Generalized Motion by Mean Curvature for Surfaces of Rotation

S. Altschuler, S. Angenent and Y. Giga

Series ♯119. August 1991

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

- ♯ 92: T. Nakazi, K. Takahashi, Homogoneous polynomials and invariant subspaces in the polydiscs II, 10 pages. 1990.
- ♯ 93: A. Arai, A theorem on essential self-adjointness with application to Hamiltonians in non-relativistic quantum field theory, 23 pages. 1990.
- ♯ 94: Y. Okabe, A. Inoue, On the exponential decay of the correlation functions for KMO-Langevin equations, 13 pages. 1990.
- ♯ 95: T. Sano, Y. Watatani, Angles between two subfactors, 62 pages. 1990.
- ♯ 96: S. Ninomiya, The Fourier-Sato transformation of pure sheaves, 22 pages. 1990.
- ♯ 97: Y. Okabe, A. Inoue, The theory of KM_2O-Langevin equations and its applications to data analysis (II): Causal analysis (1), 51 pages. 1990.
- ♯ 98: J. Lawrynowicz, S. Koshi and O. Suzuki, Dualities generated by the generalised Hurwitz problem and variation of the Yang-Mills field, 17 pages. 1991.
- ♯ 99: R. Agemi, K. Kubota and H. Takamura, On certain integral equations related to nonlinear wave equations, 52 pages. 1991.
- ♯ 100: S. Izumiya, Geometric singularities for Hamilton-Jacobi equation, 13 pages. 1991.
- ♯ 101: S. Izumiya, Legendrian singularities and first order differential equations, 16 pages. 1991.
- ♯ 102: A. Munemasa, Y. Watatani, Orthogonal pairs of $*$-subalgebras and association schemes, 11 pages. 1991.
- ♯ 103: A. Arai, O. Ogurisu, Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, 27 pages. 1991.
- ♯ 104: H. Takamura, Global existence of classical solutions to nonlinear wave equations with spherical symmetry for small data with noncompact support in three space dimensions, 14 pages. 1991.
- ♯ 105: R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, 11 pages. 1991.
- ♯ 106: T. Nakazi, Extremal problems in H^p, 13 pages. 1991.
- ♯ 107: T. Nakazi, ρ-dilations and hypo-Dirichlet algebras, 15 pages. 1991.
- ♯ 108: A. Arai, An abstract sum formula and its applications to special functions, 25 pages. 1991.
- ♯ 109: Y.-G. Chen, Y. Giga and S. Goto, Analysis toward snow crystal growth, 18 pages. 1991.
- ♯ 110: T. Hibi, M. Wakayama, A q-analogue of Capelli's identity for $GL(2)$, 7 pages. 1991.
- ♯ 111: T. Nishimori, A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem, 13 pages. 1991.
- ♯ 112: K. Matsuda, An analogy of the theorem of Hector and Duminy, 10 pages. 1991.
- ♯ 113: S. Takahashi, On a regularity criterion uo to the boundary for weak solutions of the Navier-Stokes equations, 23 pages. 1991.
- ♯ 114: T. Nakazi, Sum of two inner functions and exposed points in H^1, 18 pages. 1991.
- ♯ 115: A. Arai, De Rham operators, Laplacians, and Dirac operators on topological vector spaces, 27 pages. 1991.
- ♯ 116: T. Nishimori, A note on the classification of non-singular flows with transverse similarity structures, 17 pages. 1991.
- ♯ 117: T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, 6 pages. 1991.
- ♯ 118: R. Agemi, H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, 30 pages. 1991.

Advanced Studies in Pure Mathematics

Generalized Motion by Mean Curvature for Surfaces of Rotation

Stephen Altschuler, Sigurd Angenent and Yoshikazu Giga

§1. Introduction and main results.

This note is an announcement of our recent work on regularity of generalized evolution by mean curvature in the sense of Chen-Giga-Goto [CGG] and Evans-Spruck [ES1] when the initial (hyper) surface in $\mathbb{R}^{n+1}$ is compact and obtained by rotating the graph of a function around an axis. Our principal results are the regularity of an evolving surface except at finitely many points on the axis at finitely many times and estimates of these numbers. The details and proofs will appear in the authors' forthcoming paper [AAG].

