SINGULAR DEGENERATE PARABOLIC EQUATIONS WITH APPLICATIONS TO GEOMETRIC EVOLUTIONS

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Series ♯155. June 1992

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SINGULAR DEGENERATE PARABOLIC EQUATIONS WITH APPLICATIONS TO GEOMETRIC EVOLUTIONS

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ABSTRACT. We prove a comparison theorem for viscosity solutions of degenerate parabolic equations which is singular at finite directions of derivatives. We apply our theorem to construct a global generalized evolution for interfaces equations with a certain class of the interface energy not necessarily C^2.

1. Introduction. We are concerned with a degenerate parabolic equation of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega, \tag{1.1}$$

where Ω is a bounded domain in R^n and $T > 0$. The function $F(p, X)$ is allowed to have singularities when p belongs to finitely many half lines ℓ_i of the form

$$\ell_i = \{\eta q_i \ ; \ \eta \geq 0\}, \quad q_i \in R^n \backslash \{0\}, \quad i = 1, \cdots, m.$$

As explained later such an F naturally arises in a level set approach of motion of phase boundaries. Here $u_t = \partial u/\partial t, \nabla u$ and $\nabla^2 u$ denote, respectively, the time derivative of u, the gradient of u and the Hessian of u in space variables.

Our first goal is to establish a comparison principle for viscosity solutions of (1.1). If F has singularities only for $p = 0$, a comparison principle is established in [5] assuming that F can be extended continuously at $(p, X) = (0, O)$; See [11] for simplification of the proof. (The paper [6] includes corrections of technical errors in [5], [11]).

AMS Subject Classifications : 35K22, 35K65, 82D35 .

Although we still appeal to Crandall-Ishii's lemma [7], the method in [11] or [5] does not apply to our setting because F has singularities other than $p = 0$. By a clever choice of "test function" we shall prove a comparison principle under assumptions on the value of semicontinuous envelope of F at $(\mu q_i, \nu q_i \otimes q_i)$, $\mu > 0$, $\nu \in \mathbf{R}$, where $\otimes$ denotes the tensor product.

Our second goal is to apply our comparison results to geometric evolutions. Let Γ_t denote the hypersurface expressed as the boundary of a bounded open set D_t in $\mathbf{R}^n$ $(n \geqq 2)$ at time t. Let n denote the unit exterior normal vector field on $\Gamma_t = \partial D_t$. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The geometric evolution of Γ_t studied in [2], [3] is of the form

$$V = \frac{1}{\beta(n)}\left(-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial p_i}(n)\right) + c\right), \tag{1.2}$$

where H is positively homogeneous of degree one, β is a positive function on a unit sphere S^{n-1} in $\mathbf{R}^n$ and c is a constant.

A level set approach is to regard Γ_t as the zero-level set of an auxiliary function $u : (0,T) \times \Omega \to \mathbf{R}$ of the evolution equation

$$\begin{aligned} &u_t - \operatorname{trace}\left(A\left(\frac{\nabla u}{|\nabla u|}\right)\left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right)\nabla^2 u\left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right)\right) + B(\nabla u) = 0, \\ &A(\overline{p}) = \frac{1}{\beta(-\overline{p})}\left(\frac{\partial^2 H}{\partial p_i \partial p_j}(-\overline{p})\right), \quad \overline{p} = \frac{p}{|p|}, \\ &B(p) = -\frac{1}{\beta(-\overline{p})}|p|. \end{aligned} \tag{1.3}$$

Here Ω is taken so that Γ_t stays in Ω for $t \in (0,T)$ and u is taken so that $u > 0$ on D_t and $u < 0$ outside $\Gamma_t \cup D_t$.

A fundamental analytic question related to (1.2) and (1.3) is to construct a global in time unique generalized solution $\{\Gamma_t\}_{t \geq 0}$ for a given initial data Γ_0. Chen, Giga and Goto [5] have adapted the theory of viscosity solutions to construct unique global generalized solutions to the equation (1.3) when β is continuous and $H \in C^2(\mathbf{R}^n \backslash \{0\})$ is convex not necessarily strictly convex. Moreover, they proved the zero-level set Γ_t of u of (1.3) is determined by Γ_0 and independent of initial value of u. This yields a global unique generalized evolution to (1.2). Nearly at the same time Evans and Spruck [9] carried out this programme in a slightly different way and only for the mean curvature flow equation.

For the history of level set approach as well as its recent development we refer to [1], [4] and references therein.

In physics there is also the possibility that H is not convex as studied in [2], [3]. If H is not convex, the equation (1.3) is no longer parabolic and not well-posed. It seems to be natural to consider the convexification $\hat{H}$ when H is not convex. The problem is that the convexification $\hat{H}$ of a function H may be no longer C^2 away from zero even if H is smooth. So the equation (1.3) may have singularities other than at $\nabla u = 0$. Our comparison theory does apply to (1.3) with $H = \hat{H}$ provided that $\hat{H}$ is singular at most finitely many directions and that the derivative of $\hat{H}$ is locally Lipschitz outside zero. Once the comparison principle is established for (1.3) with $H = \hat{H}$, we can adapt the theory in [5] of constructing global unique generalized solutions of (1.2) with $H = \hat{H}$.

Angenent and Gurtin [3] solved such an equation (1.2) with $H = \hat{H}$ for $n = 2$ at least locally if each normal of initial curve (with corners) lies in the direction that the curvature of $\hat{H} = 1$ is positive. Our theory applies to their setting. Moreover we allow that normal of initial curve lies in the direction that the curvature of $\hat{H} = 1$ is zero.

In Section 2 we shall establish a comparison principle on a bounded domain for the equation (1.1). We remark the case when Ω is an unbounded domain. In Section 3 we show that the theorem in Section 2 applies to the evolution equation (1.2) so that we get a unique global solution for a given initial data Γ_0.

During this work is prepared we learned that a comparisom theorem for nonsmooth interfacial energy is obtained by Giga [12] when the interface is a graph of a function on $\mathbf{R}$. After this work was completed, we learned a recent work of Gurtin, Soner and Souganidis [13] closely related to ours. They also proved a comparison principle for (1.3) with $H = \hat{H}$, but the proof differs from ours. They also proved that generalized solution is consistent with solutions of Angenent and Gurtin [3].

