Interface evolution with Neumann boundary condition

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Interface evolution with Neumann boundary condition

MOTO-HIKO SATO

§1. **Introduction.** In this paper we aim at constructing generalized solutions for interface equation introduced in [1] and [6] for Neumann problem. We are concerned with the motion of a hypersurface in a domain Ω. The speed of the hypersurface depends on the normal vector field and its derivative. The hypersurface is assumed to intersect to the boundary $\partial\Omega$ perpendicularly. We suppose that Ω is divided into phaseI region and phaseII region by the hypersurface at time t. Let Γ_t, D_t^+ and D_t^-, respectively, denote the hypersurface, phaseI region and phaseII region at time t. The union of D_t^+, D_t^- and Γ_t equals to Ω. Let $\mathbf{n}$ denote the unit exterior normal vector on Γ_t from D_t^+ to D_t^-. It is convenient to extend $\mathbf{n}$ to a vector field (still denoted by $\mathbf{n}$) on a tubular neighborhood of Γ_t such that $\mathbf{n}$ is constant in the normal direction of Γ_t. We consider the following equation for Γ_t :

$$V = f(t, \mathbf{n}(x), \nabla \mathbf{n}(x)) \quad \text{on} \quad \Gamma_t \tag{1.1a}$$

$$\langle \gamma(x), \mathbf{n}(x) \rangle = 0 \quad \text{on} \quad b\Gamma_t = \partial\Omega \cap \bar{\Gamma}_t. \tag{1.1b}$$

Here $V = V(t, x)$ denotes the speed at $x \in \Gamma_t$ in the normal direction of Γ_t, $\gamma(x)$ is the outer unit normal of $\partial\Omega$ and f is a given function . The set $b\Gamma_t$ is the intersection of the boundary $\partial\Omega$ and the closure of Γ_t in $\mathbf{R}^n$. A typical example of (1.1a) is

$$V = -\operatorname{div} \mathbf{n}, \tag{1.2}$$

where the hypersurface Γ_t is moved by its mean curvature and (1.2) is known as the mean curvature flow equation.

Our goal is to construct a global-in-time unique generalized solution Γ_t of (1.1a)-(1.1b) for a given initial data Γ_0 provided that the problem is degenerate parabolic. In the case that the initial interface is graph of a smooth function, Huisken [11] constructed a unique global smooth evolution of interface where Ω is a cylindrical domain. In this paper we regard a surface Γ_t as a level set of an auxiliary function u as in [1] and [6]. Let us adapt their strategy to our problem. Suppose that $u > 0$ in D_t^+, $u = 0$ on Γ_t and $u < 0$ in D_t^-. If $u(t,x)$ is C^2 and $\nabla u \neq 0$ near Γ_t for $x \in \Omega$, we see

$$\mathbf{n} = -\frac{\nabla u}{|\nabla u|}, \quad \nabla \mathbf{n} = -\frac{1}{|\nabla u|}(Q_{\bar{p}}(\nabla^2 u)) \tag{1.3}$$

and $Q_{\bar{p}}(X) = R_{\bar{p}} X R_{\bar{p}}$ with $R_{\bar{p}} = I - \bar{p} \otimes \bar{p}$, $\bar{p} = \nabla u / |\nabla u|$, where $\otimes$ denotes a tensor product of vector, where ∇u and $\nabla^2 u$ denote respectively the gradient of u and the Hessian of u in space variables. It follows from (1.3) and $V = \partial_t u / |\nabla u|$ that (1.1a), (1.1b) is formally equivalent to

$$\partial_t u + F(t, \nabla u, \nabla^2 u) = 0 \quad \text{on} \quad \Omega \cap \Gamma_t \tag{1.4a}$$

$$\langle \gamma(x), \nabla u \rangle = 0 \quad \text{on} \quad b\Gamma_t \tag{1.4b}$$

with F uniquely determined by f (cf.[7]). Here $\partial_t = \partial / \partial t$ denote the time derivative of u. To construct a generalized solution we consider the level set equations (1.4a)-(1.4b) not on Γ_t but on Ω. In this paper for a bounded convex domain Ω we establish a comparison principle and construct a unique continuous viscosity solution of the level set equation (1.4a)-(1.4b) for any given initial data $u(0,x) = u_0(x)$. We also show that the zero level set Γ_t of u is independent of the choice of u_0 and is essentially determined by Γ_0. Thus we can construct a unique generalized solution Γ_t.

In [1] this programme of constructing generalized solutions was carried out when $\Omega = \mathbf{R}^n$ with no boundary condition for motion of compact hypersurfaces. Nearly at the same time Evans and Spruck [6] carried out this programme in a slightly different way and only for the mean curvature flow equation (1.2).

The first step to carry out this programme is to establish a comparison principle for the Neumann boundary value problem on a convex domain. For the Neumann problem this principle was first established by Lions [15] for Hamilton-Jacobi equations and was established by Ishii and Lions [14] for nonsingular degenerate elliptic equations. See also [4][5][13] for more general oblique boundary condition. The method of Ishii and Lions [14] does not apply to our problem since the equation is singular. So we adapt the method of [8] to this problem. We regard $\partial\Omega$ as the space infinity in the theory of [8].

The second step is to show the existence of global solution by Perron's method. We need to construct sub-and supersolution of (1.4a)-(1.4b) with a given initial data. For this purpose we use local coordinate patches near $\partial\Omega$. Our construction of sub-and supersolution does apply even if Ω is nonconvex. In the forthcoming paper [10] we shall establish the comparison principle even if Ω is nonconvex; see [9] for announcement. Thus it turns out that the level set approach does apply to the Neumann problem in arbitrary smooth domain Ω.

The idea to represent hypersurfaces as level sets goes back to Ohta, Jasnow and Kawasaki[16], who used the level set equation to derive a scaling law for dynamic structure function with "random" initial data from a physical point of view. Osher and Sethian [17] introduced a numerical method of surface evolution via level set equations. It is important to consider the Neumann problem (1.1a)-(1.1b) because the mean curvature flow equation (1.2) with (1.1b) is derived formally as a singular limit of a reaction-diffusion equation with the Neumann condition [18].

