Generalized Riesz Projections and Toeplitz Operators

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Abstract

Let $1 < p < \infty$. In this paper, for a measurable function v and a weight function w, the generalized Riesz projection P^v is defined by $P^v f = vP(v^{-1}f)$, $(f \in L^p(w))$. If P_0 is the self-adjoint projection from $L^2(w)$ onto $H^2(w)$, then $P_0 = P^\alpha$ for some outer function α satisfying $w = |\alpha|^{-2}$. In this paper, P^v on $L^p(w)$ is studied. As an application, the invertibility criterion for the generalized Toeplitz operator T^v_ϕ and the generalized singular integral operator $\phi P^v + Q^v$, $Q^v = I - P^v$ are investigated using the weighted norm inequality. The operator norm inequality for the generalized Hankel operator H^v_ϕ is also presented.

1 Introduction

Let $\mathcal{P} = \text{span}\{e^{in\theta}; n \geq 0\}$, and let $\mathcal{Q} = \text{span}\{e^{in\theta}; n < 0\}$. Then $\mathcal{P} + \mathcal{Q}$ is the set of all trigonometric polynomials. Let $dm(e^{i\theta}) = d\theta/2\pi$ be the normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let w be a positive function in $L^1 = L^1(dm)$. Let $1 \leq p < \infty$. Then $\mathcal{P} + \mathcal{Q}$ is dense in $L^p(w) = L^p(wdm)$ in norm. Let $H^p(w)$ denote the norm closure in $L^p(w)$ of $\mathcal{P}$, and let $\overline{H_0^p(w)}$ denote the norm closure in $L^p(w)$ of $\mathcal{Q}$. We will write $H^p(w) = H^p$ when $w = 1$, and then this is a usual Hardy space. The Riesz projection P from $\mathcal{P} + \mathcal{Q}$ to $\mathcal{P}$ is an operator defined by

$$(Pf)(e^{i\theta}) = \sum_{k \geq 0} \hat{f}(k) e^{ik\theta}, \quad (f \in \mathcal{P} + \mathcal{Q}),$$

where $\hat{f}(k)$ denotes the k-th Fourier coefficient of f. Hence, the Riesz projection P is a densely defined operator from $L^p(w)$ to $H^p(w)$. P may not be extended to a bounded operator. P can be extended to a bounded operator from $L^p(w)$ onto $H^p(w)$ if and only if w satisfies the condition:

$$(A_p) \qquad \sup_I \left(\frac{1}{m(I)} \int_I w dm \right) \left(\frac{1}{m(I)} \int_I w^{-1/(p-1)} dm \right)^{p-1} < \infty$$

where the supremum is over all intervals I of $\mathbf{T}$. This is the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1, p.39], [4, p.255], [11, p.209, p.450], [12, p.119]) which is a generalization of the theorem of Helson and Szegő (cf. [4, p.147], [11, p.450], [12, p.99]). Let v be a measurable function on the unit circle $\mathbf{T}$ satisfying $|v| > 0$. In this paper, the generalized Riesz projection P^v is defined by

$$(P^v f)(e^{i\theta}) = v(e^{i\theta}) P(v^{-1} f)(e^{i\theta}) = v(e^{i\theta}) \sum_{k \geq 0} (v^{-1} f)\hat{}\,(k) e^{ik\theta},$$

$(f \in v\mathcal{P} + v\mathcal{Q})$. Then $v\mathcal{P} \cap v\mathcal{Q} = \{0\}$, and P^v maps $v\mathcal{P} + v\mathcal{Q}$ onto $v\mathcal{P}$. Hence, $(P^v)^2 = P^v$. Let w be an integrable function on $\mathbf{T}$ satisfying $w > 0$. Let $1 \leq p < \infty$. If $v \in L^p(w)$, then $v\mathcal{P} + v\mathcal{Q}$ is dense in $L^p(w)$. Let $1 < p < \infty$, and let $1/p + 1/q = 1$. In Section 2, we will consider the boundedness of the generalized Riesz projection P^v. It is well known that if $p = 2$ and v is an outer function such that $|v|^2 = w$, then P^v becomes a self-adjoint projection which maps $L^2(w)$ onto $H^2(w)$ (cf. [2], [7]). In particular, $P = P^1$ is a self-adjoint projection which maps L^2 onto H^2. We will prove that if $1 < p < \infty$ and w, v satisfy some conditions, then P^v is a bounded operator on $L^p(w)$ if and only if $|v|^p w \in (A_p)$.

In Section 3, we will consider the adjoint operator for P^v. We will give the form of $(P^v)^*$, and prove that if $1 < p < \infty$ and w, v satisfy some conditions, then $(P^v)^* = P^v$ on $L^p(w) \cap L^q(w)$ if and only if $|v|^2 w$ is a constant function.

In Section 4, we will consider the invertibility of the Toeplitz operator T^v_ϕ and singular integral operator $\phi P^v + Q^v$, where $Q^v = I - P^v$. Let $1 < p < \infty$, and let $\phi \in L^\infty$. If $P^v \in B(L^p(w))$, then the operator T^v_ϕ from $\text{ran}P^v$ to $\text{ran}P^v$ is defined by

$$T^v_\phi f = P^v(\phi f), \quad (f \in \text{ran}P^v).$$

If $w \in (A_p)$, then Rochberg [13] established an invertibility criterion for the Toeplitz operator T_ϕ on $H^p(w)$ (cf. [1, p.216]). If $p = 2$ and $w = 1$, then this reduces to a theorem of Widom and Devinatz (cf. [1, p.59], [11, p.316], [12, p.250]).
In Section 5, we do not assume the boundedness of P^v on $L^p(w)$. Hence, the results in Section 5 do not follow from the theorem of Rochberg and Simonenko or the theorem of Widom and Devinatz (cf. [13], [1, p.216], [12]). We will consider the invertibility of the quotient type Toeplitz operator R^v_ϕ for an outer function v. Let $1 < p < \infty$, and let $\phi \in L^\infty$. If $\log|v| \in L^1$, then an operator R^v_ϕ is defined as a bounded operator from $H^p(w)$ to $L^p(w)/\frac{v}{\bar{v}}\overline{H^p_0(w)}$ by

$$R^v_\phi f = \phi f + \frac{v}{\bar{v}}\overline{H^p_0(w)}, \quad (f \in H^p(w)).$$

If $P^v \in B(L^p(w))$, then $\ker P^v = \frac{v}{\bar{v}}\overline{H^p_0(w)}$. R^v_ϕ is always bounded. When $v = 1$, Nakazi ([8], [9]) considered the quotient type Toeplitz operator $R_\phi = R^1_\phi$ from $H^p(w)$ to $L^p(w)/\overline{H^p_0(w)}$ and proved Lemma 5.1. We use Lemma 5.1 to prove Theorem 5.2.
In Section 6, the operator norm inequality for the generalized Hankel operator H^v_ϕ is presented. Let $1 < p < \infty$, and let $\phi \in L^\infty$. If $P^v \in B(L^p(w))$, then the Hankel operator H^v_ϕ from $\mathrm{ran} P^v$ to $\mathrm{ran} Q^v$ is defined by

$$H^v_\phi f = Q^v(\phi f), \quad (f \in \mathrm{ran} P^v).$$

If $v = w = 1$, then this reduces to a theorem of Nehari (cf. [1, p.54], [11, p.181], [12, p.181]).

2 Boundedness of P^v

In this section, we discuss the condition such that the generalized Riesz projection P^v is extended to $L^p(w)$ by continuity to a bounded operator. We will not distinguish between an operator's being bounded and being densely defined and extendable by continuity to a bounded operator. We use Lemmas 1.1 and 1.2 to prove Theorems 2.3 and 2.4.

Lemma 2.1 *Let $1 \leq p < \infty$. Let w be a positive function in L^1.*
(1) *If $|v| > 0$ and $v \in L^p(w)$, then $v\mathcal{P} + v\mathcal{Q}$ is a dense subspace of $L^p(w)$.*
(2) *If $\log w \in L^1$ and $|v| = |k|$ for some outer function k in $H^p(w)$, then $k\mathcal{P}$ is dense in $H^p(w)$.*

Proof. (1): Let $f \in L^p(w)$. Then, $v^{-1}L^p(w) = L^p(|v|^p w)$. Hence, $v^{-1}f \in L^p(|v|^p w)$. Since $\mathcal{P} + \mathcal{Q}$ is dense in $L^p(|v|^p w)$, it follows that there exists a sequence $f_n \in \mathcal{P} + \mathcal{Q}$ such that

$$\lim_{n\to\infty} \int |vf_n - f|^p w dm = \lim_{n\to\infty} \int |f_n - v^{-1}f|^p |v|^p w dm = 0.$$

(2): Let $g \in H^p(w)$. Since k is an outer function such that $|k| = |v|$, it follows that $k^{-1}g \in H^p(|v|^p w)$. Since $\mathcal{P}$ is dense in $H^p(|v|^p w)$, it follows that there exists a sequence $g_n \in \mathcal{P}$ such that

$$\lim_{n\to\infty} \int |kg_n - g|^p w dm = \lim_{n\to\infty} \int |g_n - k^{-1}g|^p |v|^p w dm = 0.$$