Acknowledgement : The authors are grateful to Professor Yoshikazu Giga who brought this problem to their attention. The authors are also grateful to Professor Panagiotis E. Souganidis for pointing out a technical error in the first version of their manuscript.

This work was done while the second author was a JSPS fellow for Japanese Junior Scientists. The work of the second author was partly supported by the Japan Ministry of Education, Science and Culture through grant no.3316.

2. Comparison theorem. Let Ω be a bounded domain in $\mathbf{R}^n$ and let T be a positive number. We consider a degenerate parabolic equation of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega. \tag{2.1}$$

For $i = 1, \cdots, m$ let ℓ_i be a half line in $\mathbf{R}^n$ of the form

$$\ell_i = \{\eta q_i; \eta \geq 0\}, \quad \text{where} \quad q_1, \cdots, q_m \in \mathbf{R}^n \backslash \{0\}.$$

We list assumptions on $F = F(p, X)$.

(F1) $$F : (\mathbf{R}^n \backslash \bigcup_{i=1}^{m} \ell_i) \times \mathbf{S}^n \longrightarrow \mathbf{R} \text{ is continuous,}$$
where $\mathbf{S}^n$ denotes the space of real $n \times n$ symmetric matrices.

(F2) F is degenerate elliptic, i.e.,
$$F(p, X+Y) \leq F(p, X) \quad \text{for all} \quad Y \geq 0.$$

(F3) $$-\infty < F_*(\mu q_i, \nu q_i \otimes q_i) = F^*(\mu q_i, -\nu q_i \otimes q_i) < +\infty$$
$$\mu > 0, \ \nu > 0 \quad i = 1, \cdots, m,$$

(F4) $$-\infty < F_*(0, O) = F^*(0, O) < +\infty,$$

where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $\mathbf{R}^n \times \mathbf{S}^n$, respectively, i.e.,

$$F_*(p, X) = \liminf_{\varepsilon \downarrow 0} \{F(r, Y); r \in (\mathbf{R}^n \backslash \bigcup_{i=1}^{m} \ell_i), \ |p - r| < \varepsilon, |X - Y| < \varepsilon\}$$

and $F^* = -(-F)_*$. Here $|X|$ denotes the operator norm of X as a self-adjoint operator on $\mathbf{R}^n$; $\otimes$ denotes a tensor product of vector in $\mathbf{R}^n$.

The assumption (F1) allows the possibility that (2.1) is singular at $\nabla u = \eta q_i$ ($i = 1, \cdots, m$). The equation (2.1) is called degenerate parabolic if (F2) holds.

We recall one of equivalent definitions of viscosity sub- and supersolutions of (2.1) (cf. [8]). A function $u : Q \to \boldsymbol{R}$ is called a *viscosity sub-(super)solution* of (2.1) in Q if $u^* < \infty$ (resp. $u_* > -\infty$) in $\overline{Q}$ and

$$\tau + F_*(p, X) \leq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}_Q^{2,+} u^*(t, x), (t, x) \in Q$$

(resp. $\tau + F^*(p, X) \geq 0$ for all $(\tau, p, X) \in \mathcal{P}_Q^{2,-} u_*(t, x), (t, x) \in Q$). Here $\mathcal{P}_Q^{2,+}$ denotes the *parabolic super 2-jet* in Q, i.e., $\mathcal{P}_Q^{2,+} u(t, x)$ is the set of $(\tau, p, X) \in \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$ such that

$$u(s, y) \leq u(t, x) + \tau(s - t) + < p, y - x > + \frac{1}{2} < X(y - x), y - x >$$
$$+ o(|s - t| + |y - x|^2) \quad \text{as} \quad (s, y) \to (t, x) \quad \text{in} \quad Q,$$

where $< \, , \, >$ denotes the Euclidean inner product; similarly, $\mathcal{P}_Q^{2,-} u = -\mathcal{P}_Q^{2,+}(-u)$. For $U = (0, T) \times D$, the set

$$\partial_p U = \{0\} \times D \cup [0, T] \times \partial D$$

is often called the parabolic boundary of U. We are now in position to state our main comparison theorem. We often suppress the word "viscosity", except in statements of theorems.

Theorem 2.1. *Suppose that Ω is bounded domain in $\boldsymbol{R}^n$ and that F satisfies (F1)-(F4). Let u and v be, respectively, viscosity sub- and supersolutions of (2.1) in $Q = (0, T) \times \Omega$. If $u^* \leq v_*$ on $\partial_p Q$, then $u^* \leq v_*$ in Q.*

We shall prove Theorem 2.1 in several steps.

The basic strategy of the proof of Theorem 2.1 is similar to the case when $F(p, X)$ has singularities only on $\{p = 0\}$ (cf. [11]). We argue by contradiction. Roughly speaking we shall find a parabolic super 2-jet of

$$w(t, x, y) = u(t, x) - v(t, y)$$

at a point (t, x, y) where $u^*(t, x) - v_*(t, y) > 0$ and x is close to y.

We should find a nice parabolic super 2-jet of w. For this purpose we introduce a test function $\Psi(t,x,y)$ and study the maximum of $w-\Psi$. When F has singularities only on $\{p=0\}$, a suitable choice of Ψ is

$$\Psi(t,x,y)=\frac{|x-y|^4}{4\varepsilon}+\frac{\sigma}{T-t}$$

with small $\varepsilon,\sigma>0$ (cf. [11]). This choice is not appropriate in our present situation because of singularities of F on half lines. We shall construct a suitable test function Ψ.

For vectors $\{q_i\}_{i=1}^m$ ($q_i\in\mathbf{R}^n\backslash\{0\}$) we take convex set M satisfying the following properties.

$$M \text{ is closed convex set in } \mathbf{R}^n \text{ and contains neighborhood of zero;} \tag{2.2a}$$

$$\text{the boundary } \partial M \text{ is } C^2; \tag{2.2b}$$

$$\text{if } n(x)=q_i/|q_i| \text{ at } x\in\partial M\ (i=1,\cdots,m), \text{ then } <\tau(x),\nabla>n(x)=0. \tag{2.2c}$$

Here n is a unit exterior normal C^1 vector field on ∂M and $\tau(x)$ is a unit tangent vector at $x\in\partial M$. We can easily construct a convex set M satisfying (2.2a)-(2.2c).