In §2 we will first establish a comparison principle on a bounded convex domain for equations including (1.4a)-(1.4b) when (1.1a) is degenerate parabolic. We remark the case when F depends on the space variable x. In §3 for a large class of geometric degenerate parabolic equations we construct a unique global viscosity solution u with initial data. Existence of viscosity solutions is based on Perron's method discussed in [12]. If Ω is convex, the idea in [1] applies to construction of viscosity sub-(super)solutions for the Neumann problem. However it does not directly apply to a general domain. In this paper we construct sub-(super)solutions in a general domain by using local coordinates near $\partial\Omega$.

In §4 we formulate a weak solution for (1.1a)-(1.1b) and apply our results in §2 and §3 to get a unique global weak solutions for a given initial data Γ_0 when Ω is bounded convex and (1.1a) is degenerate parabolic. Assumptions on f is the same as in [7], where the hypersurface Γ_t is closed with no boundary conditions.

The results in this paper have been announced in [9].

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§2. Comparison principle.

In this section we will establish a comparison principle of the Neumann problem on a bounded domain. To clarify the main idea of the proof we assume here that the equation does not depend on space variables explicitly. In the last part of this section we remark that our method applies to more general equations depending space variables.

Let Ω be a bounded domain in $\mathbf{R}^n$ and let T be a positive number. We denote by $\gamma(x)$ the outer unit normal of Ω at $x \in \partial\Omega$. We want to study the Neumann problem of the form

$$u_t + F(t, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega, \tag{2.1a}$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega, \tag{2.1b}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \bar{\Omega}. \tag{2.1c}$$

We first list assumptions on $F = F(p, X)$.

(F1) $F : (0,T) \times (\mathbf{R}^n \backslash \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous, where $\mathbf{S}^n$ denotes the space of real $n \times n$ symmetric matrices.

(F2) F is degenerate elliptic, i.e., $F(t,p,X+Y) \le F(t,p,X)$ for all $Y \ge 0, t \in (0,T)$.

(F3) $-\infty < F_*(t,0,O) = F^*(t,0,O) < \infty$ for all $t \in (0,T)$ where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $(0,T) \times \mathbf{R}^n \times \mathbf{S}^n$, respectively, i.e.,

$$F_*(t,p,X) = \liminf_{\varepsilon \downarrow 0}\{F(s,q,Y); q \neq 0, |s-t| \leq \varepsilon, |p-q| \leq \varepsilon, |X-Y| \leq \varepsilon\}$$

and $F^* = -(-F)_*$. Here $|X|$ denotes the operator the norm of X as a self- adjoint operator on $\mathbf{R}^n$.

(F4) For every $R > 0$

$$c_R = \sup\{|F(t,p,X)|; |p| \leq R, |x| \leq R, p \neq 0, t \in (0,T)\}$$

is finite.

The assumption (F1) allows the possibility that (2.1) is singular at $\nabla u = 0$. The equation (2.1a) is called degenerate parabolic if (F2) holds.

Let $Q_0 = (0,T)\times\bar{\Omega}$. A function $u : Q_0 \to \mathbf{R}$ is called a *viscosity subsolution* of (2.1a)-(2.1b) if it satisfies the following properties:

(i)
$$u^* < \infty \quad \text{on} \quad \bar{Q}_0$$

(ii)
$$\tau + F_*(t,p,X) \leq 0 \quad \text{for} \quad x \in \Omega, \quad (\tau,p,X) \in \mathcal{P}^{2,+}_{Q_0} u^*(t,x)$$
$$\langle n(x), p\rangle \wedge \{\tau + F_*(t,p,X)\} \leq 0 \quad \text{for} \quad x \in \partial\Omega, \quad (\tau,p,X) \in \mathcal{P}^{2,+}_{Q_0} u^*(t,x).$$

Similarly a function $u : Q_0 \to \mathbf{R}$ is called a *viscosity supersolution* of (2.1a)-(2.1b) if

(i)
$$u_* > -\infty \quad \text{on} \quad \bar{Q}_0$$

(ii)
$$\tau + F^*(t,p,X) \geq 0 \quad \text{for} \quad x \in \Omega, \quad (\tau,p,X) \in \mathcal{P}^{2,-}_{\bar{Q}_0} u_*(t,x)$$
$$\langle n(x), p\rangle \vee \{\tau + F^*(t,p,X)\} \geq 0 \quad \text{for} \quad x \in \partial\Omega, \quad (\tau,p,X) \in \mathcal{P}^{2,-}_{Q_0} u_*(t,x).$$

Here $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$ and $\mathcal{P}^{2,+}_{Q_0}$ denotes the *parabolic super 2-jet* in Q_0, i.e., $\mathcal{P}^{2,+}_{Q_0} u(t,x)$ is the set of $(\tau,p,X) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$ such that

$$u(s,y) \leq u(t,x) + \tau(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle$$
$$+o(|s-t| + |y-x|^2) \quad \text{as} \quad (s,y) \to (t,x) \quad \text{in} \quad Q_0$$

where $\langle , \rangle$ denotes the Euclidean inner product; similarly, $\mathcal{P}^{2,-}_{Q_0} u = -\mathcal{P}^{2,+}_{Q_0}(-u)$. In this paper we call a continuous function $m : [0,\infty) \to [0,\infty)$ a *modulus* if $m(0) = 0$ and it is nondecreasing. For $U = (0,T) \times D$, the set

$$\partial_p U = \{0\} \times D \cup [0,T) \times \partial D$$

is often called the *parabolic boundary* of U. We are now in position to state our main comparison theorem for (2.1a)-(2.1c).

