_1}$, $k = 1, 2, \ldots$, such that

$$\tilde{\delta}(2R, Q_1) \leq \alpha \leq \tilde{\delta}(R, Q_1).$$

Then, $\alpha \leq \tilde{\delta}(R, Q_1) = \tilde{\delta}(R, 2R) + \tilde{\delta}(2R, Q_1) \leq C_0 \log 2 + \tilde{\delta}(2R, Q_1)$. This implies $2C_1 \leq \tilde{\delta}(2R, Q_1)$ and hence, by Lemma 3 (2), $Q := (2R)^* \subset Q_1$. It follows by Lemma 3 again that $\alpha \leq \tilde{\delta}(R, Q_1) = \tilde{\delta}(R, Q) + \tilde{\delta}(Q, Q_1) \leq 2C_1 + \tilde{\delta}(Q, Q_1)$, and that $\tilde{\delta}(Q, Q_1) \leq \tilde{\delta}(2R, Q_1) \leq \alpha$. Thus, we have $|\tilde{\delta}(Q, Q_1) - \alpha| \leq 2C_1$. ∎

Fix $A > 3C_1$. Let $m \geq 1$ be some fixed integer and $x \in Q_0 \cap \operatorname{supp}(\mu)$. If $\tilde{\delta}(x, Q_1) > mA$, we denote by $Q_{x,m}$ a doubling cube centered at x and contained in Q_1 such that

$$|\tilde{\delta}(Q_{x,m}, Q_1) - mA| \leq 2C_1.$$

The cubes $Q_{x,m}$, $x \in Q_0 \cap \operatorname{supp}(\mu)$, are called cubes of the m-th generation. The set of all cubes with m-th generation will be denoted by D_m and the set $\bigcup_m D_m$ will be denoted by D.

LEMMA 5 *Assume that A is big enough.*

(1) *For every $Q_{x,m}$, $Q_{x,m+1} \in D$, we have $100\, Q_{x,m+1} \subset Q_{x,m}$.*

(2) *If $x, y \in \operatorname{supp}(\mu)$ are such that $Q_{x,m} \cap Q_{y,m+1} \neq \emptyset$, then $\ell(Q_{y,m+1}) \leq \frac{1}{8}\,\ell(Q_{x,m})$.*

Proof. Let us prove (1). If $Q_{x,m} \subset 100\, Q_{x,m+1}$, then

$$\begin{aligned} (m+1)A - 2C_1 &\leq \tilde{\delta}(Q_{x,m+1}, Q_1) = \tilde{\delta}(Q_{x,m+1}, Q_{x,m}) + \tilde{\delta}(Q_{x,m}, Q_1) \\ &\leq \tilde{\delta}(Q_{x,m+1}, 100Q_{x,m+1}) + mA + 2C_1 \leq C_0 \log 100 + mA + 2C_1. \end{aligned}$$

This implies $A \leq C_0 \log 100 + 4C_1$ which is not possible for sufficiently large A.

Let us prove (2). Put $P = Q_{y,m+1}$ and $P' = Q_{x,m}$. If $\ell(P) > \frac{1}{8}\ell(P')$, then $P' \subset 24P$. So, if $R := Q(x, 48\ell(P))$, we have $P, P' \subset 24P \subset R \subset 72P \subset Q_1$ and hence

$$\tilde{\delta}(P, R) \leq \delta(P, 72P) \leq C. \tag{4}$$

We now claim that

$$S := |\tilde{\delta}(P^R, Q_1) - \tilde{\delta}(R, Q_1)| \leq C. \tag{5}$$

We decompose S as

$$\begin{aligned} S &= \left| \int_{\ell(P^R)}^{\ell(P^{Q_1})} \frac{\mu(Q(y,l))}{l^n} \frac{dl}{l} - \int_{\ell(R)}^{\ell(R^{Q_1})} \frac{\mu(Q(x,l))}{l^n} \frac{dl}{l} \right| \\ &\leq \int_{\ell(P^R)}^{\ell(R)} \frac{\mu(Q(y,l))}{l^n} \frac{dl}{l} + \left| \int_{\ell(R)}^{\min\{\ell(P^{Q_1}), \ell(R^{Q_1})\}} (\mu(Q(y,l)) - \mu(Q(x,l))) \frac{dl}{l^{n+1}} \right| \\ &\quad + \int_{\min\{\ell(P^{Q_1}), \ell(R^{Q_1})\}}^{\max\{\ell(P^{Q_1}), \ell(R^{Q_1})\}} \left(\frac{\mu(Q(y,l))}{l^n} + \frac{\mu(Q(x,l))}{l^n} \right) \frac{dl}{l} \\ &=: S_1 + S_2 + S_3. \end{aligned}$$

The integrals S_1 and S_3 are easily estimated above by some constant C. So we estimate S_2. Bound S_2 from above by

$$\begin{aligned} S_2 &\le \int_{\ell(R)}^{\infty} \mu(Q(y,l)\Delta Q(x,l)) \frac{dl}{l^{n+1}} \\ &= \int_{\ell(R)}^{\infty} \int_{\mathbf{R}^d} \chi_{Q(y,l)\Delta Q(x,l)}(z)\, d\mu(z) \frac{dl}{l^{n+1}}, \end{aligned}$$

where χ_A is the indicator function of a set $A \subset \mathbf{R}^d$. A simple geometric observation tells us that $\chi_{Q(y,l)\Delta Q(x,l)}(z) = 0$ if $l \notin [\min(|z-x|_\infty, |z-y|_\infty), \max(|z-x|_\infty, |z-y|_\infty)]$, where $|z|_\infty := \max(|z_1|, \ldots, |z_d|)$. This observation and Fubini's theorem yield

$$\begin{aligned} S_2 &\le C \int_{\mathbf{R}^d \setminus P} \left| \frac{1}{|z-x|_\infty{}^n} - \frac{1}{|z-y|_\infty{}^n} \right| d\mu(z) \\ &\le C \int_{|z-y|_\infty \ge \ell(P)/2} \frac{|y-x|_\infty}{|z-y|_\infty{}^{n+1}}\, d\mu(z) \le C \frac{|y-x|_\infty}{\ell(P)} \le C. \end{aligned}$$

This proves (5).

¿From (4) and (5) we have

$$\begin{aligned} &\tilde{\delta}(P, Q_1) \le \tilde{\delta}(P, R) + \tilde{\delta}(P^R, Q_1) \\ &\le \ \tilde{\delta}(P, R) + |\tilde{\delta}(P^R, Q_1) - \tilde{\delta}(R, Q_1)| + \tilde{\delta}(R, Q_1) \le \tilde{\delta}(P', Q_1) + C \end{aligned}$$

and $(m+1)A \le mA + 4C_1 + C$, which is not possible for sufficiently large A. ∎

The weight W In what follows we shall show the simple properties of the weight W defined by (3).

By the subadditivity of N we see that W satisfies the A_1 condition:

$$NW(x) \le 2\beta\, W(x) \text{ for } \mu\text{-a.e. } x \in \mathbf{R}^d. \tag{6}$$

Indeed, we have

$$\begin{aligned} NW(x) &\le \sum_{j=1}^{\infty} (2\beta)^{1-j} N^{j+1} w(x) \\ &= 2\beta \left\{ \sum_{j=1}^{\infty} (2\beta)^{1-j} N^j w(x) - Nw(x) \right\} \le 2\beta\, W(x). \end{aligned}$$

This implies the following lemma.

LEMMA **6** *Let $\alpha \in (0,1)$ and $Q \in \mathcal{Q}(\mu, 2)$. Then, for any μ-measurable subset $A \subset Q$, we have*

$$\frac{W^\alpha(A)}{W^\alpha(Q)} \le (2\beta)^\alpha \left(\frac{\mu(A)}{\mu(Q)} \right)^{1-\alpha}.$$

Proof. It follows by Hölder's inequality and (6) that

$$\begin{aligned}
W^\alpha(A) &= \int_A W^\alpha(x)\,d\mu(x) \le \left(\int_A W\,d\mu\right)^\alpha \mu(A)^{1-\alpha} \\
&\le W(Q)^\alpha \mu(A)^{1-\alpha} = \mu(Q)\left(\frac{W(Q)}{\mu(Q)}\right)^\alpha \cdot \left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha} \\
&\le \left(\int_Q (NW(x))^\alpha\,d\mu(x)\right)\cdot\left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha} \\
&\le (2\beta)^\alpha\, W^\alpha(Q)\cdot\left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha}.
\end{aligned}$$

∎

Proof of Theorem 2 Choose A large enough so that Lemma 5 holds and fix D_m and D. Letting $F(x) := |f(x) - m_{(2Q_0)^*}(f)|$, we consider a maximal function

$$N_D F(x) := \sup_{x\in Q\in D} m_Q(F), \quad x \in Q_1.$$

If $x \in Q_1 \setminus \bigcup_{Q\in D} Q$, $N_D F(x)$ should be understood as 0.

Claim 7 *For μ-a.e. $x \in Q_0 \cap \operatorname{supp}(\mu)$ we have*

$$\begin{aligned}
|f(x) - m_{(Q_0)^*}(f)| &\le C M^\sharp f(x) + F(x), \\
F(x) &\le C\left(M^\sharp f(x) + N_D F(x)\right).
\end{aligned}$$

Proof. Since $\delta((Q_0)^*, (2Q_0)^*) \le C$, the first inequality is obvious. So we prove the second one. First of all, we notice that for μ-a.e. $x \in Q_0 \cap \operatorname{supp}(\mu)$ there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$ and (see [10])

$$\lim_{k\to\infty} m_{Q_k}(F) = F(x). \tag{7}$$

Fix some $x \in Q_0 \cap \operatorname{supp}(\mu)$. If $\tilde\delta(x, Q_1) = \infty$, $\{Q_{x,m}\}$ satisfies $\ell(Q_{x,m}) \to 0$ as $m \to \infty$ and hence $F(x) \le N_D F(x)$. If $\tilde\delta(x, Q_1) \in (mA, (m+1)A]$, for sufficiently small doubling cube Q centered at x and contained in $Q_{x,m}$, we have $\delta(Q, Q_{x,m}) \le C$. Thus, we see that

$$\begin{aligned}
m_Q(F) &= \frac{1}{\mu(Q)}\int_Q |f - m_{(2Q_0)^*}(f)|\,d\mu \\
&\le \frac{1}{\mu(Q)}\int_Q |f - m_Q(f)|\,d\mu + |m_Q(f) - m_{Q_{x,m}}(f)| + |m_{Q_{x,m}}(f) - m_{(2Q_0)^*}(f)| \\
&\le C\left(\frac{1}{\mu(Q)}\int_Q |f - m_Q(f)|\,d\mu + \frac{|m_Q(f) - m_{Q_{x,m}}(f)|}{1+\delta(Q, Q_{x,m})}\right) \\
&\quad + \frac{1}{\mu(Q_{x,m})}\int_{Q_{x,m}} |f - m_{(2Q_0)^*}(f)|\,d\mu \\
&\le C\left(M^\sharp f(x) + N_D F(x)\right).
\end{aligned}$$

If $\tilde{\delta}(x, Q_1) \leq A$, for sufficiently small doubling cube Q centered at x and contained in Q_1, we have $\tilde{\delta}(Q, (2Q_0)^*) \leq C$ and hence

$$m_Q(F) \leq \frac{1}{\mu(Q)} \int_Q |f - m_Q(f)|\, d\mu + |m_Q(f) - m_{(2Q_0)^*}(f)| \leq C\, M^\sharp f(x).$$

These observations and (7) yield the claim. ∎

¿From Claim 7 and the fact that $w(x) \leq W(x)$ for proving the theorem it suffices to show the following claim.