Let us recall the generalized evolution by mean curvature, using terminology slightly different from [CGG]. Let D and E be, respectively, an open and closed set in the space-time domain $[0,\infty)\times\mathbb{R}^{n+1}$ such that $E\supset D$. Suppose that for some $\alpha<0$ there is a viscosity solution $\psi\in K_\alpha$ of

$$\psi_t - |\nabla\psi|\operatorname{div}\left(\frac{\nabla\psi}{|\nabla\psi|}\right) = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^{n+1}, \tag{1}$$

where $\nabla\psi$ denotes the spatial gradient and $\psi_t = \partial\psi/\partial t$. Here we define

$$K_\alpha = \{\psi\in C([0,\infty)\times\mathbb{R}^{n+1}) : (\psi-\alpha) \text{ has compact support in } [0,T]\times\mathbb{R}^{n+1} \text{ for any } T>0\}.$$

If

$$D = \{(t,z)\in(0,\infty)\times\mathbb{R}^{n+1} : \psi(t,x)>0\}$$
$$E = \{(t,z)\in(0,\infty)\times\mathbb{R}^{n+1} : \psi(t,x)\geq 0\},$$

then we say that D and E is respectively, the *inner* and *outer evolution* with initial data $D(0)$ and $E(0)$ for the mean curvature flow problem.

Here $W(t)$ denotes the cross-section at time t of any set W in $[0,\infty) \times \mathbb{R}^{n+1}$.

By a main result in [CGG], for a given bounded open set D_0 and a compact set E_0 containing D_0, there is a unique inner and outer evolution with initial data D_0 and E_0, respectively. Moreover one can prove that D is completely determined by D_0 and independent of E_0. Similarly, E is completely determined by E_0. As shown in [ES1] the set $\Gamma = E \backslash D$ is also completely determined by $\Gamma_0 = E_0 \backslash D_0$. We say that Γ is the *interface* evolution with initial data Γ_0. The interface evolution equals the zero-level set of ψ solving (1). If ψ is smooth and $\nabla\psi \neq 0$ on $\Gamma(t)$, (1) implies that $\Gamma(t)$ moves by its mean curvature so our generalized formulation is natural. Evans and Spruck [ES1] showed that as long as classical smooth solution $\Gamma(t)$ exists, it coincides with the interface evolution defined above. We point out that for motion of curves (not necessarily embedded) there are other ways to track the evolution through singularities [A2], [AGr].

We are interested in the evolution of surfaces obtained by rotating the graph of a function around an axis. Suppose that

$$(2) \qquad D_0 = \{(x,y) \in \mathbb{R}^{n+1} : r < u_0(x)\}, \quad r = (y_1^2 + \cdots + y_n^2)^{1/2},$$

with $u_0 \in C(\mathbb{R})$, $u_0 \geq 0$. Suppose that D_0 is a bounded open set (not necessarily connected) with smooth boundary Γ_0. Since $D_0 \cup \Gamma_0$ is contained in a big ball, by comparison $\Gamma(t)$ must become empty in a finite time (cf. [CGG], [ES1]). In other words there is a time t_* called the *extinction time* such that $\Gamma(t) = \phi$ for $t > t_*$ and $\Gamma(t_*) \neq \phi$. We now state our main results on regularity.

Theorem 1.1. *Let Γ_0 be the smooth boundary of a bounded open set D_0 defined in (2). Let Γ be the interface evolution with initial data Γ_0. $\Gamma(t)$ is a smooth family of smooth surfaces except for possibly finitely many times $t_1 < t_2 < \cdots < t_l = t_*$ when singularities may occur along the axis of rotation.*

According to Grayson [Gr], if $D_0 \subset \mathbb{R}^3$ is a barbell with a long, thin handle, then the interface evolution $\Gamma(t)$ with initial data $\Gamma_0 = \partial D_0$ ceases to be regular in a finite time (before it shrinks to a point). Theorem 1.1 says that $\Gamma(t)$ becomes smooth instantaneously after it experiences singularities (although the number of connected components of $\Gamma(t)$ may increase).