Theorem 2.1. *Suppose that Ω is a bounded convex domain with C^2 boundary $\partial\Omega$ and that F satisfies* (F1)-(F4). *Let u and v be, respectively, sub-and supersolutions of* (2.1a)-(2.1b) *in Q_0, where $Q_0 = (0,T) \times \bar{\Omega}$. Let $u_0 = u^*(0,x)$ and $v_0 = v_*(0,x)$.*

If $u_0 \le v_0$ on $\bar{\Omega}$, then there is a modulus m such that

$$u^*(t,x) - v_*(t,y) \le m(|x-y|) \quad on \quad U_0, \tag{2.2}$$

where $U_0 = (0,T) \times \bar{\Omega} \times \bar{\Omega}$. In particular $u^* \le v_*$ on Q_0.

Remark 2.2. Since we may assume that u and v are bounded on $\bar{Q}$ (see [1]), the assumption $u_0 \le v_0$ on $\bar{\Omega}$ implies that there is a modulus function m_0 such that

$$u_0(x) - v_0(y) \le m_0(|x-y|) \quad on \quad \bar{\Omega} \times \bar{\Omega}. \tag{2.3}$$

We will state several propositions to be needed to prove Theorem 2.1. The proof is parallel to the proof of Theorem 2.1 in [8], however we give it for completeness. Assume that u and v are, respectively, upper semicontinuous and lower semicontinuous on $\bar{Q}$.

For ε, δ, $\gamma > 0$ we set

$$\begin{aligned} \Phi(t,x,y) &= w(t,x,y) - \Psi(t,x,y), \quad w(t,x,y) = u(t,x) - v(t,y), \\ \Psi(t,x,y) &= \frac{|x-y|^4}{4\varepsilon} + B(t,x,y), \\ B(t,x,y) &= \delta(\varphi(x) + \varphi(y) + 2\beta) + \frac{\gamma}{T-t}. \end{aligned} \tag{2.4}$$

The function B plays the role of a barrier for boundary and $t = T$.

Here $\varphi(x) \in C^2(\bar{\Omega})$ is taken so that $\varphi < 0$ in $\Omega, \varphi = 0$ on $\partial\Omega, \gamma(x) = \nabla_x\varphi(x)/|\nabla_x\varphi(x)|$ for all $x \in \partial\Omega, |\varphi(x)| \le \beta$ for some $\beta \ge 0$ independent of $x \in \bar{\Omega}, |\nabla_x\varphi(x)| \ge 1$ for all $x \in \partial\Omega$.

Proposition 2.3. *Suppose that u and v be, respectively, upper semicontinuous and lower semicontinuous. Assume that*

$$\alpha = \lim_{\theta\downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, (t,x,y) \in \bar{U}, t < T\} > 0. \tag{2.5}$$

Then there are positive constants δ_0 and γ_0 such that

$$\sup_{\bar{U}} \Phi(t,x,y) > \frac{\alpha}{2} \tag{2.6}$$

holds for all $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$, $\varepsilon > 0$.

Proof. Since $\bar{\Omega}$ is compact, w is bounded from above. This implies $\alpha < \infty$. By (2.5) there is a point (t_0, x_0, y_0) such that $w(t_0, x_0, y_0) > 3\alpha/4$ and $|x_0 - y_0|^4/4\varepsilon < \alpha/4$. We now observe that $\Phi(t_0, x_0, y_0) > \alpha/2$ if δ and γ is sufficiently small. ∎

Proposition 2.4. *Let* u, v, δ_0, γ_0 *be as in Proposition* 2.3. *Suppose that* w *is upper semicontinuous in* $\bar{U}$.

(i) Φ *attains a maximum over* $\bar{U}$ *at* $(\hat{t}, \hat{x}, \hat{y}) \in \bar{U}$ *with* $\hat{t} < T$.

(ii) $|\hat{x} - \hat{y}|$ *is bounded as a function of* $0 < \varepsilon < 1$, $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$.

(iii) $|\hat{x} - \hat{y}|$ *tends to zero as* $\varepsilon \to 0$; *the convergence is uniform in* $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$.

Proof.

(i) By (2.3) and the definition of B we see Φ is negative outside a compact set W in $[0, T) \times \bar{D}$, $D = \Omega \times \Omega$. Since Φ is upper semicontinuous and $\sup \Phi > 0$ by (2.5), Φ takes a maximum over $\bar{U}$ at a point of W.

(ii) From (2.5) it follows $\Phi(\hat{t}, \hat{x}, \hat{y}) > 0$ for $0 < \delta < \delta$, $0 < \gamma < \gamma_0$, $\varepsilon > 0$. This yields

$$w(\hat{t}, \hat{x}, \hat{y}) \geq \frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} + B(\hat{t}, \hat{x}, \hat{y}) \geq \frac{|\hat{x} - \hat{y}|^4}{4\varepsilon}. \tag{2.7}$$

Since $w(\hat{t}, \hat{x}, \hat{y}) \leq M$ for some $M > 0$

$$\frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} \leq M \tag{2.8}$$

which yields (ii).

(iii) Similarly, from (2.8) it follows

$$\frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} \leq M,$$

which yields (iii) as $\varepsilon \to 0$. ∎

Proposition 2.5. *Assume the hypotheses of Proposition* 2.4. *Suppose that* (2.3) *holds for* u *and* v. *Then there is* $\varepsilon_0 > 0$ *such that* Φ *attains a maximum in* $(0, T) \times \bar{\Omega} \times \bar{\Omega}$, *i.e,.* $\hat{t} \neq 0$ *for all* $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$.