CLAIM **8** *We have*

$$\left(\int_{Q_1} (N_D F(x))^q\, W(x)^\alpha\, d\mu(x)\right)^{\frac{1}{q}} \leq C \left(\int_{Q_1} (M^\sharp f(x))^q\, W(x)^\alpha\, d\mu(x)\right)^{\frac{1}{q}}.$$

Claim 8 can be obtained from the following lemma.

LEMMA **9** *For any sufficiently small $\eta > 0$, we have*

$$\begin{aligned} & W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\, M^\sharp f(x) \leq \eta\lambda\} \\ \leq\ & C\, \eta^{1-\alpha}\, W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}, \quad \lambda > 0: \end{aligned}$$

Proof. Choose $\eta > 0$ sufficiently small. We set

$$E_\lambda := \{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\, M^\sharp f(x) \leq \eta\lambda\} \text{ and } \Omega_\lambda := \{x \in Q_1 \,|\, N_D F(x) > \lambda\}.$$

For all $x \in E_\lambda$, we can select a doubling cube $Q_x = Q_{z(x), m(x)} \in D$, $Q_x \ni x$, that satisfies $m_{Q_x}(F) > (3/2)\lambda$. If $m(x) = 1$, we have $\delta(Q_x, (2Q_0)^*) < C$ and hence

$$\frac{3}{2}\lambda < m_{Q_x}(F) \leq m_{Q_x}(|f - m_{Q_x}(f)|) + |m_{Q_x}(f) - m_{(2Q_0)^*}(f)| \leq C\, M^\sharp f(x) \leq C\,\eta\,\lambda,$$

which is not possible for sufficiently small η. By replacing younger one, if necessary, we may assume that $m_{Q_{z,m}}(F) < (3/2)\lambda$ for any cube $Q_{z,m} \ni x$ with $m < m(x)$.

Let $S_x = Q_{z(x), m(x)-1}$. We claim that if η is small enough, we have $m_{S_x}(F) > \lambda$. Indeed, noticing $\delta(Q_x, S_x) \leq 2A$, we see that

$$\begin{aligned} & m_{Q_x}(F) \\ \leq\ & \frac{1}{\mu(Q_x)} \int_{Q_x} |f - m_{Q_x}(f)|\, d\mu + |m_{Q_x}(f) - m_{S_x}(f)| + |m_{S_x}(f) - m_{(2Q_0)^*}(f)| \\ \leq\ & C\, M^\sharp f(x) + m_{S_x}(F) \leq C\,\eta\,\lambda + m_{S_x}(F). \end{aligned}$$

This yields $m_{S_x}(F) \geq (3/2)\lambda - C\,\eta\lambda > \lambda$. Thus, we have

$$(3/2)\lambda > m_{S_x}(F) > \lambda. \tag{8}$$

Notice that Lemma 5 (1). By Besicovitch's covering lemma there exists a countable subset $\{x_j\}_{j\in J} \subset E_\lambda$ such that

$$E_\lambda \subset \bigcup_{j\in J} S_{x_j} \text{ and } \sum_{j\in J} \chi_{S_{x_j}} \le C\,\chi_{\Omega_\lambda}. \tag{9}$$

To simplify the notation, we write $S_j = S_{x_j}$ and $Q_j = Q_{x_j}$. Now we claim the following:

CLAIM **10** *If η is small enough, then*

$$W^\alpha(S_j \cap E_\lambda) \le C\,\eta^{1-\alpha}\,W^\alpha(S_j) \text{ for all } j \in J.$$

Accepting this claim, we finish the proof of the lemma. By using the claim and (9) we have

$$W^\alpha(E_\lambda) \le \sum_{j\in J} W^\alpha(S_j \cap E_\lambda) \le C\,\eta^{1-\alpha} \sum_{j\in J} W^\alpha(S_j) \le C\,\eta^{1-\alpha}\,W^\alpha(\Omega_\lambda).$$

Thus, the proof is over modulo the claim.

Proof of Claim 10. By Lemma 6 it suffices to show that

$$\mu(S_j \cap E_\lambda) \le C\,\eta\,\mu(S_j).$$

Let $y \in S_j \cap E_\lambda$. There exists a doubling cube $R_y = Q_{z(y),m(y)} \in D$, $R_y \ni y$, that satisfies $m_{R_y}(F) > 2\lambda$. We show that for sufficiently small η, $\ell(R_y) \le \frac{1}{8}\,\ell(S_j)$. From Lemma 5 (2) we may assume that $m(y) < m(x_j)$. By Lemma 5 (1) if $\ell(R_y) > \frac{1}{8}\,\ell(S_j)$, then we have $Q_{z(y),m(y)-1} \supset S_j \supset Q_j$. This and the assumption $m(y) - 1 < m(x_j)$ imply $(3/2)\lambda > m_{Q_{z(y),m(y)-1}}(F)$. Noticing

$$\begin{aligned} &m_{R_y}(F) \\ \le\ & \frac{1}{\mu(R_y)} \int_{R_y} |f - m_{R_y}(f)|\,d\mu \\ +\ & |m_{R_y}(f) - m_{Q_{z(y),m(y)-1}}(f)| + |m_{Q_{z(y),m(y)-1}}(f) - m_{(2Q_0)^*}(f)| \\ \le\ & C\,M^\sharp f(y) + m_{Q_{z(y),m(y)-1}}(F), \end{aligned}$$

we see that $(3/2)\lambda > m_{Q_{z(y),m(y)-1}}(F) \ge m_{R_y}(F) - C\,M^\sharp f(y) \ge 2\lambda - C\,\eta\lambda$. Hence, if $\eta < \frac{1}{3C}$, we must have $\ell(R_y) \le \frac{1}{8}\,\ell(S_j)$. Thus,

$$N_D\left(\chi_{\frac{5}{4}S_j} F\right)(y) > 2\lambda \text{ for all } y \in S_j \cap E_\lambda.$$

From (8) we obtain that $|m_{S_j}(f) - m_{(2Q_0)^*}(f)| \le (3/2)\lambda$, and that

$$N_D\left(\chi_{\frac{5}{4}S_j}(f - m_{S_j}(f))\right)(y) > \frac{\lambda}{2} \text{ for all } y \in S_j \cap E_\lambda.$$

It follows by using the weak-$(1,1)$ boundedness of N_D that

$$\mu(S_j \cap E_\lambda) \le \mu\left\{y \,|\, N_D\left(\chi_{\frac{5}{4}S_j}(f - m_{S_j}(f))\right)(y) > \lambda/2\right\} \le \frac{C}{\lambda}\int_{\frac{5}{4}S_j} |f - m_{S_j}(f)|\,d\mu.$$

Noticing that

$$\begin{aligned}
&\frac{1}{\mu\left(\frac{15}{8}S_j\right)}\int_{\frac{5}{4}S_j} |f - m_{S_j}(f)|\,d\mu \\
\le\ &\frac{1}{\mu\left(\frac{15}{8}S_j\right)}\int_{\frac{5}{4}S_j} |f - m_{(\frac{5}{4}S_j)^*}(f)|\,d\mu + \left|m_{(\frac{5}{4}S_j)^*}(f) - m_{S_j}(f)\right| \le C\,\eta\lambda,
\end{aligned}$$

we see that $\mu(S_j \cap E_\lambda) \le C\,\eta\,\mu(2S_j) \le C\,\eta\,\mu(S_j)$. ∎

Proof of Claim 8. Using Lemma 9 with $\eta > 0$ sufficiently small, for $L \gg 1$ we see that

$$\begin{aligned}
&\frac{1}{2}\left(\int_0^L q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
=\ &\left(\int_0^{L/2} q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
\le\ &\left(\int_0^{L/2} q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\ M^\sharp f(x) \le \eta\lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
+\ &\left(\int_0^{L/2} q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\ M^\sharp f(x) > \eta\lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
\le\ &\left(C\,\eta^{1-\alpha}\int_0^L q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D f(x) > \lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
+\ &\left(\int_0^\infty q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, M^\sharp f(x) > \eta\lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
=\ &\left(C\,\eta^{1-\alpha}\int_0^L q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}\,d\lambda\right)^{\frac{1}{q}} \\
&+\eta^{-1}\left(\int_{Q_1} (M^\sharp f)^q\,W^\alpha\,d\mu\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, we have obtained the following:

$$\left(\int_0^L q\lambda^{q-1}W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}\,d\lambda\right)^{\frac{1}{q}} \le C\left(\int_{Q_1} (M^\sharp f)^q\,W^\alpha\,d\mu\right)^{\frac{1}{q}}.$$

Letting $L \to \infty$, we obtain the claim. ∎

3 Applications to vector-valued inequality

Applying Theorem 2, we can obtain some vector-valued inequalities. For a vector-valued function $(f_1, f_2, \dots, f_j, \dots)$ on $\mathbf{R}^d$, we denote

$$\|f_j(x)\,|\,l^r\| := \left(\sum_{j=1}^{\infty} |f_j(x)|^r\right)^{\frac{1}{r}}, \quad r \in [1,\infty).$$

First, we need the following lemma (see [7]).

