

THE INITIAL VALUE PROBLEMS
FOR QUASI-LINEAR WAVE
EQUATIONS IN TWO SPACE
DIMENSIONS WITH SMALL DATA

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Series #176. December 1992

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**THE INITIAL VALUE PROBLEMS
FOR QUASI-LINEAR WAVE EQUATIONS
IN TWO SPACE DIMENSIONS WITH SMALL DATA**

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Abstract. The present paper studies the lifespan of solutions to quasi-linear wave equations in two space dimensions. We shall show a lower bound for the lifespan. We shall also show that if the non-linear term satisfies "null-condition", the equations have global solutions. Our basic idea is to solve ordinary differential equations which are constructed from wave equations.

1. **Introduction and Statement of Results.** We study the lifespan of solutions of quasi-linear wave equations in two space dimensions, with small initial data, as following type;

$$\square u(x, t) = a_{\alpha\beta}(u') \partial_\alpha \partial_\beta u(x, t), \quad (x, t) \in \mathbb{R}^2 \times [0, \infty), \quad (1.1)$$

$$u(x, 0) = \varepsilon f(x), \quad \partial_0 u(x, 0) = \varepsilon g(x), \quad x \in \mathbb{R}^2. \quad (1.2)$$

Here $a_{\alpha\beta}(u') = a_{\beta\alpha}(u')$ and we denote $\partial_0 = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ ($i = 1, 2$) and $\square = \partial_0^2 - \partial_1^2 - \partial_2^2$. The gradient of u is denoted by $u' = (\partial_0 u, \partial_1 u, \partial_2 u)$. We use the summation convention with subscripts $\alpha, \beta \dots$ ranging over 0, 1, 2 and i, j, \dots over 1, 2. Moreover we assume that

$$f, g \in C_0^\infty(\mathbb{R}^2) \quad \text{and} \quad f(x) = g(x) = 0, \quad \text{for} \quad |x| \geq M \quad (1.3)$$

$$a_{\alpha\beta}(p) \in C^\infty(\mathbb{R}^3) \quad (1.4a)$$

$$a_{\alpha\beta}(p) = O(|p|^2) \quad (1.4b)$$

$$|a_{\alpha\beta}(p)| < 1/2 \quad (1.4c)$$

for $|p| < \delta$, where δ is a small positive number.

The supremum of all τ for which $C^\infty(\mathbb{R}^2 \times [0, \tau))$ -solution of the Cauchy problem (1.1), (1.2) exists is called the "lifespan" T_ε . When $T_\varepsilon = \infty$, we say the Cauchy problem (1.1), (1.2) has a *global solution*.

M. Kovalyov has proved in [12] that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log(1 + T_\varepsilon) \geq C, \quad (1.5)$$

where the constant C depends on f, g and $a_{\alpha\beta}$. The first aim is to determine the constant explicitly by Friedlander radiation field. Let $U(\mathbf{x}, t)$ be a solution of linear wave equation;

$$\square U(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times [0, \infty), \quad (1.6)$$

$$U(\mathbf{x}, 0) = f(\mathbf{x}), \quad \partial_0 U(\mathbf{x}, 0) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \quad (1.7)$$

Then we can define the Friedlander radiation field $\mathcal{F}(\omega, \rho)$ by

$$\mathcal{F}(\omega, \rho) = \lim_{r \rightarrow \infty} r^{1/2} U(\mathbf{x}, t), \quad \mathbf{x} = r\omega, \quad \omega \in S^1, \quad \rho = r - t. \quad (1.8)$$

\mathcal{F} is explicitly expressed by

$$\mathcal{F}(\omega, \rho) = \frac{1}{2\sqrt{2\pi}} \int_\rho^\infty (s - \rho)^{-1/2} \{R_g(\omega, s) - \partial_s R_f(\omega, s)\} ds, \quad (1.9)$$

where R_h is Radon transform of $h \in C_0^\infty(\mathbb{R}^2)$, i.e.,

$$R_h(\omega, s) = \int_{\omega \cdot y = s} h(y) dS_y.$$

Note that \mathcal{F} satisfies

$$\mathcal{F}(\omega, \rho) = 0 \quad \text{for } \rho \geq M, \quad (1.10)$$

$$|\partial_\rho^l \mathcal{F}(\omega, \rho)| \leq C(1 + |\rho|)^{-1/2-l}. \quad (1.11)$$

For the above facts, see Hörmander [3].

We write the non-linear term in (1.1) as

$$a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u = Z_{\alpha\beta\gamma\delta}(\partial_\gamma u)(\partial_\delta u)(\partial_\alpha\partial_\beta u) + O(|u'|^3|u''|), \quad (1.12)$$

where

$$Z_{\alpha\beta\gamma\delta} = \frac{\partial^2 a_{\alpha\beta}(u')}{\partial(\partial_\gamma u)\partial(\partial_\delta u)} \Big|_{u'=0}. \quad (1.13)$$

Thus we define an important quantity

$$H = \max_{\rho \in \mathbb{R}, \omega \in S^1} \{-C(-1, \omega)\partial_\rho \mathcal{F}(\omega, \rho)\partial_\rho^2 \mathcal{F}(\omega, \rho)\}, \quad (1.14)$$

where

$$C(X) = Z_{\alpha\beta\gamma\delta}X_\alpha X_\beta X_\gamma X_\delta, \quad X = (X_0, X_1, X_2), \quad (1.15)$$

and $C(-1, \omega)$ is defined by setting $X_0 = -1$, $(X_1, X_2) = \omega \in S^1$. By (1.10) and (1.11), we find that H is well-defined and non-negative.

Theorem 1. $\liminf_{\epsilon \rightarrow 0} \epsilon^2 \log(1 + T_\epsilon) \geq \frac{1}{H}$.

Proof of this theorem is basically owed to the method in F. John [6]. When $a_{\alpha\beta}(u') = O(|u'|)$ in three space dimensions, he proved that

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log(1 + T_\epsilon) \geq \frac{1}{H^*},$$

where

$$H^* = \max_{\rho \in \mathbb{R}, \omega \in S^2} \left\{ -\frac{1}{2}C(-1, \omega)\partial_\rho^2 \mathcal{F}(\omega, \rho) \right\}.$$

We next study the interesting case $H = 0$. The condition is equivalent to the condition (i) f and g vanish identically or (ii) $C(-1, \omega) = 0$ for any $\omega \in S^1$ (see Appendix). Under the condition (i), the Cauchy problem (1.1), (1.2) has a trivial global solution $u \equiv 0$. Under the condition (ii) which is called Klainerman's null-condition, we find from (1.15) that $C(X)$

is divided by $X_0^2 - X_1^2 - X_2^2$. Hence $a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u$ is represented as a linear combination of the followings:

$$C_{\alpha\beta}(\partial_\alpha\partial_\beta u)\{(\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2\} + O(|u'|^3|u''|), \quad (1.16a)$$

$$C_{\alpha\beta}(\partial_\alpha u)\partial_\beta\{(\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2\} + O(|u'|^3|u''|), \quad (1.16b)$$

$$C_{\alpha\beta}(\partial_\alpha u)(\partial_\beta u)\square u + O(|u'|^3|u''|), \quad (1.16c)$$

$$C_\alpha(\partial_\alpha u)\{(\partial_\beta u)(\partial_\gamma\partial_\delta u) - (\partial_\gamma u)(\partial_\beta\partial_\delta u)\} + O(|u'|^3|u''|), \quad \beta, \gamma, \delta = 0, 1, 2, \quad (1.16d)$$

where $C_{\alpha\beta}$ and C_α are constants.

Theorem 2. If $H = 0$, there exists an $\varepsilon_0 > 0$ such that $T_\varepsilon = \infty$ for any $0 < \varepsilon < \varepsilon_0$.