For this convex set M we define the Minkowski function

$$P_M(x)=\inf\{\alpha\ ;\ \alpha>0,\ \alpha^{-1}x\in M\}.$$

We note that P_M has C^2 regularity outside of origin. From now on we shall suppress the subscript M. Let ε and σ be positive constants and we shall use

$$\Psi(t,x,y)=\frac{1}{4\varepsilon}(P(x-y))^4+\frac{\sigma}{T-t}$$

as a test function. We note that $c_1|x|\le P(x)\le c_2|x|$ with $0<c_1\le c_2$ by (2.2a). This implies that Ψ is C^2 even at $x=y$. Since Ψ depends on x and y through $x-y$ the following identities are trivially obtained.

Lemma2.2. *Let P be as above. Then*

$$\nabla_x(P(x-y))^4=-\nabla_y(P(x-y))^4, \tag{2.3}$$

$$\begin{aligned}\nabla_{xx}^2\Psi(t,x,y)&=\nabla_{yy}^2\Psi(t,x,y)\\&=-\nabla_{xy}^2\Psi(t,x,y)=-\nabla_{yx}^2\Psi(t,x,y),\end{aligned} \tag{2.4}$$

where $\nabla^2_{xx}, \nabla^2_{yy}, \nabla^2_{xy}, \nabla^2_{yx}$ *denote the Hessian operator in space variables* (x,x), (y,y), (x,y), (y,x), *respectively.*

We set

$$w(t,x,y) = u(t,x) - v(t,y) \quad \text{for } (t,x,y) \in \overline{U} \text{ with } U = (0,T) \times \Omega \times \Omega.$$

Proposition 2.3. *Suppose that* w *is upper semicontinuous (u.s.c) in* $\overline{U}$, $w < \infty$ *in* $\overline{U}$ *and that*

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta,\ (t,x,y) \in \overline{U}, t < T\} > 0. \tag{2.5}$$

Set $\Phi(t,x,y) = w(t,x,y) - \Psi(t,x,y)$, *then there is a positive constant* σ_0 *such that*

$$\sup_{\overline{U}} \Phi(t,x,y) > \frac{\alpha}{2} \tag{2.6}$$

holds for all $0 < \sigma < \sigma_0,\ \varepsilon > 0$.

Proof. Since w is u.s.c and $\overline{U}$ is compact, we see $\alpha < \infty$. Moreover we easily see $\sup_{\overline{U}} w(t,x,x) = \alpha$. By (2.5) there is a point (t_0, x_0, x_0) $(t_0 < T)$ such that $w(t_0,x_0,x_0) > 3\alpha/4$ and $\sigma/(T-t_0) < \alpha/4$ if σ is sufficiently small. We now observe that $\Phi(t_0,x_0,x_0) > \alpha/2$. □

Let $(\hat{t}, \hat{x}, \hat{y}) \in \overline{U}$ be a maximum point of Φ, i.e.,

$$\sup_{\overline{U}} \Phi(t,x,y) = \Phi(\hat{t}, \hat{x}, \hat{y}).$$

Proposition 2.4. *Let* σ_0 *be as in Proposition 2.3. Suppose that* w *is u.s.c in* $\overline{U}$.

(i) $(P(\hat{x} - \hat{y}))^4$ *tends to zero as* $\varepsilon \to 0$; *the convergence is uniform in* $0 < \sigma < \sigma_0$.

(ii) $|\hat{x} - \hat{y}|$ *tends to zero as* $\varepsilon \to 0$; *the convergence is uniform in* $0 < \sigma < \sigma_0$.

Proof.

(i) From (2.6) it follows $\Phi(\hat{t},\hat{x},\hat{y})>0$ for $0<\sigma<\sigma_0$, $\varepsilon>0$. This yields

$$\begin{aligned} w(\hat{t},\hat{x},\hat{y}) &\geq \frac{1}{4\varepsilon}(P(\hat{x}-\hat{y}))^4+\frac{\sigma}{T-\hat{t}} \\ &\geq \frac{1}{4\varepsilon}(P(\hat{x}-\hat{y}))^4. \end{aligned}$$

Since U is a bounded domain and since w is u.s.c, there is a positive constant M such that

$$u(t,x)-v(t,y)\leq M \quad \text{in} \quad \overline{U}.$$

We now observe

$$\frac{1}{4\varepsilon}(P(\hat{x}-\hat{y}))^4\leq M, \tag{2.7}$$

which yields (i) as $\varepsilon\to 0$.

(ii) Since $P(x)$ is comparable with $|x|$,

$$\text{if} \quad (P(x-y))^4\longrightarrow 0 \quad \text{as} \quad \varepsilon\to 0, \text{ then } \quad |\hat{x}-\hat{y}|\longrightarrow 0 \quad \text{as} \quad \varepsilon\to 0. \qquad \square$$

Proposition 2.5. *Let σ_0 be as in Proposition 2.3 and σ be $0<\sigma<\sigma_0$. Suppose that w is u.s.c in $\overline{U}$. Then*

$$\frac{1}{4\varepsilon}(P(\hat{x}-\hat{y}))\to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{2.8}$$

Proof. By (2.7) we observe that

$$\frac{1}{4\varepsilon}(P(\hat{x}(\varepsilon)-\hat{y}(\varepsilon)))^4\to\xi \quad \text{as} \quad \varepsilon\to 0 \tag{2.9}$$

for some non-negative number ξ if we take a subsequence. By Proposition 2.4 (ii) and boundedness of Ω

$$\hat{t}(\varepsilon)\to\bar{t},\ \hat{x}(\varepsilon),\ \hat{y}(\varepsilon)\to\bar{z} \quad \text{as} \quad \varepsilon\to 0 \tag{2.10}$$

for some $\bar{t}\in[0,T]$, $\bar{z}\in\overline{\Omega}$ if we take a subsequence $\varepsilon=\varepsilon_j\to 0$. By the definition of the point $(\hat{t},\hat{x},\hat{y})$ we have

$$\Phi(t,x,y)\leq\Phi(\hat{t}_j,\hat{x}_j,\hat{y}_j),$$

where $\hat{t}_j=\hat{t}(\varepsilon_j)$ and so on. Plunging $t=\bar{t}, x=y=\bar{z}$ in this inequality, we obtain

$$\begin{aligned} &u^*(\bar{t},\bar{z})-v_*(\bar{t},\bar{z})-\frac{\sigma}{T-\bar{t}} \\ &\leq u^*(\hat{t}_j,\hat{x}_j)-v_*(\hat{t}_j,\hat{y}_j)-\frac{1}{4\varepsilon}(P(\hat{x}_j-\hat{y}_j))^4-\frac{\sigma}{T-\hat{t}_j}. \end{aligned} \tag{2.11}$$