LEMMA 11 *If $q, r \in (1,\infty)$, then we have the vector-valued maximal inequality:*

$$\|\|Nf_j\,|\,l^r\|\,|\,L^q(\mu)\| \le C_{q,r}\ \|\|f_j\,|\,l^r\|\,|\,L^q(\mu)\|\,.$$

PROPOSITION 12 *Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $L^1_{loc}(\mu)$ function. Then, for every $Q_0 \in \mathcal{Q}(\mu)$, $q, r \in (1,\infty)$, there exists a constant C independent on f_j such that*

$$\left(\int_{Q_0} \|f_j(x) - m_{(Q_0)^*}(f_j)\,|\,l^r\|^q\, d\mu(x)\right)^{\frac{1}{q}} \le C\left(\int_{\frac{3}{2}Q_0} \|M^{\sharp} f_j(x)\,|\,l^r\|^q\, d\mu(x)\right)^{\frac{1}{q}}.$$

Proof. First we note that we may assume that $f_j \equiv 0$ for large j, say $j \ge N$, as long as we obtain the constants independent on N. Take $s \in (1, \min(q,r))$ and let $t = q/s$, $u = r/s$. Take $\alpha \in (0,1)$ satisfying $1 < 1/\alpha < \min(t', u')$. By using a duality argument we shall estimate

$$\left(\int_{Q_0} \|f_j(x) - m_{(Q_0)^*}(f_j)\,|\,l^r\|^q\, d\mu(x)\right)^{\frac{s}{q}} = \left(\int_{Q_0} \||f_j(x) - m_{(Q_0)^*}(f_j)|^s\,|\,l^u\|^t\, d\mu(x)\right)^{\frac{1}{t}}.$$

Take a vector-valued weight $(w_1, w_2, \dots)$ such that $\operatorname{supp} w_j \subset Q_0$ and

$$\left\|\|(w_j)^\alpha\,|\,l^{u'}\|\,|\,L^{t'}(\mu)\right\| = 1. \tag{10}$$

Then it follows by Theorem 2 and Hölder's inequality that

$$\begin{aligned}
&\int_{Q_0} \||f_j(x) - m_{(Q_0)^*}(f_j)|^s\,(w_j)^\alpha(x)\,|\,l^1\|\, d\mu(x)\\
&\le \int_{\frac{3}{2}Q_0} \|(M^{\sharp} f_j(x))^s\,(W_j)^\alpha(x)\,|\,l^1\|\, d\mu(x)\\
&\le \left(\int_{\frac{3}{2}Q_0} \|(M^{\sharp} f_j(x))^s\,|\,l^u\|^t\, d\mu(x)\right)^{\frac{1}{t}} \times \left(\int_{\frac{3}{2}Q_0} \|(W_j)^\alpha(x)\,|\,l^{u'}\|^{t'}\, d\mu(x)\right)^{\frac{1}{t'}},
\end{aligned}$$

where $W_j(x) := \sum_{j=1}^{\infty} (2\beta)^{1-j} N^j w_j(x)$. Choose β as the constant $C_{\alpha t', \alpha u'}$ in Lemma 11. Then, Lemma 11, the definition of W_j and (10) yield

$$\left(\int_{\frac{3}{2}Q_0} \|(W_j)^\alpha(x)\,|\,l^{u'}\|^{t'}\, d\mu(x)\right)^{\frac{1}{\alpha t'}} = \left(\int_{\frac{3}{2}Q_0} \|W_j(x)\,|\,l^{\alpha u'}\|^{\alpha t'}\, d\mu(x)\right)^{\frac{1}{\alpha t'}} \le C.$$

These prove the proposition. ∎

COROLLARY **13** *Let $f_j \in$RBMO. For any cube $Q_0 \in \mathcal{Q}(\mu)$ and $q, r \in (1,\infty)$, there exists a constant C independent on f_j such that*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q_0\right)}\int_{Q_0}\|f_j(x)-m_{(Q_0)^*}(f_j)\,|\,l^r\|^q\,d\mu(x)\right)^{\frac{1}{q}} \le C\,\|\|f_j\|_*\,|\,l^r\|.$$

We apply Proposition 12 to obtain a sharp maximal inequality on the Morrey spaces.

Let $k>1$ and $1\le q\le p<\infty$. We define the Morrey space $\mathcal{M}^p_q(k,\mu)$ as

$$\mathcal{M}^p_q(k,\mu) := \left\{f\in L^q_{loc}(\mu)\,|\,\|f\,|\,\mathcal{M}^p_q(k,\mu)\|<\infty\right\},$$

where

$$\|f\,|\,\mathcal{M}^p_q(k,\mu)\| := \sup_{Q\in\mathcal{Q}(\mu)}\mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q|f|^q\,d\mu\right)^{\frac{1}{q}}. \tag{11}$$

By applying Hölder's inequality to (11) it is easy to see that

$$L^p(\mu)=\mathcal{M}^p_p(k,\mu)\subset\mathcal{M}^p_{q_1}(k,\mu)\subset\mathcal{M}^p_{q_2}(k,\mu) \tag{12}$$

for $1\le q_2\le q_1\le p<\infty$. Let $k_1>k_2>1$. Then $\mathcal{M}^p_q(k_1,\mu,l^r)$ and $\mathcal{M}^p_q(k_2,\mu,l^r)$ coincide as a set and their norms are mutually equivalent. More precisely, we have (see [8])

$$\|f\,|\,\mathcal{M}^p_q(k_1,\mu)\|\le\|f\,|\,\mathcal{M}^p_q(k_2,\mu)\|\le C_d\left(\frac{k_1-1}{k_2-1}\right)^d\|f\,|\,\mathcal{M}^p_q(k_1,\mu)\|. \tag{13}$$

Nevertheless, for definiteness, we will assume $k=2$ in the definition and denote $\mathcal{M}^p_q(2,\mu)$ by $\mathcal{M}^p_q(\mu)$.

PROPOSITION **14** *Let $\{f_j\}_{j=1}^\infty$ be a sequence of $L^1_{loc}(\mu)$ function. Suppose that $1<q\le p<\infty$, $r\in(1,\infty)$ and there exist an increasing sequence of concentric doubling cubes $I_0\subset I_1\subset\ldots\subset I_k\subset\ldots$ such that*

$$\lim_{k\to\infty}m_{I_k}(f_j)=0 \text{ for all } j=1,2,\ldots \text{ and } \bigcup_k I_k=\mathbf{R}^d. \tag{14}$$

Then there exists a constant $C>0$ independent on f_j such that

$$\left\|\|f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(\mu)\right\|\le C\left\|\|M^\sharp f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(\mu)\right\|.$$

Proof. Again we may assume that $f_j\equiv 0$ for sufficiently large j. Let $R\in\mathcal{Q}(\mu)$. We shall estimate $\mu(2R)^{\frac{1}{p}-\frac{1}{q}}\left(\int_R\|f_j\,|\,l^r\|^q\,d\mu\right)^{\frac{1}{q}}$. It follows by Proposition 12 that

$$\mu(2R)^{\frac{1}{p}-\frac{1}{q}}\left(\int_R\|f_j\,|\,l^r\|^q\,d\mu\right)^{\frac{1}{q}}$$

$$\leq \mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R \|f_j - m_{R^*}(f_j)\,|\,l^r\|^q \,d\mu\right)^{\frac{1}{q}} + \mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j)\,|\,l^r\|$$

$$\leq C\,\mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\frac{3}{2}R} \|M^\sharp f_j\,|\,l^r\|^q \,d\mu\right)^{\frac{1}{q}} + \mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j)\,|\,l^r\|$$

$$\leq C \left\| \|M^\sharp f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(4/3,\mu)\right\| + \mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j)\,|\,l^r\|.$$

So we shall concentrate ourselves on estimating:

$$\mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j)\,|\,l^r\|. \tag{15}$$

We choose a cube inductively. Let $R_0 = R^*$ and $R_j = (2R_{j-1})^*$, $j = 1, 2, \ldots$. Let d be the distance between the center of I_0 and that of R. We select $K_0 \in \mathbf{N}$ so big that $\ell(R_{K_0}) \geq 2d$ and there exists some I_{K_1} such that $R_{K_0} \subset I_{K_1}$, $R_{K_0+1} \not\subset I_{K_1}$ and

$$\mu(R)^{\frac{1}{p}} \|m_{I_{K_1}}(f_j)\,|\,l^r\| \leq \left\| \|M^\sharp f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(\mu)\right\|.$$

Then simple geometric observation shows that $R_{K_0} \subset I_{K_1} \subset R_{K_0+3}$, and hence,

$$\delta(R_{K_0}, I_{K_1}) \leq \delta(R_{K_0}, R_{K_0+3}) \leq C. \tag{16}$$

We put for $i = 0, 1, \ldots$

$$J_i := \left\{ j \in \mathbf{N}_0 \cap [0, K_0]\,|\,2^i \mu(R) \leq \mu(R_j) < 2^{i+1}\mu(R) \right\}.$$

Discarding all empty sets, we obtain a finite sequence of nonnegative integers $0 \leq i_1 < i_2 < \ldots < i_{K_2}$ such that $J_{i_k} \neq \emptyset$, $\quad k = 1, 2, \ldots, K_2$ and that $J_l = \emptyset$ if $l \notin \{i_1, \ldots, i_{K_2}\}$. Set $a(i_k) := \min J_{i_k}$ and $b(i_k) := \max J_{i_k}$. Notice that $b(i_{K_2}) = K_0$. From Lemma 3 (1) and (3) we see that $\delta(R_{a(i_k)}, R_{b(i_k)} \leq C$ and $\delta(R_{b(i_k)}, R_{a(i_{k+1})}) \leq C$. This implies that

$$\mu(R)^{\frac{1}{p}} \left(\|m_{R_{a(i_k)}}(f_j) - m_{R_{b(i_k)}}(f_j)\,|\,l^r\| + \|m_{R_{b(i_k)}}(f_j) - m_{R_{a(i_{k+1})}}(f_j)\,|\,l^r\| \right)$$

$$\leq C\,2^{-\frac{i_k}{p}}\, \mu(R_{a(i_k)})^{\frac{1}{p}-\frac{1}{q}} \times \left(\int_{R_{a(i_k)}} \|M^\sharp f_j\,|\,l^r\|^q \,d\mu(x) \right)^{\frac{1}{q}}$$

$$\leq C\,2^{-\frac{i_k}{p}} \left\| \|M^\sharp f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(\mu)\right\|.$$

From (16) we also have

$$\mu(R)^{\frac{1}{p}} \left(\|m_{R_{a(i_{K_2})}}(f_j) - m_{R_{K_0}}(f_j)\,|\,l^r\| + \|m_{R_{K_0}}(f_j) - m_{I_{K_1}}(f_j)\,|\,l^r\| \right)$$

$$\leq C\,2^{-\frac{i_{K_2}}{p}} \left\| \|M^\sharp f_j\,|\,l^r\|\,|\,\mathcal{M}^p_q(\mu)\right\|.$$

Using the triangle inequality to (15), we have

$\mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j)\,|\,l^r\|$

$$
\begin{aligned}
&\leq \quad \mu(R)^{\frac{1}{p}} \sum_{k=1}^{K_2-1} \left(\|m_{R_{a(i_k)}}(f_j) - m_{R_{b(i_k)}}(f_j) \,|\, l^r\| + \|m_{R_{b(i_k)}}(f_j) - m_{R_{a(i_{k+1})}}(f_j) \,|\, l^r\| \right) \\
&+ \quad \mu(R)^{\frac{1}{p}} \left(\|m_{R_{a(i_{K_2})}}(f_j) - m_{R_{K_0}}(f_j) \,|\, l^r\| + \|m_{R_{K_0}}(f_j) - m_{I_{K_1}}(f_j) \,|\, l^r\| \right) \\
&+ \quad \mu(R)^{\frac{1}{p}} \|m_{I_{K_1}}(f_j) \,|\, l^r\| \\
&\leq \quad C \sum_{k=1}^{K_2} \left(2^{-\frac{i_k}{p}} \left\| \|M^\sharp f_j \,|\, l^r\| \,|\, \mathcal{M}^p_q(\mu) \right\| . \right) + \mu(R)^{\frac{1}{p}} \|m_{I_{K_1}}(f_j) \,|\, l^r\|.
\end{aligned}
$$

Notice that $\sum_{k=1}^{K_2} 2^{-\frac{i_k}{p}} \leq C$. This and above inequalities imply the desired inequality. ∎

References

[1] W. Chen and E. Sawyer, A note on commutators of fractional integrals with RBMO(μ) functions, Illinois J. Math. **46** (2002), no. 4, 1287-1298.