S. Klainerman [11], L. Hörmander [3], D. Christodoulou [1] and F. John [6] proved independently that the null-condition implies global existence for small data in three space dimensions. When the non-linear term is cubic form of u' in two space dimensions, P. Godin [2] proved the same results by making use of L^1 - L^∞ estimates studied in L. Hörmander [4] and S. Klainerman [9]. Theorem 2 is obtained along the same lines as in P. Godin [2].

We will prove Theorem 1 in section 2 and Theorem 2 in section 3.

2. Proof of Theorem 1. First we introduce the generalized Sobolev space. Denote by $\Gamma_1, \Gamma_2, \dots, \Gamma_7$, the vector fields

$$L_0 = t\partial_0 + x_1\partial_1 + x_2\partial_2, \quad L_i = x_i\partial_0 + t\partial_i \quad (i = 1, 2),$$

$$\Omega = x_1\partial_2 - x_2\partial_1, \quad \partial_0, \partial_1, \partial_2,$$

respectively. These operators satisfy commutation relations

$$\begin{aligned} [\Gamma_p, \square] &= \Gamma_p \square - \square \Gamma_p = 2\delta_{1p} \square, \quad p = 1, 2, \dots, 7, \\ [\Gamma, \Gamma] &= \overline{\Sigma} \Gamma, \quad [\Gamma, \partial] = \overline{\Sigma} \partial. \end{aligned} \quad (2.1)$$

$\bar{\Sigma}$ stands for finite linear combination with constant coefficients. For $\sigma \in \mathbb{Z}_+^7$ (\mathbb{Z}_+ is the set of non-negative integers), we put $\Gamma^\sigma = \Gamma_1^{\sigma_1} \Gamma_2^{\sigma_2} \cdots \Gamma_7^{\sigma_7}$. We define the norms

$$\|v(t)\|_k = \sum_{|\sigma| \leq k} \|\Gamma^\sigma v(\cdot, t)\|_{L^2_{\bar{x}}(\mathbb{R}^2)}, \quad (2.2)$$

$$|v(t)|_k = \sum_{|\sigma| \leq k} \|\Gamma^\sigma v(\cdot, t)\|_{L^\infty_{\bar{x}}(\mathbb{R}^2)}. \quad (2.3)$$

For convenience, when $k = 0$, we omit sub-index. Following propositions are very important in proving our theorems.

Proposition 2.1. (Klainerman's inequality [10]) For smooth function $v(x, t)$ ($x \in \mathbb{R}^n$, $n \geq 2$),

$$|v(x, t)| \leq C_n (1 + |x| + t)^{-(n-1)/2} (1 + |t - |x||)^{-1/2} \|v(t)\|_{[\frac{n}{2}]+1}, \quad (2.4)$$

where $[s]$ stands for the largest integer not exceeding s .

Proposition 2.2. (generalized energy estimate) If the solution u of (1.1), (1.2) exists in $C^\infty(\mathbb{R}^2 \times [0, T])$ and satisfies

$$\sup_{0 \leq s \leq t} |u'(s)|_{[\frac{k+1}{2}]} < 1, \quad \sup_{0 \leq s \leq t} |u'(s)| < \delta \quad \text{for } 0 < t < T, \quad (2.5)$$

then

$$\|u'(t)\|_k \leq C_k \|u'(0)\|_k \exp(C_k \int_0^t |u'(s)|_{[\frac{k+1}{2}]}^2 ds), \quad (2.6)$$

where δ is the one in (1.4) and $k \in \mathbb{N}$.

We prove Proposition 2.2. Multiplying Lv by $\partial_0 v$ and integrating with respect to x over \mathbb{R}^2 , we arrive at the "energy identity" for a scalar v :

$$\frac{d}{dt} \int_{\mathbb{R}^2} \{(\partial_\alpha v)(\partial_\alpha v) - a_{00}(\partial_0 v)^2 + a_{ij}(\partial_i v)(\partial_j v)\} dx = \int_{\mathbb{R}^2} J(t, x) dx,$$

where $L = \square - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta$ and

$$\begin{aligned} J = & 2(\partial_0 v)(Lv) - (\partial_0 a_{00} + 2\partial_i a_{i0})(\partial_0 v)^2 - 2(\partial_j a_{ij})(\partial_0 v)(\partial_i v) \\ & + (\partial_0 a_{ij})(\partial_i v)(\partial_j v). \end{aligned} \quad (2.7)$$

Using assumption (1.4c), we get

$$\|v'(t)\|^2 \leq 3\|v'(0)\|^2 + 2 \int_0^t ds \int_{\mathbb{R}^2} |J(s, \mathbf{x})| dx, \quad (2.8)$$

which implies

$$\|u'(t)\|^2 \leq 3\|u'(0)\|^2 + C \int_0^t |u'(s)|_{[\frac{k+1}{2}]}^2 \|u'(s)\|_k^2 ds. \quad (2.9)$$

Using (1.1) and (2.1), we verify that for $|\sigma| \leq k$

$$L\Gamma^\sigma u = \Sigma\phi(u')(\Gamma^{\xi_1}\partial_{\alpha_1}u)(\Gamma^{\xi_2}\partial_{\alpha_2}u)\cdots(\Gamma^{\xi_q}\partial_{\alpha_q}u), \quad (2.10)$$

where $\phi \in C^\infty$ in u' is formed from the $a_{\alpha\beta}(u')$, and q and the multi-indices ξ_i satisfy

$$3 \leq q \leq k+2, \quad |\xi_1| + |\xi_2| + \cdots + |\xi_q| \leq k+1. \quad (2.11)$$

By (2.9) we can assume that

$$|\xi_p| \leq \left[\frac{k+1}{2}\right] \quad \text{for } p \geq 2. \quad (2.12)$$

Therefore we find from (2.7), (2.10), (2.11), (2.12) and (2.5) that

$$\int_{\mathbb{R}^2} |J(s, \mathbf{x})| dx = O(|u'(s)|_{[\frac{k+1}{2}]}^2 \|u'(s)\|_k^2).$$

Applying (2.8) to $v = \Gamma^\sigma u$ and combining with (2.9), we get

$$\|u'(t)\|_k^2 \leq C_k(\|u'(0)\|_k^2 + \int_0^t |u'(s)|_{[\frac{k+1}{2}]}^2 \|u'(s)\|_k^2 ds).$$

Therefore Gronwall's inequality yields (2.6).

In order to prove Theorem 1, it is sufficient to show following lemma.

Lemma 2.1. For any $k \geq 9$ ($k \in \mathbb{N}$), $B > H$ and $m > 0$, there exist $J_k(B) > 0$ and $\varepsilon_k(m, B) > 0$ such that if

$$\tau < \min\{T_\varepsilon, -1 + \exp(\frac{1}{B\varepsilon^2})\} \quad (2.13)$$

and

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau, \quad (2.14)$$

then for any $0 < \varepsilon < \varepsilon_k(m, B)$

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{J_k(B)\varepsilon}{(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau, \quad (2.15)$$

where H is given in (1.14).

