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Edited by T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura

Y. Tonegawa, and K. Tsutaya

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 4 through August 6 in 2004 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broard spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 25 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,
Y. Tonegawa, and K. Tsutaya

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(第２９回偏微分方程式論札幌シンポジウム)

代表者: 小澤 徹, 儀我 美一, 神保 秀一, 中村 玄, 利根川 吉廣, 津田谷 公利
Organizers: T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura, Y. Tonegawa, K. Tsutaya

1. Period (期間) August 4 , 2004 - August 6 , 2004
2. Venues (場所)

   Department of Mathematics, Faculty of Science, Hokkaido University
   北海道大学大学院 理学研究科 数学教室

   理学部 8 号館 309 号室 (4 日)、理学部 5 号館大講義室 (5, 6 日)

   Faculty of Science Building #8 ROOM 309 (August 4)

   Faculty of Science Building #5 Large Lecture Room (August 5, 6)

   ※注意：8 号館は数学事務室のある 3 号館から通りをはさんで南西側 80 メートルの新しい建物 (3 階建)
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A variational approach to self-similar solutions for semilinear heat equations

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15:15-15:45 乾 勝也 (北大) Katsuya INUI (Hokkaido U.)
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(joint work wih Y. Giga, A. Mahalov, and S. Matsui)

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11:00-12:00 石毛 和弘 (東北大) Kazuhiro ISHIGE (Tohoku U.)
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9:30-10:30 Chun LIU (Penn State U.)
Hydrodynamics for materials with elastic properties transport and induced stresses

11:00-12:00 堤 正義 (早大) Masayoshi TSUTSUMI (Waseda U.)
Generalizations of the Landau-Lifshitz equations

12:00-13:00 Free discussion time with speakers in the coffee-tea room*

* Lecturers in each session are invited to stay in the coffee-tea room during discussion time.

Symposium Homepage: http://coe.math.sci.hokudai.ac.jp/sympo/sapporo/program040804.html

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GEOMETRIC FLOWS AND BERNOULLI PROBLEM

Olivier Ley, University of Tours, France

This is a joint work with Pierre Cardaliaguet (University of Bretagne Occidentale, Brest, France).

Statement of the problem and motivation

The aim of the work [CL] is to study nonlocal geometric flows $(\Omega(t))_{t\geq 0}$. For the moment, suppose that all sets considered are smooth enough to give sense to our calculations. For all $t \geq 0$, $\Omega(t)$ is a bounded subset of $\mathbb{R}^N$ whose boundary $\partial\Omega(t)$ evolves with a normal velocity of the type

$$\mathcal{V}_{t,x} = F(\nu_x, H_x) + \lambda h(x, \Omega(t)) \quad \text{for every } x \in \partial\Omega(t), \tag{1}$$

where $\lambda \geq 0$, ν_x is the outward unit normal to $\Omega(t)$ at x, H_x is the curvature matrix of $\partial\Omega(t)$ at x (nonpositive for convex sets), F is continuous and elliptic, i.e. nondecreasing with respect to the curvature matrix. The nonlocal term h is of Hele-Shaw type:

$$h(x, \Omega(t)) = |\nabla u(x)|^\beta \quad \text{with } \beta = 1 \text{ or } 2, \tag{2}$$

where u is the solution of an auxiliary partial differential equation

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega(t)\backslash S, \\ u = 1 & \text{on} \quad \partial S, \\ u = 0 & \text{on} \quad \partial\Omega(t). \end{cases} \tag{3}$$

The set S is a fixed source with C^2 boundary and we always assume $S \subset\subset \Omega(t)$.

In our talk, for simplicity, we will focus on two model cases:

$$\mathcal{V}_{t,x} = -1 + \lambda|\nabla u(x)|^2 \tag{4}$$

and

$$\mathcal{V}_{t,x} = H_x + \lambda|\nabla u(x)|^2. \tag{5}$$

The motivation to study such problems comes from the numerical work of Allaire, Jouve and Toader [AJT] in shape optimization. They use formally

a gradient method for the minimization of an objective function $J(\Omega)$ where Ω is a subset of $\mathbb{R}^N$.

Let us describe briefly their approach in the case related to the above velocity (4). Consider the problem of minimizing the capacity of a set under volume constraint:

$$\min_{S \subset\subset \Omega \subset\subset R^N} \{\text{cap}(\Omega) \text{ with } \text{vol}(\Omega) = \text{constant}\}, \tag{6}$$

where

$$\text{cap}(\Omega) = \int_{\Omega \setminus S} |\nabla u(x)|^2 dx, \qquad \text{vol}(\Omega) = \int_{\Omega \setminus S} dx$$

and u is the solution of (3). For any local diffeomorphism θ which maps Ω to $\theta(\Omega)$, we can compute the shape derivatives with respect to θ of the two previous quantities. By Hadamard formulas, we get

$$\text{cap}'(\Omega)(\theta) = -\int_{\partial\Omega} |\nabla u(x)|^2 \langle \theta(x), \nu_x \rangle d\sigma \quad \text{and} \quad \text{vol}'(\Omega)(\theta) = \int_{\partial\Omega} \langle \theta(x), \nu_x \rangle d\sigma,$$

where $\langle \cdot, \cdot \rangle$ is the usual euclidean inner product and $d\sigma$ is the induced measure on $\partial\Omega$. Writing the necessary condition of optimality, there exists a Lagrange multiplier $\Lambda \geq 0$ such that

$$\text{cap}'(\Omega)(\theta) + \Lambda \text{vol}'(\Omega)(\theta) = 0.$$

If we set

$$J_\lambda(\Omega) = \text{vol}(\Omega) + \lambda \text{cap}(\Omega),$$

and choose $\theta = -1 + \lambda |\nabla u(x)|^2$ as in (4), then, at least formally, we get

$$J'_\lambda(\Omega)(\theta) = -\int_{\partial\Omega} (-1 + \lambda |\nabla u(x)|^2)^2 d\sigma \leq 0.$$

Therefore $\theta = -1 + \lambda |\nabla u(x)|^2$ appears as a descent direction for the optimization problem (6). The method used in [AJT] to solve (6) is now clear: they fix an initial set Ω_0, consider the evolution $(\Omega_t)_{t \geq 0}$ with normal velocity (4) and compute the limit of Ω_t as $t \to +\infty$ which is the candidate minimizer to (6).

We end with two important remarks. At first, problem (6) is equivalent to the well-known Bernoulli exterior free boundary problem (see [FR] for a survey):

$$\text{Find a set } S \subset\subset \Omega \subset\subset \mathbb{R}^N \text{such that } |\nabla u(x)| = \frac{1}{\sqrt{\lambda}} \text{ for all } x \in \partial\Omega. \tag{7}$$

Secondly, (5) appears when considering the previous problem with perimeter constraint instead of volume constraint:

$$\min_{S \subset\subset \Omega \subset\subset R^N} \{\text{cap}(\Omega) \text{ with } \text{per}(\Omega) = \text{constant}\}, \tag{8}$$

where

$$\text{per}(\Omega) = \int_{\partial\Omega} d\sigma.$$

In this case, for any local diffeomorphism θ,

$$\text{per}'(\Omega)(\theta) = -\int_{\partial\Omega} H_x \langle \theta(x), \nu_x \rangle d\sigma$$

and $\theta = H_x + \lambda |\nabla u(x)|^2$ as in (5) looks as a descent direction for

$$J_\lambda(\Omega) = \text{per}(\Omega) + \lambda \text{cap}(\Omega).$$

Definition of solutions

As we said at the beginning, this approach relies on the assumption that all sets are smooth enough to give sense of our computations. But in reality, even for nicer velocities (as mean curvature for instance), the evolutions face a lack of regularity and singularities occur in finite time.

We intend to make the approach of [AJT] as rigorous as possible by defining generalized solutions for (4) and (5) widely inspired from the theory of viscosity solutions. Before describing our method and stating our results, let us recall some previous works on evolutions with prescribed normal velocity close to ours.

Following the numerical work of Osher and Sethian, a breakthrough was made by the articles of Chen, Giga and Goto [CGG] and Evans and Spruck [ES] in the case of local evolutions. They described the evolution as the level set of the solution of an auxiliary pde, the level set equation. This equation is solved in the sense of viscosity solutions (see [CIL]). This powerful method leads to plenty of results and was developed, in addition to the quoted mathematicians, by Barles, Ishii, Ohnuma, Sato, Soner, Souganidis and many others. We refer to Giga [G] for an overview.

When dealing with nonlocal velocities, it is not easy to write and study the level set equation. Some results in this direction were obtained recently by Kim, Slepcev and Da Lio. Our method does not use the level set approach.

Instead, we use generalized solutions which are kind of "geometric viscosity solutions" and were introduced by Cardaliaguet [C1], [C2]. Next, in [CR], these solutions were used to solve Hele-Shaw problem. The main novelty in our work is that we can deal with nonlocal Hele-Shaw terms like (2) and mean curvature as in (5) at the same time.

Before giving the definition of generalized solutions to (1), we need to introduce some notations.

Our evolution $(\Omega_t)_{t\geq 0}$ will be described by a *tube* $\mathcal{K}$ which is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$ such that $\overline{\mathcal{K}} \cap ([0,T] \times \mathbb{R}^N)$ is a compact subset of $\mathbb{R}^{N+1}$ for any $T \geq 0$. We recover the desired evolution at time t by setting $\Omega(t) := \mathcal{K}(t)$. We denote by $\widehat{\mathcal{K}} = \overline{\mathbb{R}^N - \mathcal{K}}$ the *exterior of the tube.*

If $\mathcal{K}$ is a C^1 tube (i.e. a tube whose boundary has at least C^1 regularity) then, in a natural way, at any point $(t,x) \in \partial\mathcal{K}$, the *normal velocity* $\mathcal{V}^{\mathcal{K}}_{t,x}$ to $\mathcal{K}(t)$ at x is defined by

$$\mathcal{V}^{\mathcal{K}}_{(t,x)} = -\frac{\nu_t}{|\nu_x|}. \tag{9}$$

A *regular tube* $\mathcal{K}$ is a tube with a non empty interior whose boundary has at least C^1 regularity, such that at any point $(t,x) \in \partial\mathcal{K}$, the normal velocity is finite:

$$\mathcal{V}^{\mathcal{K}}_{(t,x)} < \infty \quad \Longleftrightarrow \quad \nu_x \neq 0.$$

The above regularity assumption is generalized to nonsmooth tube as follows: we say that a tube $\mathcal{K}$ is *left lower semi-continuous* if

$$\forall t > 0,\ \forall x \in \mathcal{K}(t),\ \text{if } t_n \to t^-,\ \exists x_n \in \mathcal{K}(t_n) \text{ such that } x_n \to x\ .$$

A C^1 regular tube $\mathcal{K}_r$ is *externally tangent* to a tube $\mathcal{K}$ at $(t,x) \in \mathcal{K}$ if

$$\mathcal{K} \subset \mathcal{K}_r \text{ and } (t,x) \in \partial\mathcal{K}_r\ .$$

It is *internally tangent* to $\mathcal{K}$ at $(t,x) \in \widehat{\mathcal{K}}$ if

$$\mathcal{K}_r \subset \mathcal{K} \text{ and } (t,x) \in \partial\mathcal{K}_r\ .$$

The reason to introduce externally and internally tangent tubes is clear when making the analogy with viscosity solutions: such tubes will play the role of test-functions. With this aim, it remains to decide what regularity one has to assume for test-tubes.

Looking at (1), we see that the local term (which depends only on the curvature of $\Omega(t)$ at x) has a sense as soon as $\partial\Omega$ is C^2 in a neighborhood of (t,x). On the other hand, from classical pde theory, we know that we has to assume that $\partial\Omega(t)$ is $C^{1,1}$ to solve (3) and compute the nonlocal part in (1). Therefore, we will say that $K^s_{t,x}$ is a *smooth test-tube* at (t,x) if $(t,x) \in \partial K^s_{t,x}$ and $K^s_{t,x}$ is a $C^{1,1}$ regular tube with a C^2 boundary in a neighborhood of (t,x).

We are now ready to give the definition of generalized solutions:

Definition *(Generalized solutions) Let $\mathcal{K}$ be a tube and $S \subset\subset K_0 \subset\subset \mathbb{R}^N$ be an initial set.*

1. *$\mathcal{K}$ is* a viscosity subsolution *to (1) if $\mathcal{K}$ is left lower semi-continuous, $S \subset\subset \mathcal{K}(t)$ for any t, and if, for any smooth test-tube $K^s_{t,x}$ externally tangent to $\mathcal{K}$ at (t,x) with $t>0$, we have*

$$\mathcal{V}^{\mathcal{K}^s_{t,x}}_{(t,x)} \leq F(\nu_x, H^{\mathcal{K}^s_{t,x}}_x) + \lambda h(x, \mathcal{K}^s_{t,x}(t)),$$

   where ν_x is the spatial component of the outward unit normal and $H^{\mathcal{K}^s_{t,x}}_x$ is the curvature matrix to $\mathcal{K}^s_{t,x}(t)$ at (t,x).

   We say that $\mathcal{K}$ is a subsolution to (1) with initial position K_0 if $\mathcal{K}$ is a subsolution and if $\overline{\mathcal{K}}(0) \subset \overline{K_0}$.

2. *$\mathcal{K}$ is a* viscosity supersolution *to (1) if $\widehat{\mathcal{K}}$ is left lower semi-continuous, $S \subset\subset \mathcal{K}(t)$ for any t, and if, for any smooth test-tube $K^s_{t,x}$ internally tangent to $\mathcal{K}$ at (t,x) with $t>0$, we have*

$$\mathcal{V}^{\mathcal{K}^s_{t,x}}_{(t,x)} \geq F(\nu_x, H^{\mathcal{K}^s_{t,x}}_x) + \lambda h(x, \mathcal{K}^s_{t,x}(t)).$$

   We say that $\mathcal{K}$ is a supersolution to (1) with initial position K_0 if $\mathcal{K}$ is a supersolution and if $\widehat{\mathcal{K}}(0) \subset \overline{\mathbb{R}^N \backslash K_0}$.

3. *Finally, we say that a tube $\mathcal{K}$ is a* viscosity solution *to (1) (with initial position K_0) if $\mathcal{K}$ is a sub- and a supersolution.*

Statement of the results

Our main result is the following preservation of inclusion:

Theorem *(Inclusion principle) Let $T > 0$ and $0 < \lambda_1 < \lambda_2$ be fixed. Suppose $\mathcal{K}_1$ (respectively $\mathcal{K}_2$) is a subsolution (respectively a supersolution) to (4) or (5) with $\lambda = \lambda_1$ (respectively with $\lambda = \lambda_2$) on the time interval $[0,T)$. If*

$$\overline{\mathcal{K}_1}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset\,,$$

then

$$\forall t \in [0,T), \qquad \overline{\mathcal{K}_1}(t) \cap \widehat{\mathcal{K}_2}(t) = \emptyset\,.$$

A sketch of the proof will be given in the talk. This result corresponds to a comparison result for viscosity solutions. It implies existence and uniqueness of solutions.

Theorem *(Existence) Let $S \subset\subset K_0 \subset\subset \mathbb{R}^N$. There exists at least one solution to (4) (or (5)) with initial position K_0. More precisely, there exists a largest solution denoted by $S(K_0)$ which contains all the subsolution $\mathcal{K}$ such that $\overline{\mathcal{K}} \subset \overline{K_0}$ and there exists a smallest solution denoted by $s(K_0)$ which is contained in all the supersolution $\mathcal{K}$ such that $\overline{\mathcal{K}} \supset \overline{K_0}$.*

We continue by giving a first result of uniqueness:

Theorem *(Generic uniqueness) Let $(K_0^\lambda)_{\lambda\in(0,+\infty)}$ be a family of initial positions such that, if $\lambda' < \lambda$, then $K_0^{\lambda'} \subset K_0^\lambda$ and $\partial K_0^{\lambda'} \cap \partial K_0^\lambda = \emptyset$. Let $s(K_0^\lambda)$ (respectively $S(K_0^\lambda)$) be the smallest (respectively biggest) solution for (4) (or (5)) with λ and initial position K_0^λ. We have uniqueness in the following sense: there exists a countable subset I of $(0,+\infty)$ such that*

$$\overline{s(K_0^\lambda)} = S(K_0^\lambda) \quad \text{for all } \lambda \in (0,+\infty)\backslash I.$$

Now, we turn to the asymptotic behaviour of $\mathcal{K}(t)$ as $t \to +\infty$ as announced. From now on, we consider the evolution problem with velocity given by (4). As we said above, this problem is related to the Bernoulli exterior free boundary problem (7). We start to give a definition of generalized solutions to (7) (or equivalently (6)).

Definition *(Generalized solutions for the Bernoulli problem) A set $\Omega \subset \mathbb{R}^N$ is a solution to (7) is the constant tube $\mathcal{K} = [0, +\infty) \times \Omega$ is a solution to (4).*

There are different notions of weak solutions for (7). The one we give here is the most suitable for our purpose.

Theorem *(Existence and Uniqueness for the Bernoulli problem) Suppose that the source S is strictly starshaped. Then for any $\lambda > 0$ there exists a unique solution Ω_λ to (7).*

This result was first proved by Tepper [T]. We conclude with

Theorem *(Asymptotic behaviour) Let $\lambda > 0$ and suppose that the source S is strictly starshaped and consider $S \subset\subset K_0 \subset\subset \mathbb{R}^N$. Then every solution $\mathcal{K}$ to (4) with initial position K_0 converges (for the Hausdorff metric) to the unique solution of (7) as $t \to +\infty$.*

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A variational approach to self-similar solutions for semilinear heat equations

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1. Introduction

We consider the Cauchy problem for semilinear heat equations with singular initial data:

$$w_t = \Delta w + w^p \qquad \text{in } \mathbf{R}^N \times (0,\infty), \tag{1}$$

$$w(x,0) = \lambda a\left(x/|x|\right)|x|^{-2/(p-1)} \qquad \text{in } \mathbf{R}^N \setminus \{0\}, \tag{$2)_\lambda$}$$

where $N > 2$, $p > 1$, $a : S^{N-1} \to \mathbf{R}$, and $\lambda > 0$ is a parameter. We assume that $a \in L^\infty(S^{N-1})$ and $a \geq 0$, $a \not\equiv 0$. A typical case is $a \equiv 1$.