Proof. Suppose that the conclusion were false. Since $\hat{t} < T$ by Proposition 2.4(i), there would exist sequences $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, $\{\delta_j\} \subset (0, \delta_0)$ and

$\{\gamma_j\} \subset (0, \gamma_0)$ such that Φ attains a maximum at $(0, \hat{x}_j, \hat{y}_j)$ for the value $\varepsilon = \varepsilon_j$, $\delta = \delta_j$, $\gamma = \gamma_j$. By (2.3) and (2.6) we see

$$\frac{\alpha}{2} \leq \Phi(0, \hat{x}_j, \hat{y}_j) \leq w(0, \hat{x}_j, \hat{y}_j) \leq m_0(|\hat{x}_j - \hat{y}_j|).$$

Since $\varepsilon_j \to 0$, applying Proposition 2.4 (iii) yields $|\hat{x}_j - \hat{y}_j| \to 0$ which leads a contradiction $0 < \alpha/2 \leq 0$. ∎

Lemma 2.6 ([2]). *Let Ω_i be a locally compact set in R_i^N. Let u_i be an upper semicontinuous function with $u_i < \infty$ in $(0,T)\times\mathcal{O}_i$ for $i = 1, 2, \cdots, k$.Let $\mathcal{O}_i$ be an open set containing Ω_i for $i = 1, 2, \cdots, k$ and a function Ψ on $(0,T)\times \mathcal{O}_1 \times \cdots \mathcal{O}_k$ such that $(t, x_1, \cdots, x_k) \to \varphi(t, x_1, \cdots, x_k)$ is once continuously differentiable in t and twice continuously differentiable in $(x_1, \cdots, x_k)$. Suppose that $s \in (0,T)$,$z_i \in \Omega_i$ for $i = 1, 2, \cdots, k$ and*

$$\Phi(t, x_1, \cdots, x_k) \equiv u_1(t, x_1) + \cdots + u_k(t, x_k) - \Psi(t, x_1, \cdots, x_k) \leq \Phi(s, z_1, \cdots, z_k)$$

Assume that there is an $\omega > 0$ such that for every $M > 0$

$$\text{(2.9)} \qquad \begin{aligned} &\sigma_i \leq C \quad \textit{whenever} \quad (\sigma_i, q_i, Y_i) \in \mathcal{P}^{2,+}u(t, x_i), \\ &|x_i - z_i| + |s - t| < \omega \quad \textit{and} \quad |u_i(t, x_i)| + |q_i| + |Y_i| \leq M \end{aligned}$$

$(i = 1, \cdots, k)$, with some $C = C(M)$. Then for each $\lambda > 0$ there exists $(\tau_i, X_i) \in \mathbf{R} \times \mathbf{S}^{N_i}$ such that

$$(\tau_i, \nabla_{x_i}\Psi(s, z_1, \cdots, z_k), X_i) \in \bar{\mathcal{P}}^{2,+}u_i(s, z_i) \quad \text{for} \quad i = 1, \cdots k$$

and

$$-\left(\frac{1}{\lambda} + |A|\right) I \leq \begin{pmatrix} X_1 & \cdots & O \\ \vdots & & \vdots \\ O & \cdots & X_k \end{pmatrix} \leq A + \lambda A^2 \quad \text{and}$$

$$\tau_1 + \cdots + \tau_k = \Psi_t(s, z_1, \cdots, z_k),$$

where $A = \nabla_x^2 \Psi(s, z_1, \cdots, z_k)$.

Remark 2.7. This lemma is Theorem 8 in [2] and is considered as a local parabolic version of Crandall-Ishii's lemma [2,Theorem 6]. Here the subscript of $\mathcal{P}^{2,+}$ is suppressed. The bar over $\mathcal{P}^{2,+}$ means the closure.

Proof of Theorem 2.1. We may assume that u and v are, respectively, upper and lower semicontinuous so that

$$w(t, x, y) = u(t, x) - v(t, y)$$

is upper semicontinuous in $\bar{U}$. Suppose that (2.2) were false. Then we would have (2.5), i.e.,

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, \quad (t,x,y) \in \bar{U}, t < T\} > 0.$$

Then we see all conclusions in Propositions 2.3-2.5 would hold for Φ defined in (2.4). By Proposition 2.5 Φ attains a maximum over $\bar{U}$ at $(\hat{t},\hat{x},\hat{y}) \in U_0$ for small ε, δ, γ. Therefore we see that

$$\Phi(t,x,y) = w(t,x,y) - \Psi(\hat{t},\hat{x},\hat{y}) \leq \Phi(\hat{t},\hat{x},\hat{y}) \quad \text{in} \quad U_0. \tag{2.10}$$

We will apply Lemma 2.6 with $k = 2$, $u_1 = u$, $u_2 = -v$, $s = \hat{t}$, $z = (\hat{x},\hat{y})$. Since u and v are, respectively, sub-and supersolution of (2.1a)-(2.1b) with F satisfying (F4), we easily see the assumption (2.9) holds. We now apply Lemma 2.6 with $\Omega_i = \bar{\Omega}$ and conclude that for each $\lambda > 0$ there are (τ_1, X) and $(\tau_2, Y) \in \mathbf{R} \times \mathbf{S}^n$ such that

$$(\tau_1, \hat{\Psi}_x, X) \in \bar{\mathcal{P}}^{2,+}u(\hat{t},\hat{x}), \quad (-\tau_2, -\hat{\Psi}_y, -Y) \in \bar{\mathcal{P}}^{2,-}v(\hat{t},\hat{y}), \tag{2.11}$$

$$-\left(\frac{1}{\lambda} + |A|\right) I \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \lambda A^2, \quad \hat{\Psi}_t = \tau_1 + \tau_2, \tag{2.12}$$

where $\hat{\Psi}_t = \partial_t \Psi(\hat{t},\hat{x},\hat{y})$, $\hat{\Psi}_x = \nabla_x \Psi(\hat{t},\hat{x},\hat{y})$, etc. If $\hat{x} \in \partial\Omega$ then $\langle \gamma(\hat{x}), \hat{\Psi}_x \rangle > 0$.Indeed if $\hat{x} \in \partial\Omega$, then

$$\begin{aligned} &\langle \gamma(\hat{x}), \hat{\Psi}_x \rangle \\ =&\langle \gamma(\hat{x}), |\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}) \rangle + \langle \gamma(\hat{x}), \delta \nabla_x \varphi(\hat{x}) \rangle \\ \geq& \delta > 0 \quad \text{since} \quad \Omega \quad \text{is} \quad \text{convex.} \end{aligned}$$