From (2.9) letting $\varepsilon_j \to 0$ in (2.11) yields

$$\begin{aligned} &u^*(\bar{t},\bar{z}) - v_*(\bar{t},\bar{z}) - \frac{\sigma}{T-\bar{t}} \\ &\le \varlimsup_{\varepsilon_j\to 0} \left(u^*(\hat{t}_j,\hat{x}_j) - v_*(\hat{t}_j,\hat{y}_j) - \frac{\sigma}{T-\hat{t}_j} \right) - \xi. \end{aligned}$$

Since $u^* - v_*$ is upper semicontinuous, from (2.10) it follows $\xi \le 0$. Since the limit in (2.9) is independent of the choice of subsequence, the convergence (2.9) now yields (2.8). □

Proposition 2.6. *Assume the hypotheses of Proposition 2.4. Suppose that $u^* \le v_*$ on $\partial_p Q$. There is $\varepsilon_0 > 0$ such that Φ attains a maximum over $\overline{U}$ at an interior point $(\hat{t},\hat{x},\hat{y})$ of U, i.e., $(\hat{t},\hat{x},\hat{y}) \in (0,T)\times\Omega\times\Omega$ for all $0<\varepsilon<\varepsilon_0$ and $0<\sigma<\sigma_0$.*

Proof. Suppose that the conclusion were false. By the properties of barrier function $\sigma/(T-t)$ we see $\hat{t} < T$. There would exist sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, $\{\sigma_j\} \subset (0,\sigma_0)$ such that $\partial_p U$ contains a maximum point $(\hat{t}_j,\hat{x}_j,\hat{y}_j)$ of Φ for the value $\varepsilon = \varepsilon_j, \sigma = \sigma_j$. By (2.6) we see

$$\frac{\alpha}{2} \le \Phi(\hat{t}_j,\hat{x}_j,\hat{y}_j) \le w(\hat{t}_j,\hat{x}_j,\hat{y}_j).$$

By the definition of sub- (super- resp.) solution and the boundedness of Ω, replacing u (v resp.) by $\{\max(u(t,x),-L)\}^*$, ($\{\min(v(t,x),L)\}_*$ resp.) for sufficiently large L we may assume that u (v resp.) is bounded u.s.c (l.s.c resp.) on $\overline{Q}$. Since U is bounded, the assumption $u^* \le v_*$ on $\partial_p Q$ implies that there is a modulus function m (i.e., $m : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $m(0) = 0$) such that $u(t,x) - v(t,y) \le m(|x-y|)$ on $\partial_p U$. We have

$$w(\hat{t}_j,\hat{x}_j,\hat{y}_j) \le m(|\hat{x}_j - \hat{y}_j|).$$

Since $\varepsilon_j \to 0$, applying Proposition 2.4 (ii) yields $|\hat{x}_j - \hat{y}_j| \to 0$, which leads a contradiction $0 < \alpha/2 \le 0$. □

The following is a variant of Crandall-Ishii's lemma [7].

Lemma 2.7. *Let u_i be a viscosity solution of*

$$u_t + F_i(\nabla u, \nabla^2 u) = 0 \qquad (2.12)$$

in a neighborhood of $(s, z_i) \in (0,T) \times \boldsymbol{R}^{N_i}$ *for* $i = 1, 2, \cdots, k$, *where* $F_i : \boldsymbol{R}^{N_i} \times S^{N_i} \to \boldsymbol{R} \cup \{+\infty\}$ *is lower semicontinuous. Let* w *be a function in* $(0,T) \times \boldsymbol{R}^N$ *given by*

$$w(s,z) = \sum_{i=1}^{k} u_i(s, z_i) \quad \text{for} \quad z = (z_1, \cdots, z_k) \in \boldsymbol{R}^N,$$

where $N = N_1 + \cdots + N_k$. *Let*

$$(\tau, p, A) \in \mathcal{P}^{2,+} w(s,z),$$

where $p = (p_1, \cdots, p_k), z = (z_1, \cdots, z_k)$. *Then for each* $\lambda > 0$ *there exists* $X_i \in S^{N_i}$ *such that*

$$\tau + \sum_{i=1}^{k} F_i(p_i, X_i) \le 0$$

and

$$-\left(\frac{1}{\lambda} + |A|\right) I \le \begin{pmatrix} X_1 & & O \\ & \ddots & \\ O & & X_k \end{pmatrix} \le A + \lambda A^2,$$

where I *denotes the identity matrix.*

Remark 2.8. This lemma is Lemma 2.10 in [11]. Here and hereafter the subscript of $\mathcal{P}^{2,+}$ is suppressed. The bar over $\mathcal{P}^{2,+}$ means the closure. Although the domain considered here is $\boldsymbol{R}^{N_i}$, it is easily seen that the result is local and that $\boldsymbol{R}^{N_i}$ may be replaced by a neighborhood of $z_i \in \boldsymbol{R}^{N_i}$.