[2] J. García-Cuerva, and A. Eduardo Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, Studia Math. **162** (2004), no. 3, 245-261.

[3] D. Deng, Y. Han and D. Yang, Besov spaces with non-doubling measures, to appear in Trans. Amer. Math. Soc..

[4] Y. Han and D. Yang, Triebel-Lizorkin spaces with non-doubling measures, Studia Math. **162** (2004), no. 2, 105-140.

[5] F. Nazarov, S. Treil and A. Volberg, Cauchy integral and Carderón-Zygmund operators on nonhomogeneous spaces, Internat. Math. Res. Notices (1997), no. 15, 703-726.

[6] F. Nazarov, S. Treil and A. Volberg, Weak type estimates and Cotlar inequalities for Calderön-Zygmund operators on nonhomogeneous spaces, Internat. Math. Res. Notices (1998), 463-487.

[7] Y. Sawano, Sharp estimates of the modified Hardy-Littlewood maximal operator on the nonhomogeneous space via covering lemmas, Hokkaido Math. J. **34** (2005) p435-p458.

[8] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, to appear in Acta Math. Sin. (Engl. Ser.).

[9] Y. Sawano and H. Tanaka, Sharp maximal inequalities and commutators on Morrey spaces with non-doubling measures, to appear in Taiwanese J. Math..

[10] X. Tolsa, BMO, H^1, and Calderón-Zygmund operators for non doubling measures, Math. Ann. **319** (2001), 89-149.

[11] X. Tolsa, Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures, Adv. Math. **164** (2001), no. 1, 57-116.

[12] X. Tolsa, The space H^1 for nondoubling measures in terms of a grand maximal operator, Trans. Amer. Math. Soc.,

[13] X. Tolsa, Painlevé's problem and the semiadditivity of analytic capacity, Acta Math. **190** (2003), no. 1, 105-149.

[14] X. Tolsa, Bilipschitz maps, analytic capacity, and the Cauchy integral, to appear in Ann. of Math. **355** (2003), 315–348.

The space of Fourier multipliers as a dual space

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1 Introduction

$\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. The space $M_p(\mathbb{R}^n)$ of Fourier multipliers consists of all $m \in \mathcal{S}'(\mathbb{R}^n)$ such that T_m is bounded on $L^p(\mathbb{R}^n)$, where T_m is defined by

$$T_m f = \mathcal{F}^{-1}[m\hat{f}] = [\mathcal{F}^{-1}m] * f$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $\hat{f}$ and $\mathcal{F}^{-1}f$ denote the Fourier transform and inverse Fourier transform of f, respectively. We define the norm on $M_p(\mathbb{R}^n)$ by

$$\|m\|_{M_p} = \sup \|T_m f\|_p,$$

where the supremum is taken over all $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f\|_p = 1$.

Let $1 < p < \infty$ and p' denote the conjugate exponent of p. The space $A_p(\mathbb{R}^n)$ consists of all $f \in L^\infty(\mathbb{R}^n)$ which can be written as

$$f = \sum_{i=1}^{\infty} f_i * g_i \quad \text{in } L^\infty(\mathbb{R}^n),$$

where $\{f_i\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ and $\sum_{i=1}^\infty \|f_i\|_p \|g_i\|_{p'} < \infty$. We also define the norm on $A_p(\mathbb{R}^n)$ by

$$\|f\|_{A_p} = \inf \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{p'},$$

where the infimum is taken over all the representations for f. For $m \in M_p(\mathbb{R}^n)$, we define a linear functional φ_m on $A_p(\mathbb{R}^n)$ by

$$\varphi_m(f) = \sum_{i=1}^{\infty} T_m f_i * g_i(0)$$

for $f = \sum_{i=1}^{\infty} f_i * g_i \in A_p(\mathbb{R}^n)$. In [2], Figà-Talamanca proved the following.

Theorem 1.1. *Let $1 < p < \infty$. If $m \in M_p(\mathbb{R}^n)$, then $\varphi_m \in A_p(\mathbb{R}^n)^*$ and $\|\varphi_m\|_{(A_p)^*} = \|m\|_{M_p}$. Conversely, if $\varphi \in A_p(\mathbb{R}^n)^*$, then there exists $m \in M_p(\mathbb{R}^n)$ such that $\varphi = \varphi_m$. In this mean, we have $M_p(\mathbb{R}^n) = A_p(\mathbb{R}^n)^*$.*

The purpose of this note is to consider Figà-Talamanca's theorem for Bilinear Fourier multipliers and Fourier multipliers on Lorentz spaces.

2 Bilinear Fourier multipliers

Bilinear Fourier multipliers were studied by, for example, Coifman and Meyer [1] and Lacey and Thiele [3]. The space $M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ of bilinear Fourier multipliers consists of all $m \in \mathcal{S}'(\mathbb{R}^{2n})$ such that T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$, where T_m is defined by

$$T_m(f_1, f_2)(x) = \mathcal{F}^{-1}[m\,(\hat{f}_1 \otimes \hat{f}_2)](x,x) = [\mathcal{F}^{-1}m] * [f_1 \otimes f_2](x,x)$$

for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ and $f_1 \otimes f_2$ denotes the tensor product (that is, $f_1 \otimes f_2(x_1, x_2) = f_1(x_1)\,f_2(x_2)$). We define the norm on $M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ by

$$\|m\|_{M_{p_1,p_2}^{p_3}} = \sup \|T_m(f_1, f_2)\|_{p_3},$$

where the supremum is taken over all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$. In particular, if $m(\xi_1, \xi_2) = -i\,\mathrm{sgn}(\xi_1 - \xi_2)$ then T_m is the bilinear Hilbert transform H, where

$$H(f_1, f_2)(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y|>\epsilon} \frac{f_1(x-y)\,f_2(x+y)}{y}\,dy.$$

About thirty years ago, A. P. Calderón studied the bilinear Hilbert transform in connection with singular integrals operators on curves and posed the problem whether the bilinear Hilbert transform satisfies any L^p-boundedness. This problem was solved by Lacey and Thiele (for example, see [3]).

For appropriate functions f on $\mathbb{R}^{2n}$ and g on $\mathbb{R}^n$, we define the function $f *_2 g$ on $\mathbb{R}^{2n}$ by

$$f *_2 g(x_1, x_2) = \int_{\mathbb{R}^n} f(x_1 - y, x_2 - y)\, g(y)\, dy$$

for $x_1, x_2 \in \mathbb{R}^n$. Let $1 < p_1, p_2, p_3 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. The space $A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ consists of all $f \in L^\infty(\mathbb{R}^{2n})$ which can be written as

$$f = \sum_{i=1}^{\infty} [f_{1,i} \otimes f_{2,i}] *_2 g_i \quad \text{in } L^\infty(\mathbb{R}^{2n}),$$

where $\{f_{1,i}\}_{i=1}^\infty, \{f_{2,i}\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ and $\sum_{i=1}^\infty \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} < \infty$. We define the norm on $A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ by

$$\|f\|_{A_{p_1,p_2}^{p_3}} = \inf \sum_{i=1}^{\infty} \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'},$$

where the infimum is taken over all the representations for f. Then $A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ is a Banach space. Given $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$, we define a linear functional ψ_m on $A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ by

$$\psi_m(f) = \sum_{i=1}^{\infty} T_m(f_{1,i}, f_{2,i}) * g_i(0)$$

for $f = \sum_{i=1}^\infty [f_{1,i} \otimes f_{2,i}] *_2 g_i \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$. In [5], we proved the following.

Theorem 2.1. *Assume that* $1 < p_1, p_2, p_3 < \infty$ *and* $1/p_3 = 1/p_1 + 1/p_2$. *If* $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$, *then* $\psi_m \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})^*$ *and* $\|\psi_m\|_{(A_{p_1,p_2}^{p_3})^*} = \|m\|_{M_{p_1,p_2}^{p_3}}$. *Conversely, if* $\psi \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})^*$, *then there exists* $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ *such that* $\psi = \psi_m$. *In this mean, we have* $M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n}) = A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})^*$.

3 Fourier multipliers on Lorentz spaces

For a measurable function f, the distribution function $\mu(f, s)$, the decreasing rearrangement $f^*(t)$ and its maximal function $f^{**}(t)$ are defined by

$$\mu(f, s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}| \quad \text{for } s \geq 0,$$
$$f^*(t) = \inf\{s > 0 : \mu(f, s) \leq t\} \quad \text{for } t \geq 0,$$

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \quad \text{for } t > 0.$$

The Lorentz space $L^{(p,q)}(\mathbb{R}^n)$ consists of all f such that $\|f\|_{L^{(p,q)}} < \infty$, where

$$\|f\|_{L^{(p,q)}} = \begin{cases} \left(\int_0^\infty t^{(q/p)-1}(f^{**}(t))^q\,dt\right)^{1/q}, & 1 < p < \infty,\ 1 < q < \infty, \\ \sup_{t>0} t^{1/p} f^{**}(t), & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

The space $M(L^{(p_1,q_1)}, L^{(p_2,q_2)})$ of Fourier multipliers consists of all $m \in \mathcal{S}'(\mathbb{R}^n)$ such that T_m is bounded from $L^{(p_1,q_1)}(\mathbb{R}^n)$ to $L^{(p_2,q_2)}(\mathbb{R}^n)$. The space $A(L^{(p_0,q_0)} : L^{(p_1,q_1)}, L^{(p_2,q_2)})$ consists of all $f \in L^{(p_0,q_0)}(\mathbb{R}^n)$ which can be written as

$$f = \sum_{i=1}^{\infty} f_i * g_i \quad \text{in } L^{(p_0,q_0)}(\mathbb{R}^n),$$

where $\{f_i\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ and $\sum_{i=1}^\infty \|f_i\|_{L^{(p_1,q_1)}}\|g_i\|_{L^{(p_2,q_2)}} < \infty$. In [4], we proved the following.

Theorem 3.1. *Let* $1 < p_k < \infty$ *and* $1 \leq q_k < \infty$, $k = 0, 1, 2$. *If* $1/p_0 = 1/p_1 + 1/p_2 - 1$ *and* $1/q_0 \leq 1/q_1 + 1/q_2$, *then*

$$A(L^{(p_0,q_0)}(\mathbb{R}^n) : L^{(p_1,q_1)}(\mathbb{R}^n), L^{(p_2,q_2)}(\mathbb{R}^n))^* \cong M(L^{(p_1,q_1)}(\mathbb{R}^n), L^{(p_2',q_2')}(\mathbb{R}^n)).$$

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References

[1] R. R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque 57 (1978), 1-185.

[2] A. Figà-Talamanca, Translation invariant operators in L^p, Duke Math. J. 32 (1965), 495-501.

[3] M. T. Lacey and C. T. Thiele, L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, Ann. of Math. 146 (1997), 693-724.

[4] E. Nakai, N. Tomita and K. Yabuta, Extensions of Figà-Talamanca's multiplier theorem to Banach function spaces, submitted.

[5] N. Tomita, The space of bilinear Fourier multipliers as a dual space, Hokkaido Math. J., to appear.