Since u and v are, respectively, sub-and supersolution of (2.1a)-(2.1b) it follows from (2.11) that

$$\tau_1 + F_*(\hat{t}, \hat{\Psi}_x, X) \leq 0, \quad -\tau_2 + F^*(\hat{t}, -\hat{\Psi}_y, -Y) \geq 0,$$

which yields

$$0 \geq \hat{\Psi}_t + F_*(\hat{t}, \hat{\Psi}_x, X) - F^*(\hat{t}, -\hat{\Psi}_y, -Y). \tag{2.13}$$

The rest of the proof is almost the same in [8,Theorem 2.1]. The only difference is that we use the localized version of Crandall-Ishii's Theorem [3]. ∎

Remark 2.8. In the proof of Theorem 2.1 we use (F4) only to prove (2.10) in Lemma 2.6. However we can avoid to invoke (F4) by applying the Lemma 2.10 [8], which can be proved similarly as Lemma 2.6 (cf.[8]).

Remark 2.9. Our method applies to more general equations of the form

$$u_t + F(t,x,u,\nabla u,\nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T)\times\Omega, \tag{2.1'}$$

under the assumptions on F in [8, §4]. Our approach is basically the same as in (2.1a). However, since F depends on x, we are forced to let $\varepsilon \to 0$ in our test function Ψ of (2.11) at the end of the proof. The crucial step is to establish that $|\hat{x}-\hat{y}|^4/4\varepsilon$ converges to zero as $\varepsilon \to 0$ after we let $\delta \to 0$, $\gamma \to 0$. This can be proved as [8, Proposition 4.4]. Instead of listing all assumptions on F we give a simple example of F:

$$F(t,x,p,X) = F_1(t,p,X) + \omega(t,x)|p|.$$

Here F_1 satisfies (F1)-(F4) and ω is continuous with bound $|\nabla_x\omega|$ on $(0,T)\times\bar{\Omega}$. For this F our comparison principle can be extended. We note that our extension also applies the case when the second order term involves x-dependence.

§3. Construction of generalized solution

This section constructs a viscosity solution of initial-boundary value problem

$$u_t + F(t,x,\nabla u,\nabla^2 u) = 0 \quad \text{in} \quad Q=(0,T)\times\Omega \tag{3.1a}$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega \tag{3.1b}$$

$$u(0,x) = a_0(x) \quad \text{in} \quad \bar{\Omega}. \tag{3.1c}$$

by Perron's method [12] when F is geometric in the sense of [1] provided that $a_0 \in C(\bar{\Omega})$. Our result applies to a general bounded domain Ω with C^2 boundary $\partial\Omega$ not necessarily convex. When Ω is convex, by the comparison principle in §2, it turns out that our solution is unique and continuous. The basic strategy for constructing solutions is similar to that in [1]. First we recall basic properties of viscosity solutions.

Definition 3.1. For a sequence of functions $g_k : L \to \mathbf{R}$ $(L \subset \mathbf{R}^d)$ $(k=1,2,\cdots)$ we associate its $\Gamma^- - limit$

$$\lim_{k\to\infty} {}_*g_k : \bar{L} \to \bar{\mathbf{R}} = \mathbf{R}\cup\{\pm\infty\}$$

defined by

$$\lim_{k\to\infty} {}_*g_k(x) = \lim_{\substack{n\to\infty \\ \varepsilon\downarrow 0}} \inf_{l\geq k} \inf_{\substack{|x-y|<\varepsilon \\ y\in L}} g_l(y) \quad \text{for} \quad x\in\bar{L},$$

where $\bar{L}$ denotes the closure of L in $\mathbf{R}^d$. When $g_k = g$ for all k,

$$\lim_{k\to\infty} {}_*g_k = g_*.$$

Definition 3.2([1]). Let $E : W \to \mathbf{R}$ where $W \subset J = \Omega \times (0,T) \times \mathbf{R}^n \times \mathbf{S}^n$. We say the equation $E = 0$ is geometric in W if E satisfies for $\lambda > 0$ and $\mu \in \mathbf{R}$ there is $C_i = C_i(\lambda, \mu) > 0 \quad (i = 1,2)$ such that

$$C_1E(t,x,p,X) \leq E(t,x,\lambda p, \lambda X + \mu p \otimes p) \leq C_2E(t,x,p,X) \tag{3.2}$$

holds whenever each term is well-defined.

Here $\otimes$ denotes a tensor product of vectors in $\mathbf{R}^n$. It is easy to see that the equation $E_* = 0$ and $E^* = 0$ are geometric in $\bar{W}$ if $E = 0$ is geometric in W.

Proposition 3.3(Stability). *Let $F, F_k : J = (0,T) \times \Omega \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ and u_k be a subsolution of*

$$u_t + F_k(t,x,\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q \quad (k = 1,2,\cdots) \tag{3.3a}$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \tag{3.3b}$$

Assume that $\lim_{k\to\infty} {}_*F_k \geq F_*$ *and u_k converges to a function $u : Q_0 \to \mathbf{R}$ uniformly in each compact subset of Q_0, where $Q_0 = (0,T) \times \bar{\Omega}$. Then u is a subsolution of* (3.1a)-(3.1b) .

Proof. As usual [3] we set

$$G_{-k}(t,x,p,\tau,X) = \begin{cases} \tau + F_k(t,x,p,X) & \text{for } x \in \Omega \\ \tau + F_k(t,x,p,X) \wedge B(x,p) & \text{for } x \in \partial\Omega, \end{cases}$$

$$G_{-}(t,x,p,\tau,X) = \begin{cases} \tau + F(t,x,p,X) & \text{for } x \in \Omega \\ \tau + F(t,x,p,X) \wedge B(x,p) & \text{for } x \in \partial\Omega, \end{cases}$$

where $B(x,p) = \langle \gamma(x), p \rangle$.