Hence, $k\mathcal{P}$ is dense in $H^p(w)$. Lemma 2.1 is proved. □

Lemma 2.2 *Let $1 \le p < \infty$. Let w be a positive function in L^1. Suppose $|v| > 0$ and $v \in L^p(w)$. Then the following properties are equivalent.*
(1) P^v *is a bounded operator on* $L^p(w)$.
(2) $P^{|v|}$ *is a bounded operator on* $L^p(w)$.
(3) P *is a bounded operator* $L^p(|v|^p w)$.
If one of these conditions holds, then

$$\|P^v\|_{B(L^p(w))} = \|P^{|v|}\|_{B(L^p(w))} = \|P\|_{B(L^p(|v|^p w))}.$$

Proof. It is sufficient to prove the equivalence of (1) and (3). By (1), for all $f \in \mathcal{P}$ and $g \in \mathcal{Q}$,

$$\begin{aligned} \int |f|^p |v|^p w dm &= \int |vf|^p w dm \\ &\le \|P^v\|^p_{B(L^p(w))} \int |vf + vg|^p w dm \\ &= \|P^v\|^p_{B(L^p(w))} \int |f + g|^p |v|^p w dm. \end{aligned}$$

Hence, $\|P\|_{B(L^p(|v|^p w))} \le \|P^v\|_{B(L^p(w))} < \infty$. This implies (3). Conversely, by (3), for all $f \in \mathcal{P}$ and $g \in \mathcal{Q}$,

$$\begin{aligned} \int |vf|^p w dm &= \int |f|^p |v|^p w dm \\ &\le \|P\|^p_{B(L^p(|v|^p w))} \int |f + g|^p |v|^p w dm \\ &= \|P\|^p_{B(L^p(|v|^p w))} \int |vf + vg|^p w dm \end{aligned}$$

By Lemma 2.1(1), $v\mathcal{P} + v\mathcal{Q}$ is dense in $L^p(w)$. Hence, $\|P^v\|_{B(L^p(w))} \le \|P\|_{B(L^p(|v|^p w))} < \infty$, and hence (1) follows. Lemma 2.2 is proved. □

Suppose $w = |\alpha|^{-2}$ for some outer function α. Then $P_0 = P^\alpha$ is a self-adjoint projection from $L^2(w)$ onto $H^2(w)$. Let $Q_0 = I - P_0$. If a, b are constant functions, then $\|aP_0 + bQ_0\|_{B(L^2(w))} = \max(|a|, |b|)$ (cf. [5, Vol.I, p.79]). By the similar proof of Lemma 2.2, if $a, b \in L^\infty$, then $\|aP_0 + bQ_0\|_{B(L^2(w))} = \|aP + bQ\|_{B(L^2(|\alpha|^2 w))} = \|aP + bQ\|_{B(L^2)}$. Hence,

$$\begin{aligned} \|aP_0 + bQ_0\|_{B(L^2(w))} &= \|aP + bQ\|_{B(L^2)} \\ &= \inf_{k \in H^\infty} \left\| \frac{|a|^2 + |b|^2}{2} + \sqrt{|a\bar{b} - k|^2 + \left(\frac{|a|^2 - |b|^2}{2} \right)^2} \right\|_\infty . \end{aligned}$$

The infimum is attained (cf. [10]).

Let $1 < p < \infty$. There are many measurable functions v and w such that $v, \log v, \log w \notin L^1$, $w \in L^1$ and $P^v \in B(L^p(w))$. For example, let

$$v(e^{i\theta}) = \exp\left(\frac{1}{2\pi - \theta} \right), \quad (0 \le \theta < 2\pi),$$

and let $w = |v|^{-p}$. Since $\frac{p}{\theta - 2\pi} \le \frac{-p}{2\pi}$, it follows that $0 < w(e^{i\theta}) = \exp\left(\frac{p}{\theta - 2\pi}\right) \le \exp\left(\frac{-p}{2\pi}\right) < \infty$. Hence, $w \in L^\infty$. By Lemma 2.2 and the theorem of Gohberg, Krupnik, Hollenbeck and Verbitsky (cf. [5, Vol.II, p.102], [6]),

$$\|P^v\|_{B(L^p(w))} = \|P\|_{B(L^p)} = \frac{1}{\sin(\pi/p)} < \infty.$$

Then $\mathrm{ran} P^v \oplus \ker P^v = vH^p \oplus v\overline{H_0^p} = vL^p = L^p(w)$. If $p = 2$, then P^v is a self-adjoint projection on $L^2(w)$.

Theorem 2.3 *Let w be a positive function in L^1.*
(1) *If $|v| > 0$ and $v \in L^1(w)$, then P^v is an unbounded operator on $L^1(w)$.*
(2) *Let $1 < p < \infty$. If $|v| > 0$ and $v \in L^p(w)$, then $P^v \in B(L^p(w))$ if and only if $|v|^p w \in (A_p)$.*

Proof. (1): Suppose $P^v \in B(L^1(w))$. By Lemma 2.2, $|v| > 0$, and $P \in B(L^1(|v|w))$. By the theorem of Forelli (cf. [3]), $P \in B(L^1)$. This is a contradiction (cf. [5, Vol.I, p.78]).
(2): By Lemma 2.2, if $P^v \in B(L^p(w))$, then $P \in B(L^p(|v|^p w))$. By the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1], [4], [11], [12]), this implies $|v|^p w \in (A_p)$. The converse is also true. Theorem 2.3 is proved. □

Theorem 2.4 *Let $1 < p < \infty$. Let w be a positive function in L^1. Suppose $|v| > 0$ and $v \in L^p(w)$.*
(1) *If $P^v \in B(L^p(w))$, then*

$$\mathrm{ran} P^v = \ker Q^v = vH^p(|v|^p w) = [v\mathcal{P}]_{L^p(w)},$$

$$\ker P^v = \mathrm{ran} Q^v = v\overline{H^p_0(|v|^p w)} = [v\mathcal{Q}]_{L^p(w)},$$

where $[\ \cdot\ ]_{L^p(w)}$ *denotes the norm closure in* $L^p(w)$.
(2) *Suppose* $\log w$ *and* $\log|v|$ *are in* L^1. *Let* k *be an outer function such that* $|k| = |v|$. *Let* $Q^v = I - P^v$. *If* $P^v \in B(L^p(w))$, *then*

$$\mathrm{ran} P^v = \ker Q^v = \frac{v}{k} H^p(w) \subset L^p(w),$$

$$\ker P^v = \mathrm{ran} Q^v = \frac{v}{\bar{k}} \overline{H^p_0(w)} \subset L^p(w),$$

and

$$L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}} \overline{H^p_0(w)}.$$

(3) *If there is an outer function* k *such that* $|k| = |v|$ *and* $L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}} \overline{H^p_0(w)}$, *then* $P^v \in B(L^p(w))$.

Proof. (1): Suppose $f \in \mathrm{ran} P^v$. Then there is a $g \in L^p(w)$ such that $f = P^v g$. By Lemma 2.1(1), there is a sequence $\{t_n\}$ in $\mathcal{P} + \mathcal{Q}$ such that

$$\int |vt_n - g|^p w dm \to 0,$$

as $n \to \infty$. Since $P^v \in B(L^p(w))$, it follows that

$$\begin{aligned} \int |Pt_n - v^{-1} f|^p |v|^p w dm &= \int |vPt_n - f|^p w dm \\ &= \int |P^v(vt_n - g)|^p w dm \\ &\leq \|P^v\|^p_{B(L^p(w))} \int |vt_n - g|^p w dm. \end{aligned}$$

Hence,

$$\int |Pt_n - v^{-1} f|^p |v|^p w dm \to 0,$$

as $n \to \infty$. This implies that $v^{-1} f \in H^p(|v|^p w)$. Hence, $\mathrm{ran} P^v \subset vH^p(|v|^p w)$. Suppose $f \in vH^p(|v|^p w)$. Since $v^{-1} f \in H^p(|v|^p w)$, there is a sequence $\{g_n\}$ in $\mathcal{P}$ such that

$$\int |vg_n - f|^p w dm = \int |g_n - v^{-1} f|^p |v|^p w dm \to 0,$$

as $n \to \infty$. This implies that $f \in [v\mathcal{P}]_{L^p(w)}$. Hence, $vH^p(|v|^p w) \subset [v\mathcal{P}]_{L^p(w)}$. Therefore,

$$v\mathcal{P} \subset \mathrm{ran} P^v \subset vH^p(|v|^p w) \subset [v\mathcal{P}]_{L^p(w)}.$$

Since $(P^v)^2 = P^v$, $\mathrm{ran} P^v$ is a closed subspace of $L^p(w)$. Similarly

$$v\mathcal{Q} \subset \mathrm{ran} P^v \subset v\overline{H^p_0(|v|^p w)} \subset [v\mathcal{Q}]_{L^p(w)},$$

and ranQ^v is a closed subspace of $L^p(w)$. Hence (1) follows.
(2): By Theorem 2.3, if $P^v \in B(L^p(w))$, then $|v|^p w \in (A_p)$. Since k is an outer function such that $|k| = |v|$, it follows that

$$\begin{aligned} \mathrm{ran}P^v &= P^v L^p(w) = vP(v^{-1}h^{-1}L^p) = vPL^p(|v|^p w) \\ &= vH^p(|v|^p w) = \frac{v}{k}kH^p(|k|^p w) = \frac{v}{k}H^p(w), \end{aligned}$$

and

$$\begin{aligned} \mathrm{ran}Q^v &= Q^v L^p(w) = vQ(v^{-1}h^{-1}L^p) = vQL^p(|v|^p w) \\ &= v\overline{H_0^p(|v|^p w)} = \frac{v}{\bar{k}}\overline{kH_0^p(|k|^p w)} = \frac{v}{\bar{k}}\overline{H_0^p(w)}. \end{aligned}$$

Hence,

$$L^p(w) = \mathrm{ran}P^v + \mathrm{ran}Q^v = \frac{k}{v}H^p(w) \oplus \frac{\bar{k}}{v}\overline{H_0^p(w)}.$$