The equation (1) is invariant under the similarity transformation

$$w(x,t) \mapsto w_\mu(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

In particular, a solution w is said to be *self-similar*, when $w = w_\mu$ for all $\mu > 0$, that is,

$$w(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0. \tag{3}$$

Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1), see, e.g., [14, 15, 5, 21].

If $w(x,t)$ is a self-similar solution of (1.1) and has an initial value $A(x)$, then we easily see that A has the form $A(x) = A(x/|x|)|x|^{-2/(p-1)}$. Then the problem of existence of self-similar solutions is essentially depend on the solvablity of the Cauchy problem (1)-$(2)_\lambda$. In this talk we consider the existence of self-similar solutions of the problem (1)-$(2)_\lambda$. The idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data goes back to the study by Giga and Miyakawa [12] for the Navier-Stokes equation in vorticity form.

It is well known by Fujita [9] that if $1 < p \leq (N+2)/N$ then (1) has no time global solution w such that $w \geq 0$ and $w \not\equiv 0$. (See also [25, 14].) Then the condition $p > (N+2)/N$ is necessary for the existence of positive self-similar solutions of (1).

We briefly review some results concerning the Cauchy problem for (1) with initial date in $L^q(\mathbf{R}^N)$. Weissler [23, 24] showed that the IVP (1) with $w(x,0) = A \in L^q(\mathbf{R}^N)$ admits a unique time-local solution if $q \geq N(p-1)/2$. He also showed in [25] that the solution exists time-globally if $q = N(p-1)/2$ and if $\|A\|_{L^q(\mathbf{R}^N)}$ is sufficiently small. Giga [11] has constructed a unique local regular solution in $L^\alpha(0,T : L^\beta)$, where α and β are chosen so that the norm of $L^\alpha(0,T : L^\beta)$ is invariant under scaling. On the other hand, for $1 \leq q < N(p-1)/2$, Haraux and Weissler [13] constructed a solution $w_0 \in C([0,\infty); L^q(\mathbf{R}^N))$ of (1) satisfying $w_0(x,t) > 0$ for $t > 0$ and $\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} \to 0$ as $t \to 0$ when $(N+2)/N < p < (N+2)/(N-2)$ by seeking solutions of self-similar form. Therefore, the Cauchy problem

$$w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0,\infty) \quad \text{and} \quad w(x,0) = 0 \quad \text{in } \mathbf{R}^N \tag{4}$$

admits a non-unique solution in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \leq q < N(p-1)/2$ when $(N+2)/N < p < (N+2)/(N-2)$.

Kozono and Yamazaki [16] constructed Besov-type function spaces based on the Morrey spaces, and then obtained global existence results for the equation (1) and the Navier-Stokes system with small initial data in these spaces. Cazenave and Weissler [5] proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equations and the equation (1) with small initial data by using the weighted norms. By [16, 5] the problem (1)-$(2)_\lambda$ admits a time-global solution for sufficiently small $\lambda > 0$.

We note here that the equation (1) with $p > N/(N-2)$ has a positive singular stationary solution $W(x) = L|x|^{-2/(p-1)}$, where

$$L = \left[\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right]^{1/(p-1)}.$$

Galaktionov and Vazquez [10] investigated the uniqueness of solutions to the problem (1)-$(2)_\lambda$ in the case where $a \equiv 1$ and $\lambda = L$, and showed that the problem has a classical self-similar solution for $t > 0$ with certain values of p. In [10, p. 41] they also conjectured that the problem (1)-$(2)_\lambda$ has exactly two solutions (the minimal and maximal) when $N/(N-2) < p \leq (N+2)/(N-2)$.

Letting $\mu = t^{-1/2}$ in (3), we see that the self-similar solution w of (1) has the form

$$w(x,t) = t^{-1/(p-1)} u(x/\sqrt{t}), \tag{5}$$

where u satisfies the elliptic equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \qquad \text{in } \mathbf{R}^N. \tag{6}$$

In addition, if w satisfies $(2)_\lambda$ in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$, then u satisfies

$$(7)_\lambda \qquad \lim_{r\to\infty} r^{2/(p-1)}u(r\omega) = \lambda a(\omega) \qquad \text{for a.e. } \omega \in S^{N-1}.$$

Conversely, if $u \in C^2(\mathbf{R}^N)$ is a solution of (6) satisfying $(7)_\lambda$, then the function w defined by (5) satisfies (1)-$(2)_\lambda$ in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$. (See Lemma B.1 in [17].)

In this talk we investigate the problem (6)-$(7)_\lambda$ by making use of the methods for semilinear elliptic equations to derive the results for the Cauchy problem (1)-$(2)_\lambda$. First, we show the existence of the minimal solution by employing the comparison results based on the maximum principle. Next we apply the variational method due to [1, 6, 4] to show the existence of the second solution of the problem (6)-$(7)_\lambda$, which implies the non-uniqueness of solutions to the problem (1)-$(2)_\lambda$.

2. Existence of the minimal solution [17, Sec. 4]

For simplicity, we define $\mathcal{L}u$ by

$$\mathcal{L}u = \Delta u + \frac{1}{2}x\cdot\nabla u + \frac{1}{p-1}u$$

for $u \in C^2(\mathbf{R}^N)$. First we obtain the following results.

Lemma 1. *Let $p > (N+2)/N$. Assume that $-\mathcal{L}u \geq 0$ in $\mathbf{R}^N$, and that*

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)}u(x) \geq 0.$$

Then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$. In particular, if $-\mathcal{L}u \geq 0$ and $u \geq 0$ in $\mathbf{R}^N$ then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$.

Lemma 2. *Assume that $p > (N+2)/N$, and that $\alpha, \beta \in L^\infty(S^{N-1})$ satisfy $0 \leq \alpha(\omega) \leq \beta(\omega)$ for a.e. $\omega \in S^{N-1}$. Suppose that there exists a positive function v satisfying*

$$-\mathcal{L}v \geq v^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}v(r\omega) = \beta(\omega), \quad \text{a.e. } \omega \in S^{N-1}.$$

Then there exists a positive solution u of the problem

$$-\mathcal{L}u = u^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}u(r\omega) = \alpha(\omega), \quad \text{a.e. } \omega \in S^{N-1}.$$

By using of Lemmas 1 and 2 we obtain the following:

Theorem 1. *Assume that $p > (N+2)/N$. Then there exists a constant $\overline{\lambda} > 0$ such that*

(i) *for $0 < \lambda < \overline{\lambda}$, (6)-(7)$_\lambda$ has a positive minimal solution $\underline{u}_\lambda \in C^2(\mathbf{R}^N)$; the solution $\underline{u}_\lambda$ is increasing with respect to λ and satisfies $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $\lambda \to 0$;*

(ii) *for $\lambda > \overline{\lambda}$, there are no positive solutions $u \in C^2(\mathbf{R}^N)$ of (6)-(7)$_\lambda$.*

3. Weighted Sobolev space

Put $\rho(x) = e^{|x|^2/4}$. Then the equation (6) can be written as

$$\nabla \cdot (\rho \nabla u) + \rho \left(\frac{1}{p-1} u + u^p \right) = 0.$$

Escobedo-Kavian [8] investigated the corresponding functional

$$I_0(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} \rho dx$$

on the weighted functional spaces

$$L^q_\rho(\mathbf{R}^N) = \left\{ u \in L^q(\mathbf{R}^N) : \int_{\mathbf{R}^N} u^q \rho dx < \infty \right\} \quad \text{for } 1 \le q < \infty$$

and

$$H^1_\rho(\mathbf{R}^N) = \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2) \rho dx < \infty \right\}.$$

We recall here some results about the weighted Sobolev space $H^1_\rho(\mathbf{R}^N)$.

Lemma 3 [8, 14]. (i) *For every $u \in H^1_\rho(\mathbf{R}^N)$,*

$$\frac{N}{2} \int_{\mathbf{R}^N} u^2 \rho dx \le \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx.$$

(ii) *The embedding $H^1_\rho(\mathbf{R}^N) \subset L^{p+1}_\rho(\mathbf{R}^N)$ is continuous for $1 \le p \le (N+2)/(N-2)$, and is compact for $1 \le p < (N+2)/(N-2)$.*

It was shown by [8, 24] that there exists a solution u_0 of the problem

$$\text{(8)} \qquad \begin{cases} \Delta u + \dfrac{1}{2} x \cdot \nabla u + \dfrac{1}{p-1} u + u^p = 0 & \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N, \end{cases}$$

with $(N+2)/N < p < (N+2)/(N-2)$. Moreover, it was shown in [8] that $u_0 \in C^2(\mathbf{R}^N)$ and $u_0(x) = O(e^{-|x|^2/8})$ as $|x| \to \infty$. The uniqueness of the solution to the problem (8) was obtained by combining the results [7, 27, 19].

Now put

$$w_0(x,t) = t^{-1/(p-1)} u_0(x/\sqrt{t}), \tag{9}$$

where u_0 is the solution of the problem (8). We note that $u_0 \in L^q(\mathbf{R}^N)$ for all $q \geq 1$ and

$$\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} = t^{-1/(p-1)+N/2q} \|u_0\|_{L^q(\mathbf{R}^N)}.$$

Then w_0 solves the the Cauchy problem (4) in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \leq q < N(p-1)/2$. By the uniqueness result [19], we find that w_0 defined by (9) coincides with the non-unique solution of (4) constructed by [13].

4. Existence of the second solution: subcritical case [17, Sec. 5]

Let $\underline{u}_\lambda$ be the positive minimal solution of (6)-(7)$_\lambda$ obtained in Theorem 1. In order to find a second solution of (6)-(7)$_\lambda$ we introduce the following problem:

$$(10)_\lambda \qquad \begin{cases} \Delta u + \dfrac{1}{2} x \cdot \nabla u + \dfrac{1}{p-1} u + g(u, \underline{u}_\lambda) = 0 \quad \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 \quad \text{in } \mathbf{R}^N, \end{cases}$$

where $g(t,s) = (t+s)^p - s^p$. We easily see that, if $(10)_\lambda$ possesses a solution u_λ, then we can get another positive solution $\overline{u}_\lambda = \underline{u}_\lambda + u_\lambda$ of (6)-(7)$_\lambda$.

In this section we will show the existence of solutions of $(10)_\lambda$ in the subcritical case $(N+2)/N < p < (N+2)/(N-2)$ by using the variational method. To this end we define the corresponding functional of $(10)_\lambda$ by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \int_{\mathbf{R}^N} G(u, \underline{u}_\lambda) \rho dx$$

with $u \in H^1_\rho(\mathbf{R}^N)$, where

$$G(t,s) = \frac{1}{p+1}(t+s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t.$$

We see that the nontrivial critical point $u \in H^1_\rho(\mathbf{R}^N)$ of the functional I_λ is a weak solution of the equation in $(10)_\lambda$. Moreover, we have $u_\lambda \in C^2(\mathbf{R}^N)$ and $u_\lambda > 0$ in $\mathbf{R}^N$ by employing the bootstrap arguments and the maximum principle.

We will verify the existence of nontrivial solution of $(10)_\lambda$ by means of the Mountain Pass lemma ([1, 20]).

Lemma 4. *For $\lambda \in (0, \overline{\lambda})$ there exist some constants $\delta = \delta(\lambda) > 0$ and $\eta = \eta(\lambda) > 0$ such that $I_\lambda(u) \geq \eta$ for all $u \in H^1_\rho(\mathbf{R}^N)$ with $\|\nabla u\|_{L^2_\rho} = \delta$.*

Lemma 5. *For any $v \in H^1_\rho(\mathbf{R}^N)$ with $v \geq 0$, $v \not\equiv 0$, we have $I_\lambda(tv) \to -\infty$ as $t \to \infty$.*

Lemma 6. *The functional I_λ satisfies the Palais-Smale condition, that is, any sequence $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ such that*

$$\{I_\lambda(u_k)\} \text{ is bounded} \quad \text{and} \quad I'_\lambda(u_k) \to 0 \text{ as } k \to \infty$$

contains a convergent subsequence.

In the proofs of Lemmas 4-6, the following results play a crucial role.

Lemma 7. *Let $\underline{u}_\lambda$ be the minimal solution obtained in Theorem 1 for $\lambda \in (0, \overline{\lambda})$. Then the linearized eigenvalue problem*

$$\begin{cases} -\Delta w - \dfrac{1}{2} x \cdot \nabla w - \dfrac{1}{p-1} w = \mu p [\underline{u}_\lambda]^{p-1} w & \text{in } \mathbf{R}^N, \\ w \in H^1_\rho(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue $\mu = \mu(\lambda) > 1$. Moreover, $\mu(\lambda)$ is strictly decreasing in $\lambda \in (0, \overline{\lambda})$.

Lemma 7 follows from the fact that $\underline{u}_\lambda$ is the positive minimal solution.

As a consequence of Lemmas 4-6 we obtain the following:

Theorem 2. *Assume that $(N+2)/N < p < (N+2)/(N-2)$. Then, for $0 < \lambda < \overline{\lambda}$, there exists a positive solution $\overline{u}_\lambda$ of (6)-(7)$_\lambda$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$,*

$$\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N), \quad \text{and} \quad \overline{u}_\lambda(x) - \underline{u}_\lambda(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \to \infty.$$

Furthermore,

$$\overline{u}_\lambda - \underline{u}_\lambda \to u_0 \quad \text{in } H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \quad \text{as } \lambda \to 0,$$

where u_0 is the solution of the problem (8). In particular, $\overline{u}_\lambda \to u_0$ in $L^\infty(\mathbf{R}^N)$ as $\lambda \to 0$.

Now we consider the Cauchy problem (1)-(2)$_\lambda$. Recall that, if u is a solution of (6)-(7)$_\lambda$, then the function w defined by (5) is a solution of (1)-(2)$_\lambda$ in the sense of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$, and that w_0 defined by (9) coincides with the non-unique solution of (4) constructed by [13]. As a consequence of Theorems 1 and 2, we obtain the following results.

Corollary 1. *Assume that $p > (N+2)/N$. Then there exists a constant $\overline{\lambda} > 0$ such that*

(i) *for $0 < \lambda < \overline{\lambda}$, (1)-(2)$_\lambda$ has a positive self-similar solution $\underline{w}_\lambda$; the solution $\underline{w}_\lambda(\cdot, t)$ satisfies, for each fixed $t > 0$,*

$$\|\underline{w}_\lambda(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } \lambda \to 0;$$

(ii) *for $\lambda > \overline{\lambda}$, (1)-(2)$_\lambda$ has no positive self-similar solutions.*

Assume, furthermore, that $p < (N+2)/(N-2)$. Then (1)-(2)$_\lambda$ has a positive self-similar solution $\overline{w}_\lambda$ satisfying $\overline{w}_\lambda > \underline{w}_\lambda$ in $\mathbf{R}^N \times (0, \infty)$ for $0 < \lambda < \overline{\lambda}$. The solution $\overline{w}_\lambda$ satisfies, for each fixed $t > 0$,

$$\|\overline{w}_\lambda(\cdot, t) - w_0(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } \lambda \to 0,$$

where w_0 is the non-unique solution of (4) in $C([0, \infty); L^q(\mathbf{R}^N))$ for $1 \le q < N(p-1)/2$, which is constructed by [13].

5. Existence and nonexistence of second solutions: critical case [18]

In this section we consider the existence and nonexistence of second solutions of the problem (6)-(7)$_\lambda$ in the critical case $p = (N+2)/(N-2)$ by following the argument due to Brezis-Nirenberg [4].

For the critical growth case, there are serious difficulties in obtaining solutions by using variational methods because the Sobolev embedding $H^1 \subset L^{p+1}$ is not compact. It is well known that this lack of compactness exhibits many interesting existence and nonexistence phenomena. See, e.g., [4, 2].

Let us denote by S the best Sobolev constant of the embedding $H^1(\mathbf{R}^N) \subset L^{2N/(N+2)}(\mathbf{R}^N)$, which is given by

$$S = \inf_{u \in H^1(\mathbf{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbf{R}^N} |\nabla u|^2 dx}{\displaystyle\left(\int_{\mathbf{R}^N} |u|^{2N/(N-2)} dx \right)^{(N-2)/N}}.$$

In the critical case, the functional I_λ satisfies the following local Palais-Smale condition.

Lemma 8. *Let $p = (N+2)/(N-2)$. Then I_λ satisfies the $(PS)_c$ condition for $c \in (0, S^{N/2}/N)$, that is, any sequence $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ such that*

$$I_\lambda(u_k) \to c \in \left(0, \frac{1}{N} S^{N/2}\right) \quad \text{and} \quad I'_\lambda(u_k) \to 0 \quad \text{as } k \to \infty$$

contains a convergent subsequence.

By Lemma 8 and the Mountain Pass lemma, we obtain the following existence result.

Lemma 9. *Let $p=(N+2)/(N-2)$. Assume that there exists $v_0 \in H^1_\rho(\mathbf{R}^N)$ with $v_0 \geq 0$, $v_0 \not\equiv 0$ such that*

$$\sup_{t>0} I_\lambda(tv_0) < \frac{1}{N} S^{N/2}. \tag{11}$$

Then there exists a positive solution $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ of $(10)_\lambda$.