Proof of Theorem 2.1. We may assume that u and v are, respectively, upper and lower semicontinuous so that

$$w(t,x,y) = u(t,x) - v(t,y)$$

is upper semicontinuous in $\overline{U}$. We will deduce contradiction supposing that

$$\alpha = \lim_{\epsilon \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, \quad (t,x,y) \in \overline{U}, t < T\} > 0.$$

By (2.5) we see all conclusions Proposition 2.3 - 2.6 would hold for $\Phi = w - \Psi$. Proposition 2.6 says that Φ attains a maximum over $\overline{U}$ at $(\hat{t}, \hat{x}, \hat{y}) \in (0,T) \times \Omega \times \Omega$ for small ε, σ. In particular

$$w(t,x,y) \le w(\hat{t}, \hat{x}, \hat{y}) + \Psi(t,x,y) - \Psi(\hat{t}, \hat{x}, \hat{y}) \quad \text{in } U.$$

Expanding Ψ at $(\hat{t},\hat{x},\hat{y})$ yields

$$\left(\hat{\Psi}_t, \begin{pmatrix}\hat{\Psi}_x\\ \hat{\Psi}_y\end{pmatrix}, A\right) \in \mathcal{P}^{2,+}w(\hat{t},\hat{x},\hat{y}) \tag{2.13}$$

with $A = \begin{pmatrix}\hat{\Psi}_{xx} & \hat{\Psi}_{xy}\\ \hat{\Psi}_{yx} & \hat{\Psi}_{yy}\end{pmatrix}$, where $\hat{\Psi}_t = \partial_t\Psi(\hat{t},\hat{x},\hat{y}), \hat{\Psi}_x = \nabla_x\Psi(\hat{t},\hat{x},\hat{y}), \hat{\Psi}_{xx} = \nabla^2_{xx}\Psi(\hat{t},\hat{x},\hat{y})$ and so on.

We shall apply Lemma 2.7 with $k = 2, u_1 = u, u_2 = -v, s = \hat{t}, z = (\hat{x},\hat{y})$. Since u and v are, respectively, sub- and supersolution of (2.1) and since $(\hat{t},\hat{x},\hat{y})$ is an interior point of U, by Remark 2.8 we now apply Lemma 2.7 and conclude that for each $\lambda > 0$ there are $X, Y \in S^n$ such that

$$\hat{\Psi}_t + F_*(\hat{\Psi}_x, X) - F^*(-\hat{\Psi}_y, -Y) \le 0 \tag{2.14}$$

and

$$-\left(\frac{1}{\lambda} + |A|\right)I \le \begin{pmatrix}X & O\\ O & Y\end{pmatrix} \le A + \lambda A^2 \tag{2.15}$$

where $\hat{\Psi}_t = \partial_t\Psi(\hat{t},\hat{x},\hat{y}), \hat{\Psi}_x = \nabla_x\Psi(\hat{t},\hat{x},\hat{y})$, etc. Calculating $\hat{\Psi}_x$, $\hat{\Psi}_y$, $\hat{\Psi}_{xx}$, $\hat{\Psi}_{xy}$, $\hat{\Psi}_{yy}$, $\hat{\Psi}_t$ by using Lemma 2.2, (2.13) becomes

$$\left(\frac{\sigma}{(T-\hat{t})^2}, \begin{pmatrix}\frac{1}{\varepsilon}\hat{P}^3\hat{P}_x\\ -\frac{1}{\varepsilon}\hat{P}^3\hat{P}_x\end{pmatrix}, \frac{1}{\varepsilon}\begin{pmatrix}J & -J\\ -J & J\end{pmatrix}\right) \in \mathcal{P}^{2,+}w(\hat{t},\hat{x},\hat{y}) \tag{2.16}$$

with $J = 3\hat{P}^2(\hat{P}_x \otimes \hat{P}_x) + \hat{P}^3\hat{P}_{xx}$, where $\hat{P} = P(\hat{x}-\hat{y})$, $\hat{P}_x = (\nabla_x P)(\hat{x}-\hat{y})$, $\hat{P}_{xx} = (\nabla^2_{xx}P)(\hat{x}-\hat{y})$. We shall study (2.14). By (2.16) we have $\hat{\Psi}_x = -\hat{\Psi}_y$. We observe

$$0 \ge \frac{\sigma}{(T-\hat{t})^2} + F_*(\hat{\Psi}_x, X) - F^*(\hat{\Psi}_x, -Y).$$

Moreover, we obtain

$$0 \ge \frac{\sigma}{T^2} + F_*(\hat{\Psi}_x, X) - F^*(\hat{\Psi}_x, -Y). \tag{2.17}$$

Since F is singular where $\hat{\Psi}_x = \eta q_i$ $(i = 1, \cdots, m)$, we divide the situation in several cases.

Case I. $\hat{\Psi}_x = \eta q_i \quad (i = 1, \cdots, m)$.

Case Ia. $\hat{P} > 0$ where $\hat{P} = P(\hat{x}-\hat{y})$.

Since P is positively homogeneous of degree one, we see, by (2.2c), P is linear on a neighborhood of $\{\eta q_i\}$ ($\eta \neq 0$). This implies that $\hat{P}_{xx} = 0$ near ηq_i. Then

$$A = \frac{3\hat{P}^2}{\varepsilon}\begin{pmatrix} \hat{P}_x \otimes \hat{P}_x & -\hat{P}_x \otimes \hat{P}_x \\ -\hat{P}_x \otimes \hat{P}_x & \hat{P}_x \otimes \hat{P}_x \end{pmatrix} \quad \text{and}$$
$$A^2 = \frac{18\hat{P}^4}{\varepsilon^2}\begin{pmatrix} (\hat{P}_x \otimes \hat{P}_x)^2 & -(\hat{P}_x \otimes \hat{P}_x)^2 \\ -(\hat{P}_x \otimes \hat{P}_x)^2 & (\hat{P}_x \otimes \hat{P}_x)^2 \end{pmatrix}.$$

Moreover, since $(\hat{P}_x \otimes \hat{P}_x)^2 = |\hat{P}_x|^2(\hat{P}_x \otimes \hat{P}_x)$, we obtain

$$A^2 = \frac{18\hat{P}^4|\hat{P}_x|^2}{\varepsilon^2}\begin{pmatrix} \hat{P}_x \otimes \hat{P}_x & -\hat{P}_x \otimes \hat{P}_x \\ -\hat{P}_x \otimes \hat{P}_x & \hat{P}_x \otimes \hat{P}_x \end{pmatrix}.$$