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PARAMETERIZED LITTLEWOOD-PALEY OPERATORS

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ABSTRACT. We divide this talk into three parts. In the first part, we will recall some backgrounds of the classical Littlewood-Paley operators. Secondly, we will talk about some new results for Littlewood-Paley operators, including parameterized area integral and parameterized g_λ^* function. Some open problems will be given in the last part.

§1. THE BACKGROUND FOR LITTLEWOOD-PALEY OPERATORS

It is well known that the Littlewood-Paley operators, including the g-function, the area integral and g_λ^* function, are very important tools in harmonic analysis and other fields such as PDE. Historically, the Lusin area integral S and g_λ^* function of higher dimension were first introduced by Stein in 1958 [S1] Let $f(y,t) = (P_t * f)(y)$ be the Poisson integral of f, where $P_t(x) = \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}$ $(t>0)$ denotes the Poisson kernel in $\mathbb{R}^n$ $(n \geq 2)$. Then the operators S and g_λ^* are defined by

$$S(f)(x) = \left(\iint_{\Gamma(x)} |\nabla_{y,t} f(y,t)|^2 t^{1-n} dydt \right)^{1/2}$$

and

$$g_\lambda^*(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |\nabla_{y,t} f(y,t)|^2 t^{1-n} dydt \right)^{1/2}$$

where $\Gamma(x) = \{(t,y) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$ and $\nabla_{y,t} = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial t})$.

In [S1], Stein proved that the operator S is of weak $(1,1)$ type, and can characterize the spaces $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. More precisely, Stein gave the following result (see also [S2, p.224]).

Theorem A ([S1]). *For $1 < p < \infty$, there exists two constants A_p and B_p such that*

$$B_p \|f\|_{L^p} \leq \|S(f)\|_{L^p} \leq A_p \|f\|_{L^p}.$$

In **1972**, Fefferman and Stein [FS] proved further that the operator S characterizes the real Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p \leq 1$.

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Theorem B (**[FS]**). *For $0 < p \leq 1$, there exists two constants A_p and B_p such that*

$$B_p\|f\|_{H^p} \leq \|S(f)\|_{L^p} \leq A_p\|f\|_{H^p}.$$

In **1961**, Stein [S5] proved that if $1 < \lambda \leq 2$, then g_λ^* is L^p bounded for $2/\lambda < p < \infty$. If $\lambda > 2$, then g_λ^* is of weak type $(1,1)$. Stein [S5] also pointed out that if $1 < \lambda < 2$, then for $1 < p < 2/\lambda$, there exists an $f \in L^p$, so that $g_\lambda^*(f)(x) = \infty$,a.e. There is also an $f \in L^1$, so that $g_2^*(f)(x) = \infty$,a.e.

The weak (p,p) estimates of g_λ^* was given by Fefferman [Fe] in **1970** if $p > 1$ and $\lambda = 2/p$.

In **1975**, Calderón and Torchinsky [CT] considered more general area integrals. Suppose that ψ is a real valued function defined on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \psi(x)dx = 0, \tag{1.1}$$

then for a measurable function f on $\mathbb{R}^n$, the area integral $S_\psi(f)$ is defined by

$$S_\psi(f)(x) = \left(\iint_{\Gamma(x)} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $t > 0$ and $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t})$.

Accordingly, the generalized g_λ^* function is defined by

$$g_{\psi,\lambda}^*(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \quad \lambda > 1.$$

Calderón and Torchinsky obtained the following conclusion.

Theorem C ([CT]). *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ and satisfies (1.1), then $S_\psi(f)$ is a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $0 < p < \infty$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the class of Schwartz functions on $\mathbb{R}^n$.*

Hence, an important and interesting problem is that if the condition $\psi \in \mathcal{S}(\mathbb{R}^n)$ in Theorem C can be replaced by a weaker condition. In **1990**, Torchinsky and Wang [TW] studied this problem. Let S^{n-1} be the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Take $\psi(x) = \Omega(x)|x|^{-n+1}\chi_{B(0,1)}(x)$, where Ω is homogeneous of degree zero on $\mathbb{R}^n$ and Ω satisfies

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

and $B(0,1)$ denotes the unit ball in $\mathbb{R}^n$, then

$$S_\psi(f)(x) = \left(\iint_{\Gamma(x)} \left|\frac{1}{t}\int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} := \mu_S(f)(x)$$

and

$$g_{\psi,\lambda}^*(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|\frac{1}{t}\int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$
$$:= \mu_\lambda^*(f)(x).$$

Theorem D ([TW]). *If $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0<\alpha\le 1$, satisfies (1.2), then for $2\le p<\infty$,*

$$\|\mu_S(f)\|_{L^p}\le C_p\|\mu_\lambda^*(f)\|_{L^p}\le C_p'\|f\|_{L^p}.$$

Clearly, Theorem D is an improvement of Theorem C for $2\le p<\infty$. In **2000**, as a corollary of the L^p boundedness of the well known Marcinkiewicz integral, Y. Ding, D. Fan and Y. Pan [DFP] improved Theorem D further. In their result, $\Omega\in H^1(S^{n-1})$ has no any smoothness on S^{n-1}, where $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere (see [FP] for the definition of $H^1(S^{n-1})$).

Theorem E ([DFP]). *Suppose that $\Omega\in H^1(S^{n-1})$ satisfies (1.2), then $\mu_S(f)$ and $\mu_\lambda^*(f)$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $2\le p<\infty$.*

On the other hand, In **1985**, for $w(x)\ge 0, w\in L^1_{loc}(\mathbb{R}^n)$ and $\varphi\in C_0^\infty$ with $\int\varphi(x)=0$, Chang, Wilson and Wolff [CWW] gave the following important results, which concerns closely to the study of the Schrödinger operators,

$$\int_{\mathbb{R}^n} S(f)(x)^2 w(x)dx\le C\int_{\mathbb{R}^n}|f(x)|^2 w^*(x)dx,$$

where and in what follows, $w^*(x)$ denotes the Hardy-Littlewood maximal function of $w(x)$ which is defined by $w^*(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q w(y)dy$. In **1987**, Chanillo and Wheeden proved the following conclusions for the area integral S defined by Stein:

Theorem F ([CW]). *If $f\in\mathcal{S}$ and the nonnegative function $w\in L^1_{loc}(\mathbb{R}^n)$, then*

(a) $\int_{\{x\in\mathbb{R}^n:S(f)(x)>\beta\}} w(x)dx\le\frac{C(n)}{\beta}\int_{\mathbb{R}^n}|f(x)|w^*(x)dx, \beta>0;$

(b) $\int_{\mathbb{R}^n} S(f)(x)^p w(x)dx\le C(n,p)\int_{\mathbb{R}^n}|f(x)|^p w^*(x)dx$ *for* $1<p\le 2$;

(c) $\int_{\mathbb{R}^n} S(f)(x)^p w(x)dx\le C(n,p)\int_{\mathbb{R}^n}|f(x)|^p w^*(x)^{p/2}w(x)^{-(p/2-1)}dx$ *for* $2<p<\infty$.

L. Rosa and C. Segovia [REF, not known] considered the similar properties as in Theorem F for one sided g_λ^* function in one dimension and the kernel $\psi\in\mathcal{S}(\mathbb{R}^n)$, supported on $(-\infty,0]$.

Now let us recall some endpoint estimates for the commutators of some operators.

In **1995**, Pérez [P] obtained the weak type $L\log L$ estimate for the commutators of Calderón-Zygmund singular integral operators. Let T be a Calderón Zygmund singular integral operator, the commutator of the operator T is defined by $[b,T]f=bT(f)-T(bf)$. Denote $T_b^0=T$, then the higher order commutators of T is defined by $T_b^m=[b,T_b^{m-1}]$. [P] gave the following Theorems.

Theorem G([P]). *Let T be a Calderón Zygmund singular integral operator, $m=0,1,2,\ldots$, and b be a function in BMO. Then there exists a positive constant C such that for each smooth function with compact support f and for all $\lambda>0$,*

$$|\{y\in\mathbb{R}^n:|T_b^m(y)|>\lambda\}|\le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\Big(1+\log^+\big(\frac{|f(y)|}{\lambda}\big)\Big)^m dy.$$

In **2001**, Yong Ding, Shanzhen Lu and Pu Zhang [DLZ] obtained the similar results for the fractional integral. In 2004, when the kernel satisfies a Lip_α condition, they [DLZ1] gave the similar results for the commutators of the Marcinkiewicz integral.

Let $\Omega\in L^1(S^{n-1})$ be homogeneous of degree zero and satisfies (1.2), then the Marcinkiewicz integral operator of higher dimension is defined by

$$\mu_\Omega(f)(x)=\Big(\int_0^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\Big|^2\frac{dt}{t^3}\Big)^{\frac12}.$$

If Ω is continuous on the unit sphere of $\mathbb{R}^n$, satisfying (1.2) and the $\mathrm{Lip}_\alpha(0<\alpha\le 1)$ condition, then Torchinsky and Wang[TW] also gave the following results for the well known Marcinkiewicz integral

Theorem H ([TW]). *Suppose $1 < p < \infty$, and that $\|\mu_\Omega(f)\|_p \le k_p\|f\|_p$, then there is a constant $c_p = c_p(k_p)$ independent of f such that*

$$M^\sharp(\mu_\Omega(f)) \le c_p M_p(f)(x), \qquad for\ \ all\ \ x \in \mathbb{R}^n,$$

The definition of $M^\sharp$ and M_p will be given later.

Theorem I ([TW]). *Suppose $1 < p < \infty$, and ω be the Muckenhoupt $A_p(\mathbb{R}^n)$ class. then there is two constants $c_p(\omega)$ independent of f, $c_p'(\omega)$ independent of f such that $\|\mu_\Omega(f)\|_{L^p_\omega} \le c_p(\omega)\|f\|_{L^p_\omega}$ and $\|C_b(f)\|_{L^p_\omega} \le c_p'(\omega)\|f\|_{L^p_\omega}$, where C_b is the commutator of the operator μ_Ω.*

In connection with the well known Marcinkiewicz integral, with a strong condition assumed on the kernel, Hörmander [H] first defined and studied the L^p bounds of the following parameterized Marcinkiewicz integral in **1960**.

$$\mu_\Omega^\rho(f)(x) = \Big(\int_0^\infty |\frac{1}{t^\rho}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f(y)dy|^2\frac{dt}{t}\Big)^{\frac{1}{2}}.$$

We must point out that the Marcinkiewicz integrals and the parameterized Marcinkiewicz integrals has almost the same properties for $\rho > 0$. In **1999**, inspired by Hörmander's work [H], when the kernel satisfies a Lip_α condition, Sakamoto and Yabuta [SY] considered the following parameterized area integral μ_S^ρ and the parameterized g_λ^* function $\mu_\lambda^{*,\rho}$. Let $\rho > 0$ and $B(0,1)$ denotes the unit ball in $\mathbb{R}^n$. Take $\varphi(x) = \Omega(x)|x|^{-n+\rho}\chi_{B(0,1)}(x)$, where Ω is homogeneous of degree zero on $\mathbb{R}^n$ and Ω satisfies (1.2). Then μ_S^ρ and $\mu_\lambda^{*,\rho}$ are defined by:

$$\begin{aligned}\mu_S^\rho(f)(x) &= \Big(\iint_{\Gamma(x)}|\varphi_t * f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\\ &= \Big(\iint_{\Gamma(x)}\Big|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\mu_\lambda^{*,\rho}(f)(x) &= \Big(\iint_{\mathbb{R}^{n+1}_+}\big(\frac{t}{t+|x-y|}\big)^{\lambda n}|\varphi_t * f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\\ &= \Big(\iint_{\mathbb{R}^{n+1}_+}\big(\frac{t}{t+|x-y|}\big)^{\lambda n}\Big|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},\end{aligned}$$

where $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$, $\rho > 0$ and $\lambda > 1$.