If $\lim_{k\to\infty} {}_*F_k \geq F_*$, then $\lim_{k\to\infty} {}_*G_{-k} \geq G_{-*}$. Applying G_-, G_{-k} in place of E, E_k in the stability lemma [1, Proposition 2.4.], we conclude that u is a subsolution. ∎

In what follows we shall always assume that F satisfies (F1) and (F2) in §2.

Proposition 3.4 ([1]). *Let S be a nonempty family of subsolutions of (3.1a)(3.1b) and let u be a function defined on Q_0 by*

$$u(y) = \sup\{v(y) : v \in S\} \quad \text{for} \quad y \in Q_0.$$

Suppose that $u^*(y) < \infty$ for $y \in \bar{Q_0}$. Then u is a subsolution of (3.1a)-(3.1b).

Proposition 3.5 *Let f and $g : Q_0 \to \mathbf{R}$ be respectively a sub-(super) solution of (3.1a)-(3.1b). Suppose $f \le g$ in Q_0. Set $u(y) = \sup\{v(y) : v$ is a subsolution of (3.1a)-(3.1b) and $v \le g\}$. Then u is a viscosity solution of (3.1a)-(3.1b).*

Proposition 3.6. *Assume that F is geometric. Let u be a viscosity sub-(super)solution of (3.1a)-(3.1b). Then $\theta(u)$ is a viscosity sub-(super) solution of (3.1a)-(3.1b) whenever $\theta : \mathbf{R} \to \mathbf{R}$ is continuous nondecreasing.*

Proof. We just present a proof when u is a subsolution since the proof is the same for supersolutions. There is a sequence $\{\theta_k\}$ in $C^2(\bar{\Omega})$ of increasing function with $\theta_k{}' > 0$ such that $\theta_k \to \theta$ uniformly in $\bar{\Omega}$ as $k \to \infty$ [1, Lemma 5.4]. If F is geometric, then G_- is geometric. By [1, Lemma 5.3] we see $\theta_k(u)$ is also a subsolution of (3.1a)-(3.1b). Since $\theta_k(u)$ converges to $\theta(u)$ uniformly in Q_0, we now apply the stability Proposition 3.3 and conclude that $\theta(u)$ is a subsolution of (3.1a)-(3.1b) with $\lim_{k\to\infty} {}_*F_k \ge F_*$. ∎

Proposition 3.7. *If $u \in C^2(Q_0)$ satisfies $\langle\gamma(x), \nabla u(x)\rangle \le 0$ for $x \in \partial\Omega$ and $u_t + F(t, \nabla u, \nabla^2 u) \le 0$ for $x \in \Omega$, then u is a subsolution of (3.1a)-(3.1b)*

.

Proof. Since

$$\lambda \mapsto \langle\gamma(x), p - \lambda\gamma(x)\rangle \tag{3.4}$$

is nondecreasing in $\lambda \ge 0$ for all $x \in \partial\Omega$, $p \in \mathbf{R}^n$ applying [3, Proposition 7.2] yields the desired result. ∎

We next construct sub-(super)solution. We see the sub-(super)solution which is constructed in [1] is a sub-(super)solution of (3.1a)-(3.1b) if Ω is convex. Therefore the following Propositions 3.8-3.10 hold.

Proposition 3.8. *Suppose that F is geometric and that*

$$F_*(t, x, p, -I) \le c_-(|p|) \tag{3.5a}$$

$$F^*(t, x, p, I) \ge c_+(|p|) \tag{3.5b}$$

for some $c_\pm(\sigma) \in C^1[0,\infty)$ *and* $c_\pm(\sigma) \geq c_0 > 0$ *with some constant* c_0.

Set

$$u^\pm(t,x) = \pm(t + \omega_\pm(\rho)), \rho = |x|, \quad with \quad \omega_\pm(\rho) = \int_0^\rho \frac{\sigma}{c_\pm(\sigma)} d\sigma. \tag{3.6}$$

Then u^- *(*u^+*resp.) is a* C^2 *sub-(super-resp.) solution of (3.1a)-(3.1b) provided that* Ω *is convex with* $0 \in \bar{\Omega}$.

Proof. As in [1, Proposition 6.1] $u^\pm$ is a sub-(super)solution of (3.1a) when $x \in \Omega$(see [1, Proposition 6.1]). Therefore we only have to show that $u^\pm$ is a sub-(super)solution of (3.1b) when $x \in \partial\Omega$. Since Ω is convex, we see on the boundary $\partial\Omega$

$$\langle \gamma(x), \nabla_x u^- \rangle = \langle \gamma(x), -x/c_-(|x|) \rangle \leq 0,$$

$$\langle \gamma(x), \nabla_x u^+ \rangle = \langle \gamma(x), x/c_+(|x|) \rangle \geq 0.$$

Clearly $u^\pm$ is C^2, so Proposition 3.7 implies u^- and u^+ are, respectively, sub-(super)solution of (3.1a)-(3.1b). ∎

Proposition 3.9. *Assume that* $h : \mathbf{R} \to \mathbf{R}$ *is a continuous increasing function. Then* $U^\pm_{\xi h}(t,x) = h(u^\pm(t,x-\xi))$ *for* $\xi \in \bar{\Omega}$ *is a sub-(super)solution of (3.1a)-(3.1b)provided that* Ω *is convex.*

Proof. By Proposition 3.8 $u^\pm(t,x-\xi)$ for $\xi \in \bar{\Omega}$ is a sub-(super)solution of (3.1a)-(3.1b). By Proposition 3.6 $h(u^\pm(t,x-\xi))$ is a sub-(super)solution. ∎

Proposition 3.10. *Suppose that* Ω *is convex. Suppose that* F *satisfies* (3.5a), (3.5b) *and that* F *is geometric. Then for every* $a_0 \in C(\bar{\Omega})$ *there is a lower semicontinuous subsolution* v^- *and upper semicontinuous supersolution* v^+ *of* (3.1a)-(3.1b) *with* (3.1c) *in* Q_0 *satisfying* $v^-(t,x) \leq a_0(x) \leq v^+(t,x)$ *for all* $t \in (0,T)$ *and (3.1c) i.e.* $v^\pm = a_0$ *at* $t = 0$.