Moreover, we have $u_\lambda \in C^2(\mathbf{R}^N)$ by employing the estimate due to Brezis-Kato [3], based on the Moser's iteration technique.

In order to find a positive function $v_0 \in H^1_\rho(\mathbf{R}^N)$ satisfying (11), we set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon+|x|^2)^{(N-2)/2}} \rho^{-1/2} \quad \text{and} \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{L^{p+1}_\rho}}$$

for $\varepsilon > 0$, where $\phi \in C_0^\infty(\mathbf{R}^N)$ is a cut off function. We remark that the functional I_λ can be written as

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} \rho dx \\ &\qquad - \int_{\mathbf{R}^N} H(u, \underline{u}_\lambda) \rho dx \\ &\equiv I_0(u) - \int_{\mathbf{R}^N} H(u, \underline{u}_\lambda) \rho dx, \end{aligned}$$

where

$$H(t,s) = G(t,s) - \frac{1}{p+1} t^{p+1}.$$

Lemma 10. *For sufficient small $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $\sup_{t>0} I_\lambda(tv_\varepsilon) = I_\lambda(t_\varepsilon v_\varepsilon)$. Moreover, as $\varepsilon \to 0$ we have*

$$I_0(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} S^{N/2} + \begin{cases} O(\varepsilon), & N \geq 5 \\ O(\varepsilon |\log \varepsilon|), & N = 4 \\ O(\varepsilon^{1/2}), & N = 3 \end{cases}$$

$$\int_{\mathbf{R}^N} H(t_\varepsilon v_\varepsilon, \underline{u}_\lambda) \rho dx \geq \begin{cases} C\varepsilon^{3/4}, & N = 5 \\ C\varepsilon^{1/2}, & N = 4 \\ C\varepsilon^{1/4}, & N = 3 \end{cases}$$

with some constant $C > 0$.

As a consequence, we obtain the following:

Theorem 3. *Let $p = (N+2)/(N-2)$ and $N = 3, 4, 5$. Then, for $0 < \lambda < \overline{\lambda}$, the problem (6)-(7)$_\lambda$ has a positive solution $\overline{u}_\lambda \in C^2(\mathbf{R}^N)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$ and $\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N)$.*

On the other hand, for the case $N \geq 6$ we obtain the uniqueness result in the radial class by employing the Pohozaev type identity.

Theorem 4. *Let $p = (N+2)/(N-2)$ and $N \geq 6$. Assume that $a \equiv 1$ in (7)$_\lambda$. Then there exists a constant $\lambda_0 \in (0, \overline{\lambda})$ such that (6)-(7)$_\lambda$ has no positive radial solutions $u \in C^2(\mathbf{R}^N)$ with $u \not\equiv \underline{u}_\lambda$ for $\lambda \in (0, \lambda_0)$, that is, (6)-(7)$_\lambda$ has a unique positive radial solution $\underline{u}_\lambda$ for $0 < \lambda < \lambda_0$.*

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Uniqueness in the Cauchy problem for systems with partial analytic coefficients

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The problem of the uniqueness in the Cauchy problem is a fundamental problem in a theory of partial differential equation. The purpose of this paper is to extend, to the case of system, the recent results of Tataru[T], Hörmander[H], and Robbiano-Zuily[RZ] concerning the uniqueness of the Cauchy problem for operators with partially analytic coefficients. Concerned results in analytic coefficients is in [U]. The method of proof in [U] is the one of 'Algebraic analysis '. It is different from our proof by Carleman estimate.

We introduce our results. Let n_a, n_b be non negative integers with $n = n_a + n_b \geq 1$.We set $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$and ,for x or ξ in $\mathbb{R}^n$, $x = (x_a, x_b), \xi = (\xi_a, \xi_b)$. Let $P(x, D_x) = (p_{ij}(x, D_x))_{1 \leq i,j \leq N} = \sum_{|\alpha| \leq m} A_\alpha(x) D_x^\alpha$ be a linear differential system with principal part $P_m(x, \xi) = \sum_{|\alpha|=m} \xi^\alpha A_\alpha(x)$. Let S be a C^2 hypersurface through 0 locally given by

$$S = \{x : \varphi(x) = 0\}, \varphi(0) = 0, \varphi'(0) = (\varphi'_a(0), \varphi'_b(0)) \neq 0.$$

Our results are as follows;

Theorem 0.1
We assume that the coefficients of $P(x, D_x)$ are C^∞ in x and analytic in x_a in a neighborhood of 0 and $P_m(x, \xi)$ satisfies

1. *For any $\xi_b \in \mathbb{R}^{n_b} \backslash \{0\}$*

$$\det P_m(0, 0, \xi_b) \neq 0. \tag{1}$$

2. *For any $\xi_b \in \mathbb{R}^{n_b}$*

$$\det P_m(0, i\varphi'_a(0), i\varphi'_b(0) + \xi_b) \neq 0. \tag{2}$$

Let V be a neighborhood of 0 and $u = (u_1, u_2, \cdots, u_N) \in C^\infty(V)^N$ be such that

$$\begin{cases} P(x, D_x)u(x) = 0, \quad x \in V \\ \text{supp}u = \cup_{k=1}^N \text{supp}u_k \subset \{x \in V : \varphi(x) \leq 0\} \end{cases}$$

Then there exists a neighborhood W of 0 in which $u \equiv 0$.

We make some comments on this result. Theorem 0.1 is an extension of Tataru's results[T] to systems. Alinhac-Baouendi[AB] showed that in the case of second order hyperbolic operators $P = \partial_t^2 - A(t, x, D_x)$, the initial hypersurface is time-like and corresponding uniqueness result is false if the coefficients are merely C^∞. But in[T] Tataru showed that under the assumption that the coefficients are partially analytic , uniqueness result holds for any non-characteristic initial hypersurface. In theorem0.1, we showed the Tataru's result hold for systems. An appliation of theorem0.1 to elastic equation is as follows.

Corollary 0.2
$\alpha(t,x), \beta(t,x) \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^3)$ *satisfies*

1. $\alpha(t,x), \beta(t,x) > 0, (t,x) \in \mathbb{R}_t \times \mathbb{R}_x^3$.
2. $\alpha(t,x), \beta(t,x)$*is analytic in* t.

then elastic equations

$$\partial_t^2 u(t,x) = \alpha(t,x)\triangle u(t,x) + \beta(t,x)\text{grad div}u(t,x)$$

has a unique solution for any non-characteristic initial hypersurface.

The method used here will be basically the same as in the proof given by[RZ], that is the use of the Sjöstrand theory of FBI transformation to microlocalize the symbols and symbolic calculus and the Gårding's inequality.

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Rotating Navier-Stokes Equations with Initial Data Nondecreasing at Infinity

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We study the initial value problem for the rotating Navier-Stokes equations:

$$\text{(RNS)} \quad \begin{cases} \partial_t \mathbf{U} - \Delta \mathbf{U} + \Omega e_3 \times \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\nabla p & \text{for } 0 < t < T,\ x \in \mathbb{R}^3, \\ \nabla \cdot \mathbf{U} = 0 & \text{for } 0 < t < T,\ x \in \mathbb{R}^3, \\ \mathbf{U}(t,x)|_{t=0} = \mathbf{U}_0(x) & \text{for } x \in \mathbb{R}^3, \end{cases}$$

where $\mathbf{U} = (\mathbf{U}^1(t,x), \mathbf{U}^2(t,x), \mathbf{U}^3(t,x))$ is the velocity vector field of the fluid, $p = p(t,x)$ is the scalar pressure and $T > 0$. Here, fixed constant $\Omega \in \mathbb{R}$ is called the Coriolis parameter that represents rotating speed of the fluid around x_3 axis. The term $\Omega e_3 \times \mathbf{U}$ ($= \Omega \mathbf{J}\mathbf{U}$) is called the Coriolis term where $e_3 = (0,0,1)$ and $\mathbf{J}$ is the corresponding skew-symmetric matrix.

In periodic and cylindrical domains, Babin-Mahalov-Nicolaenko [1] and Mahalov-Nicolaenko [9] proved local existence and uniqueness of solutions uniformly in the Coriolis parameter Ω. Moreover, they proved global in time regularity of solutions when Ω is sufficienly large.

Our aim in this talk is to prove local existence uniformly in $\Omega \in \mathbb{R}$ for nondecreasing initial data $\mathbf{U}_0$ at space inifinity. For this purpose we formally transform (RNS) into the integral equation of the form:

$$\text{(I)} \quad \mathbf{U}(t) = \exp(-\mathbf{A}(\Omega)t)\mathbf{U}_0 - \int_0^t \exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\mathrm{div}(\mathbf{U} \otimes \mathbf{U})(s)\, ds \quad \text{for } t > 0.$$

Here, $\mathbf{P} = (\delta_{i,j} + R_i R_j)_{i,j}$, $1 \le i,j \le 3$, where $\delta_{i,j}$ is the Kronecker delta and R_j is the scalar Riesz operator whose symbol is $\sqrt{-1}\xi_j/|\xi|$. Since $\mathbf{P}\mathbf{U} = \mathbf{U}$ for divergence free vector field and $\mathbf{P}\Delta = \Delta\mathbf{P}$, we have

$$\begin{aligned} \mathbf{A}(\Omega) &= -\mathbf{P}\Delta + \Omega\mathbf{P}\mathbf{J} \\ &= -\Delta + \Omega\mathbf{P}\mathbf{J}\mathbf{P}, \qquad (0.1) \\ \text{hence} \quad \exp(-\mathbf{A}(\Omega)t) &= \exp(t\Delta)\exp(-\Omega\mathbf{P}\mathbf{J}\mathbf{P}t). \end{aligned}$$

In the case $\Omega = 0$, that is, on the Navier-Stokes equations (NS) without the Coriolis term, unique local existence of mild solution was proved if initial data $\mathbf{U}_0$ belongs to L^∞_σ, the space of bounded solenoidal functions, in Cannon-Knightly [2], Cannone [3] and

Giga-Inui-Matsui [5]. Of cource, the space L^∞_σ contains nondecreasing functions. The method in [5] is to use estimate for the derivative of the heat kernel in the Hardy space $\mathcal{H}^1$ obtained by Carpio [4]. For (NS) with initial data L^∞_σ, Giga-Matsui-Sawada [6] obtained unique global existence of strong solution $\mathbf{U} \in L^\infty_\sigma$ in the 2-dimensional case and J. Kato [8] proved uniqueness of weak solution $(\mathbf{U}, \nabla p)$ when $\mathbf{U} \in L^\infty$ and $p \in BMO$ in the n-dimensional case with $n \geq 2$. Here, BMO is the space of functions of bounded mean oscillations.

In the case $\Omega \neq 0$, the crucial step is to estimate the Coriolis solution operator that comes from the Coriolis term $\mathbf{PJU} = (-R_1R_1\mathbf{U}^2 + R_1R_2\mathbf{U}^1, -R_2R_1\mathbf{U}^2 + R_2R_2\mathbf{U}^1, -R_3R_1\mathbf{U}^2 + R_3R_2\mathbf{U}^1)$. The difficulty is that the term contains the Riesz operator R_j which is not bounded in L^∞. Moreover, Carpio's estimate does not apply to the term since it has no derivatives.

Recently, Hieber-Sawada [7] and Sawada [10] constructed a local solution for (RNS) with generalized Corilolis term for the solenoidal initial data $\mathbf{U}_0 \in \dot{B}^0_{\infty,1}$. Here, $\dot{B}^0_{\infty,1}$ is a Besov space including various periodic and almost periodic functions, that do not decay at space infinity. The space $\dot{B}^0_{\infty,1}$, which is a subspace of L^∞, is first used to solve Boussinesq equations by Sawada-Taniuchi [11] (see Taniuchi [12] for recent improvement). The advantage of the Besov space is boundedness of the Riesz operator in it. They are successful in estimating the Coriolis term in the Besov space.

However, their existence time estimate depends on Ω, since the Coriolis term is regarded as a perturbation. In this talk, we transformed (RNS) into (I) to estimate the linear "Heat+Coriolis" term uniformly in the Coriolis parameter Ω by using skew-symmetric structure of the operator $\mathbf{PJP}$. That is the reason that we deal rather the operator $\mathbf{PJP}$ instead of $\mathbf{PJ}$ as in (0.1). Smallness of the Coriolis term is not assumed. This is a major difference between our and their approach.

In the integral equation (I), the unboundedness problem in L^∞ arises again in the linear term. Since the Coriolis solution operator $\exp(-\Omega\mathbf{PJP}t)$ contains the Riesz transforms, one cannot expect its boundedness in L^∞. There was still a possibility that the "Heat+Coriolis" operator $\exp(t\Delta)\exp(-\Omega\mathbf{PJP}t)$ is bounded in L^∞, even if $\exp(-\Omega\mathbf{PJP}t)$ is unbounded in L^∞. Unfortunately, our exact calculation of the symbol arrived at conclusion that the solution operator is not bounded in L^∞.

In this situation we are forced to restrict initial data to a subspace of L^∞_σ. To introduce our new subspace we split initial data into 2D3C (2 dimensional 3 components) vector field part and x_3−dependent part by taking vertical average.

Definition (Vertical average). Let $\mathbf{U} \in L^\infty_\sigma(\mathbf{R}^3)$. We say that $\mathbf{U}$ admits **vertical averaging** if

$$\lim_{L\to+\infty} \frac{1}{2L}\int_{-L}^{L} \mathbf{U}(x_1, x_2, x_3)dx_3 \equiv \overline{\mathbf{U}}(x_1, x_2)$$

exists almost everywhere. The 2D3C vector field $\overline{\mathbf{U}}(x_1, x_2)$ is called **vertical average**

or **barotropic part** of $\mathbf{U}(x_1, x_2, x_3)$. Then the **baroclinic part** $\mathbf{U}^\perp$ of $\mathbf{U}$ is defined as

$$\mathbf{U}^\perp(x_1, x_2, x_3) = \mathbf{U}(x_1, x_2, x_3) - \overline{\mathbf{U}}(x_1, x_2).$$

Definition (Space for initial data). We define a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ of the form

$$L^\infty_{\sigma,a} = L^\infty_{\sigma,a}(\mathbb{R}^3) =$$
$$\{\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3);\ \mathbf{U} \text{ admits } \textbf{vertical averaging} \text{ and } \mathbf{U}^\perp \in \dot{B}^0_{\infty,1}\}.$$

The space $L^\infty_{\sigma,a}$ is a Banach space with the norm $||\mathbf{U}||_{L^\infty_{\sigma,a}} = ||\overline{\mathbf{U}}||_{L^\infty(\mathbb{R}^2;\mathbb{R}^3)} + ||\mathbf{U}^\perp||_{\dot{B}^0_{\infty,1}}$.

Theorem. *Let* $\mathbf{U}_0 \in L^\infty_{\sigma,a}$. *Then*
(1) *There exist* $T_0 > 0$ *independent of* Ω *and a unique solution* $\mathbf{U} = \mathbf{U}(t)$ *of* (I) *such that*

$$\mathbf{U} \in C([\delta, T_0]; L^\infty_\sigma) \cap C_w([0, T_0]; L^\infty_\sigma) \qquad \textit{for any } \delta > 0.$$

(2) *The solution* $\mathbf{U}$ *satisfies*

$$\sup_{t\in(0,T_0)} ||t^{1/2}\nabla\mathbf{U}||_{L^\infty_\sigma} < \infty \quad \textit{and} \quad \nabla\mathbf{U} \in C([\delta, T_0]; L^\infty_\sigma) \qquad \textit{for any } \delta > 0.$$

Remark. For a lower estimate for $T_0 > 0$ we get $T_0 \geq C/||\mathbf{U}_0||^2_{L^\infty_{\sigma,a}}$, where $C > 0$ is independent of Ω. Moreover, if in addition we assume that $\overline{\mathbf{U}_0} \in BUC$, then $\mathbf{U} \in C([0, T_0]; BUC)$ where BUC denotes the space of bounded uniformly continuous functions.

Outline of the Proof

The symbol of the Coriolis solution operator is given by

$$\text{(S)} \qquad \sigma(\exp(-\Omega\mathbf{PJP}t)) = \cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{I} - \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)\mathbf{R}(\xi)$$

where $\mathbf{I}$ is the 3×3 identity matrix, and $\mathbf{R}$ is the vector Riesz operator with its symbol

$$\mathbf{R}(\xi) = \begin{pmatrix} 0 & -\xi_3/|\xi| & \xi_2/|\xi| \\ \xi_3/|\xi| & 0 & -\xi_1/|\xi| \\ -\xi_2/|\xi| & \xi_1/|\xi| & 0 \end{pmatrix} : \text{ skew-symmetric matrix.}$$

The symbol (S) consists of the operators of the form $\exp(\alpha R_j)$ for $\alpha \in \mathbb{R}$. By virtue of splitting initial data it suffices to show boundedness only for baroclinic part belonging to $\dot{B}^0_{\infty,1}$.

It follows from Mikhlin's theorem in the Hardy space $\mathcal{H}^1$, that the spectrum set is included in the pure imaginary axis. Then by the spectrum mapping theorem, we estimate $||\exp(\alpha R_j)||_{\mathcal{H}^1\to\mathcal{H}^1} = \sup\{|\exp(-i\alpha z)|;\ z \in \mathrm{Spec}(iR_j)\} \leq 1$ since $|\exp(-i\alpha z)| = 1$ if $z \in \mathbb{R}$. Since the boundedness of the convolution type operator in the Hardy space $\mathcal{H}^1$ yields the boundedness in the Besov space $\dot{B}^0_{\infty,1}$, we conclude the boundedness $||\exp(\alpha R_j)||_{\mathcal{H}^1\to\mathcal{H}^1} \leq 1$ uniformly in $\Omega \in \mathbb{R}$.