Note the following fact: for any $x \in \mathbb{R}^n$

$$\mu_S^\rho(f)(x) \le 2^{\lambda n}\mu_\lambda^{*,\rho}(f)(x).$$

(For example, see the proof of (19) in [S2, p. 89]).

So sometimes we only list the results for $\mu_\lambda^{*,\rho}$, by the above inequality, we know that similar results can be obtained directly for the parametric area integral.

Theorem J ([SY]). *If $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, satisfies (1.2). Then*

(i) *for $\rho > 0$ and $2 \le p < \infty$, $\|\mu_\lambda^{*,\rho}(f)\|_{L^p} \le C_{n,p,\rho,\alpha}\|f\|_{L^p}$;*
(ii) *for $0 < \rho \le n/2$ and $2n/(n+2\rho) < p < 2$, $\|\mu_\lambda^{*,\rho}(f)\|_{L^p} \le C_{n,p,\rho,\alpha}\|f\|_{L^p}$;*
(iii) *for $\rho > n/2$ and $1 < p < 2$, $\|\mu_\lambda^{*,\rho}(f)\|_{L^p} \le C_{n,p,\rho,\alpha}\|f\|_{L^p}$;*
(iv) *for $0 < \rho \le n/2$, $1 \le p \le 2n/(n+2\rho)$, there exists a function $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ satisfies (1.2), such that μ_S^ρ and $\mu_\lambda^{*,\rho}$ are not bounded on $L^p(\mathbb{R}^n)$.*

Here (i) holds for $\lambda > 1$, *(ii) and (iii) hold for* $\lambda > 2/p$.

Notice that if taking $\rho = 1$, then the conclusion (i) in Theorem J is just Theorem D. Therefore, Theorem J is an extension of Theorem D.

In **2002**, Y. Ding, S. Lu and K. Yabuta [DLY] gave the following L^p boundedness of the parametric functions $\mu_\lambda^{*,\rho}$.

Theorem K [DLY]. *Suppose that* $\Omega \in L\log^+L(S^{n-1})$ *satisfies* (1.2). *Then for* $\rho > 0, \lambda > 1$ *and* $2 \le p < \infty, \|\mu_\lambda^{*,\rho}(f)\|_p \le C/\sqrt{\rho}\|f\|_p$,

In [DLY], the authors also gave the weighted L^2 boundedness of $\mu_\lambda^{*,\rho}$.

Theorem L ([DLY]). *Suppose that* $\Omega \in L\log^+ L(S^{n-1})$ *satisfies* (1.2). *Then exists a constant* $C > 0$ *such that for* $\rho > 0$, $\lambda > 1$ *and any nonnegative locally integrable function* ϕ,

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^2\phi(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^2\phi^*(x)dx.$$

Since for any $0 < \alpha \le 1$ and $1 < q \le \infty$, the including relationship

$$\text{Lip}_\alpha(S^{n-1}) \subsetneq L^q(S^{n-1}) \subsetneq L\log^+L(S^{n-1}) \subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}), \tag{1.3}$$

holds. Hence, Theorem K improves essentially the conclusion (i) of Theorem J. If take $\rho = 1$, $\varphi \in C^\infty$, has compact support and satisfies $\int \varphi(x) = 0$, denote $\varphi_t(y) = \frac{1}{t^n}\varphi(\frac{y}{t})$, in **1991**, D. Yang [YDC] obtained some results analog with the results of [CW] and the following Theorem for g_λ^*.

Theorem M [YDC]. *If* $\lambda > 3 + 1/n$, *then* $\|g_\lambda^*\|_{L^1} \le C\|f\|_{H^1}$.

In **2002**, J. Duoandikoetxea and E. Seijo [DS] studied the weighted inequalities for the parameterized area integrals and the parameterized Littlewood-Paley function $\mu_\lambda^{*,\rho}$,

Theorem N [DS]. *Suppose* $\Omega \in L^q(S^{n-1})(q > 1)$, $p \ge 2$ *and* $\omega \in AW_p(M_\Omega) \cap A_{p/2}$. *then both operators* $\mu_{\Omega,S}^\rho$ *and* $\mu_\lambda^{*,\rho}$ *are bounded on* $L^p(\omega)$.

(see [DS] for the defintion of $AW_p(M_\Omega)$).

§2. Main Results for parametric operators

In order to state our results, first we give one definition and some notations.

Definition 1. L^q-Dini condition: For $\Omega(x') \in L^q(S^{n-1})$, the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta) = \sup_{|\gamma|\le\delta}\left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q d\sigma(x')\right)^{1/q},$$

where γ is a rotation on S^{n-1}, $|\gamma| = \sup_{x'\in S^{n-1}}|\gamma x' - x'|$. If $\omega_q(\delta)$ satisfies the following inequality

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}d\delta < \infty,$$

we say that $\Omega(x')$ satisfies the L^q-Dini condition.

Here and after, we always assume that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and satisfies

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.2}$$

(1.3)-(1.7) always represents the same condition as follows:

$$\mathrm{Lip}_\alpha(S^{n-1}) \subsetneq L^q(S^{n-1}) \subsetneq L\log^+L(S^{n-1}) \subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}). \tag{1.3}$$

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta < \infty, \qquad \sigma > 1. \tag{1.4}$$

$$\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \qquad 0 < \alpha \le 1. \tag{1.5}$$

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta < \infty, \qquad \sigma > 1. \tag{1.6}$$

$$|\Omega(x) - \Omega(y)| \le C|x-y|^\alpha, \quad \text{for any } x, y \in S^{n-1}, 0 < \alpha \le 1. \tag{1.7}$$

We say that Ω satisfies L^q-log σ-Dini condition if (1.6) holds. If (1.5) holds, we say that Ω satisfies L^q-α-Dini condition.

Our main results are as follows:

2.1 The weak (1,1) estimates and weighted L^p boundedness

Theorem 1 (Y. Ding and Q. Xue, [DX1]). *Let $\Omega \in L^2(S^{n-1})$ satisfies (1.4)($\sigma > 1$). Then for $\rho > n/2$ and $\lambda > 2$, there exists a constant $C = C_{n,\rho,\sigma}$ such that for all $\beta > 0$ and $f \in L^1(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : \ \mu_\lambda^{*,\rho}(f)(x) > \beta\}| \le \frac{C}{\beta}\|f\|_1.$$

By (1.3), applying the Marcinkiewicz interpolation theorem (see [S2]) between Theorem 1 and Theorem K, we may obtain immediately the $L^p(\mathbb{R}^n)$ boundeds of the operator $\mu_\lambda^{*,\rho}$ for $1 < p < 2$.

Corollary 1. *Let $\Omega \in L^2(S^{n-1})$ satisfies (1.4)($\sigma > 1$). Then for $\rho > n/2$ and $\lambda > 2$, $\mu_\lambda^{*,\rho}$ is of type (p,p) for $1 < p < 2$.*

Remark 1. By [DLX], we know that the condition in Theorem 1 is (1.4) L^2-Dini condition, which is much weaker than (1.7) Lip_α condition, so these results substantially improved the results of E.M.Stein ([S1] for area integral and [S5] for g_λ^* function) with Poisson kernel.

In the above Corollary, $\lambda > 2$, of course is not the best condition for $p > 1$, one may guess it is still hold for $\lambda > 2/p$.

In fact, by applying Banach space valued version of Stein's interpolation theorem of analytic families of linear operators. We have

Theorem 2 (Y. Ding, Q. Xue and K. Yabuta, [DXY1]). *Suppose that $w(x)$ is a nonnegative locally integrable function to satisfy the doubling property. Let $\Omega \in L^2(S^{n-1})$ satisfies (1.4) ($\sigma > 1$). Then*

[i] *If $\rho > n/2$ and $\lambda > 2$, then*

$$\int_{\{x\in\mathbb{R}^n:\ \mu_\lambda^{*,\rho}(f)(x)>\beta\}} w(x)dx \le C/\beta \int_{\mathbb{R}^n} |f(x)|w^*(x)dx \quad \text{for any} \quad \beta > 0;$$

[ii] *If $\rho > n/2$, $1 < p < 2$ and $\lambda > 2/p$, then*

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^p w^*(x)dx;$$

[iii] *If $0 < \rho \le n/2$, $\frac{2n}{n+2\rho} < p < 2$ and $\lambda > 2/p$, then*

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^p w^*(x)dx.$$

For $p > 1$ and $\lambda > 2/p$, one can see it clearly that the L^p boundedness still holds.

Remark 2. Theorem 1 and the conclusions (i) in Theorem 2 don't hold for $0 < \rho < n/2$ when $n > 2$. Otherwise, just take $w = 1$, by interpolation between Theorem K and (i), we get the L^p $(1 < p < 2)$ boundedness of $\mu_\lambda^{*,\rho}$, which contradicts with the conclusions (iv) in Theorem J.

Note that the function Ω in Theorem 2 needs to satisfy the condition (1.3), although this is a very weak smoothness condition. However, below we will see that for the case $2 \le p < \infty$, in the results of weighted L^p-boundedness for the operators $\mu_\lambda^{*,\rho}$ and μ_S^ρ, the function Ω has no any smoothness on the unit sphere.

Theorem 3 ([DXY1]). *Suppose that $2 \le p < \infty$ and $w(x) \ge 0$ is a locally integrable function on $\mathbb{R}^n$. If $\Omega \in L\log^+ L(S^{n-1})$, then for $\rho > 0$ and $\lambda > 1$,*

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx \le C \int_{\mathbb{R}^n} |f(x)|^p w^*(x)^{p/2} w(x)^{-(p/2-1)} dx,$$

where C is a constant independent of f and w.

Remark 3. The condition assumed on Ω in Theorem 2 and 3 also is much weaker than that in [CW] and [CWW].

Remark 4. The similar results for area integral μ_S^ρ in Theorem 2 essentially improved the results of S. Chanillo and R.L. Wheeden [CW], since the condition assumed on the kernel in this Corollary is L^2-Dini condition, while the results in [CW] was obtained with Poisson kernel.