Proof. For each $\xi \in \bar{\Omega}$ the continuity of a_0 guarantees that there is a continuous nondecreasing function $h = h_\xi : \mathbf{R} \to \mathbf{R}$ with $h(0) = a_0(\xi)$ such that $U^-_{\xi h}(0,x) \leq u_0(x), U^+_{\xi h}(0,x) \geq a_0(x)$. We set

$$v^-(t,x) = \sup\{U^-_{\xi h}(t,x); h = h_\xi, \xi \in \mathbf{R}^n\}$$
$$v^+(t,x) = \inf\{U^+_{\xi h}(t,x); h = h_\xi, \xi \in \mathbf{R}^n\}.$$

By Propositions 3.4 and 3.9 v^- and v^+ are sub-(super)solution. Other properties of $v^\pm$ can be proved similarly as in [1, Proposition 6.4] ∎

Even for nonconvex Ω, we can construct sub-(super)solution by using local coordinates $\chi_\xi = (\chi_\xi^1, \cdots, \chi_\xi^n)$ near boundary point $\xi \in \partial\Omega$ with $\chi_\xi^n(x) = \mathrm{dist}(x, \partial\Omega)$ for $x \in \bar{\Omega}$.

Proposition 3.11. *For a nonconvex domain Ω, the statement of Proposition 3.10 still holds provided that $\partial\Omega$ is C^2.*

Proof. For each $\xi \in \partial\Omega$ there is a C^2 diffeomorphism $\chi_\xi = (\chi_\xi^1, \cdots, \chi_\xi^n)$: $B_r(\xi) \mapsto \overline{\chi(B_r(\xi))}$ such that $\chi_\xi^n = \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$ and $\chi_\xi(\xi) = 0$, where $B_r(\xi)$ denotes the open ball in $\mathbf{R}^n$ of radius r centered at ξ. Under the mapping $y = \chi_\xi(x)$, let $\tilde{u}(y) = u(x)$ and $F(t, x, \nabla u, \nabla^2 u) = \tilde{F}(t, y, \nabla_y \tilde{u}, \nabla_y^2 \tilde{u})$ in $\chi_\xi(B_r)$. Then (3.1a), (3.1b) become

$$\tilde{u}_t + \tilde{F}(t, y, \nabla_y \tilde{u}, \nabla_y^2 \tilde{u}) = 0 \quad \text{in} \quad \chi_\xi(\mathrm{B_r} \cap \Omega) \tag{3.1'a}$$

$$\langle \tilde{\gamma}(y), \nabla_y \tilde{u} \rangle = 0 \quad \text{on} \quad \chi_\xi(B_r \cap \partial\Omega). \tag{3.1'b}$$

Here $\tilde{\gamma} = (0, \cdots, 0, -1)$ is the unit exterior normal vector to the halfspace $\{y_n > 0\}$. We easily observe that $\tilde{F}$ is still degenerate elliptic and geometric. By Propositions 3.7 and 3.8 the function $u_\xi^-(t, \chi_\xi(x))$ is a subsolution of

$$u_t + F(t, x, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad B_r(\xi) \cap \Omega$$

$$\partial u / \partial \gamma = 0 \quad \text{on} \quad B_r(\xi) \cap \partial\Omega.$$

Let M be a constant such that $\inf_{\bar{\Omega}} a_0 > M$. We next take a continuous increasing function $h_\xi : \mathbf{R} \to \mathbf{R}$ satisfying

(i) $\quad h_\xi(0) = a_0(\xi)$

(ii) $\quad h_\xi(-\omega_-(|\chi_\xi(x)|)) \leq a_0(x), x \in B_r(\xi) \cap \Omega$

(iii) $\quad h_\xi(-\omega_-(|\chi_\xi(x)|)) \leq M, x \in \partial B_r(\xi) \cap \Omega,$

where $\omega_-(\rho)$ is defined by (3.6). Of course, this is possible since a_0 is continuous on $\bar{\Omega}$ and ω_- is increasing. We then set

$$V_\xi^-(t, x) = h_\xi(u^-(t, \chi_\xi(x)) \vee M. \tag{3.7}$$

Since the equation is geometric, the property (iii) implies that $V_\xi^-(t, x)$ is a global subsolution of (3.1a)-(3.1b). By (i) and (ii)

$$V_\xi^-(t, x) \leq a_0(x) \quad \text{for all} \quad t \geq 0 \tag{3.8}$$

$$V_\xi^-(0, \xi) = a_0(\xi). \tag{3.9}$$

If ξ is an interior point of Ω, then we take $B_r(\xi)$ small so that $B_r(\xi)$ is contained in Ω. We may assume that $\xi = 0 \in \Omega$ by a translation. We take h_ξ satisfying (i)-(iii) where χ_ξ is replaced by the identity mapping id. We define V_ξ^- by (3.7) with $\chi_\xi = \mathrm{id}$. Similarly to the case $\xi \in \partial\Omega$ we observe that V_ξ satisfies (3.8), (3.9) and is a global subsolution of (3.1a)-(3.1b). By Proposition 3.4, it follows from (3.8) that

$$v^-(t,x) = \sup\{V_\xi^-(t,x); \xi \in \bar{\Omega}\}$$

is a lower semicontinuous subsolution of (3.1a)-(3.1b). By (3.9) we also observe $v_*^-(0,x) = a_0(x)$. One can construct an upper semicontinuous supersolution satisfying $v^+(t,x) \geq a_0(x)$ and $v^{+*}(0,x) = a_0(x)$ as in the same way. ∎