Since $|k| = |v|$, it follows that

$$L^p(w) = \frac{k}{v}L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}}\overline{H_0^p(w)}.$$

(3): Since $|v|^p w \in L^1$ and $L^p(w) = H^p(w) \oplus \frac{k}{\bar{k}}\overline{H_0^p(w)}$, it follows that

$$\begin{aligned} L^p(|v|^p w) = k^{-1}L^p(w) &= k^{-1}H^p(w) \oplus \overline{k^{-1}H_0^p(w)} \\ &= H^p(|v|^p w) \oplus \overline{H_0^p(|v|^p w)}. \end{aligned}$$

By the closed graph theorem, this implies that $P \in B(L^p(|v|^p w))$. Theorem 2.4 is proved. $\square$

Let $1 \le p < \infty$. If $f \in L^p(w)$ and $w \in L^1$, then $fw \in L^1$. Let

$$K^p(w) = \{f \in L^p(w) \ ; \ (fw)\hat{}\,(n) = 0, \ (n < 0)\},$$

and let

$$K_0^p(w) = \{f \in L^p(w) \ ; \ (fw)\hat{}\,(n) = 0, \ (n \le 0)\}.$$

Hence, $K^p(w)$ and $K_0^p(w)$ are closed subspaces of $L^p(w)$ satisfying $K^p(w) = L^p(w) \cap w^{-1}H^1$. The shift operator maps $K^p(w)$ onto $K_0^p(w)$. If $p = 2$, then we have the orthogonal decomposition:

$$L^2(w) = H^2(w) \oplus \overline{K_0^2(w)}.$$

If $w = 1$, then $K^p(w) = H^p$. According to the Riesz representation theorem, for every bounded linear functional $\phi \in H^p(w)^*$, $1 < p < \infty$, there exists a unique function $g \in K^q(w)$, $1/p + 1/q = 1$, such that

$$\phi(f) = \int f\bar{g}w\,dm, \ (f \in H^p(w)).$$

We use Lemmas 2.5 and 2.6 to prove Theorem 2.7.

Lemma 2.5 *Let $1 \le p < \infty$, and let $1/p + 1/q = 1$. Let h be an outer function satisfying $w = |h|^p$.*

(1)
$$K^p(w) = \frac{h^{p-1}}{w} H^p = \frac{h^p}{w} H^p(w).$$

(2)
$$K^q(w) = \frac{h}{w} H^q = \frac{h^p}{w} H^q(w).$$

(3) *$K^p(w) = H^p(w)$ if and only if w is a constant function.*

Proof. (1): Suppose $f \in K^p(w)$. Then $f \in L^p(w) \cap w^{-1}H^1$. Then $fh \in L^p$ and $(fw)/(h^{p-1}) \in H^p$. Then $f \in \frac{h^{p-1}}{w} H^p$. The converse is also true. Hence, $K^p(w) = \frac{h^{p-1}}{w} H^p$. Since $H^p(w) = H^p(|h|^p) = h^{-1}H^p$, it follows that $K^p(w) = \frac{h^p}{w} H^p(w)$.
(2): Suppose $f \in K^q(w)$. Then $f \in L^q(w) \cap w^{-1}H^1$. Then $fh^{p-1} \in L^q$ and $fw/h \in H^q$. Then $f \in \frac{h}{w} H^q$. The converse is also true. Hence, $K^q(w) = \frac{h}{w} H^q$. Since $H^q(w) = H^q(|h^p|) = h^{1-p}H^q$, it follows that $K^q(w) = \frac{h^p}{w} H^q(w)$.
(3): Suppose $K^p(w) = H^p(w)$. Since $1 \in H^p(w)$, $1 \in K^p(w)$. By (1), $1 \in \frac{h^{p-1}}{w} H^p$. Hence, there is an $f \in H^p$ such that $\frac{fh^{p-1}}{w} = 1$. Since $h^{p-1} \in H^q$, fh^{p-1} is a positive function in H^1. Hence, f and h are constant functions. Hence, w is a constant function. The converse is clear. Lemma 2.5 is proved. □

Lemma 2.6 *Let $w, \log w, w^{(2-p)/2} \in L^1$.*
(1) *$H^p(w) \oplus \overline{K_0^p(w)} = L^p(w)$ if and only if $H^p(w^{(2-p)/2}) \oplus \overline{H_0^p(w^{(2-p)/2})} = L^p(w^{(2-p)/2})$.*
(2) *There is a constant C such that*

$$\int |f|^p w dm \le C \int |f+g|^p w dm, \ \ (f \in H^p(w), \ g \in \overline{K_0^p(w)})$$

if and only if there is a constant C such that

$$\int |f|^p w^{(2-p)/2} dm \le C \int |f+g|^p w^{(2-p)/2} dm, \ \ (f \in \mathcal{P}, \ g \in \mathcal{Q}).$$

Proof. By the closed graph theorem, it is sufficient to prove (1). Since $\log w \in L^1$, there is an outer function h satisfying $w = |h|^p$. Let $p = 2a$. Then $w = h^a \overline{h^a}$ and $w^{(2-p)/2} = w^{1-a}$. By Lemma 2.5,

$$\begin{aligned} h^a \left(H^p(w) \oplus \overline{K_0^p(w)} \right) &= h^{a-1} H^p \oplus \overline{h^{a-1} H_0^p} \\ &= H^p(|h^{1-a}|^p) \oplus \overline{H_0^p(|h^{1-a}|^p)} \\ &= H^p(w^{1-a}) \oplus \overline{H_0^p(w^{1-a})} \\ &= H^p(w^{(2-p)/2}) \oplus \overline{H_0^p(w^{(2-p)/2})}. \end{aligned}$$

Since $L^p(w^{(2-p)/2}) = L^p(w^{1-a}) = L^p(|h^{1-a}|^p) = h^{a-1}L^p = h^a L^p(w)$, this implies (1). Lemma 2.6 is proved. □

Theorem 2.7 *Let $w \in L^1$. Suppose $w = |\alpha|^{-2}$, for some outer function α.*
(1) $P^\alpha \in B(L^p(w))$ *if and only if* $w^{(2-p)/2} \in (A_p)$. *Then* $\|P^\alpha\|_{B(L^p(w))} = \|P\|_{B(L^p(w^{(2-p)/2}))}$.
(2) $\text{ran}P^\alpha = H^p(w), \quad \ker P^\alpha = \overline{K_0^p(w)}, \quad \text{ran}P^{\bar\alpha} = K^p(w), \quad \ker P^{\bar\alpha} = \overline{H_0^p(w)}$.
(3) *If* $w^{(2-p)/2} \in (A_p)$, *then* $L^p(w) = H^p(w) \oplus \overline{K_0^p(w)}$, *and* P^α *is a bounded projection from* $L^p(w)$ *onto* $H^p(w)$ *such that*

$$P^\alpha(f+g) = f, \quad (f \in H^p(w),\ g \in \overline{K_0^p(w)}).$$

(4) P^α *(resp.* $I - P^\alpha$*) is a self-adjoint projection from* $L^2(w)$ *onto* $H^2(w)$ *(resp.* $\overline{K_0^2(w)}$*).*
(5) $P^{\bar\alpha}$ *(resp.* $I - P^{\bar\alpha}$*) is a self-adjoint projection from* $L^2(w)$ *onto* $K^2(w)$ *(resp.* $\overline{H_0^2(w)}$*).*

Proof. (1): By Theorem 2.3(2), if $P^\alpha \in B(L^p(w))$, then $|\alpha|^p w \in (A_p)$. Hence, $w^{(2-p)/2} = w^{-p/2}w = |\alpha|^p w \in (A_p)$. The converse is also true.
(2): By Lemma 2.5(1), $H_0^p(w) = \frac{\alpha^2}{|\alpha|^2}K_0^p(w)$. By Theorem 2.4(2), $\text{ran}P^\alpha = H^p(w)$ and

$$\ker P^\alpha = \frac{\alpha}{\bar\alpha}\overline{H_0^p(w)} = \frac{\alpha}{\bar\alpha}\frac{|\alpha|^2}{\alpha^2}\overline{K_0^p(w)} = \overline{K_0^p(w)}.$$

Similarly, $\ker P^{\bar\alpha} = \overline{H_0^p(w)}$ and

$$\text{ran}P^{\bar\alpha} = \frac{\bar\alpha}{\alpha}\overline{H^p(w)} = \frac{\bar\alpha}{\alpha}\frac{\alpha^2}{|\alpha|^2}K^p(w) = K^p(w).$$

(3): If $w^{(2-p)/2} \in (A_p)$, then $L^p(w^{(2-p)/2}) = H^p(w^{(2-p)/2}) \oplus \overline{H_0^p(w^{(2-p)/2})}$. By Lemma 2.6(1), $L^p(w) = H^p(w) \oplus \overline{K_0^p(w)}$. Since $(P^\alpha)^2 = P^\alpha$, (3) follows.
(4): Since

$$\int f\bar g w dm = 0, \quad (f \in H^2(w),\ g \in \overline{K_0^2(w)}),$$

it follows that $L^2(w) = H^2(w) \oplus \overline{K_0^2(w)}$ is the orthogonal decomposition. Since $\text{ran}P^\alpha = H^2(w)$, and $\ker P^\alpha = \overline{K_0^2(w)}$, it follows that P^α is a self-adjoint projection.
(5): Since

$$\int f\bar g w dm = 0, \quad (f \in K^2(w),\ g \in \overline{H_0^2(w)}),$$

it follows that $L^2(w) = K^2(w) \oplus \overline{H_0^2(w)}$ is the orthogonal decomposition. Since $\text{ran}P^{\bar\alpha} = K^2(w)$, and $\ker P^{\bar\alpha} = \overline{H_0^2(w)}$, it follows that $P^{\bar\alpha}$ is a self-adjoint projection from $L^2(w)$ onto $K^2(w)$. Theorem 2.7 is proved. $\square$

3 Adjoint operators for P^v

Let $1 < p < \infty$, and let $1/p + 1/q = 1$. In this section P^v is supposed to be a bounded operator on $L^p(w)$. For functions $f \in L^p(w)$ and $g \in L^q(w)$, let

$$\langle f, g \rangle_w = \int f \bar{g} w dm.$$

To each $P^v \in B(L^p(w))$ corresponds a unique $(P^v)^* \in B(L^q(w))$ that satisfies

$$\langle P^v f, g \rangle_w = \langle f, (P^v)^* g \rangle_w, \quad (f \in L^p(w),\ g \in L^q(w)).$$

We use Lemmas 3.1 and 3.2 to prove Theorem 3.3.