Now, we need to give some denotes and definitions. Put

$$M_p f(x) = \sup_{x\in Q} \left(|Q|^{-1} \int_Q |f(y)|^p dy \right)^{1/p}$$

where Q is a cube containing x with sides parallel to the coordinate axes; The generalized sharp function $M_p^\sharp f$ is given by

$$M_p^\sharp f(x) = \sup_{x\in Q} \left(|Q|^{-1} \int_Q |f(y) - f_Q|^p dy \right)^{1/p},$$

where f_Q is the average of f over Q, we simply denote $M_1 f = Mf$, $M_1^\sharp f = M^\sharp f$. then set $BMO(\mathbb{R}^n) = \{f : \|f\|_* = \|M^\sharp f\|_\infty < \infty\}$, as is pointed in [TW], the expressions $\|M_p^\sharp f\|_\infty$ all give equivalent BMO norms for a given functions f.

We also get some similar results which is analog to the Marcinkiewicz integral,

Theorem 4 (Y. Ding and Q. Xue, [DX]). *Let Ω satisfies (1.4)($\sigma > 1$). Then for $\rho > n/2, \lambda > 2$ and $1 < p < \infty$,*

$$M^\sharp(\mu_\lambda^{*,\rho} f)(x) \le C_p M_p f(x) \quad \text{for all} \quad x \in \mathbb{R}^n$$

where C_p is a constant independent of f.

Let ω be a weight in the Muckenhoupt A_p class, then we have

Corollary 2. *Let Ω satisfies the same condition as in Theorem 7. For $\rho > n/2$ and $\lambda > 2$, $1 < p < \infty$, then there is a constant C independent of f such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{L^p_\omega} \le C\|f\|_{L^p_\omega}.$$

2.2 Boundedness on Hardy and weak Hardy spaces.

First, we need to give the definition of the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$, which was first introduced by Fefferman and Soria [FSo] in **1987**. A well known fact is that $L^1 \subset H^{1,\infty}$.

Definition 2. Suppose that $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\int \phi \neq 0$. Denote $f_+^*(x) = \sup_{t>0} |(\phi_t * f)(x)|$, where $\phi_t(x) = t^{-n}\phi(x/t)$. A function f is said to belong to the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ if $f_+^* \in L^{1,\infty}(\mathbb{R}^n)$, i.e., there exists a constant $C > 0$ such that for any $\beta > 0$

$$\sup_{\beta>0} \beta|\{x \in \mathbb{R}^n : f_+^*(x) > \beta\}| \leq C.$$

The smallest constant C satisfying the above inequality is called the $H^{1,\infty}(\mathbb{R}^n)$ norm of f, which is denoted by $\|f\|_{H^{1,\infty}}$.

We have the following conclusions for $\mu_\lambda^{*,\rho}$ on Hardy spaces and the weak Hardy spaces.

Theorem 5 (Y. Ding, S. Lu and Q. Xue, [DLX2]). *Suppose $\Omega \in L^2(S^{n-1})$ satisfies (1.4)($\sigma > 1$), for $f(x) \in H^1(\mathbb{R}^n)$, $\rho > n/2$ and $\lambda > 2$, then*

$$\|\mu_\lambda^{*,\rho}\|_{L^1} \leq C\|f\|_{H^1}.$$

Theorem 6 ([DLX2]). *Suppose $\Omega \in L^2(S^{n-1})$ satisfies (1.5)($0 < \alpha \leq 1$), for $f(x) \in H^{1,\infty}(\mathbb{R}^n)$, $\rho > n/2$ and $\lambda > 2$, then*

$$|\{x : |\mu_\lambda^{*,\rho}(f)(x)| > \beta\}| \leq C\|f\|_{H^{1,\infty}}/\beta \qquad \forall\ \beta > 0.$$

Remark 5. Compare with the standard theory of g_λ^* function, the conditions of Ω in our results are much weaker than Theorem M, and since $2 < 3 + 1/n$, so Theorem 5 is better than that in Theorem M.

It is easy to see that the condition (1.4) is weaker than the condition (1.5). However, the relationship between the condition (1.5) and the Lipschitz condition (1.7) is not clear up to now. But note that Theorem 6 holds for any $\alpha > 0$, we have

Corollary 3. *If $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$. Then for $\rho > n/2$, $\lambda > 2$, there exists a constant $C = C_{n,\rho,\alpha}$ such that for any $\beta > 0$, $|\{x : |\mu_\lambda^{*,\rho}(f)(x)| > \beta\}| \leq C\beta^{-1}\|f\|_{H^{1,\infty}}$.*

2.3 Boundedness of the Commutators

In order to state clear, now we give the definitions of the Commutators of $\mu_\lambda^{*,\rho}$ and μ_S^ρ. Let $\mathcal{H}_i$ be the Hilbert space defined by

$$\mathcal{H}_1 = \left\{h : \quad \|h\|_{\mathcal{H}_1} = \left(\int_0^\infty \int_{|y|<1} |h(t,y)|^2 \frac{dydt}{t}\right)^{1/2} < \infty\right\}.$$

$$\mathcal{H}_2 = \left\{h : \quad \|h\|_{\mathcal{H}_2} = \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{1}{1+|y|}\right)^{\lambda n} |h(t,y)|^2 \frac{dydt}{t}\right)^{1/2} < \infty\right\}.$$

Then the parametric operators can be looked as vector valued functions in the following Hilbert spaces.

$$\begin{aligned}\mu_S^\rho(f)(x) &= \left(\int_0^\infty \int_{|y|<1} \left|\int t^{-n}\phi\left(\frac{x-z}{t} - y\right) f(z)dz\right|^2 \frac{dydt}{t}\right)^{1/2} \\ &:= \|\phi_{t,y}(f)(x)\|_{\mathcal{H}_1}\end{aligned}$$

and

$$\begin{aligned}\mu_\lambda^{*,\rho}(f)(x) &= \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{1}{1+|y|}\right)^{\lambda n} \left|\int t^{-n}\phi\left(\frac{x-z}{t} - y\right) f(z)dz\right|^2 \frac{dydt}{t}\right)^{1/2} \\ &:= \|\phi_{t,y}(f)(x)\|_{\mathcal{H}_2},\end{aligned}$$

where $\phi(x) = \frac{\Omega(x)}{|x|^{n-\rho}}\chi_{\{|x|<1\}}$,then the commutators of $\mu_\lambda^{*,\rho}$ and μ_S^ρ are defined by

$$\mu_{S,b}^\rho(f)(x) = \|b(x)\phi_{t,y}(f)(x) - \phi_{t,y}(bf)(x)\|_{\mathcal{H}_1}$$

$$= \left(\iint_{\Gamma(x)}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

$$\mu_{\lambda,b}^{*,\rho}(f)(x) = \|b(x)\phi_{t,y}(f)(x) - \phi_{t,y}(bf)(x)\|_{\mathcal{H}_2}$$

$$= \left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

If $m > 1$, then the higher order commutators is defined by

$$\mu_{S,b^m}^\rho(f)(x) = \left(\iint_{\Gamma(x)}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$\mu_{\lambda,b^m}^{*,\rho}(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

We have the following weighted endpoint estimates for the commutators.

Theorem 7 (Y. Ding and Q. Xue, [DX2]). *Let $b \in BMO$, $m = 0, 1, 2, ...$, Ω satisfies* (1.4) *for $\sigma > 2$ and $\lambda > 2$. ω is an A_1 weight. Then there exists a positive constant C, such that for any $\beta > 0$ and each smooth function with compact support f,*

$$\omega\{v \in \mathbb{R}^n : |\mu_{\lambda,b^m}^{*,\rho}(f)(v)| > \beta\} \le C_{\|b\|_*^m}\int_{\mathbb{R}^n}\frac{|f(v)|}{\beta}\left(1+\log^+\left(\frac{|f(v)|}{\beta}\right)\right)^m\omega(v)dv.$$

Remark 6. Note that in [P], the condition assumed on the kernel is the usual C-Z conditions; In [DLZ1], the kernel satisfies the Lip_α condition. While in the above Theorem, only the condition (1.4) assumed on Ω is needed. As we know this condition even is much weaker than Lip_α condition indeed.

2.4 Boundedness on Campanato spaces

We recall also the definition of Campanato spaces For $1 \le p < \infty$ and $-n/p \le \alpha \le 1$, the Campanato space $\mathcal{E}^{\alpha,p}$ is defined by the set of functions for which

$$\|f\|_{\mathcal{E}^{\alpha,p}} = \sup_{x_0\in\mathbb{R}^n}\sup_B\frac{1}{|B|^{\alpha/n}}\left(\frac{1}{|B|}\int_B|f(x)-f_B|^p\,dx\right)^{1/p} < \infty,$$

where B moves over all balls centered at x_0, and f_B is the average of f over B, $(1/|B|)\int_B f(t)\,dt$.

It is well known that for $0 < \alpha \le 1$, $\mathcal{E}^{\alpha,p} = \mathrm{Lip}_\alpha$: the Banach space of Lipschitz continuous functions of exponent α, and the norms are equivalent. If $\alpha = 0$, $\mathcal{E}^{\alpha,p}$ coincides with BMO: the space of functions of bounded mean oscillation. And if $\alpha < 0$, $CalE^{\alpha,p}$ is equivalent to the Morrey space $L^{p,n+p\alpha}$, and these norms are equivalent. We note that balls can be replaced by cubes with sides parallel to the coordinate axes and the norms are equivalent.

In **1984**, S. Wang [SW1] showed that the BMO boundedness of Littlewood-Paley's g-function follows from its finiteness on a set of positive measure. Since then, many authors considered such problems in BMO, Lipschitz spaces, and Morrey spaces i.e. in Campanato spaces. In **1990**, S. Wang and J. Chen [SC] showed that the BMO boundedness follows from its finiteness at only one point for Littlewood-Paley's g-function, Lusin's area function and Littlewood-Paley's g^*-function, and Marcinkiewicz function. Recently, Y. Sun [Su] improves and extends their results to the case of Campanato spaces. Further, K. Yabuta [Ya] improves Sun's result and also treats the case of parametrized Marcinkiewicz integrals. With Y. Ding and K. Yabuta, we improved the results in [Ya] with more rough kernels and also get the following results for the operators $\mu_\lambda^{*,\rho}$ Our results are as follows.