We shall prove Theorem 1 by assuming Lemma 2.1. Let $U(x, t)$ be the solution of (1.6), (1.7). We find from (1.4b) that

$$\Gamma^\sigma u|_{t=0} = \varepsilon \Gamma^\sigma U|_{t=0} + O(\varepsilon^3) \quad \text{for any } \sigma \in \mathbb{Z}_+^7. \quad (2.16)$$

Therefore by (2.16)

$$\|u'(0)\|_k \leq C_k \varepsilon.$$

Moreover by $k \geq 9$ and (2.4)

$$|u'(0)|_{[\frac{k+1}{2}]} \leq |u'(0)|_{k-2} \leq C_k \|u'(0)\|_k \leq C_k \varepsilon. \quad (2.17)$$

Letting $m > \max\{2J_k(B), C_k\}$, we get for sufficiently small τ

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau. \quad (2.18)$$

If (2.18) holds for any τ , then $T_\varepsilon = \infty$. Hence there exists a τ ($0 < \tau < T_\varepsilon$) such that (2.18) holds and

$$|u'(\tau)|_{[\frac{k+1}{2}]} = \frac{m\varepsilon}{(1+\tau)^{1/2}}. \quad (2.19)$$

Suppose that $\tau < -1 + \exp(\frac{1}{B\varepsilon^2})$, then we can apply Lemma 2.1 and obtain

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{J_k(B)\varepsilon}{(1+t)^{1/2}} < \frac{m\varepsilon}{2(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau.$$

This contradicts (2.19). Therefore we have

$$T_\varepsilon > \tau \geq -1 + \exp(\frac{1}{B\varepsilon^2}) \quad \text{for } 0 < \varepsilon < \varepsilon_k(m, B).$$

Since B is arbitrary except for condition " $B > H$ ", Theorem 1 follows.

Now we prove Lemma 2.1. First we verify that (2.15) holds for $0 \leq t \leq \varepsilon^{-1}$. By (2.17) we get

$$\|u'(0)\|_k < C_k \varepsilon.$$

Then for sufficiently small t

$$\|u'(t)\|_k < 2C_k^2 \varepsilon,$$

also by (2.4) and $k \geq 9$

$$|u'(t)|_{[\frac{k+1}{2}]} < \frac{2C_k^3 \varepsilon}{(1+t)^{1/2}} < \delta,$$

for sufficiently small ε . Using Proposition 2.2, we find that these inequality will continue to hold as long as

$$4C_k^7 \varepsilon^2 \log(1+t) < \log 2.$$

Therefore if we take ε such that

$$4C_k^7 \varepsilon^2 \log(1+\varepsilon^{-1}) < \log 2,$$

(2.15) holds for $0 \leq t < \varepsilon^{-1}$.

Moreover we have $u(x, t) = 0$ for $|x| \geq t + M$ (see [8], Appendix 1). Therefore we can restrict ourselves in the region

$$\varepsilon^{-1} \leq t < \tau, \quad |x| \leq t + M. \quad (2.20)$$

In order to show (2.15) in the region (2.20), we introduce "pseudo characteristic rays" in (r, t) -plane, which is given by solutions of ordinary differential equations;

$$\frac{dr}{dt} = \kappa(r, t), \quad (2.21)$$

where $\omega \in S^1$ is fixed and

$$\kappa(r, t) = 1 + \frac{1}{2}C(-1, \omega)(\partial_0 u)^2. \quad (2.22)$$

For each point (r, t) with $r \geq 0$, $\varepsilon^{-1} \leq t < \tau$, there exists such a curve through this point. Continuing this curve backwards, we arrive at a point (r_1, t_1) for which either $r_1 = 0, t_1 > \varepsilon^{-1}$ or $r_1 \geq 0, t_1 = \varepsilon^{-1}$. We call S_λ the solution of (2.21) with $t_1 - r_1 = \lambda$.

Along S_λ , we find that

$$\left| \frac{d(t-r-\lambda)}{dt} \right| = |1-\kappa| \leq C|u'|^2 \leq \frac{Cm^2\varepsilon^2}{1+t}, \quad (2.23)$$

$$|t-r-\lambda| \leq Cm^2\varepsilon^2 \log(1+t) \leq \frac{Cm^2}{B}, \quad (2.24)$$

where we have used (2.21), (2.22), (2.14) and (2.13). We take $\varepsilon_k(m, B)$ such that $\varepsilon_k(m, B) < \delta m^{-1}$, then Proposition 2.2, (2.14) and (2.13) yield

$$\|u'(t)\|_k < C_k \varepsilon \exp\left(C_k \int_0^t \frac{m^2 \varepsilon^2}{1+s} ds\right) < C_k \varepsilon \exp\left(\frac{C_k m^2}{B}\right).$$

Therefore by $k \geq 9$ and Proposition 2.1

$$|u'(t)|_{\lceil \frac{k+1}{2} \rceil + 2} \leq |u'(t)|_{k-2} \leq \frac{C_k \varepsilon \exp\left(\frac{C_k m^2}{B}\right)}{(1+t)^{1/2}(1+|t-r|)^{1/2}}. \quad (2.25)$$

We set

$$\lambda_0 = \frac{Cm^2}{B} + \exp\left(\frac{2C_k m^2}{B}\right). \quad (2.26)$$

Then by (2.24), (2.25) and (2.26) along S_λ with $\lambda \geq \lambda_0$

$$|u'(t)|_{\lceil \frac{k+1}{2} \rceil} \leq \frac{C_k \varepsilon \exp\left(\frac{C_k m^2}{B}\right)}{(1+t)^{1/2}(1+\exp\left(\frac{2C_k m^2}{B}\right))^{1/2}} \leq \frac{C_k \varepsilon}{(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau.$$

This implies that (2.15) is valid along S_λ with $\lambda \geq \lambda_0$, then it is sufficient to show (2.15) along S_λ with $-M \leq \lambda \leq \lambda_0$, $\varepsilon^{-1} \leq t < \tau$.

For functions $\varphi(x, t; \varepsilon), \psi(x, t; \varepsilon)$ we write $\varphi = O^*(\psi)$, if for any $k \geq 9, B > H$ and $m > 0$, there exist $J_k(B) > 0$ and $\varepsilon_k(m, B) > 0$ such that

$$|\varphi(x, t; \varepsilon)| < J_k(B)\psi(x, t; \varepsilon) \quad \text{for } 0 < \varepsilon < \varepsilon_k(m, B),$$

as long as (2.13), (2.14) hold, along S_λ with $-M \leq \lambda \leq \lambda_0$ and $\varepsilon^{-1} \leq t < \tau$. Then our purpose is to prove

$$|u'(t)|_{\lceil \frac{k+1}{2} \rceil} = O^*(\varepsilon(1+t)^{-1/2}). \quad (2.27)$$

If we take $\varepsilon_k(m, B) < \lambda_0^{-1/p}$, then we find

$$\lambda_0 = O^*(\varepsilon^{-p}) \quad \text{for fixed } p > 0. \quad (2.28)$$

For later we shall assume that $p < \frac{1}{8}$. We find from (2.28) and (2.24) that

$$t = r + O^*(\varepsilon^{-p}), \quad r^{-1} = t^{-1} + O^*(\varepsilon^{-p}t^{-2}), \quad r^{1/2} = t^{1/2} + O^*(\varepsilon^{-p}t^{-1/2}). \quad (2.29)$$

Then it follows from (2.25), (2.29) and (2.28) that

$$|u'(t)|_{[\frac{k+1}{2}]_+2} = O^*(\varepsilon^{1-p}t^{-1/2}). \quad (2.30)$$

Since

$$\begin{aligned} |\Gamma^\sigma u(x, t)| &= \left| - \int_r^{t+M} \omega_i \partial_i \Gamma^\sigma u(s\omega, t) ds \right| \\ &= O\left(\int_r^{t+M} |u'(t)|_{[\frac{k+1}{2}]_+2} ds \right) = O^*(\varepsilon^{1-2p}t^{-1/2}), \end{aligned}$$

for $|\sigma| \leq [\frac{k+1}{2}] + 2$, we have

$$|u(t)|_{[\frac{k+1}{2}]_+2} = O^*(\varepsilon^{1-2p}t^{-1/2}). \quad (2.31)$$

The operator ∂_α can be written as

$$\partial_i = -\omega_i \partial_0 + \frac{1}{t} L_i + \frac{\omega_i}{t+r} L_0 - \frac{r\omega_i \omega_j}{t(t+r)} L_j, \quad \partial_0 = \frac{1}{t^2 - r^2} (tL_0 - x_i L_i),$$

(see [6], Appendix 2 and [7]) and these representations yield

$$\partial_\alpha v = -\omega_\alpha \partial_0 v + O(t^{-1}|v|_1) = -\omega_\alpha \partial_r v + O(t^{-1}|v|_1), \quad (2.32)$$

$$\partial_\alpha v = O\left(\frac{1}{|t-r|} |v|_1\right), \quad (2.33)$$

$$\partial_\alpha \partial_\beta v = \omega_\alpha \omega_\beta \partial_0^2 v + O(t^{-1}|v'|_1) = \omega_\alpha \omega_\beta \partial_r^2 v + O(t^{-1}|v'|_1), \quad (2.34)$$

$$(\partial_0 + \partial_r)v = O(t^{-1}|v|_1), \quad (\partial_0 + \partial_r)^2 v = O(t^{-2}|v|_2), \quad (2.35)$$

where $\partial_r = \omega_i \partial_i$.