Lemma 3.1 *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let $w \in L^1$, $w > 0$, and let v be a measurable function.*
(1) $|v|^p w \in (A_p)$ *if and only if* $|v|^{-q} w^{1-q} \in (A_q)$.
(2) $w^{(2-p)/2} \in (A_p)$ *if and only if* $w^{(2-q)/2} \in (A_q)$.

Proof. (1): If $|v|^p w \in (A_p)$, then $(|v|^p w)^{-1/(p-1)} \in (A_q)$. Since $(p-1)(q-1) = 1$, it follows that $|v|^{-q} w^{1-q} \in (A_q)$. The converse is also true.
(2): If $w^{(2-p)/2} \in (A_p)$, then $(w^{(2-p)/2})^{-1/(p-1)} \in (A_q)$. Since $(p-1)(q-1) = 1$, it follows that

$$\left(\frac{2-p}{2}\right)\left(\frac{-1}{p-1}\right) = \frac{p}{2(p-1)} - \frac{1}{p-1} = \frac{q}{2} - (q-1) = \frac{2-q}{2}.$$

Hence, $w^{(2-q)/2} \in (A_q)$. The converse is also true. Lemma 3.1 is proved. □

Lemma 3.2 *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let $w \in L^1$, $w > 0$, and let $|v|^p w \in (A_p)$. Then $(P^v)^* \in B(L^q(w))$, $((P^v)^*)^2 = (P^v)^*$, and*

(1) $$(P^v)^*(g) = \frac{1}{\bar{v}w} P(\bar{v} w g), \quad (g \in L^q(w)).$$

(2) *If* $\log w,\ \log|v| \in L^1$, *then*

$$(P^v)^*(g_1 + g_2) = g_1, \quad \left(g_1 \in \frac{1}{k\bar{v}w} H^q(w),\ g_2 \in \frac{1}{\overline{kvw}} \overline{H_0^q(w)}\right),$$

where k is an outer function satisfying $|k| = |vw|^{-1}$.

Proof. (1): By Theorem 2.3(2), $P^v \in B(L^p(w))$. Hence, $(P^v)^* \in B(L^q(w))$. If $|v|^p w \in (A_p)$, then there is a constant $\delta > 0$ satisfying $(|v|^p w)^{1+\delta} \in L^1$ (cf. [4, p.262]). Since $1/p + 1/q = 1$, there is a constant $r > 1$ satisfying

$$\frac{1}{p(1+\delta)} + \frac{1}{q} = \frac{1}{r}.$$

Then

$$\int |vgw|^r dm \leq \left(\int (|v|^p w)^{1+\delta} dm\right)^{\frac{r}{p(1+\delta)}} \left(\int |g|^q w dm\right)^{\frac{r}{q}}.$$

For all $f \in v\mathcal{P} + v\mathcal{Q}$ and all $g \in L^q(w)$,

$$\begin{aligned}
\langle f,\ (P^v)^* g\rangle_w &= \langle P^v f,\ g\rangle_w = \int (P^v f)\bar{g} w dm \\
&= \int v P(v^{-1} f)\bar{g} w dm = \int P(v^{-1} f)\overline{P(\bar{v} g w)} dm \\
&= \int v^{-1} f \overline{P(\bar{v} g w)} dm = \int \frac{1}{vw} f \overline{P(\bar{v} g w)} w dm \\
&= \left\langle f,\ \frac{1}{\bar{v} w} P(\bar{v} w g)\right\rangle_w .
\end{aligned}$$

By Lemma 2.1(1), $v\mathcal{P} + v\mathcal{Q}$ is dense in $L^p(w)$.
(2): By Lemma 3.1(1), if $|v|^p w \in (A_p)$, then $|v|^{-q} w^{1-q} \in (A_q)$. Hence, $|(vw)^{-1}|^q w = |v|^{-q} w^{1-q} \in L^1$. By (1) and Theorem 2.4(2),

$$\operatorname{ran}(P^v)^* = \frac{1}{k\bar{v}w} H^q(w), \quad \ker(P^v)^* = \frac{1}{\overline{k}vw} \overline{H_0^q(w)}.$$

Since $((P^v)^*)^2 = (P^v)^*$,

$$L^q(w) = \operatorname{ran}(P^v)^* \oplus \ker(P^v)^* = \frac{1}{k\bar{v}w} H^q(w) \oplus \frac{1}{\overline{k}vw} \overline{H_0^q(w)}.$$

Lemma 3.2 is proved. □

By Lemma 3.2, if $v = 1$ and w satisfies the Muckenhoupt condition (A_p), then $P^* f = P^{1/w} f$, $(f \in L^q(w))$.

Theorem 3.3 *Let* $1 < p < \infty$, *and let* $1/p + 1/q = 1$. *Suppose* $v,\ w \in L^1$, $w > 0$, $|v|^p w \in (A_p)$ *and* $|v|^q w \in L^1$. *Then the following two properties are equivalent.*
(1) $(P^v)^* g = P^v g, \quad (g \in L^p(w) \cap L^q(w))$.
(2) $|v|^2 w$ *is a constant function.*

Proof. Suppose (1) holds. Since $v \in L^p(w) \cap L^q(w)$, $(P^v)^* v = P^v v = vP1 = v$. By Lemma 3.2, $(\bar{v}w)^{-1} P(\bar{v} w v) = v$. Hence, $P(|v|^2 w) = |v|^2 w$. By Lemma 3.1(1), if $|v|^p w \in (A_p)$, then there is a constant $\delta > 0$ satisfying $(|v|^p w)^{1+\delta} \in L^1$ (cf. [4, p.262]). Since $1/p + 1/q = 1$, there is a constant $r > 1$ satisfying

$$\frac{1}{p(1+\delta)} + \frac{1}{q} = \frac{1}{r}.$$

Then

$$\begin{aligned}\int (|v|^2 w)^r dm &= \int |v|^r w^{r/p} |v|^r w^{r/q} dm \\ &\leq \left(\int (|v|^p w)^{1+\delta} dm\right)^{\frac{r}{p(1+\delta)}} \left(\int |v|^q w dm\right)^{\frac{r}{q}} < \infty.\end{aligned}$$

Hence, $|v|^2 w$ is a positive function satisfying $|v|^2 w \in H^r$, $r > 1$. This implies (2). Conversely, suppose (2) holds. By Lemma 3.2, for all $g \in L^p(w) \cap L^q(w)$,

$$(P^v)^* g = \frac{1}{\bar{v}w} P(\bar{v}wg) = \frac{v}{|v|^2 w} P\left(\frac{|v|^2 w}{v} g\right) = \frac{v}{c} P\left(\frac{c}{v} g\right) = vP(v^{-1}g) = P^v g.$$

Theorem 3.3 is proved. □

By Theorem 3.3, if $v = 1$ and w satisfies the Muckenhoupt condition (A_p), then $P^* = P$ on $L^p(w) \cap L^q(w)$ if and only if w is a constant.

Corollary 3.4 *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let α be an outer function such that $w = |\alpha|^{-2}$. If $w^{(2-p)/2} \in (A_p)$, then P^α is a bounded operator on $L^p(w)$, and $(P^\alpha)^*$ is a bounded operator on $L^q(w)$ such that*

$$(P^\alpha)^*(g_1 + g_2) = g_1, \quad (g_1 + g_2 \in H^q(w) \oplus \overline{K_0^q(w)}).$$

and

$$(P^\alpha)^* = P^\alpha, \quad \text{on} \quad L^p(w) \cap L^q(w).$$

Proof. By Theorem 2.7, if $w^{(2-p)/2} \in (A_p)$, then $P^\alpha \in B(L^p(w))$ and $H^p(w) \oplus \overline{K_0^p(w)} = L^p(w)$. By Lemma 3.1(2), if $w^{(2-p)/2} \in (A_p)$, then $w^{(2-q)/2} \in (A_q)$, and hence $H^q(w) \oplus \overline{K_0^q(w)} = L^q(w)$. For all $f_1 + f_2 \in H^p(w) \oplus \overline{K_0^p(w)}$, and all $g_1 + g_2 \in H^q(w) \oplus \overline{K_0^q(w)}$,

$$\begin{aligned}\langle f_1 + f_2,\ (P^\alpha)^*(g_1 + g_2)\rangle_w &= \langle P^\alpha(f_1 + f_2),\ g_1 + g_2\rangle_w \\ &= \langle f_1,\ g_1 + g_2\rangle_w \\ &= \langle f_1,\ g_1\rangle_w \\ &= \langle f_1 + f_2,\ g_1\rangle_w.\end{aligned}$$