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Blow-up profile for a nonlinear heat equation with the Neumann boundary condition

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This talk is concerned with the nonlinear diffusion equation

$$\begin{cases} u_t = \triangle u + u^p & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$, ν is the unit outward normal vector on $\partial\Omega$, $p > 1$ is a constant and $u_0 \in L^\infty(\Omega)$ is a nonnegative function with $\|u_0\|_\infty \neq 0$. For the solution $u(x,t)$ of the nonlinear diffusion equation, the *blow-up time* T is defined by

$$T = \sup\{\tau > 0 \,|\, u(x,t) \text{ is bounded in } \bar{\Omega} \times (0,\tau)\}.$$

Then, $0 < T < +\infty$ and $\overline{\lim}_{t\to T}\|u(x,t)\|_{C(\bar{\Omega})} = +\infty$ hold. The *blow-up set* of the solution $u(x,t)$ is defined as the set

$$\{x \in \bar{\Omega} \,|\, \text{there is a sequence } (x_n,t_n) \text{ in } \bar{\Omega} \times (0,T) \text{ such that}$$

$$(x_n,t_n) \to (x,T) \text{ and } u(x_n,t_n) \to +\infty \text{ as } n \to \infty\}.$$

This set is a nonempty closed set in $\bar{\Omega}$. From standard parabolic estimates, we can obtain the *blow-up profile*, which is a smooth function defined by

$$u_*(x) = \lim_{t\to T} u(x,t)$$

outside the blow-up set.

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The blow-up problem has been studied by many authors since the pioneering work due to Fujita [13]. There are a number of results for the nature of the blow-up set. For the Cauchy problem with $(N-2)p < N+2$, Velázquez [34] showed that the $(N-1)$-dimensional Hausdorff measure of the blow-up set is bounded in compact sets of $\mathbf{R}^N$ whenever the solution is not the constant blow-up one $(p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}$. For the Cauchy problem or the Cauchy-Dirichlet problem in a convex domain with $(N-2)p < N+2$, Merle and Zaag [25] showed that for any finite set $D \subset \Omega$, there exists u_0 such that the blow-up set is D (See also [1] and [3]). For the Cauchy problem with $N = 1$, Herrero and Velázquez [17] showed that for any point $\bar{x}$ in the blow-up set of a solution $\bar{u}$ and $\varepsilon > 0$, there exists u_0 with $\|u_0 - \bar{u}_0\|_C \leq \varepsilon$ such that the blow-up set of u consists of a single point x with $|x - \bar{x}| \leq \varepsilon$. For the Cauchy-Dirichlet problem in an ellipsoid centred at the origin with $(N-2)p < N$, Filippas and Merle [10] showed that if the blow-up time is large, then the blow-up set consists of a single point near the origin. Also, for the Cauchy or Cauchy-Dirichlet problem with $(N-2)p \leq N+2$, Mizoguchi [27] showed the following. For any nonnegative function $\phi \in C(\bar{\Omega})$ and $\delta > 0$, if $\varepsilon > 0$ is small, then any point x in the blow-up set satisfies $\phi(x) \geq \max_y \phi(y) - \delta$ for $u_0 = \varepsilon^{-1}\phi$. For the Cauchy-Neumann problem, the first author [18] showed the following. Suppose that $\Omega = (0,\pi) \times \Omega_0$ is a cylindrical domain with a bounded smooth domain Ω_0 in $\mathbf{R}^{N-1}$ and that a nonnegative function $\phi \in L^\infty(\Omega)$ satisfies $\int_\Omega \phi(x_1, x_2, \cdots, x_N) \cos x_1 dx > 0$. If $\varepsilon > 0$ is small, then the blow-up set is contained in the base plane $\{0\} \times \bar{\Omega}_0$ for $u_0 = \varepsilon\phi$. Recently, for the Cauchy-Neumann problem with $(N-2)p \leq N+2$, Mizoguchi and the first author [20] obtained the following. Let P be the orthogonal projection in $L^2(\Omega)$ onto the eigenspace corresponding to the second eigenvalue of the Laplace operator with the Neumann condition. For any nonnegative function $\phi \in L^\infty(\Omega)$ and neighborhood W of $\{x \in \bar{\Omega} \mid (P\phi)(x) = \max_{y\in\bar{\Omega}}(P\phi)(y)\} \cup \partial\Omega$, if $\varepsilon > 0$ is small, then the blow-up set is contained in W for $u_0 = \varepsilon\phi$. See, e.g., the references in this note for related results or other studies on blow-up formation in $u_t = \triangle u + u^p$.

In this talk, we study the blow-up profile.

For large initial data $u_0^\varepsilon = \varepsilon^{-1}\phi$, we have the following.

Theorem 1 ([35]) *Let $\phi \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the blow-up set of the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon^{-1}\phi$ is contained in the set $S := \{x \in \bar{\Omega} | \, \phi(x) \geq \max_{y\in\bar{\Omega}} \phi(y) - \delta\}$ and the blow-up profile u_*^ε satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(\phi(x)^{-(p-1)} - (\max_{y\in\bar{\Omega}}\phi(y))^{-(p-1)}\right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Theorems 2 and 3 are instability results for constant blow-up solutions.

Theorem 2 ([36]) *Let $f \in C(\bar{\Omega})$ be a positive function, and let δ and T_0 be positive constants. Then, there exist C and $\varepsilon_0 > 0$ satisfying the following: For any $\varepsilon \in (0, \varepsilon_0]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ satisfying $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ and*

$$\left\| u_0^\varepsilon(x) - (p-1)^{-\frac{1}{p-1}} T_0^{-\frac{1}{p-1}} \right\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$$

such that the blow-up time of the solution u^ε with initial data $u^\varepsilon(x,0) = u_0^\varepsilon(x)$ is larger than T_0 and the inequality

$$\|\varepsilon u^\varepsilon(x, T_0) - f(x)\|_{C(\bar{\Omega})} \leq \delta$$

holds.

Theorem 3 ([36]) *Let $f \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$, and let δ and c be positive constants. Then, there exist C and $\varepsilon_0 > 0$ satisfying the following: For any $\varepsilon \in (0, \varepsilon_0]$, there exists $u_0^\varepsilon \in C^2(\bar{\Omega})$ with $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ and $\|u_0^\varepsilon - c\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$ such that the blow-up set of the solution u^ε with the initial data u_0^ε is contained in the set $S := \{x \in \bar{\Omega} \,|\, f(x) \geq \max_{y\in\bar{\Omega}} f(y) - \delta\}$ and the blow-up profile u_*^ε satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left(f(x)^{-(p-1)} - (\max_{y\in\bar{\Omega}} f(y))^{-(p-1)}\right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Let λ_i be the i-th eigenvalue of $-\triangle\varphi = \lambda\varphi$ with the Neumann boundary condition $\frac{\partial\varphi}{\partial\nu} = 0$, where $0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. We denote the orthogonal projection in $L^2(\Omega)$ onto the eigenspace X_i corresponding to the i-th eigenvalue by P_i. Here, we remark that $P_1\phi = \frac{1}{|\Omega|}\int_\Omega \phi\, dx$ is a constant.

For small initial data $u_0^\varepsilon = \varepsilon\phi$, Mizoguchi and the first author already showed Propositions 4 and 5 below.

Proposition 4 ([20]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$. Then, there exist a constant $\varepsilon_0 > 0$ and a family $\{(t^\varepsilon, \delta^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0]} \subset \mathbf{R}^2$ such that the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon\phi$ and its blow-up time T^ε satisfy $\lim_{\varepsilon\to+0} t^\varepsilon = 1$, $\lim_{\varepsilon\to+0} \varepsilon^{p-1}T^\varepsilon = (p-1)^{-1}(P_1\phi)^{-(p-1)}$, $\lim_{\varepsilon\to+0} \varepsilon^{p-1}e^{\lambda_2 T^\varepsilon}\delta^\varepsilon = (p-1)^{-1}(P_1\phi)^{-p}$ and*

$$\lim_{\varepsilon\to+0} \left\| \frac{t^\varepsilon}{\delta^\varepsilon}\left(1 - (p-1)^{\frac{1}{p-1}} t^{\varepsilon\frac{1}{p-1}} u^\varepsilon(x, T^\varepsilon - 1)\right) \right.$$

$$\left. -e^{\lambda_2}\left((\max_{y\in\bar\Omega}(P_2\phi)(y)) - (P_2\phi)(x)\right) \right\|_{L^\infty(\Omega)} = 0.$$

Proposition 5 ([19]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$. Then, there exist C and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon\phi$ and its blow-up time T^ε satisfy $u^\varepsilon(x,t) \le C(T^\varepsilon - t)^{-\frac{1}{p-1}}$ for all $(x,t) \in \bar\Omega \times [T^\varepsilon - 1, T^\varepsilon)$.*

We obtain the following as a corollary of the propositions above.

Theorem 6 ([21]) *Let $\phi \in L^\infty(\Omega)$ be a nonnegative function with $\|\phi\|_\infty \neq 0$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the blow-up set of the solution u^ε with the initial data $u_0^\varepsilon = \varepsilon\phi$ is contained in the set $S := \{x \in \bar\Omega |\, (P_2\phi)(x) \ge \max_{y\in\bar\Omega}(P_2\phi)(y) - \delta\}$. Further, the blow-up time T^ε and the blow-up profile u_*^ε satisfy the inequality*

$$\left|\varepsilon^{p-1}T^\varepsilon - (p-1)^{-1}(P_1\phi)^{-(p-1)}\right| + \left\| \varepsilon^{-1}e^{-\frac{\lambda_2 T^\varepsilon}{p-1}} u_*^\varepsilon(x) \right.$$

$$\left. -(p-1)^{-\frac{1}{p-1}}(P_1\phi)^{\frac{p}{p-1}}\left((\max_{y\in\bar\Omega}(P_2\phi)(y)) - (P_2\phi)(x)\right)^{-\frac{1}{p-1}} \right\|_{C(\bar\Omega\setminus S)} \le \delta.$$

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Nonlinear wave equations in exterior domains

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1 Introduction

In this talk we consider the initial -boundary value problem for linear and nonlinear wave equations in an exterior domain Ω in $\mathbf{R}^N$ with the homogeneous Dirichlet boundary condition. Roughly speaking we derive local and total energy decay estimates for the linear wave equation with some localized dissipation like $a(x)u_t$ and apply these estimates to the existence problem of global decaying solutions of nonlinear wave equations.

The dissipation $a(x)u_t$ is intended to be as weak as possible , and if the obstacle $V = \mathbf{R}^N/\Omega$ is star-shaped some of our results hold even if $a(x) \equiv 0$. Throughout the works we employ only standard multiplier methods originated by Protter [47] and Morawetz [27] , and in this sense our arguments are elementary.(Cf.Chen [4], Komornik [17], Lions [19], Zuazua [58] etc.)

To specify our assumption on $a(x)$ we introduce a part of the boundary $\partial\Omega$ following Russell [50] and Lions [19]:

$$\Gamma(x_0) = \{x \in \partial\Omega | \nu(x) \cdot (x - x_0) > 0\}, \quad x_0 \in \mathbf{R}^N. \tag{1.1}$$

where $\nu(x)$ is the outward normal at $x \in \partial\Omega$. We note that V is star-shaped with respect to x_0 if and only if $\Gamma(x_0) = \emptyset$. We shall assume

Hyp.A. $a(x) > \varepsilon_0 > 0$ on a neighbourhood ω of $\Gamma(x_0)$ for some x_0.

$a(x) \equiv 0$ (no dissipation) is allowed if V is star-shaped.
First ,following [32]we consider the local energy decay to the linear problem:

$$u_{tt} - \Delta u + a(x)u_t = 0 \text{ in } \Omega \times [0, \infty), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ and } u|_{\partial\Omega} = 0. \tag{1.3}$$

We shall prove an algebraic decay of the local energy $E_R(t) = \int_{\Omega_R} |u_t(t)|^2 + |\nabla u(t)|^2) dx, \Omega_R = \Omega \cap B_R$ for the finite energy solutions $u(t)$ of (1.2)-(1.3), where B_L denotes the ball centered at the origin with the radius R. When N is odd we can further apply the method due to Morawetz [28] to conclude the exponential decay of $E_R(t)$. When $a(x) \equiv 0$ and V is not

star-shaped we can not generally expect any uniform decay rate like $E_R(t) \leq C(E(0))g(t)$ with $\lim_{t\to\infty} g(t) = 0$ (Ralston [49]) . But, in our case , due to the dissipation $a(x)u_t$ we need not assume any geometrical condition on V. Our result is a natural extension of the classical one due to Morawetz [28] to a general domain. The same result has been proved by Aloui and Khenissi [1] by a different method based on Lax-Phillips Theory. Ikawa[9],[10] proved an exponential decay of local energy under a derivative-loss for some special situation with several convex obstacles. Iwasaki[12] proved a local energy decay for a rather general hyperbolic system with a general boundary condition which yields a contraction semi-group, but, no decay rate is given there.

Next, we shall derive (total) energy decay like $E(t) \leq CI_0^2(1+t)^{-1}$ where $I_0^2 = \|u_0\|_{H^1}^2 + \|u_1\|^2$ for the same problem (1.2)-(1.3). For this, however, we must assume further

Hyp.A'. $a(x) \geq \varepsilon_0 > 0$ for $|x| >> 1$.

The result is well known when $\Omega = R^N$and $a(x) \geq \varepsilon_0 > 0$ on R^N and our result extends it to a more delicate situation.

We use the local energy decay to derive L^p estimates for the linear wave equation in exterior domains. For this we use the so-called 'cut-off' method as in Shibata and Tsutsumi[53].

We apply the estimates to the existence problem of small amplitude global solutions for the semilinear equations:

$$u_{tt} - \Delta u + a(x)u_t = f(u) \text{ in } \Omega \times [0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ and } u|_{\partial\Omega} = 0, \tag{1.5}$$

where $f(u)$ is a nonlinear resource term like $f(u) = |u|^\alpha u, \alpha > 0$. We consider this problem under two types of assumption on $a(x)$ (1) Hyp.A and Hyp.A' and (2) Hyp.A. For the first case we require only the regularity on the initial data as $(u_0, u_1) \in H_0^1 \times L^2$ or $(u_0, u_1) \in H^2 \cap H_0^1 \times H_0^1$, while for the second case we need much more regularity as $(u_0, u_1) \in H^{2M} \times H^{2M-1}, M = [N/2] + 1$. Note that the latter is applied to $N \geq 3$. Our restrictions on the exponent α do not seem to be optimal compared with the nondissipative case or the dissipative case as u_t in the whole space (Pecher [46], Georgiev [6], Todorova-Yordanov [56], Nishihara [43],Ikehata[11] Narazaki [42] etc.) and it is desirable to refine our results by making additional assumptions on the initial data. See Smith and Sogge[55] for the critical exponent of $f(u) = -|u|^\alpha u$ outside a convex obstacle in R^3.

We also consider the quasilinear wave equation:

$$u_{tt} - \text{div}\{\sigma(|\nabla u|^2)\nabla u\} + a(x)u_t = 0 \text{ in } \Omega \times [0,\infty) \tag{1.6}$$

$$u(x,0) = u_0(x) \quad and \quad u_t(x,0) = u_1(x) \text{ and } u|_{\partial\Omega} = 0 \tag{1.7},$$

where $\sigma(v^2)$ is a function like $\sigma(v^2) = 1/\sqrt{1+v^2}$. Under two types of assumptions, respectively, we prove the existence of smooth global solutions for the small initial data. The essential idea of the proof is the same as semilinear cases. But, More careful analysis will be required. When $a(x) \equiv 1$ Matsumura [21] proved the global existence of smooth

solutions for (1.6)-(1.7) with $\Omega = \mathbf{R}^N$ and this result was generalized by Shibata [52] to the exterior problems with $N \geq 3$. Our first result establishes a global existence under a weaker assumption on $a(x)$ which admits $a(x)$ to vanish in large area. For some very delicate dissipations in bounded domains see Shibata and Zheng[54] and Nakao[29, 34].

When $a(x) \equiv 0$ and $N = 1, 2$ we can not generally expect the global existence of smooth solutions of (1.6)-(1.7) even if the initial-data are small and smooth. Indeed, when $\Omega = \mathbf{R}^N$ nonexistence was proved by Lax [18] and John [13] for the case $N = 1$ and Hoshiga [8] for the case $N = 2$. For the case $N \geq 3$ Kleinermann and Ponce [16], Shatah [51] proved global existence of small amplitude solutions when $\Omega = R^N$ and Shibata and Tsutsumi [53] proved similar results for exterior problems under the assumption that the obstacle $V \equiv R^N/\Omega$ is non-trapping, especially ,convex. Recently, Keel,Smith and Sogge [14],[?] have developed the theory in this direction. However, if Ω is a general domain no result on global existence has been known.

Finally , following [39],we consider the initial boundary value problem (1.4)-(1.5) with the linear dissipation $a(x)u_t$ replaced by nonlinear one $\rho(x, u_t)$. It is an interesting problem to discuss the energy decay property for the nonlinear dissipation like $\rho(x, u_t) = a(x)|u_t|^r u_t$. Indeed, such a problem has been fully studied for the case of bounded domains (see Nakao [31],Tcheugoue' Te'bou [?], Martinez [20] and the references cited there). Further, corresponding problem for the Klein-Gordon equation was studied by Nakao [29, 33], Mochizuki and Motai [25]. But, for the wave equation under consideration there seems to be little results. Mochizuki and Motai [25] treated the case $\rho(x, u_t) = |u_t|^r u_t$ and derived a logarithmic decay rate. Ono [44] treated the case $\rho = u_t + |u_t|^r u_t$ and derived an algeblaic decay of energy $E(t)$. In [44] the linear term u_t plays an essential role. Some related topics are discussed also in Matsuyama [22]. The difficulty for the whole or exterior domains comes from the facts that (1) Poincare's inequality fails and (2) we have very few means to control $L^2(\Omega)$ norm well. Here we consider the case like that $\rho(x, u_t) = a(x)|u_t|^r u_t$ on some bounded domain Ω_R and $\rho(x, u_t) = a(x)u_t$, linear, for large $|x|$ with $a(x)$ satisfying Hyp.A and Hyp.A'. We also present a result on the global existence for the semilinear equations (1.4)-(1.5) with a source term $f(u)$. For detailed proofs see [39].