The operator L can be written in the form

$$\begin{aligned} Lv &= 2t^{-1/2}(\partial_0 + \kappa \partial_r)(t^{1/2} \partial_0 v) - (\partial_0 + \partial_r)^2 v + \frac{t-r}{tr} \partial_0 v \\ &\quad - 2(\kappa - 1)(\partial_0 + \partial_r) \partial_0 v - \frac{\delta_{ij} - \omega_i \omega_j}{t^2} L_i L_j v + \frac{\omega_i}{tr} L_i v \\ &\quad - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta v + C(-1, \omega) \partial_0^2 v. \end{aligned}$$

Then by (2.29), (2.33), (2.34), (2.35) and (2.30)

$$\frac{d}{dt}(t^{1/2}\partial_0 v) = \frac{1}{2}t^{1/2}Lv + O^*(t^{-3/2}|v|_2). \quad (2.36)$$

We apply (2.36) to $v = \Gamma^\sigma u$ with $|\sigma| \leq [\frac{k+1}{2}]$ below. When $v = \Gamma^\sigma u$

$$t^{1/2}\partial_0\Gamma^\sigma u(t)|_{t=\varepsilon^{-1}} = O(\varepsilon^{-1/2}|u'(\varepsilon^{-1})|_{[\frac{k+1}{2}]}) = O(\varepsilon). \quad (2.37)$$

Now we show (2.27) by induction. We first show

$$|u'(t)| = O^*(\varepsilon t^{-1/2}). \quad (2.38)$$

Let $v = u$ in (2.36). Then it follows from (2.31) and $Lu = 0$ that

$$\frac{d}{dt}(t^{1/2}\partial_0 u) = O^*(\varepsilon^{1-2p}t^{-2}). \quad (2.39)$$

Integrating (2.39) from ε^{-1} to t , we find from (2.37) that

$$|t^{1/2}\partial_0 u(t)| \leq |s^{1/2}\partial_0 u(s)|_{s=\varepsilon^{-1}} + O^*\left(\int_{\varepsilon^{-1}}^t \varepsilon^{1-2p}s^{-2}ds\right) = O^*(\varepsilon), \quad (2.40)$$

which implies

$$\partial_0 u(t) = O^*(\varepsilon t^{-1/2}). \quad (2.41)$$

Using (2.32), (2.31) and (2.41),

$$\begin{aligned} \partial_i u(t) &= -\omega_i \partial_0 u + O(t^{-1}|u|) \\ &= O^*(\varepsilon t^{-1/2}) + O^*(\varepsilon^{1-2p}t^{-3/2}) = O^*(\varepsilon). \end{aligned}$$

Next we shall show

$$|u'(t)|_1 = O^*(\varepsilon t^{-1/2}). \quad (2.42)$$

We begin the proof of (2.42) by showing

$$\partial_\alpha \partial_\beta u(t) = O^*(\varepsilon t^{-1/2}). \quad (2.43)$$

Letting $v = \partial_0 u$ in (2.36), it follows from $Lu = 0$, (1.15), (2.32), (2.34), (2.30) and (2.31) that

$$\begin{aligned}
L(\partial_0 u) &= \square(\partial_0 u) - a_{\alpha\beta}(u')\partial_\alpha\partial_\beta\partial_0 u \\
&= (\partial_0 a_{\alpha\beta}(u'))\partial_\alpha\partial_\beta u \\
&= Z_{\alpha\beta\gamma\delta}\partial_0\{(\partial_\gamma u)(\partial_\delta u)\}\partial_\alpha\partial_\beta u + O(|u'|^2|u''|^2) \\
&= 2C(-1, \omega)(\partial_0^2 u)^2\partial_0 u + O(|u'|^2|u''|^2 + t^{-1}|u|_1|u'|_1^2) \\
&= 2C(-1, \omega)t^{-1}(t^{\frac{1}{2}}\partial_0^2 u)^2\partial_0 u + O^*(\varepsilon^{3-4p}t^{-2}).
\end{aligned}$$

Then we write the differential equation (2.36) as

$$\frac{d}{dt}W_1(t) = C(-1, \omega)t^{-1/2}W_1(t)^2\partial_0 u(t) + O^*(\varepsilon^{1-p}t^{-3/2}), \quad (2.44)$$

where

$$W_1(t) = t^{1/2}\partial_0^2 u(t). \quad (2.45)$$

Notice that by (2.30),

$$W_1(t) = O^*(\varepsilon^{1-p}). \quad (2.46)$$

The following facts play an important role in our proof.

$$|\varepsilon\partial_0^\ell U(\mathbf{x}, t) - (-1)^\ell\partial_\rho^\ell \mathcal{F}(\omega, \rho)| = O(\varepsilon r^{-3/2}), \quad (2.47)$$

$$|\partial_0^\ell u(\mathbf{x}, \varepsilon^{-1}) - \varepsilon\partial_0^\ell U(\mathbf{x}, \varepsilon^{-1})| = O(\varepsilon^3), \quad (2.48)$$

with $\ell = 1, 2$. (2.47) can be proved by using Lemma 2.1.1 in [3]. We prove (2.48) for $\ell = 1$ because another case can be proved in the similar way. By (2.16), the function $u - \varepsilon U$ satisfies

$$\square(u - \varepsilon U) = a_{\alpha\beta}(u')\partial_\alpha\partial_\beta u, \quad (2.49)$$

$$\|\partial_0 u(0) - \varepsilon\partial_0 U(0)\| = O(\varepsilon^3). \quad (2.50)$$

Applying Proposition 2.1 and classical energy estimate to (2.49), (2.50), we find that

$$\begin{aligned}
\|\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})\| &\leq C(1 + \varepsilon^{-1})^{-1/2}\|\partial_0 u(\varepsilon^{-1}) - \varepsilon\partial_0 U(\varepsilon^{-1})\| \\
&\leq C\varepsilon^{1/2}(\|\partial_0 u(0) - \varepsilon\partial_0 U(0)\| + \int_0^{\varepsilon^{-1}} \|a_{\alpha\beta}(u'(s))\partial_\alpha\partial_\beta u(s)\| ds).
\end{aligned}$$

Since $0 \leq s \leq \varepsilon^{-1}$, we find that

$$\|a_{\alpha\beta}(u'(s))\partial_\alpha\partial_\beta u(s)\| \leq C \left(\int_{\mathbb{R}^2} |u'(s)|_{[\frac{k+1}{2}]}^6 ds \right)^{1/2} = O(\varepsilon^3 s^{-1/2}).$$

Therefore we have

$$|\partial_0 u(\varepsilon^{-1}) - \varepsilon \partial_0 U(\varepsilon^{-1})| = O(\varepsilon^{7/2} + \varepsilon^{1/2} \int_0^{\varepsilon^{-1}} \varepsilon^3 s^{-1/2} ds) = O(\varepsilon^3).$$

This implies (2.44) with $\ell = 1$.