By Perron's method we find at least one viscosity solution for (3.1a)-(3.1c). The uniqueness and continuity of the solution follows from the comparison theorem in §2 when Ω is convex. We consider the equation independent of space variables for simplicity.

$$u_t + F(t, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega \tag{3.1"a}$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \tag{3.1"b}$$

As in [7] if F is geometric, (F1)-(F4) deduces (3.5a) and (3.5b) so we obtain:

Theorem 3.11. *Assume that F satisfies* (F1)-(F4) *and is geometric. Then for $a \in C(\bar{\Omega})$ there is a viscosity solution u_a of* (3.1"a)-(3.1"b) *with $u_a(0,x) = a(x)$. If Ω is convex, u_a is unique and $u_a \in C(\bar{Q}_0)$. Moreover if $b \leq a$ with $b \in C(\bar{\Omega})$, then $u_b \leq u_a$ in Q_0.*

Remark 3.12. Recently in [9] we find a comparison principle for a general C^2 bounded domain so it turns out that the uniqueness and continuity of u holds even for a general domain.

§4. Interface evolution.

In this section we apply our results in §2 §3 to construct a unique generalized evolution to (1.1a)-(1.1b). Once we have established Proposition 3.6 and Theorem 3.11, we obtain a unique generalized evolution as in [7] and [1]. We shall briefly state our results for the reader's convenience. Assume that Ω is a bounded convex domain and $\partial\Omega$ is of C^2 class. We consider (1.1a)-(1.1b) i.e.,

$$V = f(t, \mathbf{n}(x), \nabla \mathbf{n}(x)) \quad \text{on} \quad \Gamma_t \tag{4.1a}$$

$$\langle \gamma(x), \mathbf{n}(x) \rangle = 0 \quad \text{on} \quad b\Gamma_t. \tag{4.1b}$$

Suppose that $u > 0$ in D_t^+, $u = 0$ on Γ_t and $u < 0$ in D_t^-. If $u(t,x)$ is C^2 and $\nabla u \neq 0$ near Γ_t for $x \in \Omega$, we see

$$\mathbf{n} = -\frac{\nabla u}{|\nabla u|}, \quad \nabla \mathbf{n} = -\frac{1}{|\nabla u|}(Q_{\bar{p}}(\nabla^2 u)) \tag{4.2}$$

and $Q_{\bar{p}}(X) = R_{\bar{p}}XR_{\bar{p}}$ with $R_{\bar{p}} = I - \bar{p} \otimes \bar{p}$, $\bar{p} = \nabla u/|\nabla u|$. It follows from (4.2) and $V = \partial_t u/|\nabla u|$ that (4.1a), (4.1b) is formally equivalent to

$$\text{(4.3a)} \qquad \partial_t u + F(t, \nabla u, \nabla^2 u) = 0 \quad \text{if} \quad x \in \Omega \cap \Gamma_t$$

$$\text{(4.3b)} \qquad \langle \gamma(x), \nabla u \rangle = 0 \quad \text{if} \quad x \in b\Gamma_t$$

with

$$\text{(4.4)} \qquad F(t, p, X) = -|p| f(t, -\bar{p}, -\frac{1}{|p|}(Q_{\bar{p}}(X)).$$

Similarly to [7] we first define a weak solution $\{\Gamma_t, D_t^+\}_{t \geq 0}$ of (4.1a)-(4.1b) through a viscosity solution of (4.3a)-(4.3b) with (4.4).

Definition 4.1. *Let D_0^+ and D_0^- be disjoint open sets and Γ_0 be a closed set in Ω such that $D_0^+ \cup D_0^- = \Omega \setminus \Gamma_0$. Suppose that there is a viscosity solution $u \in C([0,T) \times \bar{\Omega})$ of (4.3a)-(4.3b) in $(0,T) \times \bar{\Omega}$ with $u(0,x) = a(x)$ such that zero level surface of $u(t,\cdot)$ at time $t \geq 0$ equals Γ_t and that the set of $u(t,x) > 0$ equals D_t^+. If $\{\Gamma_t, D_t^+\}|_{t=0} = \{\Gamma_0, D_0^+\}$, we say $\{\Gamma_t, D_t^+\}_{t \geq 0}$ is a weak solution of (4.1a)-(4.1b) with initial data $\{\Gamma_0, D_0^+\}$.*

As in [7] we list assumption of f to which our theory applies.

$$\text{(f1)} \qquad f : [0,T) \times E \to \mathbf{R} \quad \text{is} \quad \text{continuous,}$$

where $E = \{(\bar{p}, Q_{\bar{p}}(X)); \bar{p} \in S^{n-1}, X \in S^n\}$ (see [7]).

$$\text{(f2)} \qquad f(t, -\bar{p}, -Q_{\bar{p}}(X) \geq f(t, -\bar{p}, -Q_{\bar{p}}(Y))$$

for $X \geq Y$, $\bar{p} \in S^{n-1}$ and $t \geq 0$.

$$\text{(f3)} \qquad \lim_{\rho \downarrow 0} \rho \inf_{\bar{p}=1} (-f(s, -\bar{p}, \frac{I - \bar{p} \otimes \bar{p}}{\rho})) > -\infty,$$

$$\text{(f4)} \qquad \lim_{\rho \downarrow 0} \rho \sup_{\bar{p}=1} (-f(s, -\bar{p}, \frac{-I + \bar{p} \otimes \bar{p}}{\rho})) < +\infty.$$

Theorem 4.2. *Assume that f satisfies (f1)-(f4). Let ${D_0}^+$ and ${D_0}^-$ be disjoint open sets and Γ_0 be a closed set in Ω such that ${D_0}^+ \cup {D_0}^- = \Omega \setminus \Gamma_0$. Then there is a unique global weak solution $\{\Gamma_t, {D_t}^+\}_{t \geq 0}$ of (4.1a)-(4.1b) with initial data $\{\Gamma_0, {D_0}^+\}$.*

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