For Kirchhoff type quasilinear wave equations in exterior domains see Racke [48], Mochizuki [23], Yamazaki [57], Bae and Nakao [2] and the references cited in these papers.

Basic identities

Let $f \in L^2_{loc}([0, \infty); L^2(\Omega))$ and let $u(t) \in C([0, \infty); H^1_0(\Omega)) \cap C^1([0, \infty); L^2(\Omega)$ be a solution of the problem

$$u_{tt} - \Delta u + a(x)u_t = f \text{ in } \Omega \times [0, \infty), \tag{2.1}$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ and } u|_{\partial\Omega} = 0, \tag{2.2}$$

Let $\eta(x) \in W^{1,\infty}(\Omega))$ and $\mathbf{h}(x) = (h_1(x), \cdots, h_N(x)) \in W^{1,\infty}(\Omega)$. Then multiplying the equation by $u_t, \eta(x)u$ and $\mathbf{h}(x) \cdot \nabla u$ and integrating by parts we obtain the following identities:

$$\frac{d}{dt}E(t) + \int_\Omega a(x)|u_t|^2 dx = \int_\Omega f u_t dx, \tag{A}$$

$$\frac{d}{dt}\int_\Omega \eta(x)u_t u dx - \int_\Omega \eta(x)|u_t|^2 dx + \int_\Omega \nabla u\cdot\nabla(\eta u)dx$$
$$+\int_\Omega \eta(x)a(x)u_t u dx = \int_\Omega f\eta(x)u dx \qquad (B)$$

and

$$\frac{d}{dt}\left\{\int_\Omega u_t h(x)\cdot\nabla u dx\right\} + \frac{1}{2}\int_\Omega \nabla\cdot\mathbf{h}(x)(|u_t|^2 - |\nabla u|^2)dx + \int_\Omega \sum_{i,j=1}^N \frac{\partial h_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx$$
$$-\frac{1}{2}\int_{\partial\Omega}|\frac{\partial u}{\partial \nu}|^2\nu\cdot\mathbf{h}(x)dS + \int_\Omega a(x)u_t\mathbf{h}(x)\cdot\nabla u dx = \int_\Omega f\mathbf{h}(x)\cdot\nabla u dx. \qquad (C)$$

We also use some variations of the above identities. These are the main tools.

2 Local energy decay

Consider (1.2)-(1.3). Concerning the initial data we assume that (u_0, u_1) belongs to $H_0^1(\Omega)\times L^2(\Omega)$ and has a compact support, that is,

$$\text{supp } u_0 \cup \text{ supp } u_1 \subset B_L = \{x\in R^N|\quad |x|\le L\}$$

for some $L>0$.

We assume further that supp $a(\cdot)\subset B_{\tilde{L}}$ for some $\tilde{L}>0$. We may assume $\tilde{L}\le L$.

Our main result reads as follows.

Theorem 2.1 *Under Hyp.A the solutions* $u(t)\in C([0,\infty);H_0^1(\Omega))\cap C^1([0,\infty);L^2(\Omega))$ *of the problem (3.1.1)-(3.1.2) satisfy the estimate*

$$E_{loc}^\varepsilon(t)\le C_{\varepsilon,\delta}E(0)(1+t)^{-1+\delta} \qquad (2.3)$$

with any $0<\varepsilon,\delta<1$, *where we set*

$$E_{loc}^\varepsilon(t)\equiv\frac{1}{2}\int_{\Omega\cap B_{L+\varepsilon t}}(|u_t|^2+|\nabla u|^2)dx.$$

The constant $C_{\varepsilon,\delta}$ *depends on* ε,δ *and* L.

Since $a(x)$ has compact support we can apply the argument in Morawetz[28] to get the exponential decay for the case of odd dimensions.

Corollary 2.1 *Let* $N\ge 3$ *be odd. Then, under the conditions of Theorem 2.1, we have further*

$$E_{loc}^\varepsilon(t)\le C_{\varepsilon,\delta}E(0)e^{-\lambda t}$$

for some $\lambda=\lambda(\varepsilon,\delta)>0$.

Remark. When $a(x) \equiv 1$ Dan and Shibata[5] proved by a spectral method that

$$E_{loc}(t) CE(0)(1+t)^{-N}.$$

Remark . When $a(x) \equiv 0$ and V consists of several convex bodies in some location Ikawa[9][10] proved for the case $N = 3$,

$$E_{loc}(t) \leq C(\|u_0\|_{H_2} + \|u_1\|_{H_1})e^{-\lambda t}$$

with some $\lambda > 0$.

For aproof of Theorem 2.1 see [32]

3 Quasilinear wave equations

We consider (1.6)-(1.7). As in the previous section ,under two types of assumptions Hyp.A and Hyp.$\tilde{\text{A}}$ on the dissipation $a(x)u_t$, we give theorems of global existence of the problem (1.6)-(1.7).

Concerning $\sigma(\cdot)$ we make the following assumptions.

Hyp.D. $\sigma(\cdot)$ is a differentiable function on $R^+ = [0, \infty]$ and satisfies the conditions:

$$\sigma(v^2) \geq k_0 > 0 \text{ and } \sigma(v^2) - 2|\sigma'(v^2)|v^2 \geq k_0 > 0, \text{ if } |v| \leq L$$

where $L > 0$ is an arbitrarily fixed constant and $k_0 \equiv k_0(L)$ is a positive constant. (We may assume $\sigma(0) = 1$ for simplicity.)

Our first result in this section reads as follows.

Theorem 3.1 *Let N be any integer ≥ 1 and assume that $\sigma(\cdot) \in C^{m+1}(R^+)$ and $a(\cdot) \in C^{m+1}(\bar{\Omega})$ with an integer $m > [N/2] + 1$. Then, under Hyp.$\tilde{A}$ and Hyp.D, there exists $\delta > 0$ such that if $(u_0, u_1) \in H^{m+1} \times H^m$ satisfy the compatibility condition of the $m-th$ order and smallness condition $I_m \equiv \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} < \delta$, the problem (1.6)-(1.7) admits a unique solution $u(t)$ in the class X_m. Further, the following estimates hold:*

$$\|D_t^{k+1}u(t)\|^2_{H^{m-k}} + \|D_t^k \nabla u(t)\|^2_{H^{m-k}} \leq CI_m^2(1+t)^{-k-1} \quad \text{for } 0 \leq k \leq m$$

and

$$\|\nabla u(t)\|^2_{H^{m-k}} \leq CI_m^2(1+t)^{-1} \quad \text{for } 0 \leq k \leq m.$$

Remark. When Ω is a bounded domain we can consider a more delicate situation where $a(x)$ is localized and further degenerate on any $(N-1)$ submanifolds in $\bar{\Omega}$. See [34].

The result by the second approach are stated separately in the cases $N \geq 4$ and $N = 3$.

Theorem 3.2 *Let $N \geq 4$. When N is even we assume that V is convex. Assume that σ and $a(\cdot)$ are of C^{3M} class. We assume that (u_0, u_1) belongs to $H^{3M+1} \cap W^{2M+1,1} \times H^{3M} \cap W^{2M,1}$ and satisfies the compatibility conditions of the $3M$ th order associated with the quasilinear problem (1.6)-(1.7) and also the linear problem with $\sigma \equiv 1$. Further, we assume that $a(\cdot)$ satisfies Hyp.A and suppa$(\cdot)$ is compact . Then, under Hyp.D, there exists $\delta > 0$ such that if*

$$I_{3M} \equiv \|u_0\|_{H^{3M+1}} + \|u_0\|_{W^{2M+1,1}} + \|u_1\|_{H^{3M}} + \|u_1\|_{W^{2M,1}} \leq \delta,$$

there exists a unique solution $u(t)$ in the class

$$Y_{3M} \equiv$$

$$\bigcap_{k=0}^{3M} C^k([0,\infty); H^{3M+1-k} \cap H_0^1) \bigcap C^{3M+1}([0,\infty); L^2) \bigcap W^{k,\infty}([0,\infty); W^{M+1-k,\infty}(\Omega))$$

, satisfying

$$\sum_{k=0}^{3M} \|D_t^k \nabla u(t)\|_{H^{3M-k}} \leq C I_{3M} < \infty$$

and

$$\sum_{k=0}^{M} \|D_t^k \nabla u(t)\|_{W^{M-k,\infty}} \leq C I_{3M} (1+t)^{-d}$$

with $d = (N-1)/2$.

More interesting is the case $N = 3$, where the situation is also more delicate .

Theorem 3.3 *Let $N = 3$. Assume that σ and $a(\cdot)$ are of C^{4M+2}class. We assume that (u_0, u_1) belongs to $H^{4M+3} \cap W^{4M+2,q} \times H^{4M+2} \cap W^{4M+1,q}$ and satisfies the compatibility conditions of the $4M+2-th$ order associated with the quasilinear problem (1.6-(1.7) and also the linear problem with $\sigma \equiv 1$. We assume that $a(\cdot)$ satisfies Hyp.A and supp $a(\cdot)$ is compact . Then, under Hyp.D, there exists δ such that if*

$$\tilde{I}_{4M+2} \equiv \|u_0\|_{H^{4M+3}} + \|u_0\|_{W^{4M+2,q}} + \|u_1\|_{H^{4M+2}} + \|u_1\|_{W^{4M+1,q}} \leq \delta,$$

there exists a unique solution $u(t)$ in the class

$$Y_{4M+3} \equiv \bigcap_{k=0}^{4M+2} C^k([0,\infty); H^{4M+3-k} \cap H_0^1) \bigcap C^{4M+2}([0,\infty); L^2)$$

, satisfying

$$\sum_{k=0}^{4M+3} \|D_t^k \nabla u(t)\|_{H^{4M+3-k}} \leq C \tilde{I}_{4M+2} < \infty$$

and

$$\sum_{k=0}^{M+1} \|D_t^k \nabla u(t)\|_{W^{M+1-k,p}} \leq C \tilde{I}_{4M+2} (1+t)^{-d(p)}$$

with $d(p) = (p-2)(1-\varepsilon)/p, 0 < \varepsilon << 1$, where we should take $6 \leq p < \infty$ and $q = p/(p-1)$

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Movement of Hot Spots of the Exterior Domain of a Ball

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1 Introduction

We consider the movement of the maximum points of the solutions of the Cauchy-Neumann problem and the Cauchy-Dirichlet of the heat equation,

$$\text{(1.1)} \qquad \begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \Omega \end{cases}$$

and

$$\text{(1.2)} \qquad \begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \Omega, \end{cases}$$

where Ω is a domain with the smooth boundary $\partial\Omega$, and $\partial_t = \partial/\partial t$, $\partial_\nu = \partial/\partial\nu$, $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$. Let u be a solution of (1.1) or (1.2). Under suitable assumptions on the initial data ϕ, the set of the maximum points of $u(\cdot, t)$,

$$H(t) = \left\{ x \in \overline{\Omega} \,:\, u(x,t) = \max_{y \in \overline{\Omega}} u(y,t) \right\}$$

is not empty for all $t > 0$. Then we call $H(t)$ the hot spots of the solution u at the time t. In this talk we study the movement of hot spots $H(t)$ of the solution u as $t \to \infty$. We next give some known results on the movement of the hot spots $H(t)$ as $t \to \infty$.

(1) Ω : bounded domains

Let Ω be a bounded domain with the smooth boundary $\partial\Omega$. By the Fourier expansions of the solutions of (1.1) and (1.2), we see that, for "almost all" initial data $\phi \in L^2(\Omega)$, the hot spots of the solutions tend to the maximum points of the first nonconstant eigenfunctions of Δ (see [12]).

For the zero Neumann boundary condition, Kawohl [10] conjectured that, for any convex domains Ω, the set of the maximum points of the first nonconstant eigenfunction is a subset of $\partial\Omega$. If this conjecture is true, then the hot spots of the solution of (1.1) in a bounded convex domain Ω tend to the boundary of the domain Ω for "almost all" initial data $\phi \in L^2(\Omega)$. It is known that this conjecture holds for parallelepipeds, balls, (see [10]), and two dimensional, thin convex polygonal domain with some symmetries (see [1] and [8]). For any non-convex domain Ω, we have an example of the domain where any first nonconstant eigenfunction does not take its maximum on the boundary of the domain (see [2]).

(2) Ω : unbounded domains

Chavel and Karp [3] studied the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds, and obtained some asymptotic properties of solutions concerning the movement of hot spots of the solution. In particular, for the Euclidean space $\mathbf{R}^N$, they proved that, for any nonzero, nonnegative initial data $\phi \in L_c^\infty(\mathbf{R}^N)$, the hot spots $H(t)$ of the solution at each time $t > 0$ are contained in the closed convex hull of the support of ϕ, and the hot spots $H(t)$ tend to the center of mass of ϕ as $t \to \infty$. Subsequently, Jimbo and Sakaguchi [9] studied the movement of hot spots of the solution of the heat equation in the half space $\mathbf{R}_+^N$ under the Dirichlet, Neumann, and Robin boundary conditions. In particular, they proved that the hot spots $H(t)$ of the solution of (1.1) in the half space $\mathbf{R}_+^N$ with the nonzero, nonnegative initial data $\varphi \in L_c^\infty(\mathbf{R}_+^N)$ satisfies

$$H(t) \subset \partial\mathbf{R}_+^N = \{x = (x', x_N) \in \mathbf{R}^N : x_N = 0\} \tag{1.3}$$

for all sufficiently large t. We may obtain their results for the cases $\Omega = \mathbf{R}^N$ and $\Omega = \mathbf{R}_+^N$ by using the fundamental solution of the heat equation.

Next we consider the simplest exterior domain of a compact set,

$$\Omega = \{x \in \mathbf{R}^N \ : \ |x| > 1\}.$$

Even for this simple exterior domain, it is difficult to know the sign of differential of the Neumann and Dirichlet heat kernels. So it seems difficult to

stydy the movement of hot spots by using the the Neumann and Dirichlet heat kernels directly. Jimbo and Sakaguchi [9] assumed the radially symmetry of the initial data ϕ, and studied the movement of the hot spots $H(t)$ of the solutions of (1.1) and (1.2). For the Cauchy-Neumann problem (1.1), they proved that the hot spots $H(t)$ satisfies

$$H(t) \subset \partial\Omega = \partial B(0,1) \tag{1.4}$$

for all sufficiently large t. Furthermore, for the Cauchy-Dirichlet problem (1.2), they proved that there exist a constant T and a function $r = r(t) \in C^\infty([T,\infty))$ such that

$$H(t) = \{x \in \mathbf{R}^N : |x| = r(t)\}, \qquad \lim_{t\to\infty} t^{-1} r(t)^3 = 2 \tag{1.5}$$

if $N = 3$. Their proofs of (1.4) and (1.5) heavily depend on the properties of zero sets of the heat equation in $\mathbf{R}$, and it seems so difficult to apply their proofs to the solutions without the radially symmetry.

In this talk we study the movement of hot spots of the solutions of (1.1) and (1.2) in the exterior domain of a ball as $t \to \infty$, without the radially symmetry of the initial data ϕ.

2 The Cauchy-Neumann Problem

In this section we consider the Cauchy-Neumann problem (1.1), and study the movement of hot spots $H(t)$ of the solution of (1.1) in the exterior domain Ω of a ball. Throughout this section we assume that

$$\Omega = \{x \in \mathbf{R}^N : |x| > L\}, \quad \phi \in L^2(\Omega, \rho dx), \quad \int_\Omega \phi(x)dx > 0, \tag{2.1}$$

where $L > 0$ and $\rho(x) = e^{|x|^2/4}$. We first give a sufficient condition for the hot spots $H(t)$ to exist only on the boundary $\partial\Omega$ for all sufficiently large t.

Theorem 2.1 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Put*

$$A_N(\phi) = \int_\Omega x\phi(x)\left(1 + \frac{L^N}{N-1}|x|^{-N}\right)dx \Big/ \int_\Omega \phi(x)dx.$$

Assume

$$A_N(\phi) \in B(0,L) = \mathbf{R}^N \setminus \overline{\Omega}. \tag{2.2}$$

Then there exists a positive constant T such that

$$H(t) \subset \partial\Omega = \{x \in \mathbf{R}^N : |x| = L\} \tag{2.3}$$

for all $t \geq T$.

In particular, we see that, under the condition (2.1), the hot spots $H(t)$ of the radial solution of (1.1) exists only on the boundary of the domain Ω for all sufficiently large t.

Remark 2.1 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Let $C(u(t))$ a center of mass of $u(t)$, that is,*

$$C(u(t)) = \int_\Omega xu(x,t)dx \Big/ \int_\Omega u(x,t)dx.$$

Then it does not necessarily hold that $C(u(t)) = C(\phi)$ for all $t > 0$. On the other hand, we put

$$A_N(u(t)) \equiv \int_\Omega xu(x,t)\left(1 + \frac{L^N}{N-1}|x|^{-N}\right)dx \Big/ \int_\Omega u(x,t)dx, \quad t > 0.$$

Then we have $A_N(u(t)) = A_N(\phi)$ for all $t > 0$, and $\lim_{t\to\infty} C(u(t)) = A(\phi)$.

Next we give a result on the limit set of $H(t)$ as $t \to \infty$.