We take $\lambda = \varepsilon/18\hat{P}^2|\hat{P}_x|^2$ in (2.15) to get

$$A + \lambda A^2 = \frac{4\hat{P}^2}{\varepsilon}\begin{pmatrix} \hat{P}_x \otimes \hat{P}_x & -\hat{P}_x \otimes \hat{P}_x \\ -\hat{P}_x \otimes \hat{P}_x & \hat{P}_x \otimes \hat{P}_x \end{pmatrix}$$

and

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \frac{4\hat{P}^2}{\varepsilon}\begin{pmatrix} \hat{P}_x \otimes \hat{P}_x & -\hat{P}_x \otimes \hat{P}_x \\ -\hat{P}_x \otimes \hat{P}_x & \hat{P}_x \otimes \hat{P}_x \end{pmatrix},$$

which yields

$$X, Y \leq \frac{4\hat{P}^2}{\varepsilon}\hat{P}_x \otimes \hat{P}_x.$$

Since $\hat{P} > 0$ we see $\hat{\Psi}_x \neq 0$. From $\hat{\Psi}_x = \dfrac{\hat{P}^3}{\varepsilon}\hat{P}_x = \mu q_i$, we obtain

$$\hat{P}_x = \frac{\varepsilon\mu}{\hat{P}^3}q_i \quad \text{and} \quad \hat{P}_x \otimes \hat{P}_x = \frac{\varepsilon^2\mu^2}{\hat{P}^6}q_i \otimes q_i,$$

which yields

$$X, Y \leq Z \quad \text{with } Z = \frac{4\varepsilon\mu^2}{\hat{P}^4}q_i \otimes q_i.$$

We shall study (2.17). By (F2) we obtain

$$0 \geq \frac{\sigma}{T^2} + F_*(\mu q_i, Z) - F^*(\mu q_i, -Z).$$

By (F3) this yields $\sigma \leq 0$, which contradicts $\sigma > 0$.

Case Ib. $\hat{P} = 0$ $(i.e., \hat{x} = \hat{y})$.

In this case $\hat{\Psi}_x = 0$ and $A = O$. From (2.15) we have

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \begin{pmatrix} O & O \\ O & O \end{pmatrix},$$

which yields $X \leq O, Y \leq O$. We prove similarly as Case Ia using (F4) instead of (F3).

$$\text{Case II}. \quad \hat{\Psi}_x \neq \eta q_i \quad (i = 1, \cdots, m).$$

This can be treated by a standard argument. We give a proof for completeness.

From (2.16) we have

$$A = \frac{1}{\varepsilon} \begin{pmatrix} J & -J \\ -J & J \end{pmatrix}$$

with $J = 3\hat{P}^2(\hat{P}_x \otimes \hat{P}_x) + \hat{P}^3\hat{P}_{xx}$. We take $\lambda = \varepsilon$ in (2.15) to get

$$A + \lambda A^2 = \frac{1}{\varepsilon} \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}$$

with $K = J + 2J^2$. Moreover, we obtain

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \frac{1}{\varepsilon} \begin{pmatrix} K & -K \\ -K & K \end{pmatrix},$$

which yields $X + Y \leq O$. We shall study (2.17). By (F2) we obtain

$$0 \geq \frac{\sigma}{T^2} + F(\hat{\Psi}_x, X) - F(\hat{\Psi}_x, X).$$

This yields $\sigma \leq 0$, which contradicts $\sigma > 0$. We thus prove Theorem 2.1. □

Remark 2.9. When Ω is an unbounded domain, we suppose additional assumption on F

For every $R > 0$,

(F5) $$C_R = \sup\{|F(p, X)|; |p| \leq R, |X| \leq R, p \in \mathbf{R}^n \setminus \bigcup_{i=1}^{m} \ell_i\} < +\infty.$$

Since $c_1|x - y| \leq P(x - y) \leq c_2|x - y|$ for some positive constants c_1 and c_2, we can prove the following theorem on an unbounded domain in the same way as in [11].

Theorem 2.10. *Suppose that F satisfies (F1)-(F5). Let u and v be, respectively, viscosity sub- and supersolutions of (2.1) in $Q = (0,T) \times \Omega$. Assume that*

(A1) $u(t,x) \le K(|x|+1)$, $v(t,x) \ge -K(|x|+1)$ *for some* $K > 0$ *independent of* $(t,x) \in Q$;

(A2) *there is a modulus* m_T *such that* $u^*(t,x) - v_*(t,y) \le m_T(|x-y|)$ *for all* $(x,t,y) \in \partial_p U$, *where* $U = (0,T) \times \Omega \times \Omega$;

(A3) $u^*(t,x) - v_*(t,y) \le K(|x-y|+1)$ *on* $\partial_p U$ *for some* $K > 0$ *independent of* $(t,x,y) \in \partial_p U$.

Then there is a modulus m such that

$$u^*(t,x) - v_*(t,y) \le m(|x-y|) \quad \text{in } U.$$

In particular $u^* \le v_*$ in Q.

Remark 2.11. When Ω is unbounded the choice of γ, δ in [11, Proposition 2.4] may actually depend on ε, although it is not explicitly mentioned. But the proof of [11, Theorem 2.1] works even if γ, δ depends on ε.

3. Application. Let D_t be a bounded open set in $R^n (n \ge 2)$ at time t. Let Γ_t denote the hypersurface of the boundary of D_t. Let n denote the unit exterior normal vector field on $\Gamma_t = \partial D_t$. We extend n to a vector field (still denote by n) on a tubular neighborhood of Γ_t so that n is constant in the normal direction. Let $V = V(t,x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. We are concerned with the evolution equation for Γ_t :

$$V = \frac{1}{\beta(n)}\left(-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial p_i}(n)\right) + c\right), \tag{3.1}$$

where H is positively homogeneous of degree one, i.e.,

$$H(\lambda p) = \lambda H(p) \quad \text{for } \lambda > 0,\ p \in R^n \backslash \{0\},$$

β is a positive function on a unit sphere S^{n-1} in $\mathbf{R}^n$ and c is a constant.

This evolution equation is derived by Gurtin describing motion of phase-boundaries [2,3]. When H is convex, the equation (3.1) is a degenerate parabolic equation. However, if H is not convex, (3.1) is no longer parabolic and not well-posed. In this case we should consider the convexification of H so that (3.1) is parabolic. Angenent and Gurtin [3] solve such evolution equation for $n = 2$ at least locally if each normal of initial curve (with corners) lies in the direction that the curvature of $H = 1$ is positive.

We aim at constructing global generalized solution of (3.1) by a level set approach when H is the convexification. The problem is that the convexification H of a function h may be no longer C^2 even if h is smooth. The theory of [5] does not apply because their theorem needs C^2 regularity of H. As we discuss below, our comparison theorem in Section 2 does apply to $H \in C^1(\mathbf{R}^n \setminus \{0\})$ provided that $H \in C^2(\mathbf{R}^n \setminus \bigcup_{i=1}^m \ell_i)$ and ∇H is locally Lipschitz on $\mathbf{R}^n \setminus \{0\}$, where ℓ_i $(i = 1, \cdots, m)$ is a half line. Although not all h give such regularity for H, our theory applies to the equation (3.1) studied in [3] for $n = 2$.