On the other hand, by Theorem 3.3, $(P^\alpha)^* = P^\alpha$. Corollary 3.4 is proved. □

4 Invertibility of T_ϕ^v and $\phi P^v + Q^v$

In this section, the invertibility criterion for the generalized Toeplitz operator T_ϕ^v and the generalized singular integral operator $\phi P^v + Q^v$, $Q^v = I - P^v$ are investigated using the weighted norm inequality. By the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1], [4], [11], [12]), $w \in (A_p)$ if and only if P is a bounded projection from $L^p(w)$ onto $H^p(w)$. For $\phi \in L^\infty$, the Toeplitz operator T_ϕ is defined as a bounded operator from $H^p(w)$ to $H^p(w)$ by

$$T_\phi f = P(\phi f), \quad (f \in H^p(w)).$$

By Theorem 2.3, if $|v|^p w \in (A_p)$, then $P^v \in B(L^p(w))$. Since $(P^v)^2 = P^v$, $\mathrm{ran} P^v$ is a closed subspace of $L^p(w)$. For $\phi \in L^\infty$, the generalized Toeplitz operator T_ϕ^v is defined as a bounded operator from $\mathrm{ran} P^v$ to $\mathrm{ran} P^v$ by

$$T_\phi^v f = P^v(\phi f), \quad (f \in \mathrm{ran} P^v).$$

We use Lemma 4.1 to prove Lemma 4.2.

Lemma 4.1 *Let $1 < p < \infty$. Suppose $\phi \in L^\infty$, $w, \log w \in L^1$, and $|v|^p w \in (A_p)$. Then the following properties are equivalent.*
(1) *T_ϕ^v is a left invertible operator on* $\mathrm{ran} P^v$.
(2) *T_ϕ is a left invertible operator on $H^p(|v|^p w)$.*
(3) *$\phi P + Q$ is a left invertible operator on $L^p(|v|^p w)$.*
(4) *$\phi P^v + Q^v$ is a left invertible operator on $L^p(w)$.*

Proof. Let $w' = |v|^p w$. By Theorem 2.3, T_ϕ^v, T_ϕ, $\phi P + Q$, $\phi P^v + Q^v$ are bounded operators on each spaces. Suppose (1) holds. Then there is an $\varepsilon_1 > 0$ such that

$$\int |T_\phi^v f|^p w dm \geq \varepsilon_1 \int |f|^p w dm, \quad (f \in \mathrm{ran} P^v).$$

Suppose $f \in H^p(w')$. Since $\log w \in L^1$, there is an outer function h satisfying $w = |h|^p$. Since $\log |v| \in L^1$, there is an outer function k satisfying $|k| = |v|$. Since $w' = |v|^p w$, $H^p(w') = H^p(|kh|^p) = \frac{1}{kh} H^p = k^{-1} H^p(w)$. By Theorem 2.4, $\mathrm{ran} P^v = \frac{v}{k} H^p(w) = v H^p(w')$. Hence, there is a $g \in \mathrm{ran} P^v$ such that $g = vf$. By (1), there is an $\varepsilon_1 > 0$ such that

$$\begin{aligned} \int |T_\phi f|^p w' dm &= \int |P(\phi f)|^p |v|^p w dm \\ &= \int |v P(\phi v^{-1} g)|^p w dm \\ &= \int |P^v(\phi g)|^p w dm \end{aligned}$$

$$
\begin{aligned}
&= \int |T_\phi^v g|^p w dm \\
&\geq \varepsilon_1 \int |g|^p w dm \\
&= \varepsilon_1 \int |f|^p w' dm.
\end{aligned}
$$

This implies (2). Suppose (2) holds. Then there is an $\varepsilon_2 > 0$ such that

$$\int |T_\phi f|^p w' dm \geq \varepsilon_2 \int |f|^p w' dm, \quad (f \in H^p(w')).$$

Suppose $f \in L^p(w')$. Let $g = (I + Q\phi P)f$. Since $\phi \in L^\infty$ and $w' \in (A_p)$, it follows that $g \in L^p(w')$, and there is a $C_1 > 0$ such that

$$
\begin{aligned}
\int |Qg|^p w' dm &= \int |Q(P\phi P + Q)g|^p w' dm \\
&\leq C_1 \int |(P\phi P + Q)g|^p w' dm.
\end{aligned}
$$

By the theorem of Hunt, Muckenhoupt and Wheeden (cf. [1], [4], [11], [12]), if $w' \in (A_p)$, then $P, Q \in B(L^p(w'))$, and there is a $C_2 > 0$ such that

$$
\begin{aligned}
\varepsilon_2 \int |Pg|^p w' dm &\leq \int |T_\phi P g|^p w' dm \\
&= \int |P(P\phi P + Q)g|^p w' dm \\
&\leq C_2 \int |(P\phi P + Q)g|^p w' dm.
\end{aligned}
$$

Since $Q = I - P$, it follows that $Q \in B(L^p(w'))$, and there is a $C_3 > 0$ such that

$$\int |g|^p w' dm \leq C_3 \int |(P\phi P + Q)g|^p w' dm.$$

Since $P, Q \in B(L^p(w'))$, it follows that $(I + Q\phi P)f \in L^p(w')$, and there is a $C_4 > 0$ such that

$$
\begin{aligned}
\int |f|^p w' dm &= \int |(I - Q\phi P)(I + Q\phi P)f|^p w' dm \\
&\leq C_4 \int |(I + Q\phi P)f|^p w' dm \\
&\leq C_3 C_4 \int |(P\phi P + Q)(I + Q\phi P)f|^p w' dm \\
&= C_3 C_4 \int |(\phi P + Q)f|^p w' dm.
\end{aligned}
$$

This implies (3). Suppose (3) holds. Then there is an $\varepsilon_3 > 0$ such that

$$\int |(\phi P + Q)f|^p w' dm \geq \varepsilon_3 \int |f|^p w' dm, \quad (f \in L^p(w')).$$

Suppose $f \in L^p(w)$. Then $vf \in L^p(|v|^p w) = L^p(w')$. Since $P^v f = vP(v^{-1}f)$ and $Q^v f = vQ(v^{-1}f)$, it follows that

$$\begin{aligned}\int |(\phi P^v + Q^v)f|^p w dm &= \int |v(\phi P + Q)(v^{-1}f)|^p w dm \\ &= \int |(\phi P + Q)(v^{-1}f)|^p |v|^p w dm \\ &\geq \varepsilon_3 \int |v^{-1}f|^p |v|^p w dm = \varepsilon_3 \int |f|^p w dm.\end{aligned}$$

This implies (4). Suppose (4) holds. Then there is an $\varepsilon_4 > 0$ such that

$$\int |(\phi P^v + Q^v)f|^p w dm \geq \varepsilon_4 \int |f|^p w dm, \quad (f \in L^p(w)).$$

By Theorem 2.3, $P^v \in B(L^p(w))$. Suppose $f \in \mathrm{ran} P^v$. Since $Q^v = I - P^v$, it follows that $P^v f = f$, $Q^v f = 0$, and there is an $\varepsilon_5 > 0$ such that

$$\begin{aligned}\int |T^v_\phi f|^p w dm &= \int |P^v(\phi f)|^p w dm \\ &= \int |(P^v \phi P^v + Q^v) f|^p w dm \\ &= \int |(\phi P^v + Q^v)(I - Q^v \phi P^v) f|^p w dm \\ &\geq \varepsilon_4 \int |(I - Q^v \phi P^v) f|^p w dm \\ &\geq \varepsilon_5 \int |(I + Q^v \phi P^v)(I - Q^v \phi P^v) f|^p w dm \\ &= \varepsilon_5 \int |f|^p w dm.\end{aligned}$$

This implies (1). Lemma 4.1 is proved. □

We use Lemma 4.2 to prove Theorem 4.3.

Lemma 4.2 *Let $1 < p < \infty$. Suppose $w, \log w \in L^1$ and $|v|^p w \in (A_p)$. Suppose $w = |h|^p$ and $|v| = |k|$ for some outer functions h and k. Let ϕ be a nonzero function in L^∞ and let*

$$\psi = \phi \frac{\overline{kh}}{kh}.$$

Then the following properties are equivalent.
(1) *T^v_ϕ is a left invertible operator on* $\mathrm{ran} P^v$.
(2) *T_ψ is a left invertible operator on H^p.*

Proof. Suppose (1) holds. By Lemma 4.1,

$$\begin{aligned}\int |(\phi P+Q)f|^p|v|^p w dm &\geq \varepsilon_1 \int |f|^p|v|^p w dm \\ &\geq \varepsilon_2 \int |Pf|^p|v|^p w dm, \quad (f \in L^p(|v|^p w)).\end{aligned}$$

Hence,

$$\int |\phi f_0+\overline{g_0}|^p|v|^p w dm \geq \varepsilon_2 \int |f_0|^p|v|^p w dm, \quad (f_0 \in H^p(|v|^p w),\ g_0 \in H_0^p(|v|^p w)).$$

Hence,

$$\int \left|\phi \frac{\overline{kh}}{kh} kh f_0+\overline{kh g_0}\right|^p dm \geq \varepsilon_2 \int |kh f_0|^p\, dm, \quad (f_0 \in H^p(|kh|^p),\ g_0 \in H_0^p(|kh|^p)).$$

Since $khH^p(|kh|^p)=H^p$, it follows that

$$\int |\psi f_1+\overline{g_1}|^p dm \geq \varepsilon \int |f_1|^p dm, \quad (f_1 \in H^p,\ g_1 \in H_0^p).$$

Hence,

$$\int |(\psi P+Q)f|^p dm \geq \varepsilon_3 \int |f|^p dm, \quad (f \in L^p).$$

By Lemma 4.1 with $v=w=1$,

$$\int |T_\psi f|^p dm \geq \varepsilon_4 \int |f|^p dm, \quad (f \in H^p).$$

This implies (2). The converse is also true. Lemma 4.2 is proved. □

If $P^v \in B(L^p(w))$, then T_ϕ^v is an invertible operator on $\operatorname{ran}P^v$ if and only if $P^v\phi P^v+Q^v$ is an invertible operator on $L^p(w)$ if and only if ϕP^v+Q^v is an invertible operator on $L^p(w)$, since $P^v\phi P^v+Q^v = T_\phi^v P^v+Q^v$, $(\phi P^v+Q^v)(I-Q^v\phi P^v) = P^v\phi P^v+Q^v$, and $(I-Q^v\phi P^v)^{-1}=I+Q^v\phi P^v$ (cf. [11, p.393], [12, Vol.1, p.274]). Hence, we consider only the invertibility of T_ϕ^v. Corollary 4.4 is the theorem of Rochberg and Simonenko (cf. [13], [1, p.216], [12]). Their proof did not use the theorem of Widom and Devinatz. We use the theorem of Widom and Devinatz to prove Theorem 4.3.