We also obtain a rough picture of the interface evolution by counting the number of "necks" and "bulges" of $D(t)$. Suppose that an open set D_0 is of the form (2). By the number k of *necks* of D_0 we mean the number of local minima of u_0 on the open set where u_0 is positive. Similarly the number h of *bulges* of D_0 is defined as the number of local maxima of u_0.

Theorem 1.2. *Let Γ_0 and D_0 be as in Theorem 1.1. Let Γ and D be, respectively, the interface evolution and inner evolution with initial data Γ_0 and D_0.*

(i) *There is a $u \in C(\mathbb{R} \times [0,\infty))$ with $u \geq 0$ which is smooth where u is positive such that*

$$D = \{(t,x,y) \in [0,\infty) \times \mathbb{R}^{n+1} : r < u(x,t)\}$$
$$\Gamma = \{(t,x,y) \in [0,\infty) \times \mathbb{R}^{n+1} : r = u(x,t), (x,t) \in \text{spt } u\},$$

(ii) *Let $k(t)$ and $h(t)$ be, respectively, the number of necks and bulges of $D(t)$. Then $k(t)$ and $h(t)$ are nonincreasing functions which are finite for $t > 0$. At the singular times t_j $(1 \leq j \leq \ell)$ the sum $k + h$ must drop. In particular $\ell \leq k(\delta) + h(\delta) \leq k(0) + h(0)$ for δ sufficiently small.*

(iii) *Near $t = t_j$ the set Γ is smooth except at finitely many points on the x-axis at the time t_j. The number of such points is not greater than $(h+k)(t_j - 0) - (h+k)(t_j)$, where $h(t-0) = \lim_{s\uparrow t} h(s)$.*

Let $p(t)$ be the number of connected components of $D(t)$. Since $h(t) = k(t) + p(t)$ by the finiteness of h and k, Theorem 1.2 (ii) now yields:

Corollary 1.3. *Let t_j $(1 \leq j \leq \ell)$ be the set of singular times in Theorem 1.1. If $h(t)$ is constant near t_j, then $p(t)$ must increase across $t = t_j$. If $k(t)$ is constant near t_j, then $p(t)$ must decrease across $t = t_j$.*

For example suppose that D_0 is a barbell with a long, thin handle with $k(0) = 1$, $h(0) = 2$. The number of singular times $\ell \leq 3$. For Grayson's barbell, at the first singular time t_1, it pinches off in the middle (at one point). In other words $k(t_1 + 0) = 0$ while $h(t_1 + 0) = 2$ so that $p(t_1 + 0) = 2$. Each connected component of $\Gamma(t)$, ($t > t_1$ near $t = t_1$) shrinks to a point at $t = t_2$ or t_3. If $\ell = 1$, $\Gamma(t)$ shrinks to a point at $t_1 = t_*$ (Proposition 2.5 (iii) and Theorem 2.8). It may happen that $\Gamma(t)$ stays nonconvex for all $t < t_1$.