Theorem 8 ([DXY2]). *Let $\rho > 0$, $\lambda > 1$, and suppose that $\Omega \in L^q(S^{n-1})$ for some $q > 1$ and satisfies the cancellation condition. Then, if $f \in BMO(\mathbb{R}^n)$ and $\mu_\lambda^{*,\rho}(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Theorem 9 ([DXY2]). *Let $\sigma > 0$ and $\Omega \in L^1(S^{n-1})$ satisfies L^1-β Dini condition for some $0 < \beta \leq 1$ and the cancellation condition. Suppose that $f \in Lip_\alpha(\mathbb{R}^n)$ for $0 < \alpha < \max\{\frac{1}{2}, \min\{\beta, \sigma\}\}$ and $\mu_\lambda^{*,\rho}(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$ $\lambda > \lambda_0$, where $\lambda_0 = 1$ for $0 < \alpha < 1/2$, and $\lambda_0 = 1 + 2\alpha/n$ for $1/2 \leq \alpha < 1$. Then $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}.$$

Theorem 10 ([DXY2]). *Let $1 < p < \infty$, $-n/p \leq \alpha < 0$. Suppose a positive number σ and a function Ω on S^{n-1} satisfy one of the following conditions:*

(i) *$\sigma > -\alpha$, $\max\{1, \frac{2n}{n+2\sigma}\} < p$, $\lambda > \max\{1, 2/p\}$, $\Omega \in L^{\max\{2,p'\}}(S^{n-1})$ satisfies the cancellation condition. In the case $p < 2$, Ω moreover satisfies L^2-$\log\beta$-Dini condition for some $0 < \beta \leq 1$.*

(ii) *$\sigma > n/2$, $\lambda > 2$, $\Omega \in L^2(S^{n-1})$ and Ω satisfies L^2-$\log\beta$-Dini condition for some $0 < \beta \leq 1$ and the cancellation condition. Then, if $f \in \mathcal{E}^{\alpha,p}$ and $\mu_\lambda^{*,\rho}(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu_\lambda^{*,\rho}(f)(x) < \infty$ a.e. on $\mathbb{R}^n$, and there is a constant C independent of f, such that*

$$\|\mu_\lambda^{*,\rho}(f)\|_{\mathcal{E}^{\alpha,p}} \leq C\|f\|_{\mathcal{E}^{\alpha,p}}.$$

§3 Open Problems

Problem 1. Most of our results hold for $\rho > n/2$ and not hold for $\rho < n/2$, but unknown for $\rho = n/2$, in particular, the boundedness from H^1 to L^1 holds or not is unknown.

Problem 2. For $\rho > n/2$ and $\lambda > 2$, we guess if the kernel $\Omega \in L^2(S^{n-1})$, Both $\mu_\lambda^{*,\rho}$ and μ_S^ρ are of weak $(1,1)$ type. Is this true?

Problem 3. Can we improve that condition $\Omega \in L^2(S^{n-1})$ by a weaker one, such as $L^q(S^{n-1})$-Dini condition for $1 < q < 2$?

Problem 4. We assumed the weights with the doubling properties in Theorem 2, which is unnatural condition compared with the results of Chanillo and Wheeden (Theorem F). Can we remove this condition?

References

[CT] A. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, Adv. Math.,* **16**,*1-64 (1975)..*

[CW] S. Chanillo and R. Wheeden, *Some weighted inequalities for the area integral,* Indiana Univ. Math. J., **36** (1987), 277-294.

[CWW] S. Y. Chang, J. Wilson and T. Wolff, *Some weighted inequalities concerning the Schröding operators,* Comment. Math. Helv., **60** (1985), 217-246.

[DFP] Y. Ding, D. Fan and Y. Pan, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel,* Acta. Math. Sinica., (English series) **16** (2000), 593-600.

[DLX] Y. Ding, S. Lu and Q. Xue, *Marcinkiewicz Integral on Hardy spaces,* Integr. Equ. Oper. Theory., **42** (2002), 174-182..

[DLY] Y. Ding, S. Lu and K. Yabuta, *A problem on rough parametric Marcinkiewicz functions,* J. Austral. Math. Soc., **72** (2002), 13-21.

[DLY1] Y. Ding, S. Lu and K. Yabuta, *On commutators of Marcinkiewicz integrals with rough kernel*, J. Math. Anal. Appl., **275** (2002), 60-68.

[DLZ] Y. Ding, S. Lu and P. Zhang, *Weak estimates for commutators of fractional integral operators*, Sci.China. (series A)., **44** (2001), 877-888.

[DLZ1] Y. Ding, S. Lu and P. Zhang, *Weighted Weak type estimates for commutators of Marcinkiewicz integrals*, Sci.China. (series A)., **47** (2004), 83-95.

[DXY1] Y. Ding, Q. Xue and K. Yabuta, *Weighted estimate for a class of Littlewood-Paley operators (to appear in Taiwanese J. Math.)*.

[DXY2] Y. Ding, Q. Xue and K. Yabuta, *Existence and boundedness of Parameterized Marcinkiewicz integral with rough kernel on Campanato space (to appear in Nagoya Math. J.)*.

[DLX2] Y. Ding, S. Lu and Q. Xue, *Parameterized Littlewood-Paley operators on Hardy and weak Hardy spaces (to appear in Math. Nachr.)*.

[DX] Y. Ding and Q. Xue, *Weighted boundedness for parameterized Littlewood-Paley operators (to appear in Taiwanese J. Math.)*.

[DX1] Y. Ding and Q. Xue, *Weak type (1, 1) bounds for a class of the Littlewood-Paley operators*, J. Math. Soc. Japan., **57, no. 1** (2005), 184-194.

[DX2] Y. Ding and Q. Xue, *Endpoint estimates for commutators of parametrized Littlewood-Paley operators, submitted.*

[DS] J. Duoandikoetxea and E. Seijo, *Weighted inequalities for rough square functions through extrapolation,*, Studia. Math., **149** (2002).

[FP] D. S. Fan and Y. B. Pan, *Singular integral operator with rough kernel supported by subvarieties*, Amer. J. Math., **119** (1997), 799-839.

[FSa] D. Fan and S. Sato, *Weak type (1,1) estimates for Marcinkiewicz integrals with rough kernels*, Tôhoku Math. J., **53** (2001), 265-284.

[Fe] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math., **124** (1970), 9-36.

[FS] C. Fefferman and E. Stein, *Some maximal inequalities*, Amer. J. Math., **93** (1971), 107-115.

[FSo] R. Fefferman and F. Soria, *The Weak spaces H^1*, Studia Math., **85** (1987), 1-16.

[FR] G. Fazio and M. Ragusa, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients,*, J. Funt. Anl., **112** (1993), 241-256.

[G] G. J. Gasper, *On the Littlewood-Paley and Lusin function in higher dmensions*, Proc. Nat. Acad. Sci. U.S.A., **57** (1967), 25-28.

[GR] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities Related Topics*, Amsterdam: North-Holland. (1985).

[Han] Y. Han, *Some properties of S-function and Marcinkiewicz integrals,*, Acta Scientiarium Naturation Univ., Pekingniss. **5** (1987), 21-34.

[H] L.Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math., **104** (1960), 93-140.

[K] C. E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, (1991), CBMS,* **83**, Amer. Math. Soc..

[Ku] D. Kurtz, *Littlewood-Paley operators on BMO,*, Proc. Amer. Math. Soc., **99** (1987), 657-666.

[KW] D. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343-362.

[LH] H. Liu, *The Weak H^p space on homogeneous groups*, Lecture notes in Math., **1494 Springer** (1991), 113-118.

[LY] S. Z. Lu and D. C. Yang, *The central BMO spaces and Littlewood-Paley operators*, Approx. Th. and its Appl., **11** (1995), 72-94.

[MW] B. Muckenhoupt and R. Wheeden, *Norm inequalities for the Littlewood-Paley function g_λ^**, Trans. Amer. Math. Soc., **191** (1974), 95-111.

[P] C. Pérez, *Endpoint estimate for Commutators of Singular integral operators*, J. Function. Analysis., **128** (1995), 163-185.

[Q] S. Qiu, *Boundedness of Littlewood-Paley operators and Marcinkiewicz integral on $\mathcal{E}^{\alpha,p}$*, J.Math. Res.Exposition., **12** (1992), 41-50.

[SY] M. Sakamoto and K. Yabuta, *Boundedness of Marcinkiewicz functions*, Studia Math., **135** (1999), 103-142.

[S1] E. M. Stein, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc., **88** (1958), 430-466.

[S2] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

[S3] E. M. Stein, *The development of square functions in the work of A. Zygmund*, Bull. Amer. Math. Soc., (N.S.) **7** (1982), 359-376.

[S4] E. M. Stein, *Harmonic Analysis:Real-Variable Methods, Orthogonality, and Oscillatory Integrals (1993) Princeton Univ. Press, Princeton, N.J.*

[S5] E. Stein, *On some function of Littlewood-paley and Zygmund*, Bull. Amer. Math. Soc., **67** (1961), 99-101.

[Su] Y. Sun, *On the existence and boundedness of square function operators on Campanato spaces*, Nagoya Math. J., **173** (2004), 139–151.

[TW] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloquium Math., **60/61** (1990), 235-243.

[T] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis. (1986), Academic Press, New York.*

[TW] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloq.Math., **60/61** (1990), 235-243.

[SW] Silei. Wang, *Some properities of the Littlewood-Paley's g-function* ,, Scientia Sinica, Series A., **28** (1985).

[SW1] Silin Wang, *Boundedness of the Littlewood-Paley g-function on* $Lip_\alpha(\mathbb{R}^n)(0 < \alpha < 1)$,, Illinois. J. Math., **33**, 1989, 531-541.

[SC] S. Wang and J. Chen, *Some notes on square function operator*,, Annals of Mathematics (Chinese), Series A., **11** (1990), 630–638.

[Ya] K. Yabuta, *Existence and boundedness of* g_λ^**- function and Marcinkiewicz functions on Campanato spaces*,, Sci. Math. Jpn., **59** (2004), 93-112.

[YDC] D. C. Yang, *On the boundedness of the generalized Littlewood-paley function*,, Chinese Annals of Mathematics, Series A., **12** (1991), 736-744.

Harmonic Analysis and its Applications at Sapporo

August 22(Mon)–24(Wed), 2005
Room 309, Building ♯8,
Faculty of Science, Hokkaido University

Organizers : Akihiko Miyachi (Tokyo Woman's Christian Univ.), Kazuya Tachizawa (Hokkaido Univ.)

Program

August 22(Mon)

9:30-10:30 : Michael Lacey (Georgia Institute of Technology)
The Nehari Problem in Several Complex Variables, I

10:45-11:45 : Katsushi Fukuyama (Kobe Univ.)
On the law of the iterated logarithm for gap series

12:00-13:00 : Yonggeun Cho (Hokkaido Univ.)
Boundedness of Fourier multiplier operator defined by elliptic type function

15:00-16:00 : Gustavo Garrigós (Universidad Autonoma de Madrid)
Bergman projections and Wolff's local smoothing inequalities in light-cones

16:15-17:15 : Loukas Grafakos (Univ. of Missouri)
Two counterexamples in the theory of singular integrals, I

August 23(Tue)

9:30-10:30 : Izabella Laba (Univ. British Columbia)
Distance sets: combinatorics and Fourier analysis, I

10:45-11:45 : Qingying Xue (Kwansei Gakuin Univ.)
Parameterized Littlewood-Paley Operators

12:00-13:00 : Michael Lacey (Georgia Institute of Technology)
The Nehari Problem in Several Complex Variables, II

15:00-16:00 : Naohito Tomita (Osaka Univ.)
The space of Fourier multipliers as a dual space

16:15-17:15 : Loukas Grafakos (Univ. of Missouri)
Two counterexamples in the theory of singular integrals, II

18:00-20:00 Banquet at Faculty House "Enreisou"

August 24(Wed)

9:30-10:30 : Hitoshi Tanaka (Univ. of Tokyo)
Morrey spaces for non-doubling measures

10:45-11:45 : Izabella Laba (Univ. British Columbia)
Distance sets: combinatorics and Fourier analysis, II

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