Theorem 4.3 *Let $1<p<\infty$. Suppose $w, \log w \in L^1$ and $|v|^p w \in (A_p)$. Let ϕ be a nonzero function in L^∞. Then the following properties are equivalent.*
(1) *T_ϕ^v is an invertible operator on* $\operatorname{ran}P^v$.
(2) *$\phi=\gamma\exp(U-i\tilde{V})$, where γ is a constant with $|\gamma|=1$, U is a bounded real function, V is a real function in L^1 and $|v|^p w\exp(pV/2) \in (A_p)$. ($\tilde{V}$ denote the harmonic conjugate function of V.)*

Proof. Suppose (1) holds. Since $\log w \in L^1$, there is an outer function h satisfying $w = |h|^p$. Since $\log|v| \in L^1$, there is an outer function k satisfying $|k| = |v|$. By Theorem 2.4, $\text{ran}P^v = \frac{v}{k}H^p(w) = vH^p(|v|^p w)$ and $L^p(w) = H^p(w) \oplus \frac{k}{k}\overline{H_0^p(w)} = \text{ran}P^v \oplus \frac{v}{k}\overline{H_0^p(w)}$. Since $1 \in H^p(|v|^p w)$, $v \in vH^p(|v|^p w) = \text{ran}P^v$. Since T_ϕ^v is invertible, there is an $f \in \text{ran}P^v$ such that $T_\phi^v f = v$. Hence, $P^v(\phi f) = v$. Hence, $\phi f - v = Q^v(\phi f)$. Hence, there is a $g \in \text{ran}Q^v$ such that $\phi f = v + g$. Let

$$\psi = \phi \frac{\overline{kh}}{kh}.$$

Then $\psi f kh = \phi f \overline{kh} = (v+g)\overline{kh}$. Since $f \in \text{ran}P^v = \frac{v}{k}H^p(w)$, it follows that $\frac{fk}{v} \in H^p(w) = h^{-1}H^p$. Hence, $\frac{fkh}{v} \in H^p$. Since $g \in \text{ran}Q^v = \frac{v}{k}\overline{H_0^p(w)} = \frac{v}{\overline{kh}}\overline{H_0^p}$, it follows that $\frac{g\overline{kh}}{v} \in \overline{H_0^p}$. Let $F_0 = \frac{fkh}{v}$. Then $F_0 \in H^p$, and

$$\psi F_0 - \overline{kh} = \frac{g\overline{kh}}{v} \in \overline{H_0^p}.$$

Let c be the 0th Fourier coefficient of kh. Since kh is an outer function, $c \neq 0$. Then $\psi F_0 - \bar{c} \in \overline{H_0^p}$. Hence, $T_\psi F_0 = \bar{c}$. Hence, $1 \in \text{ran}T_\psi$. Hence, there is an $F \in H^p$ such that $\psi F - 1 \in \overline{H_0^p}$. Hence, $\psi z F - z \in \overline{H^p}$. Hence, $T_\psi(zF) - z$ is a constant. Since $1 \in \text{ran}T_\psi$, this implies that $z \in \text{ran}T_\psi$. Suppose $1, z, ..., z^n \in \text{ran}T_\psi$ and there are constants $c_1, c_2, ..., c_n$ such that $\psi z^n F - z^n - (c_1 z^{n-1} + c_2 z^{n-2} + ... + c_n) \in \overline{H_0^p}$. Then

$$\psi z^{n+1} F - z^{n+1} - (c_1 z^n + c_2 z^{n-1} + ... + c_n z) \in \overline{H^p}.$$

Let c_{n+1} be the 0th Fourier coefficient of this function. Then

$$\psi z^{n+1} F - z^{n+1} - (c_1 z^n + c_2 z^{n-1} + ... + c_n z + c_{n+1}) \in \overline{H_0^p}.$$

Hence,

$$T_\psi(z^{n+1}F) - z^{n+1} - (c_1 z^n + c_2 z^{n-1} + ... + c_n z + c_{n+1}) = 0.$$

Since $1, z, ..., z^n \in \text{ran}T_\psi$, it follows that $z^{n+1} \in \text{ran}T_\psi$. Hence, $1, z, z^2, ... \in \text{ran}T_\psi$. Hence, $\text{ran}T_\psi$ is dense in H^p (cf. [9]). By Lemma 4.2, T_ψ is left invertible. Hence, T_ψ is an invertible operator on H^p. By the theorem of Widom and Devinatz (cf. [1], [11], [12]), $\psi = \gamma_1 \exp(U - i\tilde{V}_0)$, where γ_1 is a constant with $|\gamma_1| = 1$, U is a bounded real function, V_0 is a real function in L^1 and $\exp(pV_0/2) \in (A_p)$. Hence,

$$\phi \frac{\overline{kh}}{kh} = \psi = \gamma_1 \exp(U - i\tilde{V}_0).$$

There are constants γ_2 and γ_3 with $|\gamma_2| = |\gamma_3| = 1$ such that

$$h^p = \gamma_2 \exp\left(\log w + i(\log w)\tilde{}\,\right),$$

$$k = \gamma_3 \exp\left(\log|v| + i(\log|v|)\tilde{}\,\right).$$

Hence, there is a constant γ_4 with $|\gamma_4| = 1$ such that

$$\phi = \gamma_1 \frac{(kh)^2}{|kh|^2} \exp(U - i\tilde{V_0}) = \gamma_4 \exp\left(U - i(V_0 - \log|v|^2 - \log w^{2/p})\tilde{}\right).$$

Let $V = V_0 - \log|v|^2 - \log w^{2/p}$. Then $\phi = \gamma_4 \exp(U - i\tilde{V})$ and $|v|^p w = \exp(p(V_0 - V)/2)$. Hence, $|v|^p w \exp(pV/2) = \exp(pV_0/2) \in (A_p)$. This implies (2).
Conversely, suppose (2) holds. By the similar calculation, (2) implies that $\psi = \gamma_1 \exp(U - i\tilde{V_0})$, where γ_1 is a constant with $|\gamma_1| = 1$, U is a bounded real function, V_0 is a real function in L^1 and $\exp(pV_0/2) \in (A_p)$. By the theorem of Widom and Devinatz (cf. [1], [11], [12]), T_ψ is an invertible operator on H^p. By Lemma 4.2, T_ϕ^v is a left invertible operator on ranP^v. It is sufficient to prove that ranT_ϕ^v is dense in ranP^v. Let n be a nonnegative integer. Then there is an $F \in H^p$ such that $T_\psi F = P(z^n \overline{kh})$. Since $P(\psi F - z^n \overline{kh}) = 0$, it follows that $\psi F - z^n \overline{kh} = \phi \frac{\overline{kh}}{kh} F - z^n \overline{kh} \in \overline{H_0^p}$. Hence, $\frac{\phi \bar{k} F}{kh} - z^n \bar{k} \in \overline{H_0^p(w)}$. By Theorem 2.4, $\frac{\phi v F}{kh} - z^n v \in \frac{v}{k} \overline{H_0^p(w)} = \ker P^v$. Let $G = \frac{vF}{kh}$. Then $G \in \frac{v}{k} H^p(w) = \text{ran}P^v$. Since $\phi G - z^n v \in \ker P^v$, it follows that $T_\phi^v G = P^v(\phi g) = P^v(z^n v) = z^n v$. Hence, $z^n v \in \text{ran}T_\phi^v$, $(n = 0, 1, 2...)$. Let $g \in \text{ran}P^v$. Then $v^{-1} g \in k^{-1} H^p(w) = H^p(|v|^p w)$. Hence, there is a sequence of analytic polynomials f_n such that $\|f_n - v^{-1} g\|_{L^p(|v|^p w)} \to 0, (n \to \infty)$. Hence, $\|v f_n - g\|_{L^p(w)} \to 0$. Therefore ran$T_\phi^v$ is dense in ranP^v. This implies (1). Theorem 4.3 is proved. □

By Theorem 4.3, T_ϕ^v is invertible on ranP^v if and only if T_ϕ is invertible on $H^p(|v|^p w)$. Hence, it is proved that the condition "T_ψ is an invertible operator on H^p" is also equivalent in the theorem.