It follows from (2.40), (2.47) and (2.48) that

$$\begin{aligned} t^{1/2} \partial_0 u(t) &= (\varepsilon^{-1})^{1/2} \partial_0 u(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) \\ &= (\varepsilon^{-1})^{1/2} \varepsilon \partial_0 U(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) \\ &= (\varepsilon^{-1} - \lambda)^{1/2} \varepsilon \partial_0 U(\varepsilon^{-1}) + O^*(\varepsilon^{3/2}) + O(\{(\varepsilon^{-1})^{1/2} - (\varepsilon^{-1} - \lambda)^{1/2}\} \varepsilon) \\ &= -\varepsilon \partial_\rho \mathcal{F}(\omega, -\lambda) + O^*(\varepsilon^{3/2}). \end{aligned}$$

On the other hand, by (2.47) and (2.48)

$$\begin{aligned} W_1(\varepsilon^{-1}) &= (\varepsilon^{-1})^{1/2} \partial_0^2 u(\varepsilon^{-1}) \\ &= (\varepsilon^{-1})^{1/2} \varepsilon \partial_0^2 U(\varepsilon^{-1}) + O(\varepsilon^{5/2}) \\ &= (\varepsilon^{-1} - \lambda)^{1/2} \varepsilon \partial_0^2 U(\varepsilon^{-1}) + O(\varepsilon^{5/2}) + O(\{(\varepsilon^{-1})^{1/2} - (\varepsilon^{-1} - \lambda)^{1/2}\} \varepsilon) \\ &= \varepsilon \partial_\rho^2 \mathcal{F}(\omega, -\lambda) + O(\varepsilon^{3/2}). \end{aligned} \tag{2.52}$$

Then by (2.46), (2.51) and (2.52) we rewrite (2.44)

$$\frac{d}{dt} W_1(t) = -\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) t^{-1} W_1(t)^2 + O^*(\varepsilon^{1-p} t^{-3/2} + \varepsilon^{7/2-2p} t^{-1}), \tag{2.53}$$

$$W_1(\varepsilon^{-1}) = \varepsilon \partial_\rho^2 \mathcal{F}(\omega, -\lambda) + O(\varepsilon^{3/2}). \tag{2.54}$$

If the solution $W_1(t)$ of (2.53), (2.54) satisfies

$$W_1(t) = O^*(\varepsilon), \tag{2.55}$$

then we obtain (2.43). Indeed, using (2.45), we have

$$\partial_0^2 u(t) = O^*(\varepsilon t^{-1/2}), \tag{2.56}$$

then (2.34), (2.30) and (2.56) yield (2.43).

We shall show (2.55). Multiplying $\text{sgn}W_1$ to both sides of (2.53),

$$\frac{d}{dt}|W_1(t)| \leq -\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) t^{-1} |W_1(t)|^2 + |L(t)|, \quad (2.57)$$

where

$$L(t) = O^*(\varepsilon^{1-p} t^{-3/2} + \varepsilon^{7/2-2p} t^{-1}).$$

Note that

$$\int_{\varepsilon^{-1}}^t |L(s)| ds = O^*(\varepsilon^{5/4}). \quad (2.58)$$

Replacing if necessary W_1 by $-W_1$, we may assume

$$W_1(\varepsilon^{-1}) \geq 0.$$

We set

$$\beta(t) = W_1(\varepsilon^{-1}) + J_k(B) \varepsilon^{5/4} - \varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) \int_{\varepsilon^{-1}}^t |W_1(s)|^2 s^{-1} ds,$$

then we find that

$$0 \leq |W_1(t)| \leq \beta(t).$$

If $C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) \geq 0$, by (2.57)

$$\frac{d}{dt}|W_1(t)| \leq |L(t)|.$$

Integrating this inequality from ε^{-1} to t , we obtain by (2.37)

$$|W_1(t)| \leq W_1(\varepsilon^{-1}) + O^*(\varepsilon^{5/4}) = O^*(\varepsilon).$$

Therefore we assume $C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) < 0$. In this case,

$$\begin{aligned} \frac{d}{dt}\beta(t) &= -\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) t^{-1} |W(t)|^2 \\ &\leq -\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) t^{-1} \beta(t)^2. \end{aligned} \quad (2.59)$$

Now we consider a Cauchy problem;

$$\frac{d}{dt}Z(t) = -\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) t^{-1} Z(t)^2, \quad (2.60)$$

$$Z(\varepsilon^{-1}) = \beta(\varepsilon^{-1}) = W_1(\varepsilon^{-1}) + J_k(B) \varepsilon^{5/4}. \quad (2.61)$$

Then by (2.59) and (2.60)

$$\begin{aligned} & \frac{d}{dt} \left\{ (Z(t) - \beta(t)) \exp(\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) \int_{\varepsilon^{-1}}^t (z(s) + \beta(s)) s^{-1} ds) \right\} \\ &= \left\{ \frac{d}{dt} Z(t) - \frac{d}{dt} \beta(t) + \varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) t^{-1} (Z(t)^2 - \beta(t)^2) \right\} \\ & \quad \times \exp(\varepsilon C(-1, \omega) \partial_\rho \mathcal{F}(\omega, -\lambda) \int_{\varepsilon^{-1}}^t (Z(s) + \beta(s)) s^{-1} ds) \geq 0. \end{aligned}$$

Since $Z(\varepsilon^{-1}) = \beta(\varepsilon^{-1})$, we have

$$\beta(t) \leq Z(t).$$

Solving (2.60), (2.61) explicitly, we have by (2.13) and (2.45)

$$\begin{aligned} Z(t) &= \frac{Z(\varepsilon^{-1})}{1 - \varepsilon Z(\varepsilon^{-1}) (-C(-1, \omega)) \partial_\rho \mathcal{F}(\omega, -\lambda) \log t} \\ &= \frac{O^*(\varepsilon)}{1 - \varepsilon (\varepsilon \partial_\rho^2 \mathcal{F}(\omega, -\lambda) + O^*(\varepsilon^{5/4})) (-C(-1, \omega)) \partial_\rho \mathcal{F}(\omega, -\lambda) \log t} \\ &= \frac{O^*(\varepsilon)}{1 - \frac{1}{B} \{ (-C(-1, \omega)) \partial_\rho \mathcal{F}(\omega, -\lambda) \partial_\rho^2 \mathcal{F}(\omega, -\lambda) + O^*(\varepsilon^{1/4}) \}} \\ &\leq \frac{O^*(\varepsilon)}{\frac{1}{B} (B - H + O^*(\varepsilon^{1/4}))} = O^*(\varepsilon). \end{aligned}$$

Hence we have

$$0 \leq |W_1(t)| \leq \beta(t) \leq Z(t) = O^*(\varepsilon),$$

then (2.55) holds.