Theorem 2.2 *Let u be a solution of the Cauchy-Neumann problem* (1.1) *under the condition* (2.1). *Assume $A_N(\phi) \neq 0$. Put*

$$x_\infty = L\frac{A_N(\phi)}{|A_N(\phi)|} \quad \text{if} \quad A_N(\phi) \in B(0,L) \quad \text{and} \quad x_\infty = A_N(\phi) \quad \text{if} \quad A_N(\phi) \in \overline{\Omega}$$

Then

$$\lim_{t\to\infty} \sup\{|x_\infty - y| : y \in H(t)\} = 0.$$

By Theorem 2.2, we see that the hot spots $H(t)$ tends to one point x_∞ as $t \to \infty$ if $A_N(\phi) \neq 0$, and see that (2.3) does not hold if $A_N(\phi) \in \Omega$ (compare with (1.3) and (1.4)).

3 The Cauchy-Dirichlet Problem

In this section we consider the Cauchy-Dirichlet problem (1.2), and study the movement of hot spots $H(t)$ of the solution of (1.2) in the exterior domain Ω of a ball. Throughout this section we assume that

$$\Omega = \{x \in \mathbf{R}^N : |x| > L\}, \quad \phi \in L^2(\Omega, e^{|x|^2/4}dx), \qquad m_\phi > 0, \tag{3.1}$$

where $L > 0$ and

$$m_\phi = \begin{cases} \displaystyle\int_\Omega \phi(x)\left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right)dx & \text{if } N \geq 3, \\ \displaystyle\int_\Omega \phi(x)\log\frac{|x|}{L}dx & \text{if } N \geq 2. \end{cases}$$

We first give the following theorems on the asymptotic behavior of the solution u of (1.2), which implies that the hot spots $H(t)$ run away from the boundary $\partial\Omega$ as $t \to \infty$.

Theorem 3.1 *Let u be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1) *and $N \geq 3$. Then*

$$\lim_{t\to\infty}\int_\Omega u(x,t)dx = m_\phi > 0 \tag{3.2}$$

and

$$\lim_{t\to\infty} t^{\frac{N}{2}}u(x,t) = (4\pi)^{-\frac{N}{2}}m_\phi\left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) \tag{3.3}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Theorem 3.2 *Let u be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1) *and $N = 2$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq C(\log t)^{-1}\|\phi\|_{L^2(\Omega,\rho dx)} \tag{3.4}$$

for all $t \geq 1$. Furthermore

$$\lim_{t\to\infty}(\log t)\int_\Omega u(x,t)dx = 2m_\phi \tag{3.5}$$

and

$$\lim_{t\to\infty} t(\log t)^2 u(x,t) = \frac{1}{\pi}m_\phi\log\frac{|x|}{L} \tag{3.6}$$

uniformly for all x on any compact set in $\overline{\Omega}$.

Remark 3.1 *Collet, Martínes, and Martín* [4] *proved the asymptotic behavior of the Dirichlet heat kernel* $G = G(x, y, t)$ *on the exterior domain of a compact set as* $t \to \infty$. *In particular, for the exterior domain* $\mathbf{R}^N \setminus \overline{B(0, L)}$, *they obtained that*

$$\lim_{t\to\infty} t^{\frac{N}{2}} G(x, y, t) = (4\pi)^{-\frac{N}{2}} \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right)\left(1 - \frac{L^{N-2}}{|y|^{N-2}}\right) \quad \text{if } N \geq 3, \tag{3.7}$$

$$\lim_{t\to\infty} t(\log t)^2 G(x, y, t) = \frac{1}{\pi} \log \frac{|x|}{L} \log \frac{|y|}{L} \qquad \text{if } N = 2, \tag{3.8}$$

for all $x, y \in \Omega$ (*see also* [6]). *By* (3.3) *and* (3.6), *we may obtain* (3.7) *and* (3.8), *and the proof of this paper is complete different from the one of* [4]. *Furthermore we remark that Herraiz* [7] *applied the comparison method to the Cauchy-Dirichlet problem* (1.2) *in general exterior domains and obtained the similar results to Theorems* 3.1 *and* 3.2 *for nonnegative initial data* ϕ.

Next we give a result on the rate for the hot spots $H(t)$ to run away from the boundary Ω as $t \to \infty$.

Theorem 3.3 *Let* u *be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1). *Put*

$$\zeta(t) = 2(N-2)L^{N-2}t \quad \text{if } N \geq 3, \quad \zeta(t) = 2t(\log t)^{-1} \quad \text{if } N = 2.$$

Then

$$\lim_{t\to\infty} \sup_{x\in H(t)} \left| \zeta(t)^{-1}|x|^N - 1 \right| = 0. \tag{3.9}$$

Furthermore there exists a positive constant T *such that, if* $x \in H(t)$ *and* $t \geq T$, *then*

$$H(t) \cap l_x = \{x\}, \tag{3.10}$$

where $l_x = \{x \in \mathbf{R}^N : kx/|x|,\ k \geq 0\}$.

Remark 3.2 *Let* u *be a radial solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1). *Then we see that there exists a smooth curve* $r = r(t) \in (L, \infty)$ *such that* $H(t) = \{x \in \mathbf{R}^M : |x| = r(t)\}$ *for all sufficiently large* t.

Next we give a sufficient condition for the hot spots $H(t)$ to consists of one point $x(t)$ after a finite time. Furthermore we give the limit of $x(t)/|x(t)|$ as $t \to \infty$.

Theorem 3.4 *Let u be a solution of the Cauchy-Dirichlet problem* (1.2) *under the condition* (3.1). *Assume that*

$$A_D(\phi) \equiv \int_\Omega x\phi(x)\left(1-\frac{L^N}{|x|^N}\right)dx \neq 0.$$

Then there exist a positive constant T and a smooth curve $x = x(t) \in C^\infty([T,\infty) : \Omega)$ such that $H(t) = \{x(t)\}$ for all $t \geq T$ and

$$\lim_{t\to\infty}\frac{x(t)}{|x(t)|} = \frac{A_D(\phi)}{|A_D(\phi)|}. \tag{3.11}$$

4 Outline of the Proofs

In this section we give the outline of the proofs of Theorems 2.1 and 2.2 only. In order to prove Theorems 2.1 and 2.2, we consider the asymptotic behavior of the radial solution v_k of the Cauhy-Neumann problem (L_k):

$$(L_k)\qquad \begin{cases} \partial_t v_k = \mathcal{L}_k v_k \equiv \Delta v_k - \dfrac{\omega_k}{|x|^2}v_k & \text{in } \Omega\times(0,\infty), \\ \partial_\nu v_k = 0 & \text{on } \partial\Omega\times(0,\infty), \\ v_k(x,0) = \phi_k(x) & \text{in } \Omega, \end{cases}$$

where $k \in \mathbf{N}\cup\{0\}$ and ϕ_k is a radial function belonging to $L^2(\Omega,\rho dx)$ with $\rho(y) = \exp(|y|^2/4)$. Here $\{\omega_k\}_{k=0}^\infty$ be the eigenvalues of

$$-\Delta_{\mathbf{S}^{N-1}}Q = \omega Q \qquad \text{on } \mathbf{S}^{N-1}, \tag{4.1}$$

such that $0 = \omega_0 < \omega_1 = N-1 < \omega_2 = 2N < \omega_3 < \cdots$, where $\Delta_{\mathbf{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbf{S}^{N-1}$. Furthermore we define a rescaled function w_k of the solution v_k as follows:

$$w_k(y,s) = (1+t)^{\frac{N+k}{2}}v_k(x,t), \quad y = (1+t)^{-\frac{1}{2}}x, \quad s = \log(1+t). \tag{4.2}$$

Then the function w_k satisfies

$$(P_k)\qquad \begin{cases} \partial_s w_k = P_k w_k + \dfrac{N+k}{2}w_k & \text{in } W, \\ \partial_\nu w_k = 0 & \text{on } \partial W, \\ w_k(y,0) = \phi(y) & \text{in } \Omega, \end{cases}$$

where

$$P_k w = \Delta_y w + \frac{y}{2} \cdot \nabla_y w - \frac{\omega_k}{|y|^2} w = \frac{1}{\rho} \mathrm{div}\,(\rho \nabla_y w) - \frac{\omega_k}{|y|^2} w,$$

$$\Omega(s) = e^{-s/2}\Omega,\ W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}),\ \partial W = \bigcup_{0<s<\infty} (\partial\Omega(s) \times \{s\}).$$

We study the asymptotic behavior of the first eigenvalue and the first eigenfunction of the operator P_k, and obtain the asymptotic behavior of the solution w_k in the space L^2 with weight ρ. Furthermore, for $k = 0, 1, 2$, by using the radially symmetry of v_k, the equations (L_k) and (P_k), and the Ascoli-Arzera theorem, we study the asymptotic behavior of v_k, $\partial_r v_k$, and $\partial_r^2 v_k$ as $t \to \infty$. Finally we study the asymptotic behavior of u, ∇u, and $\nabla^2 u$ as $t \to \infty$ by using the results on v_k, $\partial_r v_k$, and $\partial_r^2 v_k$, and prove Theorems 2.1 and 2.2.

For the case $k = 0$, we extend the domain of w_0 to $\mathbf{R}^N$, and apply the Ascoli-Arzera theorem to w_0. Then, by using the results on the asymptotic behavior of w_0 in the space L^2 with weight ρ, we obtain a result on the asymptotic behavior of v_0 and $\partial_r v_0$, where $r = |x|$. Furthermore we obtain a result on the asymptotic behavior of $\partial_r^2 v_0$ as $t \to \infty$ by using the ones of v_0 and $\partial_r v_0$. On the other hand, for the case $k = 1$, the inequality

$$\sup_{s>1} \|\nabla_y^2 w_1(\cdot, s)\|_{C(\Omega(s))} < \infty$$

does not necessarily holds, and $w(y, s)$ tends to 0 uniformly for all y with $|y| \le Re^{-s/2}$ with any $R > L$. So it is not useful to apply the Ascoli-Arzera theorem to w_1 for the aim at studying the asymptotic behavior of w_1 and $\partial_r w_1$ in the domain $\{y \in \Omega(s) : |y| \le Re^{-s/2}\}$, as $s \to \infty$. To overcome this difficulty, we may apply the Ascoli-Arzera theorem w_1 in the any annulus $D(\epsilon, R) = \{y \in \mathbf{R}^N : \epsilon \le |y| \le R\}$ with $0 < \epsilon < R$, and obtain the asymptotic behavior of w_1 in the annulus $D(\epsilon, R)$. Furthermore we use the equation (L_1) effectively, and study the asymptotic behavior of v_1, $\partial_r v_1$ and $\partial_r^2 v_1$ as $t \to \infty$. For $k = 2$, we apply the similar arguments to in w_1 to w_2, and study the asymptotic behavior of v_2, $\partial_r v_2$ and $\partial_r^2 v_2$ as $t \to \infty$.

For the Cauchy-Dirichlet problem (1.2), we follow the strategy for the proofs of Theorems 2.1 and 2.2, and study the asymptotic behavior of the solutions of (1.2) to prove Theorems 3.1–3.4.

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Dirichlet Inhomogeneous Boundary Value Problem for the n+1 Complex Ginzburg-Landau Equation

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1 ABSTRACT

The classical Ginzburg-Landau equation (GLE) when $n = 1$

$$u_t = (a + i\alpha)\Delta u - (b + i\beta)|u|^2 u \tag{1.1}$$

was found for a general class of nonlinear evolution problems including several classical problems from hydrodynamics and other applications in chemistry and physics. It was derived from the Navier-Stocks equations via multiple scaling methods in convection. This equation and its variations with additional nonlinear terms have been extensively studied. For example, a mathematically rigorous proof of the validity of this equation was given for a general solution of one space variable and a quadratic nonlinearity [1]. For a sample of references, see [2]-[7].

The Ginzburg-Landau equation has an intimate relation to the nonlinear Schrödinger equation (NLS). By taking $a = b = 0$ in (1.1), the GLE formally becomes the NLS

$$v_t = i\alpha\Delta v - i\beta|v|^2 v. \tag{1.2}$$

Frequently it is asked if the solution u of the GLE approaches to the solution v of NLS in an appropriate space norm as a, b tends to zero. If the answer is positive, then what is the convergence rate? The inviscid limit itself is an interesting topic because of its importance in both mathematical theory and physical applications.

Although there is a very large literature on the Ginzburg- Landau equations (classical or generalized), most of them are concerned with initial value or homogeneous boundary value problems. For inhomogeneous boundary value problem of GLE, we are only aware of certain results in one space dimensions. Existence, uniqueness and well-posedness of a global solution are proved (Bu [8]) when $\alpha\beta > 0$ or $|\beta| < \sqrt{3}b$. Global strong solutions to a more generalized version of the GLE with either Dirichlet or Neumann boundary data are found in [9]-[10]. There is a complete lack of publication

regarding well-posedness and inviscid limit of the GLE with inhomogeneous boundary data in higher dimension ($n \geq 2$).

We study the following inhomogeneous boundary value problem for the $n+1$ complex Ginzburg-Landau equation:

$$u_t = (a + i\alpha)\Delta u - (b + i\beta)|u|^2 u \tag{1.3}$$

$$u(x, 0) = h(x) \text{ for } x \in \Omega \tag{1.4}$$

$$u(x, t) = Q(x, t) \text{ on } \partial\Omega \tag{1.5}$$

where Ω is an open bounded set in R^n with C^∞ boundary and h, Q are given smooth functions. The boundary condition is inhomogeneous and of Dirichlet type. Under suitable conditions, we prove the existence of a unique global solution in H^1. Further, this solution approaches to the solution of the corresponding NLS limit under identical initial and boundary conditions as $a, b \to 0^+$.

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The upper bound of the best constant of Trudinger-Moser inequality and its application to Gagliardo-Nirenberg inequality

Hidemitsu Wadade

Abstract

We have considered so-called Trudinger-type inequality which was proved in [11] , and obtained the upper bound of the best constant of the Trudinger-type inequality , and by changing normalization from homogeneous-norm to inhomogeneous norm , we have gotten the best constant of Trudinger-type inequality in parts.

1 Introduction

We can say that Trudinger's inequality is one of Sobolev's inequality , and originally , Trudinger had led this inequality to argue the regularity of the solutions of the elliptic partial differential equation. At first , we state the classical Sobolev's imbedding theorem which is the following;

Sobolev's imbedding theorem . *Let n be dimension , and $p \in (1, \infty)$. Then the following continuous imbeddings hold;*

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \ (1/q = 1/p - s/n, \ 0 < s < n/p),$$
$$H^{s,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \ (n/p < s < \infty),$$

and when $s = n/p$, it is called the limiting case ,

$$H^{n/p,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \ (p \leqq \forall q < \infty), \tag{1.1}$$

however ,

$$H^{n/p,p}(\mathbb{R}^n) \subsetneq L^\infty(\mathbb{R}^n).$$

Remark . (i) *For $p \in (1,\infty)$, we let $L^p(\mathbb{R}^n)$ denote the classical Banach space consisting of measurable functions on $\mathbb{R}^n$ that are p-integrable. The norm in $L^p(\mathbb{R}^n)$ is defined by*

$$\|u\|_p = \|u\|_{L^p(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |u(x)|^p dx\Big)^{1/p}.$$

(ii) *For $s > 0$, and $p \in (1,\infty)$, we let $H^{s,p}(\mathbb{R}^n)$ denote the fractional Sobolev space;*

$$H^{s,p}(\mathbb{R}^n) := (I-\Delta)^{-s/2} L^p(\mathbb{R}^n) \quad (I; \textit{the identity operator}),$$

and the norm in $H^{s,p}(\mathbb{R}^n)$ is defined by

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} := \begin{cases} \|(I-\Delta)^{s/2}u\|_{L^p(\mathbb{R}^n)}, \\ \textit{or} \\ \|u\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{s/2}u\|_{L^p(\mathbb{R}^n)}. \end{cases}$$

These norms are equivalent each other.

Trudinger-type inequality which was proved in [11] is the following Proposition 1.1. We can say that Proposition 1.1 is the inequality which generalizes the limiting case in Sobolev's imbedding theorem. In fact , we can lead the limiting case (1.1) by applying Proposition 1.1.

Proposition 1.1 ([11]). *Let n be dimension , and $p \in (1,\infty)$. Then there exists a positive constant $\alpha_{p,n}$ satisfying the following;*
for all $\alpha \in (0, \alpha_{p,n})$, there exists a positive constant $C_{\alpha,p,n}$, such that the following inequality holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)}u\|_p \leqq 1$,

$$\int_{\mathbb{R}^n} \Phi_p(\alpha|u(x)|^{p'})dx \leqq C_{\alpha,p,n}\|u\|_p^p,$$

where

$$\Phi_p(\xi) = \exp(\xi) - \sum_{\substack{0 \leqq j < p-1 \\ j \in \mathbb{N}\cup\{0\}}} \frac{\xi^j}{j!} \quad (\xi \in \mathbb{R}),\ p' = p/(p-1).$$

Though two positive constants $\alpha_{p,n}$ and $C_{\alpha,p,n}$ appear in Proposition 1.1 , one of our goal is to investigate the best constant of $\alpha_{p,n}$ in the left-hand side.

2 Main results

We have obtained the upper bound of the best constant of Trudinger-type inequality in Proposition 1.1 , which is the following Theorem 2.1;

Theorem 2.1. *Let n be dimension , and $p \in [2, \infty)$. Then for all $\alpha \in (A_{p,n}, \infty)$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{n/p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)} u_k\|_p \leqq 1$ $(k \in \mathbb{N})$, such that*

$$\frac{1}{\|u_k\|_p^p} \int_{\mathbb{R}^n} \Phi_p(\alpha |u_k(x)|^{p'}) dx \to \infty \ \ as \ \ k \to \infty,$$

where

$$A_{p,n} := \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^{n/p} \Gamma(n/(2p))}{\Gamma(n/(2p'))} \right]^{p'},$$

$$\omega_{n-1} := \frac{2\pi^{n/2}}{\Gamma(n/2)} \ (the \ surface \ area \ of \ the \ unit \ ball \ in \ \mathbb{R}^n),$$

and Γ is the Gamma function.