Corollary 4.4 *Let $1 < p < \infty$. Suppose $w \in (A_p)$. Let ϕ be a nonzero function in L^∞. Then the following properties are equivalent.*
(1) T_ϕ is an invertible operator on $H^p(w)$.
(2) $\phi = \gamma \exp(U - i\tilde{V})$, where γ is a constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w \exp(pV/2) \in (A_p)$.

Proof. Let $v = k = 1$. By Theorem 2.4, ran$P$ = ran$P^v = \frac{v}{k} H^p(w) = H^p(w)$. Theorem 4.3 proves Corollary 4.4. □

Corollary 4.5 *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Let α be an outer function such that $w = |\alpha|^{-2}$. If $w^{(2-p)/2} \in (A_p)$, then P^α is a bounded projection from $L^p(w)$ onto $H^p(w)$ such that $(P^\alpha)^* = P^\alpha$, on $L^p(w) \cap L^q(w)$. Then the following properties are equivalent.*

(1) T^α_ϕ *is an invertible operator on* $H^p(w)$.
(2) $\phi = \gamma \exp(U - i\tilde{V})$, *where* γ *is a constant with* $|\gamma| = 1$, U *is a bounded real function,* V *is a real function in* L^1 *and* $w^{(2-p)/2}\exp(pV/2) \in (A_p)$.

Proof. By Corollary 3.4, P^α is a bounded projection from $L^p(w)$ onto $H^p(w)$ such that $(P^\alpha)^* = P^\alpha$. In the proof of Theorem 4.3, let $v = k = \alpha$. Then

$$\psi = \phi \frac{\overline{kh}}{kh}.$$

By Theorem 4.3, T^α_ϕ is invertible on $H^p(w)$ if and only if $\phi = \gamma \exp(U - i\tilde{V})$, where γ is a constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w^{(2-p)/2}\exp(pV/2) = |v|^p w \exp(pV/2) \in (A_p)$. Corollary 4.5 is proved. □

Corollary 4.6 *Let* ϕ *be a nonzero function in* L^∞. *Let* $w \in L^1$. *Suppose* $w = |\alpha|^{-2}$ *for some outer function* α. *Then* P^α *is a self-adjoint projection from* $L^2(w)$ *onto* $H^2(w)$. *Then the following properties are equivalent.*
(1) T^α_ϕ *is an invertible operator on* $H^2(w)$.
(2) T_ϕ *is an invertible operator on* H^2.
(3) $\phi = \gamma \exp(U - i\tilde{V})$, *where* γ *is a constant with* $|\gamma| = 1$, U *is a bounded real function,* V *is a real function in* L^1 *and* $e^V \in (A_2)$.

Proof. By Theorem 4.3, T^v_ϕ is invertible on $\mathrm{ran}P^v$ if and only if T_ϕ is invertible on $H^p(|v|^p w)$. Hence (1) is equivalent to (2). By Theorem 2.7, P^α is a self-adjoint projection from $L^2(w)$ onto $H^2(w)$. Since $p = 2$, it follows that $w^{(2-p)/2}\exp(pV/2) = e^V \in (A_p)$. By Corollary 4.5, (1) is equivalent to (3). Corollary 4.6 is proved. □

By the theorem of Widom (cf. [1, p.68], [12, p.260]), the spectrum of $T^\alpha_\phi \in B(H^2(w))$ is connected.

5 Invertibility of R^v_ϕ

In this section, we assume that v is an outer function. We do not assume that $P^v \in B(L^p(w))$. Hence, the results in this section do not follow from the theorem of Rochberg and Simonenko or the theorem of Widom and Devinatz (cf. [13], [1, p.216], [12]). Let $1 < p < \infty$. Let w, $\log w \in L^1$. Let $\phi \in L^\infty$. The operator R^v_ϕ is defined as a bounded operator from $H^p(w)$ to $L^p(w)/\frac{v}{\overline{v}}\overline{H^p_0(w)}$ by

$$R^v_\phi f = \phi f + \frac{v}{\overline{v}}\overline{H^p_0(w)}, \quad (f \in H^p(w)).$$

If $P^v \in B(L^p(w))$, then $\ker P^v = \frac{v}{\overline{v}}\overline{H_0^p(w)}$. If $w = |\alpha|^{-2}$ for some outer function α, then R_ϕ^α is a bounded operator from $H^p(w)$ to $L^p(w)/\overline{K_0^p(w)}$ such that

$$R_\phi^\alpha f = \phi f + \overline{K_0^p(w)}, \quad (f \in H^p(w)).$$

If $P^v \in B(L^p(w))$, then T_ϕ^v is an invertible operator on ranP^v if and only if R_ϕ^v is an invertible operator from $H^p(w)$ onto $L^p(w)/\frac{v}{\overline{v}}\overline{H_0^p(w)}$. Theorem 4.3 for an outer function v follows from Theorem 5.2. Theorem 5.2 with $P^v \in B(L^p(w))$ follows from Theorem 4.3. We use Lemma 5.1 to prove Theorem 5.2. Nakazi [9] considered the case when $v = 1$, and proved Lemma 5.1. We use Lemma 5.1 to prove Theorem 5.2.

Lemma 5.1 *Let $1 < p < \infty$. Suppose $w = |h|^p$ for some outer function $h \in H^p$, $\phi \in L^\infty$ and v is an outer function. Then the following conditions are equivalent.*
(1) *R_ϕ^1 is an invertible operator from $H^p(w)$ onto $L^p(w)/\overline{H_0^p(w)}$.*
(2) *$\phi = k_0(\overline{h_0}/h_0)(h/\overline{h})$, where k_0 is an invertible function in H^∞ and h_0 is an outer function in H^p with $|h_0|^p \in (A_p)$.*
(3) *$\phi = \gamma \exp(U - i\tilde{V})$, where γ is a constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $w \exp(pV/2) \in (A_p)$.*

Theorem 5.2 *Let $1 < p < \infty$. Suppose $w = |h|^p$ for some outer function $h \in H^p$, $\phi \in L^\infty$ and v is an outer function. Let*

$$\psi = \phi \frac{\overline{v}}{v}.$$

Then the following conditions are equivalent.
(1) *R_ϕ^v is an invertible operator from $H^p(w)$ onto $L^p(w)/\frac{v}{\overline{v}}\overline{H_0^p(w)}$.*
(2) *$\phi = \gamma \exp(U - i\tilde{V})$, where γ is a constant with $|\gamma| = 1$, U is a bounded real function, V is a real function in L^1 and $|v|^p w \exp(pV/2) \in (A_p)$.*

Proof. If R_ϕ^v is left invertible, then for any $f \in H^p(w)$ and $g \in H_0^p(w)$,

$$\int \left| \phi \frac{\bar{k}}{k} f + \overline{g} \right|^p wdm = \int \left| \phi f + \frac{v}{\overline{v}} \overline{g} \right|^p wdm \geq \varepsilon \int |f|^p wdm,$$

where ε is a positive constant. This implies that $R_{\phi\bar{k}/k}^1$ is left invertible. The converse is also true. Hence, R_ϕ^v is left invertible if and only if $R_{\phi\bar{k}/k}^1$ is left invertible. Since

$$\left(L^p(w)/\frac{v}{\overline{v}}\overline{H_0^p(w)} \right)^* = \frac{v}{\overline{v}} K^q(w), \quad (H^p(w))^* = L^q(w)/\overline{K_0^q(w)},$$

it follows that $(R^v_\phi)^*$ is a bounded operator from $\frac{v}{\bar v}K^q(w)$ to $L^q(w)/\overline{K^q_0(w)}$. For all $F \in K^q(w)$ and all $g \in H^p(w)$,

$$\begin{aligned}\left\langle (R^v_\phi)^*\left(\frac{v}{\bar v}F\right),\ g\right\rangle &= \left\langle \frac{v}{\bar v}F,\ R^v_\phi g\right\rangle \\ &= \int \frac{v}{\bar v}F\overline{\phi g}wdm \\ &= \left\langle \bar\phi\frac{v}{\bar v}F + \overline{K^q_0(w)},\ g\right\rangle.\end{aligned}$$

Hence,

$$(R^v_\phi)^*\left(\frac{\bar v}{v}F\right) = \bar\phi\frac{v}{\bar v}F + \overline{K^q_0(w)}, \quad (F \in K^q(w)).$$

If R^v_ϕ is a right invertible operator from $H^p(w)$ to $L^p(w)/\overline{\frac{v}{\bar v}H^p_0(w)}$, then $(R^v_\phi)^*$ is a left invertible operator from $\frac{v}{\bar v}K^q(w)$ to $L^q(w)/\overline{K^q_0(w)}$. Hence,

$$\int \left|\bar\phi\frac{v}{\bar v}F + G\right|^q wdm \geq \varepsilon\int |F|^q wdm, \quad (F \in K^q(w),\ G \in \overline{K^q_0(w)}).$$

Hence $(R^1_{\phi\bar v/v})^*$ is a left invertible operator from $K^q(w)$ to $L^q(w)/\overline{K^q_0(w)}$. Hence, $R^1_{\phi\bar v/v}$ is a right invertible operator from $H^p(w)$ to $L^p(w)/\overline{H^p_0(w)}$. The converse is also true. Hence, R^v_ϕ is right invertible if and only if $R^1_{\phi\bar v/v}$ is right invertible. Hence, R^v_ϕ is invertible if and only if $R^1_{\phi\bar v/v}$ is invertible. By Lemma 5.1, $R^1_{\phi\bar v/v}$ is invertible if and only if

$$\phi\frac{|v|^2}{v^2} = \phi\frac{\bar v}{v} = \gamma_0\exp(U - i\tilde V_0),$$

where γ_0 is a constant with $|\gamma| = 1$, U is a bounded real function, V_0 is a real function in L^1 and $w\exp(pV/2) \in (A_p)$. Since v is an outer function,

$$v^2 = \gamma_1\exp(\log|v|^2 + i(\log|v|^2)\tilde{}\ ).$$

Hence,

$$\phi = \gamma_2\exp\left(U - i(V_0 - \log|v|^2)\tilde{}\ \right).$$

Let $V = V_0 - \log|v|^2$. Then $\phi = \gamma_2\exp(U - i\tilde V)$, and

$$|v|^p w\exp(pV/2) = w\left(|v|^2e^V\right)^{p/2} = w\exp(pV_0/2) \in (A_p).$$

Theorem 5.2 is proved. □

By Theorem 4.3 and Theorem 5.2, if $P^v \in B(L^p(w))$ and v is an outer function, then T^v_ϕ is invertible if and only if R^v_ϕ is invertible.