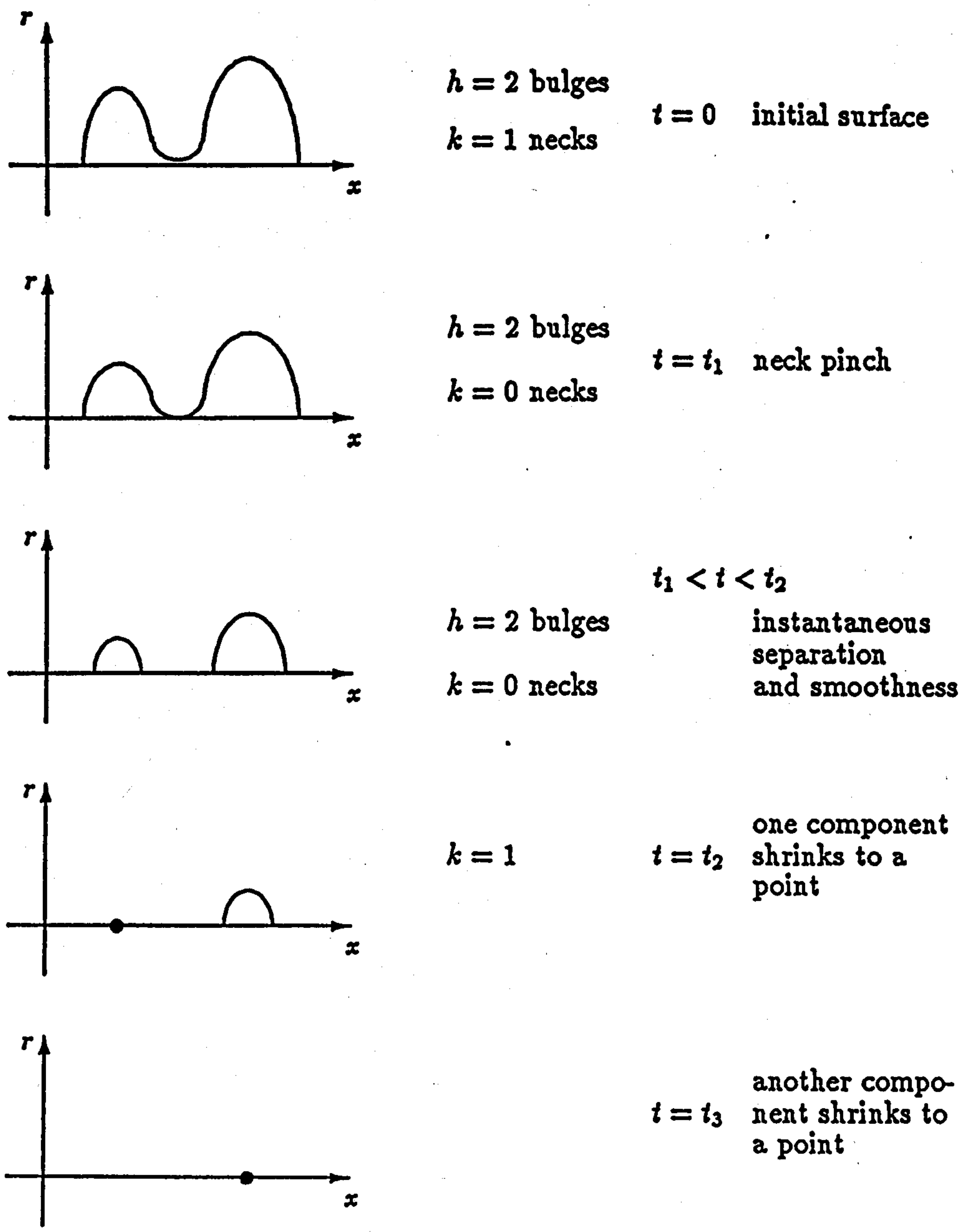
In fact applying a topological argument with our main theorems and Huisken's results [H1], [H2] one can construct such a surface. The following surface Γ_0^λ is pointed out by R. Hamilton.

Corollary 1.4. *Let Γ_0^λ $(0 \le \lambda \le 1)$ be a surface defined by*

$$r^2 = (1 - x^2)(1 - \lambda + \lambda x^4).$$

Let Γ^λ be the interface evolution with initial data Γ_0^λ. There is a μ, $0 < \mu < 1$ such that

barbell evolution



(i) $\Gamma^\mu(t)$ stays nonconvex and smooth for all $t < T_\mu$ and shrinks to a point at time $t = T_\mu$.

(ii) $\lim_{t \uparrow T_\mu} \sup(\sup_{\Gamma^\mu(t)} |A(t)|)(T_\mu - t)^{1/2} = \infty$

where A denotes the second fundamental form of $\Gamma^\mu(t)$.