Remark . (i) *We let $\alpha_{p,n}$ be the best constant of Trudinger-type inequality in Proposition* 1.1;

$$\alpha_{p,n} := \sup\{\alpha > 0; \ There \ exists \ C > 0 \ which \ is \ independent \ of \ u,$$
$$such \ that \ for \ all \ u \in H^{n/p,p}(\mathbb{R}^n) \ with \ \|(-\Delta)^{n/(2p)} u\|_p \leqq 1,$$
$$\int_{\mathbb{R}^n} \Phi_p(\alpha |u(x)|^{p'}) dx \leqq C \|u\|_p^p .\},$$

then, Theorem 2.1 *implies that* $\alpha_{p,n} \leqq A_{p,n}$ $(p \in [2, \infty))$.

(ii) *Unfortunately, when $p \in (1, 2)$, we couldn't obtain the same result. It is the technical problem of the proof.*

Next, we state Trudinger-type inequality the normalization of which is different from the one of Proposition 1.1. For this Trudinger-type inequality, we have obtained the following result;

Theorem 2.2. (i) *Let n be dimension, $p \in (1,\infty)$, and $\epsilon > 0$. Then for all $\alpha \in (0, A_{p,n}/B_p)$, there exists a positive constant $C_{\alpha,p,n}$, such that the following inequality holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(\epsilon I - \Delta)^{n/(2p)}u\|_p \leqq 1$,*

$$\int_{\mathbb{R}^n} \Phi_p(\alpha|u(x)|^{p'})dx \leqq C_{\alpha,p,n}\epsilon^{-n/2},$$

where $A_{p,n}$ is the same value as the one in Theorem 2.1,

$$B_p := (p-1)^{p'}\left[\sup_{\substack{f\in L^p(\mathbb{R}^n)\\ \|f\|_p\leqq 1}}\left\{\int_0^\infty (f^{**}(t)-f^*(t))^p dt\right\}\right]^{1/(p-1)},$$

f^* is the rearrangement of f, and
f^{**} is the average function of f^*; $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)ds$ $(t>0)$.

(ii) *For all $\alpha \in (A_{p,n},\infty)$, there exists a sequence $\{u_k\}_{k=1}^\infty \subset H^{n/p,p}(\mathbb{R}^n)$ with $\|(\epsilon I-\Delta)^{n/(2p)}u_k\|_p \leqq 1$ $(k\in\mathbb{N})$, such that*

$$\int_{\mathbb{R}^n} \Phi_p(\alpha|u_k(x)|^{p'})dx \to \infty \text{ as } k \to \infty.$$

Remark . (i) *For the constant appeared in B_p, we can easily see that*

$$\frac{1}{(p-1)^p} \leqq \sup_{\substack{f\in L^p(\mathbb{R}^n)\\ \|f\|_p\leqq 1}}\left\{\int_0^\infty (f^{**}(t)-f^*(t))^p dt\right\} < \infty \ (p\in(1,\infty)),$$

therefore the contradiction doesn't arise in Theorem 2.2 because $1 \leqq B_p < \infty$.

(ii) *When $p \in [2,\infty)$, we have been able to obtain the exact value of B_p;*

$$\sup_{\substack{f\in L^p(\mathbb{R}^n)\\ \|f\|_p\leqq 1}}\left\{\int_0^\infty (f^{**}(t)-f^*(t))^p dt\right\} = \frac{1}{p-1} \ (p\in[2,\infty)),$$

therefore, $B_p = p-1\,(p\in[2,\infty))$. So, when $p=2$, we can say that $A_{2,n}$ is the best constant of Trudinger-type inequality in Theorem 2.2 because $B_2 = 1$.

Next, we state the application to Gagliardo-Nirenberg inequality proved in [11] which is the following;

Proposition 2.1. ([11]) *Let n be dimension, and $p \in (1, \infty)$. Then there exists a positive constant $M_{p,n}$, such that for all $u \in H^{n/p,p}(\mathbb{R}^n)$, and for all $q \in [p, \infty)$, the following inequality holds*;

$$\|u\|_q \leqq M_{p,n} q^{1/p'} \|(-\Delta)^{n/(2p)} u\|_p^{1-p/q} \|u\|_p^{p/q}. \tag{2.1}$$

Remark . (i) *It was proved that Proposition* 1.1 *and Proposition* 2.1 *were equivalent in* [11].

(ii) *We can say that Proposition* 2.1 *is the precise estimate of Sobolev's imbedding theorem in the limiting case. Actually, we can get the fact* (1.1) *from* (2.1). *Moreover, since $H^{n/p,p}(\mathbb{R}^n)$ is never imbedded to $L^\infty(\mathbb{R}^n)$, we can see that the right-hand side in* (2.1) *diverges as q tends to infty. It was also proved in* [11] *that this order $q^{1/p'}$ of divergence was optimal.*

By applying the relation proved in [11] between the positive constant $\alpha_{p,n}$ of Trudinger-type inequality in Proposition 1.1 and the positive constant $M_{p,n}$ of Gagliardo-Nirenberg inequality in Proposition 2.1, and Theorem 2.1, we can obtain the lower bound of the best constant for $M_{p,n}$ appeared in (2.1) which is the following;

Theorem 2.3. *Let n be dimension, $p \in [2, \infty)$, and*

$$\begin{aligned} M_{p,n} := \inf\{M > 0;\ & \textit{For all } u \in H^{n/p,p}(\mathbb{R}^n) \textit{ and for all } q \in [p, \infty), \\ & \textit{the following inequality holds}; \\ & \|u\|_q \leqq M q^{1/p'} \|(-\Delta)^{n/(2p)} u\|_p^{1-p/q} \|u\|_p^{p/q}.\}, \end{aligned}$$

then,

$$M_{p,n} \geqq \left(\frac{1}{p' e A_{p,n}}\right)^{1/p'}.$$

Trudinger-type inequality and Gagliardo-Nirenberg inequality are equivalent. Therefore, to investigate the best constant of the former is useful to investigate the one of the latter.

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On the singular limit of anisotropic Allen-Cahn equation approximating anisotropic mean curvature flow with driving force term

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1 Introduction

We consider an anisotropic Allen–Cahn equation and the motion of its internal transition layers. It is well-known that its internal transition layer converges to the interface moving under a corresponding anisotropic mean curvature flow. The aim is to show that the convergence is uniform with respect to any derivatives of the surface energy density.

An anisotropic Allen–Cahn equation is proposed by [MWBCS]. We consider the functional of the form

$$F_\varepsilon(v) = \int_{\mathbf{R}^n} \left[\frac{1}{2}\gamma(\nabla v)^2 + \frac{1}{\varepsilon^2}(W(v) - \varepsilon\lambda f v)\right] dx,$$

Here $\gamma \in C^2(\mathbf{R}^n \setminus \{0\})$ is positive in S^{n-1}, convex, positively homogeneous of degree one. Moreover, we assume that γ^2 is strictly convex. The function W is a double-well potential of the form $W(v) = (v^2-1)^2/2$. The quantity λ is a constant depending only on W. The quantity f is a constant. We consider a weighted L^2-gradient flow of this functional, and obtain an anisotropic Allen–Cahn equation. Its explicit form is

$$\beta(\nabla v)\partial_t v - \mathrm{div}\gamma(\nabla v)\xi(\nabla v) + \frac{1}{\varepsilon^2}(W'(v) - \varepsilon\lambda f) = 0. \tag{1.1}$$

Here $\beta \in C(\mathbf{R}^n \setminus \{0\})$ is a positive in S^{n-1} and positively homogeneous of degree zero, and $\xi = D\gamma = (\partial_{p_1}\gamma(p), \ldots, \partial_{p_n}\gamma(p))$ for $p = (p_1, \ldots, p_n)$. A formal asymptotic analysis provided by [MWBCS], [WM] and [BP1] (the case $\beta \equiv 1$) says that the

internal transition layer of (1.1) approximates the evolving interface $\{\Gamma_t\}_{t\geq 0}$ under the evolution law of the form

$$\beta(\mathbf{n})V = -\gamma(\mathbf{n})\{\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) + f\} \quad \text{on } \Gamma_t, \tag{1.2}$$

where $\mathbf{n}$ denotes the outer unit normal vector field of Γ_t, V denotes the velocity for the direction $\mathbf{n}$, and the divergence operator in this equation denotes the surface divergence on Γ_t. Physically, the function γ is called surface energy density, which induces an anisotropy of the equilibrium form of interfaces. The function ξ is called Cahn–Hoffman vector. The function β expresses an anisotropy of kinetics. The quantity f is a driving force of the evolution. The quantity γ/β is called mobility.

For the solution of (1.1) with initial data such that $v(x,0)$ is positive in inside of Γ_0 and negative in outside of Γ_0, the fact that

$$v \longrightarrow \left\{ \begin{array}{ll} +1 & \text{in inside of } \Gamma_t \\ -1 & \text{in outside of } \Gamma_t \end{array} \right\} \text{ locally uniformly as } \varepsilon \to 0 \tag{1.3}$$

is rigorously proved by [ElS1] at least if the initial interface is smooth, [ElPS] and [ElS2] for more general cases than [ElS1], these include the double obstacle problem. In [ElPS] and [ElS2], they introduce a generalized solution of (1.2) by using a level set method due to [CGG1] or [ES]. They consider a signed distance function from Γ_t and construct a sub- and supersolution of (1.1) for the estimate of convergence. In [ElS2], they use an anisotropic distance function that is induced by Finsler geometry as outlined by [BP2] (see section 3).

Here we note that their convergence results depend on the smoothness of γ. One can find in [ElS2] that how to determine ε for the estimate to obtain (1.3) at least depends on the 2nd derivatives of γ. Physically, however, there is a situation such that γ is not smooth, i.e., an equilibrium form of interface may have a flat portion called facet. If one tries to consider such a situation by (1.1) with γ_ε approximating nonsmooth γ, their results is not enough.

In this paper, therefore, we will show the convergence of internal transition layer is in some sense '*uniformly*' with respect to any derivatives of γ provided that $\gamma, 1/\gamma, \beta, 1/\beta$ on the unit sphere is bounded. No control of derivatives of γ is necessary. This gives a way to approximate crystalline motion [T], [AG] in the plane by an anisotropic Allen-Cahn type equation in conjunction with a general level set method for nondifferentiable γ in [GG1], [GG2]. Recently, [BGN] proved the convergence of (1.2) to crystalline motion (even with double obstacle form) when

$\{\gamma \leq 1\}$ is a convex polygon. We call such γ crystalline. However, they assume $\beta \equiv 1$ which is very restrictive. Moreover, our uniform convergence result itself holds for arbitrary dimensional spaces.

The difficulty treating (1.1) directly is that (1.1) does not enjoy a comparison principle. This problem is overcome in [ElS2] by adjusting a definition of solution to have a comparison principle. In this paper we consider a modified equation of (1.1) instead of (1.1) to remove singularities which are due to nonconstant kinetic factor β. The advantage of our idea over [ElS2] is that the usual theory of viscosity solutions is available for a modified equation. We prove that the solution of a modified equation satisfies (1.3) and the convergence is '*uniformly*' with respect to any derivatives of γ.

The proof of (1.3) is completed by following the method in [ElS2] and adjust some properties in [ElS2] for our problems. We construct a viscosity sub- and supersolution of (1.1) for estimate to obtain the convergence result by combining a distance function induced by Finsler geometry as in [BP2], the method as in [ESS] and the traveling wave as in [BSS]. The key estimate why we can prove the convergence result without respect to the derivative of γ is in an estimate of the time derivative of a distance function from Γ_t. The similar estimate is in [ElS2]. However they obtain the estimate by using the quantity $\|\gamma\|_{C^{2,1}(B_2(0)\setminus B_{1/2}(0))}$. In this paper we will prove that by using a duality between γ and a support function of $\{p \in \mathbf{R}^n; \gamma(p) \leq 1\}$.

2 Main Result

2.1 Equations

We now recall an anisotropic mean curvature flow. Let $\{\Gamma_t\}_{t\geq 0}$ be a family of closed hypersurfaces in $\mathbf{R}^n$. We consider an evolution law for Γ_t of the form

$$\beta(\mathbf{n})V = -\gamma(\mathbf{n})\{\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) + f\} \quad \text{on } \Gamma_t, \tag{2.1}$$

where V denotes the normal velocity of the surface Γ_t and $\mathbf{n}$ denotes the outer unit normal vector field of Γ_t. In this paper we assume that

(A1) $\beta \in C(\mathbf{R}^n \setminus \{0\})$, $\gamma \in C^2(\mathbf{R}^n \setminus \{0\})$, f is a constant,

(A2) there exist positive constants Λ_β and Λ_γ satisfying

$$\Lambda_\beta^{-1} \leq \beta \leq \Lambda_\beta,\ \Lambda_\gamma^{-1} \leq \gamma \leq \Lambda_\gamma \quad \text{on } S^{n-1},$$

(A3) β and γ is positively homogeneous of degree 0 and 1, respectively,

(A4) γ is convex, and γ^2 is strictly convex,

where S^{n-1} is a unit sphere. The vector field ξ is the gradient field of γ i.e., $\xi = D\gamma = (\partial_{p_1}\gamma, \ldots, \partial_{p_n}\gamma)$, $\partial_{p_i}\gamma = \partial\gamma/\partial p_i$, $1 \leq i \leq n$. The divergence operator in (2.1) denotes the surface divergence on Γ_t. In this paper, we only consider the driving force term f is constant in order to remove the technical difficulties.

We are interested in the motion of Γ_t started from some compact Γ_0 in finite time interval $(0,T)$. Then we may assume that there exists a big cube $\prod_{j=1}^n [a_j, b_j]$ satisfying $\Gamma_t \subset \prod_{j=1}^n [a_j, b_j]$ for $t \in [0,T)$. Therefore we consider the all equation on $\mathbf{T}^n = \prod_{j=1}^n \mathbf{R}/(b_j - a_j)\mathbf{Z}$ $(j = 1,\ 2,\ \ldots,\ n)$ with the periodic boundary condition, i.e., $u(x + (b_j - a_j)e_j, t) = u(x,t)$ $(j = 1,\ 2,\ \ldots,\ n)$ for $(x,t) \in \mathbf{R}^n \times [0,T)$.

A generalized notation of the motion of Γ_t is given by using the level set method (See [CGG1], [G2]). We introduce an auxiliary function function $u\colon \mathbf{T}^n \times [0,T) \to \mathbf{R}$ and define

$$\Gamma_t = \{x \in \mathbf{R}^n;\ u(x,t) = 0\}. \tag{2.2}$$

The level set equation obtained from (2.1) is of the form

$$\begin{cases} \beta(\nabla u)\partial_t u - \gamma(\nabla u)\{\mathrm{div}\xi(\nabla u) + f\} = 0 & \text{in } \mathbf{T}^n \times (0,T), \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbf{T}^n. \end{cases} \tag{2.3}$$

It is well-known that, for the periodic initial data, there exists a unique global periodic viscosity solution of (2.3) (See [CGG1] or [G2]). We define that Γ_t is a generalized solution of (2.1) if Γ_t is given by (2.2) for a viscosity solution u of (2.3) with initial data u_0 satisfying $\Gamma_0 = \{x;\ u_0(x) = 0\}$.

As the way to analyze the motion of Γ_t, there is the approximation of Γ_t by the internal layer of Allen–Cahn type equation induced by [MWBCS]. The explicit form is

$$\begin{cases} \beta(\nabla v)\partial_t v - \mathrm{div}\{\gamma(\nabla v)\xi(\nabla v)\} - \dfrac{1}{\varepsilon^2}(W'(v) - \varepsilon\lambda f) = 0 & \text{in } \mathbf{T}^n \times (0,T), \\ v(\cdot,0) = v_0(\cdot) & \text{on } \mathbf{T}^n. \end{cases} \tag{2.4}$$

Here W is a double-well potential of the form $W(\sigma) \coloneqq (\sigma^2 - 1)^2/2$, and λ is a constant depends on W, in our case $\lambda = 2/3$. We choose a suitable v_0 to approximate an interfaces moving by (2.1). See section 2.4 and Theorem 2.1 to know how to choose

v_0. The internal transition layers of (2.4) approximates the motion of Γ_t. It is already proved rigorously by [ElS2] et al.

Our aim in this paper is to clarify quantities which determine the speed of the convergence of internal layers.

Unfortunately, (2.4) has singularities so that we cannot apply the usual theory of viscosity solutions. To overcome this difficulty, we modify the equation. We introduce a cut-off function $\zeta \in C^\infty([0,\infty))$ satisfying

$$\zeta(\sigma) = \begin{cases} 1 & \text{if } \sigma \le 1/2, \\ 0 & \text{if } \sigma \ge 3/4, \end{cases}$$

and $\lambda' \le 0$. Let $\tilde{\beta}$ be a function defined by

$$\tilde{\beta}(p) = (1 - \zeta(|p|))\beta(p) + \Lambda_\beta \zeta(|p|). \tag{2.5}$$

We introduce a modified equation of (2.4) of the form

$$\begin{cases} \tilde{\beta}(\nabla v)\partial_t v - \operatorname{div}\{\gamma(\nabla v)\xi(\nabla v)\} + \dfrac{1}{\varepsilon^2}(W'(v) - \varepsilon\lambda f) = 0 & \text{in } \mathbf{T}^n \times (0,T), \\ v(\cdot,0) = v_0(\cdot) \quad \text{on } \mathbf{T}^n. \end{cases} \tag{2.6}$$

The same type of modification appears in [ElPS]. The main advantage of (2.6) over (2.4) is that the singularities at $\nabla v = 0$ is reduced since $\tilde{\beta}$ is positive so that we can apply the usual theory of viscosity solutions, in particular the comparison principle. We treat (2.6) as the approximation model of an anisotropic mean curvature flow instead of (2.4).