Now we prove (2.42). Let $v = \Gamma u$ in (2.36) (Γ is an arbitrary one) and set

$$W(t) = t^{1/2} \partial_0 \Gamma u(t). \quad (2.62)$$

It follows from (2.1), (2.32), (2.38), (2.43) and (2.62) that

$$\begin{aligned}
L(\Gamma u) &= \square \Gamma u - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta \Gamma u \\
&= \square \Gamma u - \Gamma \square u + \Gamma(a_{\alpha\beta}(u') \partial_\alpha \partial_\beta u) - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta \Gamma u \\
&= C \square u + (\Gamma a_{\alpha\beta}(u')) \partial_\alpha \partial_\beta u + a_{\alpha\beta}(u') (\Gamma \partial_\alpha \partial_\beta u - \partial_\alpha \partial_\beta \Gamma u) \\
&= C a_{\alpha\beta}(u') \partial_\alpha \partial_\beta u + (\Gamma a_{\alpha\beta}(u')) \partial_\alpha \partial_\beta u + a_{\alpha\beta}(u') \partial_\gamma \partial_\delta u \\
&= O(|u'|^2 |u''| + |\Gamma u'| |u'| |u''|) \\
&= O(|u'|^2 |u''| + |\partial_\alpha \Gamma u| |u'| |u''|) \\
&= O(|u'|^2 |u''| + |\partial_0 \Gamma u| |u'| |u''| + t^{-1} |u|_2 |u'| |u''|) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2} |W| + \varepsilon^{3-2p} t^{-5/2}) \\
&= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2} |W|).
\end{aligned}$$

Therefore by (2.36) and (2.31)

$$\frac{d}{dt} W(t) = O^*(\varepsilon^3 t^{-1} + \varepsilon^2 t^{-1} |W(t)| + \varepsilon^{1-2p} t^{-2}).$$

Integrating this equation from ε^{-1} to t and using (2.13) and (2.37), we have

$$\begin{aligned}
|W(t)| &\leq |W(\varepsilon^{-1})| + O^* \left(\int_{\varepsilon^{-1}}^t (\varepsilon^3 s^{-1} + \varepsilon^2 |W(s)| s^{-1} + \varepsilon^{1-p} s^{-2}) ds \right) \\
&= O^*(\varepsilon) + O^* \left(\varepsilon^2 \int_{\varepsilon^{-1}}^t |W(s)| s^{-1} ds \right).
\end{aligned}$$

Hence Gronwall's inequality and (2.13) lead to

$$|W(t)| = O^*(\varepsilon \exp(\varepsilon^2 \log t)) = O^*(\varepsilon e^{1/B}) = O^*(\varepsilon),$$

i.e.,

$$\partial_0 \Gamma u(t) = O^*(\varepsilon t^{-1/2}). \quad (2.63)$$

We also find from (2.1), (2.32), (2.31), (2.38) and (2.63) that (2.42) holds.

Finally we prove (2.27). It is sufficient to show that

$$|u'(t)|_t = O^*(\varepsilon t^{-1/2}), \quad (2.64)$$

under the assumption

$$|u'(t)|_{\ell-1} = O^*(\varepsilon t^{-1/2}), \quad (2.65)$$

where $1 \leq \ell \leq \lfloor \frac{k+1}{2} \rfloor$. Let $v = \Gamma^\ell u$ (for $\ell \in \mathbb{Z}_+$, Γ^ℓ stands for $\sum_{|\sigma|=\ell} \Gamma^\sigma$) in (2.36) and set

$$W(t) = t^{1/2} \partial_0 \Gamma^\ell u(t). \quad (2.66)$$

Using (2.1), (2.32), (2.38), (2.42) and (2.65), we have

$$\begin{aligned} L(\Gamma^\ell u) &= \square \Gamma^\ell u - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta \Gamma^\ell u \\ &= \Gamma^\ell \square u + \sum_{\mu < \ell} C_\mu \Gamma^\mu \square u - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta \Gamma^\ell u \\ &= \Gamma^\ell (a_{\alpha\beta}(u') \partial_\alpha \partial_\beta u) + \sum_{\mu < \ell} C_\mu \Gamma^\mu (a_{\alpha\beta}(u') \partial_\alpha \partial_\beta u) - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta \Gamma^\ell u \\ &= \sum_{\nu < \ell} \binom{\ell}{\nu} (\Gamma^{\ell-\nu} a_{\alpha\beta}(u')) (\Gamma^\nu \partial_\alpha \partial_\beta u) + a_{\alpha\beta}(u') \Gamma^\ell \partial_\alpha \partial_\beta u \\ &\quad + \sum_{\mu < \ell} C_\mu \Gamma^\mu (a_{\alpha\beta}(u') \partial_\alpha \partial_\beta u) - a_{\alpha\beta}(u') \partial_\alpha \partial_\beta \Gamma^\ell u \\ &= O(|u'|_\eta |u'|_\zeta |u'|_\xi) \quad (\eta, \zeta, \xi \leq \ell, \eta + \zeta + \xi \leq \ell + 1) \\ &= O(|u'|_{\ell-1}^3 + |u'|_0 |u'|_1 |u'|_\ell) \\ &= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-1} |u'|_\ell) \\ &= O^*(\varepsilon^3 t^{-3/2} + \varepsilon^2 t^{-3/2} V(t)), \end{aligned}$$

where

$$V(t) = t^{1/2} |u'(t)|_\ell. \quad (2.67)$$

Therefore by (2.36) and (2.31)

$$\frac{d}{dt} W(t) = O^*(\varepsilon^3 t^{-1} + \varepsilon^2 t^{-1} V(t) + \varepsilon^{1-2p} t^{-2}).$$

Integrating this equation from ε^{-1} to t and using (2.13) and (2.37), we have

$$\begin{aligned} |W(t)| &\leq |W(\varepsilon^{-1})| + O^* \left(\int_{\varepsilon^{-1}}^t (\varepsilon^3 s^{-1} + \varepsilon^2 V(s) s^{-1} + \varepsilon^{1-2p} s^{-2}) ds \right) \\ &= O^*(\varepsilon) + O^* \left(\varepsilon^2 \int_{\varepsilon^{-1}}^t V(s) s^{-1} ds \right). \end{aligned}$$

Then by (2.66), (2.1), (2.32), (2.37), (2.65) and (2.31)

$$V(t) = O^*(\varepsilon) + O^*(\varepsilon^2 \int_{\varepsilon^{-1}}^t V(s)s^{-1}ds).$$

Gronwall's inequality and (2.13) yield

$$V(t) = O^*(\varepsilon \exp(\varepsilon^2 \log t)) = O^*(\varepsilon \varepsilon^{1/B}) = O^*(\varepsilon). \quad (2.68)$$

We find from (2.67) and (2.68) that (2.64) holds and this complete the proof of Theorem 1.

3. Proof of Theorem 2. For a function $h \in C^\infty(\mathbb{R}^2 \times [0, T])$, we denote by $E_T(h)$ the solution of the Cauchy problem;

$$\begin{aligned} \square E_T(h)(x, t) &= h(x, t), \quad (x, t) \in \mathbb{R}^2 \times [0, T], \\ E_T(h)(x, 0) &= 0, \quad \partial_0 E_T(h)(x, 0) = 0, \quad x \in \mathbb{R}^2. \end{aligned}$$

Using an L^1 - L^∞ estimate in Corollary 6.2 in [4] (also see [2], Lemma 4.1), we can prove;

Proposition 3.1 Suppose that $b \in \mathbb{R}$, and $h_1, h_2 \in C^\infty(\mathbb{R}^2 \times [0, T])$ have a compact support in x for fixed t , then there exists $C > 0$ such that

$$(1+t)|E_T(h_1 h_2)(x, t)|^2 \leq C \left(\sum_{|\theta| \leq 1} \int_0^t \frac{\|\Gamma^\theta h_1(s)\|^2}{(1+s)^b} ds \right) \left(\sum_{|\theta| \leq 1} \int_0^t \frac{\|\Gamma^\theta h_2(s)\|^2}{(1+s)^{1-b}} ds \right), \quad (3.1)$$

for $0 \leq t < T$. We also need following proposition which is proved in [13].