We shall show the comparison theorem in Section 2 applies to degenerate parabolic equations associated with (3.1) through a level set approach when H is convex C^2 outside $\bigcup_{i=1}^m \ell_i$ and ∇H is locally Lipschitz on $\mathbf{R}^n \setminus \{0\}$. As a result we can apply the level set approach to (3.1) and construct a unique global generalized solution.

Suppose that the hypersurface Γ_t is expressed as a zero-level of an auxiliary function $u = u(t, x)$ and that $u(t, x) > 0$ if and only if $x \in D_t$. As in [10] (3.1) is equivalent to

$$u_t + F(\nabla u, \nabla^2 u) = 0 \tag{3.2}$$

on Γ_t with

$$\begin{aligned} F(p, X) &= -\text{trace}\,(A(\overline{p}) R_{\overline{p}} X R_{\overline{p}}) + B(p), \\ A(\overline{p}) &= \frac{1}{\beta(-\overline{p})} \left(\frac{\partial^2 H}{\partial p_i \partial p_j}(-\overline{p}) \right), R_{\overline{p}} = I - \overline{p} \otimes \overline{p}, \\ \overline{p} &= \frac{p}{|p|}, B(p) = -\frac{c}{\beta(-\overline{p})} |p|. \end{aligned} \tag{3.3}$$

By the definition of A and B we have the following proposition.

Proposition 3.1.

(i) *If $H \in C^2(\mathbf{R}^n \setminus \bigcup_{i=1}^m \ell_i)$, then $A \in C(S^{n-1} \setminus \bigcup_{i=1}^m \overline{q_i})$,* *where* $\ell_i = \{\eta\overline{q_i} \in \mathbf{R}^n; \eta \geq 0\}$ *and* $|\overline{q_i}| = 1$.

(ii) *If* ∇H *of* $H \in C^1(\mathbf{R}^n\backslash\{0\})$ *is locally Lipschitz on* $\mathbf{R}^n\backslash\{0\}$, *then* A *is bounded.*

(iii) *If* $\beta > 0$ *is continuous, so is* B *on* $\mathbf{R}^n$.

Lemma 3.2. *Let* $H \in C^2(\mathbf{R}^n)$ *be positively homogeneous of degree one. Then*

$$pA(\overline{p}) = 0, \quad A(\overline{p})^t p = 0,$$

provided that H *is* C^2 *near* $-\overline{p} = -p/|p|$ *and* $p \neq 0$, *where* p *is a row vector.*

Proof. From the homogeneity of H we have

$$\sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(\lambda p) = H(p).$$

Plunging $\lambda = 1$ and differentiating this identity in p_j, we obtain

$$\sum_{i=1}^{n} \frac{\partial^2 H}{\partial p_i \partial p_j}(p)p_i = 0 \qquad j = 1, \cdots, m.$$

which easily yields $pA(\overline{p}) = 0$ by replacing p by $-\overline{p}$. Since $A(\overline{p})$ is a symmetric matrix, we have

$${}^t(pA(\overline{p})) = {}^tA(\overline{p})^t p = A(\overline{p})^t p = 0. \qquad \square$$

Corollary 3.3. *Assume the hypotheses of Lemma 3.2. Then*

$$A(\overline{p})p \otimes p = O, \quad p \otimes pA(\overline{p}) = O.$$

Proof. By Lemma 3.2, we have

$$A(\overline{p})p \otimes p = A(\overline{p})^t pp = O,$$

$$p \otimes pA(\overline{p}) = {}^t ppA(\overline{p}) = O. \qquad \square$$

Since trace $(A(\overline{p})R_{\overline{p}}XR_{\overline{p}}) =$ trace $(R_{\overline{p}}A(\overline{p})R_{\overline{p}}X)$, Corollary 3.3 yields

$$\text{trace } (A(\overline{p})R_{\overline{p}}XR_{\overline{p}}) = \text{trace } (A(\overline{p})X).$$

This yields

$$F(p, X) = -\text{trace } (A(\overline{p})X) + B(p). \tag{3.4}$$

Applying Corollary 3.3 to (3.4), we obtain the following lemma.

Lemma 3.4. *Let F be defined by (3.4). Suppose that $A \in C(S^{n-1}\setminus\bigcup_{i=1}^m \overline{q_i})$ is bounded and that B is continuous on $\mathbf{R}^n$. Then*

$$F(p, \sigma p \otimes p) = B(p),$$

where $p \in \mathbf{R}^n\setminus\bigcup_{i=1}^m \ell_i$ and $\sigma \in \mathbf{R}$.

Theorem 3.5. *Let F be defined by (3.4). Suppose that $A \in C(S^{n-1}\setminus\bigcup_{i=1}^m \overline{q_i})$ is bounded and that B is continuous on $\mathbf{R}^n$. Then*

$$F^*(\mu q_i, \nu q_i \otimes q_i) = B(\mu q_i), \tag{3.5}$$

$$F_*(\mu q_i, \nu q_i \otimes q_i) = B(\mu q_i) \tag{3.6}$$

$$\mu > 0,\ \nu \in \mathbf{R}.$$

Proof. Without the lost of generality we may assume $\mu = 1$ by setting μq_i as a new vector $\tilde{q}_i$. Since B is continuous,

$$F^*(q_i, \nu q_i \otimes q_i) - B(q_i) = G^*(q_i, \nu q_i \otimes q_i)$$

with $G(p, X) = F(p, X) - B(p)$. We shall show $G^*(q_i, \nu q_i \otimes q_i) = 0$. By the definition of G^* we have

$$G^*(q_i, \nu q_i \otimes q_i)$$
$$= \lim_{\varepsilon \downarrow 0} \sup\{G(\xi, Y); |\xi - q_i| \leq \varepsilon, |Y - \nu q_i \otimes q_i| \leq \varepsilon, (\xi, Y) \in (\mathbf{R}^n\setminus\bigcup_{i=1}^m \ell_i) \times S^n\}.$$