2.2 Anisotropic distance

To state our main result it is convenient to introduce a anisotropic *distance* function induced by Finsler (Minkowski) metric as in [BP2]. We introduce the support function γ° of the convex set $\{p \in \mathbf{R}^n; \gamma(p) \le 1\}$ defined by

$$\gamma^\circ(p) = \sup\{\langle p, q\rangle; \gamma(q) \le 1\}.$$

Here we remark that $\gamma^\circ \in C^2(\mathbf{R}^n \setminus \{0\})$, γ° is convex, positively homogeneous of degree 1. Moreover we observe that, for each $p \in \mathbf{R}^n \setminus \{0\}$, there exists uniquely $q \in \{p \in \mathbf{R}^n; \gamma(p) \le 1\}$ satisfying $\gamma^\circ(p) = \langle p, q\rangle$ since γ^2 is strictly convex. By using this, we define an anisotropic *distance* Ξ by

$$\Xi(x,y) = \gamma^\circ(x - y).$$

We remark that, while the definition of distance, only the symmetry does not hold for Ξ. For the subset $\Gamma \subset \mathbf{R}^n$ we define

$$\Xi(x, \Gamma) = \inf\{\Xi(x, y);\ y \in \Gamma\}.$$

We remark that the orientation of x and a set in Ξ is suitable to our proof. Our argument in hereafter also apply to the reversed oriented version of the anisotropic distance function such as $\Xi(\Gamma, x) = \inf\{\Xi(y, x);\ y \in \Gamma\}$. However, we remark that the sign of the derivatives of d is also reversed if we use reversed version of d. In this paper we only use our orientation with respect to x and a set.

2.3 Traveling wave

To construct layers around Γ_t it is convenient to introduce a traveling wave. Let $\bar{\varepsilon}$ satisfy that $\sigma \to W'(\sigma) - \varepsilon\lambda f$ has exactly three zeros $h_- < h_0 < h_+$ provided that $\varepsilon \in (0, \bar{\varepsilon})$. For $\varepsilon \in (0, \bar{\varepsilon})$ we now consider an ODE of the form

$$\begin{aligned} &Q'' + cQ' = W'(Q) - \varepsilon\lambda f \quad \text{in } \mathbf{R}, \\ &\lim_{\sigma \to \pm\infty} Q(\sigma) = h_\pm,\ Q(0) = h_0, \end{aligned} \tag{2.7}$$

where c is constant determined by $h_\pm$, h_0. We remark that Q and c depend on ε. Here and hereafter we suppress the dependence of ε. The explicit form of Q is in [BSS].

2.4 Main result

We now determine the moving interfaces by (2.1). Let O_0 be an open subset in $\mathbf{T}^n$ and $\Gamma_0 = \partial O_0$. Let d_0 be a signed anisotropic distance function from an initial interface Γ_0 defined by

$$d_0(x) = \begin{cases} \Xi(x, \Gamma_0) & \text{if } x \in O_0 \cup \Gamma_0, \\ -\Xi(x, \Gamma_0) & \text{otherwise.} \end{cases} \tag{2.8}$$

We now note that d_0 is continuous on $\mathbf{T}^n$ and spatially periodic. Let u be a periodic viscosity solution of (2.3) with initial data $u_0 = d_0$. Then we obtain a generalized solution Γ_t of (2.1) started from Γ_0 by (2.2) with a viscosity solution u of (2.3).

We assume that $\Gamma_t \neq \emptyset$ for $t \in [0, T)$. We define a signed anisotropic distance function $d\colon \mathbf{T}^n \times [0, T) \to \mathbf{R}$ by

$$d(x, t) = \begin{cases} \Xi(x, \Gamma_t) & \text{if } x \in \{y \in \mathbf{T}^n;\ u(y, t) \geq 0\}, \\ -\Xi(x, \Gamma_t) & \text{if } x \in \{y \in \mathbf{T}^n;\ u(y, t) < 0\}. \end{cases} \tag{2.9}$$

We are now in position to state our main result.

Theorem 2.1 *Let O_0 be an open set in $\mathbf{T}^n$ and $\Gamma_0 = \partial O_0$. Let $d_0(x)$ and $d(x,t)$ be the anisotropic signed distance function from Γ_0 and Γ_t defined by (2.8) and (2.9), respectively. Let v be a viscosity solution of (2.6) with initial data $v_0(x) = Q(d_0(x)/\varepsilon)$. For each $\theta > 0$, there exist positive constants $\delta = \delta(\theta)$, $\varepsilon_0 = \varepsilon_0(\theta, \Lambda_\beta, \Lambda_\gamma)$ and $C = C(\theta, \Lambda_\beta, \Lambda_\gamma)$ satisfying*

$$v(x,t) \le -1 + C_1 \exp\left(-\frac{C_2\delta}{\varepsilon}\right) + C\varepsilon \tag{2.10}$$

if $(x,t) \in \{(y,s) \in \mathbf{T}^n \times (0,T); d(y,s) \le -\theta\}$ provided that $\varepsilon \in (0, \varepsilon_0)$, where C_1 and C_2 are numerical constants.

We remark that this result is a refined version of [ElS2] since the constants C_1, C_2, C and ε_0 are independent of first and 2nd derivatives of γ. It is useful to treat the approximating problem of (2.3) and (2.6) for nonsmooth γ.

The main strategy of the proof stems from [ESS] and [ElS2]. We construct a function $\psi = \psi_{\varepsilon,\delta}$ satisfying

1. for each $\theta > 0$, there exist positive constants $\delta = \delta(\theta)$ and $C = C(\theta, \Lambda_\beta, \Lambda_\gamma)$ such that $\psi(x,t)$ satisfies (2.10) for $(x,t) \in \{(y,s) \in \mathbf{T}^n \times (0,T);\ d(y,s) < -\theta\}$,
2. for each δ there exists a positive constant ε_0 such that ψ is a supersolution of (2.6) provided that $\varepsilon \in (0, \varepsilon_0)$,
3. the inequality $\psi(x,0) \ge Q(d_0(x)/\varepsilon)$ holds for $x \in \mathbf{T}^n$.

Then, by the comparison principle, we obtain Theorem 2.1. Unfortunately the construction by [ESS] and [ElS2] is suitable only to construct a supersolution of (2.4). It is not enough to construct a supersolution of (2.6). We now give a formal calculation. Set

$$R_\varepsilon = \beta(\nabla\psi) - \operatorname{div}\{\gamma(\nabla\psi)\xi(\nabla\psi)\} + \frac{1}{\varepsilon^2}(W'(\psi) - \varepsilon\lambda f),$$
$$\tilde{R}_\varepsilon = \tilde{\beta}(\nabla\psi) - \operatorname{div}\{\gamma(\nabla\psi)\xi(\nabla\psi)\} + \frac{1}{\varepsilon^2}(W'(\psi) - \varepsilon\lambda f).$$

Then we observe that

$$\tilde{R}_\varepsilon = R_\varepsilon + (\Lambda_\beta - \beta(\nabla\psi))\zeta(|\nabla\psi|)\partial_t\psi.$$

So it suffices to obtain the suitable estimate for $\partial_t \psi$. This estimate also is a crucial one for the dependence of the derivative of γ to the estimate of the convergence.

We only give a few words on the estimate of $\partial_t \psi$. One of key observation is the new proof of the estimate

$$-\mathrm{div}\{\gamma(\nabla d)\xi(\nabla d)\} \geq -\frac{n-1}{d}$$

in the viscosity sense for any distance function $d(x, \Gamma)$ where Γ is closed set. The traditional way to calculate the left hand side is to differentiate γ twice. However, if we do so we need the modulus of the second derivative of γ. We use the duality property of γ and γ° to obtain the key estimate instead of the direct calculation.

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Hydrodynamics For Materials With Elastic Properties Transport and Induced Stresses

Chun Liu

Complex fluids such as polymeric solutions, liquid crystal solutions, pulmonary surfactant solutions, electro-rheological fluids, magneto-rheological fluids and blood suspensions exhibit many intricate rheological and hydrodynamic features that are very important to biological and industrial processes. Applications include the treatment of airway closure disease by surfactant injection; polymer additive to jets in inkjet printers, fuel injection, fire extinguishers; magneto-rheological damping of structural vibrations etc. The segregation, migration and aggregation of the particles and the stretching, coiling and entanglement of the molecules in the complex fluids that endows them with the unique rheological and hydrodynamic properties required for specific biological, physiological and industrial needs.One good example is the migration of blood cells in arteries towards the center axis (the Fahreus-Lynquist effect). This segregation leaves a low viscosity plasma marginal layer that helps reduces the overall resistance to blood flow. This complex physiological rheology has important implications in blood pressure, clotting, plaque formation and other cardiovascular diseases. An important goal of the large and multi-disciplinary field of fluid mechanics is to derive continuum partial differential equations (field equations) to describe the rheology of these various fluids and to solve these equations to explain and predict their macroscopic behavior.

The most common origin and manifestation of anomalous phenomena in complex fluids are different "elastic" effects. They can be the elasticity of deformable particles, elastic repulsion between charged liquid crystals, polarized colloids or multi-component phases, elasticity due to microstructures, or bulk elasticity endowed by polymer molecules in viscoelastic complex fluids. The physical properties are purely determined by the interplay of entropic and structural intermolecular elastic forces and interfacial interactions. These elastic effects can be represented in terms of certain internal variables, for example, the orientational order parameter in liquid crystals (related to their microstructures), the distribution density function in the dumb-bell model for polymeric materials, the magnetic field in magneto-hydrodynamic fluids, the volume fraction in mixture of different materials etc. The different rheological and hydrodynamic properties can be attributed to the special coupling between the transport of the internal variable and the induced elastic stress. In our energetic formulation, this represents a competition between the kinetic energy and the elastic energy. We look at the following system (a simplified Ericksen-Leslie system modeling the flow of nematic liquid crystals) as an example for such complex fluids:

$$u_t + (u \cdot \nabla)u + \nabla p - \nu \Delta u + \lambda \nabla \cdot (\nabla d \odot \nabla d) = 0, \quad (1)$$

$$d_t + (u \cdot \nabla)d - \gamma(\Delta d - f(d)) = 0, \quad (2)$$

with $\nabla \cdot u = 0$, where u represents the flow velocity, p the pressure, d represents the *normed* director, $f(d) = F'(d)$ where $F(d)$ is the bulk part of the elastic energy. It is the coupling between the transport of d (material derivative here) and the induced elastic stress $(\nabla d \odot \nabla d)_{ij} = \sum_{k=1}^{n} (\nabla_i d_k)(\nabla_j d_k)$ that yields the following energy law, which presents the

dissipative nature of the system:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|u|^2 + \lambda|\nabla d|^2 + 2\lambda F(d))dx = -\int_\Omega (\nu|\nabla u|^2 + \lambda\gamma|\Delta d - f(d)|^2)dx. \tag{3}$$

On the other hand, the force balance (momentum equation) can be derived by the Least Action Principle, using the total energy functional and the way the internal variable d is transported. The competition between kinetic and elastic energy also produces the specific properties of the system, such as the stability and regularity of the hydrostatic configurations. When applied to micro-particles or molecules, the elastic energy determines the microstructures formation and how they interact with the fluid. The understanding of such underlying structures is also crucial in designing the accurate numerical algorithms in order to simulate the system, especially when the solutions involve singularities.

Most complex fluid behavior results from the multi-scale properties of the fluid material at the micro-structure scales. Hence, understanding complex fluid rheology and hydrodynamics must necessarily begin at the molecular and particulate level. The Fokker-Planck, Ginzburg-Landau or Liouville type statistical equations describing the nanoscale molecular dynamics or the microscale particulate dynamics are used to obtain rheological constitutive equations through least action principles, as have been done for viscoelastic polymeric fluids and liquid crystal solutions. The systems will satisfy the energy law (Second Law of Thermodynamics). The resulting partial differential equation system will involve multiple scales. In order to obtain the effective continuum equations at the macroscopic scale, mean field theories are often invoked to obtain closure in such field theoretic approaches. When these constitutive equations are inserted back into the Cauchy equation for force balance, the desired partial differential equation results.

The Navier-Stokes equation for Newtonian fluids is the simplest of these, and fortunately, it does obey an energy law. On the other hand, the dumbbell model equation for polymeric materials loses the energy law after closure, even for the simple cases such as FENE models. Recently, more and more studies show that this classical approach is inadequate due to several deficiencies. Pertinent physics at the particulate and molecular level remains elusive for many complex fluids. Even when the physics are known, some microscale phenomena remain unexplored due to mathematical and/or numerical difficulties. For example, defects in the liquid crystal have been shown to produce bulk flow (back flow). The resulting flow can also destroy the defects and hence change the bulk rheology.

In this talk, we intend to introduce some of the mathematical tools, modelling, analysis and numerics, that are useful in studying these important and complicated materials.

Generalizations of Landau-Lifshitz equations

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It is known that several geometric flows on Adjoint G-orbits are completely integrable and equivalent to soliton equations. One of them is the Heisenberg (or Landau-Lifshitz) ferromagnetic model for $\psi : \mathbb{R}^{1+1} \to \mathbf{S}^2$,

$$\psi_t = \psi \times \psi_{xx}$$

which is transformed into the $SU(2)/U(1)$ form, say,

$$S_t = \frac{1}{2}[\, S, S_{xx}\,]$$

with $S \in \mathbf{su}(2)$. In this talk we generalize it in some directions.

Let G be a Lie group and $\mathbf{g}$ is its Lie algebra. Assume that there is an Ad-invariant inner product $< \cdot, \cdot >$ in $\mathbf{g}$. Let $A \in \mathbf{g}$ and M_A be the adjoint G-orbit at $A \in \mathbf{g}$ and H_A be the isotropic subgroup at $A \in \mathbf{g}$ Then, $M_a \simeq G/H_A$ is a homogeneous space. The minimal polynomial $\phi_A(\lambda)$ of $A \in \mathbf{g}$ is invariant for the adjoint G-orbit and yields the constraint for S.

Consider a one parameter family of $\mathbf{g}$-valued connection 1-form

$$\omega = \lambda S dx + (\, \lambda^2 S + \lambda p(S, S_x)\,)\, dt \tag{1}$$

where $\lambda \in \mathbb{C}$, $S \in M_A$ and $p(S, S_x)$ denotes a polynomial of S and S_x.

The flatness assumption of the form ω for all $\lambda \in \mathbb{C}$ yields a variety of complete integrable equations for appropriate $p(S, S_x)$ as follows:

1 . For $\phi_A(\lambda) = \lambda^2 + a\lambda + b \quad (a^2 - 4b \neq 0)$ we have

$$S_t = \frac{1}{4b - a^2}[\, S, S_{xx}\,].$$

2 . When $\phi_A(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$, if $a^2 = 3b$ and $ab - 9c \neq 0$, we have

$$S_t = \frac{1}{9c - ab}[\, S, 2aS_{xx} + 3(SS_{xx} + S_x^2 + S_{xx}S)\,].$$

If $(a^2 - 4b)(a^2 - b) + bc = 0$ and $(a^2 - 4b)(ab - 3c) + 2a^2c \neq 0$, then we have

$$\begin{aligned} S_t \;=\; & \frac{1}{(a^2 - 4b)(ab - 3c) + 2a^2c}\{\, 2a[\, S, SS_x^2 + SS_{xx}S + S_x^2S - S_xSS_x\,] \\ & +(a^2 - 4b)[\, S, SS_{xx} + S_{xx}S + S_x^2\,]\,\}. \end{aligned}$$

If $a(a^2 - 4b) + 9c = 0$ and $b(a^2 - 4b) + 3ac \neq 0$, we have

$$S_t = \frac{1}{b(a^2 - 4b) + 3ac}\{\, 4b - a^2[\, S, S_xx\,] - 3[\, S, SS_{xx} + SS_x^2 + S_x^2S - S_xSS_x\,]\,\}.$$

3 For $\phi_A(\lambda) = \lambda^4 - 1$, we have

$$S_t = \frac{1}{8}[\, S, 3S^2S_x + 4SS_xS + 3S_xS^2\,]_x.$$

Instead of (1) we consider

$$\omega = \lambda S dx + (\, \lambda^n S + \sum_{k=1}^{n-1} \lambda^k p_k(S, S_x, \cdots, \partial^{n-k}S)\,)\, dt \tag{2}$$

where p_k is a polynomial of the indicated variables. Then, the flatness assumption gives the hierarchies of higher order generalized Landau-Lifsitz equations. For example $S^2 = -I$ we have

$$\begin{aligned} S_t = \frac{1}{4}S_{xxx} - \frac{3}{8}(SS_x^2)_x \qquad & (n = 3) \\ S_t + \frac{1}{16}[\, S, S_{xxxx}\,] + \frac{1}{32}[\, S, S_x^2S_{xx} + S_xS_{xx}S_x + S_{xx}S_x^2\,] = 0 \qquad & (n = 4) \\ S_t = \frac{1}{16}S_{5x} + \frac{5}{32}(S_x^2S_{xx} + S_xS_{xx}S_x + S_{xx}S_x^2)_x \qquad & (n = 5). \end{aligned}$$

We discuss the Gauge equivalence of the generalized Landau-Lifshitz equations and the counterparts of generalized nonlinear Schrödinger eqations.

Finally in this talk we also mention the local well-posedness of the Cauchy problem for the higher dimensional gneneralized Landau-Lifshitz flows on G-Adjoint orbits.