Proposition 3.2 If functions u, v are smooth and

$$u(x, t) = v(x, t) = 0 \quad \text{for } |x| \geq t + M,$$

then we have

$$\|u(t)v'(t)\| \leq C_M \sum_{|\theta|=1} |\Gamma^\theta v(t)| \|u'(t)\|. \quad (3.2)$$

Let k be fixed and $k \geq 9$. For functions $\varphi(\mathbf{x}, t; \varepsilon), \psi(\mathbf{x}, t; \varepsilon)$, we denote $\varphi = O^*(\psi)$ if for any $m > 0$, there exist $J > 0$ and $\varepsilon(m) > 0$ such that for $0 < \varepsilon < \varepsilon(m)$

$$|\varphi(\mathbf{x}, t; \varepsilon)| < J\psi(\mathbf{x}, t; \varepsilon) \quad \text{for } 0 \leq t < \tau,$$

as long as $\tau < T_\varepsilon$ and

$$|u(t)|_{[\frac{k+1}{2}]+1} < \frac{m\varepsilon}{(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau. \quad (3.3)$$

By the similar argument in Section 2, it is sufficient to prove;

Lemma 3.1. $|u(t)|_{[\frac{k+1}{2}]} = O^*(\varepsilon(1+t)^{-1/2})$.

We conclude this section by proving Lemma 3.1. Denote by F the right-hand side of (1.1). By (2.1) we have

$$\square \Gamma^\sigma u = \sum_{\lambda \leq \sigma} C_\lambda \Gamma^\lambda F \quad (C_\sigma = 1).$$

Then we can write

$$\Gamma^\sigma u = W^\sigma + \sum_{\lambda \leq \sigma} C_\lambda E_\tau(\Gamma^\lambda F),$$

where W^σ is a solution of linear wave equation;

$$\begin{aligned} \square W^\sigma(\mathbf{x}, t) &= 0, \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times [0, \tau), \\ W^\sigma(\mathbf{x}, 0) &= \Gamma^\sigma u(\mathbf{x}, 0), \quad \partial_0 W^\sigma(\mathbf{x}, 0) = \partial_0 \Gamma^\sigma u(\mathbf{x}, 0), \quad \mathbf{x} \in \mathbb{R}^2. \end{aligned}$$

Since, as well known,

$$|W^\sigma(\mathbf{x}, t)| \leq \frac{C_\sigma \varepsilon}{(1+t)^{1/2}} \quad \text{for } 0 \leq t < \tau,$$

it is sufficient to show

$$E_\tau(\Gamma^\lambda F) = O^*(\varepsilon(1+t)^{-1/2}) \quad \text{for } |\lambda| \leq \left[\frac{k+1}{2}\right] + 1. \quad (3.4)$$

As stated in introduction, we may assume that F has a form in (1.16a-d). We verify (3.4) for each case.

Case (1.16a). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha\partial_\beta uQ(u'))) = O^*(\varepsilon(1+t)^{-1/2}), \quad (3.5)$$

$$E_\tau(\Gamma^\lambda(|u'|^3|u''|)) = O^*(\varepsilon(1+t)^{-1/2}), \quad (3.6)$$

where $Q(u') = (\partial_0 u)^2 - (\partial_1 u)^2 - (\partial_2 u)^2$. It follows from Proposition 2.2 and (3.3) that

$$\begin{aligned} \|u'(t)\|_k &\leq C\varepsilon \exp\left(C \int_0^t |u'(s)|_{[\frac{k+1}{2}]}^2 ds\right) \\ &\leq C\varepsilon \exp(Cm^2\varepsilon^2 \log(1+t)) \\ &\leq C\varepsilon(1+t)^p \quad \text{for } 0 \leq t < \tau, \end{aligned}$$

if $Cm^2\varepsilon^2 < p$. For later we shall assume that $p < \frac{1}{4}$. Hence we have

$$\|u'(t)\|_k = O^*(\varepsilon(1+t)^p). \quad (3.7)$$

Now we shall prove (3.5). Using Proposition 3.1 with $b = -1$, we get

$$(1+t)^{1/2}|E_\tau(\Gamma^\lambda(\partial_\alpha\partial_\beta uQ(u')))| \leq C \sum_{\mu+\nu=\lambda} I_\mu(t)J_\nu(t), \quad (3.8)$$

where

$$\begin{aligned} I_\mu(t) &= \left(\sum_{|\theta| \leq 1} \int_0^t \|\Gamma^\theta \Gamma^\mu \partial_\alpha \partial_\beta u(s)\|^2 (1+s)^{-2} ds \right)^{1/2}, \\ J_\nu(t) &= \left(\sum_{|\theta| \leq 1} \int_0^t \|\Gamma^\theta \Gamma^\nu Q(u'(s))\|^2 (1+s) ds \right)^{1/2}. \end{aligned}$$

Since $k \geq 9$, $|\theta| + |\mu| + 1 \leq [\frac{k+1}{2}] + 3 \leq k$. Therefore, by (3.7) we get

$$I_\mu(t) = O^* \left(\left(\int_0^t \varepsilon^2 (1+s)^{2p-2} ds \right)^{1/2} \right) = O^*(\varepsilon). \quad (3.9)$$

On the other hand, since

$$Q(u') = t^{-1}(L_0 u \partial_0 u - L_i u \partial_i u),$$

$\Gamma^\theta \Gamma^\nu Q(u')$ is represented as the sum of

$$\Gamma^\rho(t^{-1})\Gamma^\eta(L_\alpha u)\Gamma^\xi(\partial_\alpha u), \quad \rho + \eta + \xi = \theta + \nu, \quad \alpha = 0, 1, 2.$$

We can verify in the support of u

$$\Gamma^\rho(t^{-1}) \leq Ct^{-1}(1+t^{-|\rho|}). \quad (3.10)$$

Moreover we find that

$$\|\Gamma^\eta(L_\alpha u(t))\Gamma^\xi(\partial_\alpha u(t))\| \leq C|u(t)|_{[\frac{\ell+1}{2}]} \|u'(t)\|_\ell, \quad (3.11)$$

where $\ell = [\frac{k+1}{2}] + 3$. Indeed, if we set $\zeta = \eta + \chi$ ($|\chi| = 1$, $\Gamma^\chi = L_\alpha$), we obtain $|\zeta + \xi| \leq [\frac{k+1}{2}] + 3 = \ell$. If $|\xi| \geq [\frac{\ell+1}{2}]$, then $|\zeta| \leq [\frac{\ell+1}{2}]$ and (3.11) holds. If $|\xi| < [\frac{\ell+1}{2}]$, i.e., $|\xi| \leq [\frac{\ell+1}{2}] - 1$, we have by using Proposition 3.2

$$\begin{aligned} \|\Gamma^\zeta u(t)\Gamma^\xi \partial_\alpha u(t)\| &\leq C \sum_{|\iota|=1} |\Gamma^{\zeta+\iota} u(t)| \|\Gamma^\zeta u'(t)\| \\ &\leq C|u(t)|_{[\frac{\ell+1}{2}]} \|u'(t)\|_\ell, \end{aligned}$$

which implies (3.11). Since $k \geq 9$, we get $\ell < k$ and $[\frac{\ell+1}{2}] \leq [\frac{k+1}{2}]$. Then, by (3.3) and (3.7)

$$\begin{aligned} \|\Gamma^\zeta u(t)\Gamma^\xi \partial_\alpha u(t)\| &\leq C|u(t)|_{[\frac{k+1}{2}]} \|u'(t)\|_k \\ &= O^*(\varepsilon(1+t)^{p-1/2}). \end{aligned} \quad (3.12)$$

Combining (3.10) and (3.12), we get

$$\|\Gamma^\theta \Gamma^\nu Q(u'(t))\| = O^*(\varepsilon t^{-1}(1+t^{-|\theta+\nu|})(1+t)^{p-1/2}). \quad (3.13)$$

On the other hand, as shown in section 2,

$$\|\Gamma^\theta \Gamma^\nu Q(u'(t))\| \leq C\varepsilon \quad \text{for } 0 \leq t \leq 1. \quad (3.14)$$

Hence by (3.13) and (3.14)

$$\begin{aligned} J_\nu(t) &= \left(\sum_{|\theta| \leq 1} \int_0^1 \|\Gamma^\theta \Gamma^\nu Q(u'(s))\|^2 (1+s) ds + \sum_{|\theta| \leq 1} \int_1^t \|\Gamma^\theta \Gamma^\nu Q(u'(s))\|^2 (1+s) ds \right)^{1/2} \\ &\leq \left(C\varepsilon + O^* \left(\int_1^t \varepsilon^2 (1+s)^{2p-2} ds \right) \right)^{1/2} = O^*(\varepsilon). \end{aligned} \quad (3.15)$$

Therefore (3.5) follows from (3.9) and (3.15).