From Lemma 3.4 and (3.4) it follows that

$$\begin{aligned} G(\xi, Y) &= G(\xi, \nu\xi \otimes \xi) + G(\xi, Y) - G(\xi, \nu\xi \otimes \xi) \\ &= -\text{trace } (A(\overline{\xi})(Y - \nu\xi \otimes \xi)) \\ &= -\text{trace } (A(\overline{\xi})(Y - \nu q_i \otimes q_i)) - \text{trace } (A(\overline{\xi})(\nu q_i \otimes q_i - \nu\xi \otimes \xi)). \end{aligned}$$

Since $|Y - \nu q_i \otimes q_i| \leq \varepsilon$ and $|\xi - q_i| \leq \varepsilon$, we obtain

$$\begin{aligned} |G(\xi, Y)| &\leq |-\text{trace } (A(\overline{\xi})(Y - \nu q_i \otimes q_i))| + |-\text{trace } (A(\overline{\xi})\nu(q_i \otimes q_i - \xi \otimes \xi))| \\ &\leq \varepsilon \sup_{|\overline{\xi}|=1} |\text{trace } A(\overline{\xi})| + c|\nu|\varepsilon \sup_{|\overline{\xi}|=1} |\text{trace } A(\overline{\xi})| \to 0 \end{aligned}$$

as $\varepsilon \to 0$, which yields (3.5), where c is a constant depending only on n. The proof of (3.6) parallels (3.5). □

Theorem 3.6. *Assume that $H \in C^1(R^n \setminus \{0\})$ is convex and positively homogeneous of degree one. Assume that $H \in C^2(R^n \setminus \bigcup_{i=1}^m \ell_i)$ and that ∇H is locally Lipschitz on $R^n \setminus \{0\}$. Assume that β is positive continuous. Let F be defined by (3.4). Then F satisfies (F3) and (F4) in Section 2.*

Proof. Applying Proposition 3.1 and Theorem 3.5, we observe that F satisfies (F3). It remains to prove

$$F_*(0,O) = F^*(0,O) \quad (=0).$$

By the definition of F^*

$$\begin{aligned} &F^*(0,O) \\ &= \lim_{\varepsilon \downarrow 0} \sup\{F(\xi, Y)\ ;\ |\xi| \le \varepsilon,\ |Y| \le \varepsilon,\ (\xi, Y) \in (R^n \setminus \bigcup_{i=1}^m \ell_i) \times S^n\} \end{aligned}$$

From (3.4) it follows that

$$F(\xi, Y) = -\text{trace}\ (A(\bar{\xi})Y) + B(\xi).$$

Since $|Y| \le \varepsilon$ and $|\xi| \le \varepsilon$, we obtain

$$\begin{aligned} |F(\xi, Y)| &\le |-\text{trace}\ (A(\bar{\xi})Y)| + |B(\xi)| \\ &\le \varepsilon \sup_{|\bar{\xi}|=1} |\text{trace}\ A(\bar{\xi})| + |B(\xi)| \end{aligned}$$

$\to 0$ as $\varepsilon \to 0$, which yields $F^*(0,O) = 0$. Similarly we prove $F_*(0,O) = 0$. Thus we show that F satisfies (F3) and (F4). □

Remark 3.7. We shall show another short proof of Theorem 3.5 observing that F defined by (3.4) is geometric in the sense of [5], i.e.,

$$\begin{aligned} &F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X) \\ &\qquad \text{for all } \lambda > 0,\ \sigma \in R,\ p \in R^n \setminus \bigcup_{i=1}^m \ell_i,\ X \in S^n. \end{aligned}$$

Indeed it is easy to check that F^* and F_* are geometric provided that F is geometric. Note that values of F^* and F_* are finite. Thus we observe that

$$F^*(\mu q_i, \nu q_i \otimes q_i) = F^*(\mu q_i, O) = B(\mu q_i),$$

which yields (3.5). The proof of (3.6) is the same.

We apply our theory to (3.1) to construct a global weak solution. The equation (3.2) is clearly geometric. We can see that $\theta(u)$ is viscosity sub(super)solution of (3.2) if u is sub(super)solution of (3.2), where θ is a continuous nondecreasing function (cf. [5, Theorem 5.6]). So we can construct viscosity solution of (3.2) with initial data similarly as in [5]. Similarly to [5] we define a weak solution $\{(\Gamma_t, D_t)\}_{t\geq 0}$ of (3.1) through a viscosity solution of (3.2) with initial data $u(0,x) = a(x)$.

Definition 3.8. D_0 denotes a bounded open set and $\Gamma_0(\subset R^n \backslash D_0)$ denotes a compact set containing ∂D_0. $\{(\Gamma_t, D_t)\}_{t\geq 0}$ denotes a family of compact sets and bounded open sets in R^n. Suppose that for some $\alpha > 0$ there is a viscosity solution $u \in C_\alpha([0,T] \times R^n)$ for (3.2) with initial data $u(0,x) = a(x)$ in $(0,\infty) \times R^n$ such that zero-level surface of $u(t,\cdot)$ at time $t \geq 0$ equals Γ_t and that the set D_t where $u > 0$ is bounded open. If $(\Gamma_t, D_t)|_{t=0} = (\Gamma_0, D_0)$, we say $\{(\Gamma_t, D_t)\}_{t\geq 0}$ is a weak solution of (3.1) with initial data (Γ_0, D_0). Here $T > 0$ is arbitrary and $v \in C_\alpha(A)$ means $v - \alpha$ is continuous and has compact support in A.

Similarly as in [10, §3], applying the comparison theorem 2.1 yields:

Theorem 3.9. *Suppose that $\beta > 0$ is continuous and that $H \in C^1(R^n \backslash \{0\})$ is convex and positively homogeneous of degree one. Suppose that $H \in C^2(R^n \backslash \bigcup_{i=1}^m \ell_i)$ and that ∇H is locally Lipschitz on $R^n \backslash \{0\}$. Let D_0 be a bounded open set in R^n and let $\Gamma_0(\subset R^n \backslash D_0)$ be a compact set containing ∂D_0. Then there is a unique global solution $\{(\Gamma_t, D_t)\}_{t\geq 0}$ of (3.1) with initial data (Γ_0, D_0)* (cf. [5, Theorem 7.3], [10, Proposition 3.3]).

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