Theory of optimal reshaping of functions and its applications

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Approximating an object with a "simpler" object is is common in many field of mathematics, and a basic operation for solving many scientific problems. Especially, simplification of a function is a popular problem. Consider a families $\mathbf{X}$ and $\mathbf{Y}$ of functions, where $\mathbf{Y} \subset \mathbf{X}$, and $\mathbf{Y}$ is the set of functions in $\mathbf{X}$ satisfying a condition S. We consider a problem of approximating a a function f in $\mathbf{X}$ (or a function f in a subfamily $\mathbf{Z}$ of $\mathbf{X}$) into a function $\phi(f)$ in $\mathbf{Y}$.

> Given an input function f, compute a function $\phi(f) \in \mathbf{Y}$ such that $d(f, \phi(f))$ is minimized, where d is a fixed distance measure in the function space.

Weierstrauss's approximation theorem givens approximation of continuous function by a polynomial function, and approximation using Chebyshev's polynomials give an optimized version. Taylor expansion and Fourier expansion are popular tools to transform function-approximating problems into vector approximating problems.

Those are analytic methods, and we consider discrete and geometric method for the function approximation problem, where the key condition S concerns the *shape* of the output function. The easiest example is the Gauss's least square method approximating a discrete-valued function f by a linear (or low-degree polynomial) function $\phi(f)$ to minimize the L_2-distance (under a discrete measure). Digital monochromatic (resp. color) image can be considered as a function f $n \times n$ pixel grid to nonnegative real values (resp. color space) representing the brightness value (resp. colors). Therefore, most of problems in computer vision can be formulated into function approximation problems:

1. Digital halftoning (monochromatic version): $\phi(f)$ is a $\{0, 1\}$-valued function. A discrepancy measure is considered [3].

2. Color quantization: $\phi(f)$ is a function into a discrete set corresponding to a fixed number of colors.

3. Image Segmentation: $\phi(f)$ is a locally constant function with a constant number of pieces [2].

4. Image compression: $\phi(f)$ is a linear sum of of base functions (typically in a wavelet basis) with a bounded number of nonzero coefficients.

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Function reshaping problem is a kind of geometric reshaping problem in computational geometry[8]. Convex hull and the optimal k-link approximation are typical examples. We briefly survey previous results, and then present recent results by authors where we consider conditions on the number of peaks or linear pieces of the output function.

In particular, given a piecewise linear function f in one variable, we can compute the function $\phi(f)$ with at most k maximal peaks to minimize the L_p distance $|f - \phi(f)|_p$ in $O(n \log^2 n)$ time, and $O(n \log n)$ time if $p = 2$ [5, 7].

On the other hand, if we want to compute the continuous function $\phi(f)$ with at most k linear pieces, the problem for minimizing L_2 distance seems to be very difficult. Indeed, even if we replace f to be a discrete function, we can only design a PTAS algorithm [1]. Higher dimensional version of the problem will be also considered[4, 6].

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