Next we shall prove (3.6). Using Proposition 3.1 with $b = \frac{1}{2}$, we get

$$(1+t)^{1/2} |E_\tau(\Gamma^\lambda(|u'|^3|u''|))| \leq C \sum_{\mu+\nu=\lambda} I_\mu(t) J_\nu(t), \quad (3.16)$$

where

$$I_\mu(t) = \left(\sum_{|\theta| \leq 1} \int_0^t \|\Gamma^\theta \Gamma^\mu(|u'(s)|^2)\|^2 (1+s)^{-1/2} ds \right)^{1/2},$$

$$J_\nu(t) = \left(\sum_{|\theta| \leq 1} \int_0^t \|\Gamma^\theta \Gamma^\mu(|u'(s)| |u''(s)|)\|^2 (1+s)^{-1/2} ds \right)^{1/2}.$$

Since $|\theta + \mu| \leq [\frac{k+1}{2}] + 2 \leq k$, we can verify

$$\|\Gamma^\theta \Gamma^\mu(|u'(s)|^2)\| \leq C |u(s)|_{[\frac{k+1}{2}]+1} \|u'(s)\|_k.$$

Using (3.3) and (3.7), we get

$$\|\Gamma^\theta \Gamma^\mu(|u'(s)|^2)\| \leq C m \varepsilon^2 (1+s)^{p-1/2} = O^*(\varepsilon (1+s)^{p-1/2}).$$

Then we have

$$I_\mu(t) = O^* \left(\left(\int_0^t \varepsilon^2 (1+s)^{2p-3/2} ds \right)^{1/2} \right) = O^*(\varepsilon). \quad (3.17)$$

Similarly we also find that

$$J_\nu(t) = O^*(\varepsilon). \quad (3.18)$$

Therefore (3.16), (3.17) and (3.18) imply (3.6).

Case (1.16b). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u \partial_\beta Q(u'))) = O^*(\varepsilon(1+t)^{-1/2}).$$

This can be obtained similarly to (3.5), by using (3.10), (3.11) and Proposition 3.1, 3.2 but $\ell = [\frac{k+1}{2}] + 4$ in (3.11).

Case (1.16c). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u \partial_\beta u \square u)) = O^*(\varepsilon(1+t)^{-1/2}).$$

Using (1.1), we get

$$\begin{aligned} \Gamma^\lambda(\partial_\alpha u \partial_\beta u \square u) &= \Gamma^\lambda(\partial_\alpha u \partial_\beta u (\partial_\gamma u \partial_\delta u \square u + O(|u'|^3 |u''|))) \\ &= O(\Gamma^\lambda(|u'|^4 |u''|)). \end{aligned}$$

Therefore this case can be reduced to (3.6).

Case (1.16d). We have to show

$$E_\tau(\Gamma^\lambda(\partial_\alpha u (\partial_\beta u \partial_\gamma \partial_\delta u - \partial_\gamma u \partial_\beta \partial_\delta u))) = O^*(\varepsilon(1+t)^{-1/2}).$$

Using (2.32) and (2.34), we have

$$\begin{aligned} \partial_\alpha u (\partial_\beta u \partial_\gamma \partial_\delta u - \partial_\gamma u \partial_\beta \partial_\delta u) &= O(t^{-1}(|u'|^2 |u'|_1 + |u'| |u''| |u|_1)) \\ &= O(t^{-1} |u'| |u|_1 |u'|_1). \end{aligned}$$

Therefore this case can be verified similarly to (3.5) as $Q = t^{-1} |u'| |u|_1$, then the proof of Theorem 2 is complete.

Appendix. If $H = 0$, we find from the definition of H (1.14) that

$$C(-1, \omega) \partial_\rho \mathcal{F}(\omega, \rho) \partial_\rho^2 \mathcal{F}(\omega, \rho) = \frac{1}{2} C(-1, \omega) \partial_\rho ((\partial_\rho \mathcal{F}(\omega, \rho))^2) \geq 0,$$

for any $\omega \in S^1$ and $\rho \in \mathbb{R}$. For fixed ω , if $C(-1, \omega) \neq 0$, then $\partial_\rho ((\partial_\rho \mathcal{F}(\omega, \rho))^2)$ is of constant sign in ρ . Using (1.10) and (1.11), we find that $\partial_\rho \mathcal{F}(\omega, \rho) = 0$ for any $\rho \in \mathbb{R}$, i.e., $\mathcal{F}(\omega, \rho) \equiv \text{const}$ in ρ . Therefore (1.10) implies $\mathcal{F}(\omega, \rho) = 0$ for any $\rho \in \mathbb{R}$.

We set

$$\Omega = S^1 \setminus \{\omega \in S^1 \mid C(-1, \omega) = 0 \text{ and } \mathcal{F}(\omega, \rho) = 0 \text{ for any } \rho \in \mathbb{R}\}.$$

We claim that Ω is either the set of ω such that $C(-1, \omega) = 0$, or the set of ω such that $\mathcal{F}(\omega, \rho) = 0$ for any $\rho \in \mathbb{R}$. Set

$$F = \{\omega \in \Omega \mid C(-1, \omega) \neq 0\}.$$

Clearly F is open in Ω . On the other hand, by the above argument we have

$$F = \{\omega \in \Omega \mid \mathcal{F}(\omega, \rho) = 0 \text{ for any } \rho \in \mathbb{R}\}.$$

Then we find that F is also closed in Ω . Therefore F is equal to either \emptyset or Ω . When $F = \emptyset$, Ω is the set of ω such that $C(-1, \omega) = 0$. When $F = \Omega$, Ω is the set of ω such that $\mathcal{F}(\omega, \rho) = 0$ for any $\rho \in \mathbb{R}$.

Since

$$S^1 = \Omega \cup \{\omega \in S^1 \mid C(-1, \omega) = 0 \text{ and } \mathcal{F}(\omega, \rho) = 0 \text{ for any } \rho \in \mathbb{R}\},$$

either " $C(-1, \omega) = 0$ for any $\omega \in S^1$ " or " $\mathcal{F}(\omega, \rho) = 0$ for any $\omega \in S^1$ and $\rho \in \mathbb{R}$ ", when $H = 0$. Moreover if $\mathcal{F}(\omega, \rho) = 0$ for any $\omega \in S^1$ and $\rho \in \mathbb{R}$, then, by (1.9),

$$\begin{aligned} \mathcal{F}(\omega, \rho) + \mathcal{F}(-\omega, -\rho) &= \frac{1}{\sqrt{2\pi}} \int_{\rho}^{\infty} (s - \rho)^{-1/2} R_g(\omega, s) ds = 0, \\ \mathcal{F}(\omega, \rho) - \mathcal{F}(-\omega, -\rho) &= -\frac{1}{\sqrt{2\pi}} \int_{\rho}^{\infty} (s - \rho)^{-1/2} \partial_s R_f(\omega, s) ds = 0. \end{aligned}$$

Using Tichmarsh's Theorem (see [14], p.166), Radon problem (see [5], p.162) and integrating by parts, we find that f and g vanish identically. Therefore the condition " $H = 0$ " implies either " $C(-1, \omega) = 0$ for any $\omega \in S^1$ " or " f and g vanish identically".

Acknowledgement. The author is grateful to Professor Rentaro Agemi for his suggestion of this problem and his useful comments.

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