6 Norms of Hankel Operators H_ϕ^v

In this section, the operator norm inequality for the generalized Hankel operator H_ϕ^v is presented. Let $Q^v = I - P^v$. By Theorem 2.3, if $Q^v \in B(L^p(w))$, then $|v|^p w \in (A_p)$. For $\phi \in L^\infty$, the generalized Hankel operator H_ϕ^v is defined as a bounded operator from $\mathrm{ran} P^v$ to $\ker P^v$ by

$$H_\phi^v f = Q^v(\phi f), \quad (f \in \mathrm{ran} P^v).$$

If $w \in (A_p)$, $Q = I - P$, and $\phi \in L^\infty$, then the original Hankel operator H_ϕ is defined as a bounded operator from $H^p(w)$ to $H^p(w)$ by

$$H_\phi f = Q(\phi f), \quad (f \in H^p(w)).$$

We use Lemma 6.1 to prove Theorem 6.2.

Lemma 6.1 *Let $1 < p < \infty$, and let $1/p + 1/q = 1$. Suppose $w \in L^1, w > 0, \log w \in L^1$. For a function k, the following two properties are equivalent.*
(1) $k \in H_0^1$, and $\|k\|_1 \le 1$.
(2) There are $f \in H^p(w)$ and $g \in K_0^q(w)$ such that $\|f\|_{p,w} = \|g\|_{q,w} \le 1$, and $k = fgw$.

Proof. Suppose (1) holds. By the factorization theorem, there exists an inner function j and an outer function $k_0 \in H^1$ such that $k = zjk_0$. Let $h \in H^p$ be an outer function such that $w = |h|^p$. If $f = h^{-1} j k_0^{1/p}$, then $f \in H^p(w)$. By Lemma 2.5, if $g = w^{-1} h z k_0^{1/q}$, then

$$g \in \frac{h^p}{w} h^{1-p} H_0^q = \frac{h^p}{w} H_0^q(w) = K_0^q(w),$$

$\|f\|_{p,w} = \|g\|_{q,w} = \|k\|_1 \le 1$, and $k = fgw$. This implies (2). Conversely, suppose (2) holds. Since $K_0^q(w) = \frac{h^p}{w} H_0^q(w)$, it follows from (1) that $gw \in h^p H_0^q$. Hence, $fgw \in h^p H^p(w) H_0^q(w) = h^p H_0^1(w) = H_0^1$. By the Hölder inequality, $\|k\|_1 = \|fgw\|_1 \le \|f\|_{p,w} \|g\|_{p,w}$. This implies (1). Lemma 6.1 is proved. □

Theorem 6.2 *Let $1 < p < \infty$. Suppose $\phi \in L^\infty$, and w is a positive function such that w, $\log w \in L^1$. Let $\log |v| \in L^1$. If $|v|^p w \in (A_p)$, then the following inequality holds.*

$$\|H_\phi\|_{B(L^2)} \le \|H_\phi^v\|_{B(L^p(w))} \le \|Q^v\|_{B(L^p(w))} \|H_\phi\|_{B(L^2)}.$$

Proof. By Theorem 2.3, if $|v|^p w \in (A_p)$, then $P^v \in B(L^p(w))$. We shall prove the first inequality. Let k be an outer function such that $|v| = |k|$. Hence,

$$\begin{aligned}
\|H_\phi^v\|_{B(L^p(w))} &= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|H_\phi^v f\|_{p,w} \\
&\ge \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1, g\in K_0^q(w), \|g\|_{q,w}\le 1} \left|\int (H_\phi^v f)\frac{k}{v} gwdm\right| \\
&= \sup_{f,g} \left|\int Q^v(\phi f)\frac{k}{v} gwdm\right|.
\end{aligned}$$

By Theorem 2.4, $\mathrm{ran}P^v = \frac{v}{k}H^p(w)$. Hence, $P^v(\phi f)\frac{k}{v} \in H^p(w)$. By Lemma 2.5, if $g \in K_0^q(w)$, then $g \in \frac{h^p}{w}H_0^q(w)$. Hence, $gw \in h^p H_0^q(w)$. Hence, $P^v(\phi f)\frac{k}{v}gw \in h^p H_0^1(w) = H_0^1$. Hence,

$$\|H_\phi^v\|_{B(L^p(w))} \ge \sup_{f,g} \left|\int \phi \frac{k}{v} f gwdm\right|.$$

Let $F = \frac{k}{v}f$. Since $\mathrm{ran}P^v = \frac{v}{k}H^p(w)$, it follows that $f \in \mathrm{ran}P^v$ if and only if $F \in H^p(w)$, and $\|f\|_{p,w} = \|F\|_{p,w}$. Hence,

$$\|H_\phi^v\|_{B(L^p(w))} \ge \sup_{F\in H^p(w), \|F\|_{p,w}\le 1, g\in K_0^q(w), \|g\|_{q,w}\le 1} \left|\int \phi F gdm\right|.$$

By Lemma 6.1 and the theorem of Nehari (cf. [1], [11], [12]),

$$\|H_\phi^v\|_{B(L^p(w))} \ge \sup_{k\in H_0^1, \|k\|_1\le 1} \left|\int \phi k dm\right| = \mathrm{dist}(\phi, H^\infty).$$

Next we shall prove the second inequality. If $f \in \mathrm{ran}P^v$ and $G \in H^\infty$, then $Gf \in \mathrm{ran}P^v = \frac{v}{k}H^p(w)$. Hence, $v^{-1}Gf \in k^{-1}H^p(w) = H^p(|v|pw)$. Since $|v|^p w \in (A_p)$, it follows that $P^v(Gf) = vP(v^{-1}Gf) = vv^{-1}Gf = Gf$. Hence, $Q^v(Gf) = (I - P^v)(Gf) = 0$. Hence,

$$\begin{aligned}
\|H_\phi^v\|_{B(L^p(w))} &= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|H_\phi^v f\|_{p,w} \\
&= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|Q^v(\phi f)\|_{p,w} \\
&= \sup_{f\in \mathrm{ran}P^v, \|f\|_{p,w}\le 1} \|Q^v((\phi - G)f)\|_{p,w} \\
&\le \|Q^v\|_{B(L^p(w))}\|\phi - G\|_\infty.
\end{aligned}$$

Hence,

$$\|H_\phi^v\|_{B(L^p(w))} \le \|Q^v\|_{B(L^p(w))} \inf_{G\in H^\infty} \|\phi - G\|_\infty = \|Q^v\|_{B(L^p(w))}\mathrm{dist}(\phi, H^\infty).$$

Hence,

$$\text{dist}(\phi, H^\infty) \leq \|H_\phi^v\|_{B(L^p(w))} \leq \|Q^v\|_{B(L^p(w))}\text{dist}(\phi, H^\infty).$$

If $v = w = 1$, then the equalities hold, and hence we have the Nehari theorem: $\|H_\phi\|_{B(L^2)} = \|H_\phi^1\|_{B(L^2)} = \text{dist}(\phi, H^\infty)$. Theorem 6.2 is proved. □

Corollary 6.3 *Let* $1 < p < \infty$. *Suppose* $\phi \in L^\infty$, *and* w *is a positive function such that* w, $\log w \in L^1$.
(1) *If* $w = |v|^{-p}$ *for some function* v, *then* H_ϕ^v *is a bounded operator from* $\text{ran}P^v$ *to* $\ker P^v$ *satisfying*

$$\|H_\phi\|_{B(L^2)} \leq \|H_\phi^v\|_{B(L^p(w))} \leq \frac{1}{\sin(\pi/p)}\|H_\phi\|_{B(L^2)}.$$

(2) *If* $w = |\alpha|^{-2}$ *for some outer function* α, *then* H_ϕ^α *is a bounded operator from* $H^2(w)$ *to* $\overline{K_0^2(w)}$ *satisfying*

$$\|H_\phi^\alpha\|_{B(L^2(w))} = \|H_\phi\|_{B(L^2)}.$$

Proof. It is sufficient to prove (1). By Lemma 2.2, if $|v|^p w$ is a constant, then $\|P^v\|_{B(L^p(w))} = \|P\|_{B(L^p)}$. By the similar proof, it follows that $\|Q^v\|_{B(L^p(w))} = \|Q\|_{B(L^p)}$. It is known that $\|P\|_{B(L^p)} = \|Q\|_{B(L^p)}$ (cf. [5, Vol.I, p.79]). By the theorem of Gohberg, Krupnik, Hollenbeck and Verbitsky (cf. [5], [6]), $\|P\|_{B(L^p)} = \frac{1}{\sin(\pi/p)}$. By Theorem 6.2, Corollary 6.3 is proved. □

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