This gives a concrete example of a so-called type II singularity in dimension greater than one (curve shortening). The singularity demonstrated above is sometimes referred to as a 'degenerate neck princh'. For convex curves, type II singularities were studied in detail in [A2] and for general curves by [Alt]. Once $\Gamma(t)$ becomes convex, according to Huisken [H1], it stays convex and shrinks to a point with finite limit in (ii).

§2. Until necks pinch

We begin with various properties of a smooth family of surfaces $\Gamma(t)$ $(0 < t < T)$ which evolves by their mean curvature. We assume that $\Gamma(t)$ is compact, connected, and that it is obtained by rotating the graph of function

$$r = u(x,t), \quad a_1(t) < x < a_2(t), \quad 0 \le t < T$$

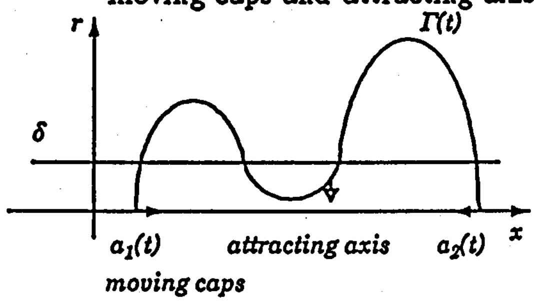
around the axis. The mean curvature flow equation is written as the *horizontal graph equation* for u:

$$u_t = \frac{u_{xx}}{1+(u_x)^2} - \frac{n-1}{u}. \tag{3}$$

Theorem 2.1 (Attracting axis). *For any $\tau > 0$, there is a $\delta = \delta(\Gamma(0), \tau) > 0$ such that the mean curvature H (measured with respect to the outward unit normal) is nonpositive at any point on $\Gamma(t)$ with $r < \delta$ and $t > \tau$.*

In other words the motion is monotone near x-axis.

moving caps and attracting axis



Theorem 2.2 (Interior gradient estimates). *Let $u(x,t)$ be a smooth solution of the graph equation, defined for $|x-x_0| < \varepsilon$, $0 < t < T$, and let*

$$\delta = \inf_{|x-x_0|<\varepsilon} u(x,0)$$

be positive. Then there is a constant $C = C(\delta, \varepsilon)$ such that

$$|u_x(x,t)| \leq \exp C(\frac{u(x,t)^2}{t} + t)$$

for all $|x - x_0| < \varepsilon/2$ and $0 < t < T$ with $u(x,t) > \delta$.

The proofs depend on the Sturmian theorem concerning the zeroes of classical solution of a linear parabolic PDE (see [A1], [A2]). To prove Theorem 2.1 we need the following lemma which is also obtained by the Sturmian theorem [A4] but for the singular equation "vertical graph equation":

$$v_t = \frac{v_{rr}}{1+(v_r)^2} + \frac{n-1}{r} v_r$$

where $\Gamma(t)$ is expressed as $x = v(r,t)$ near the x-axis.

Lemma 2.3 (Moving caps). $a_1'(t) > 0 > a_2'(t)$ *for* $0 < t < T$.

To get Theorem 2.1 we count the intersections of the graphs of $u(x,t)$ and the Catenoid ($n = 2$) (and its generalizations for $n \geq 3$) which are stationary solutions of the mean curvature flow equations, i.e., minimal surfaces. To prove Theorem 2.2 we use other special solutions instead of the Catenoid. Similar applications of the Sturmian theorem are found in [A2, A3].

The following lemma on intersections is a variant of the Sturmian theorem.

Lemma 2.4 (Intersections). *Suppose that $\Gamma^i(t)$ $(i = 1,2)$ smoothly evolves by its mean curvature $(0 \leq t < T)$. Suppose that $\Gamma^i(t)$ is obtained by rotating a smooth simple curve γ_t^i in the $x-r$ plane $(r \geq 0)$ around the x-axis and that $\Gamma^i(t)$ is smooth. Suppose that $\gamma_t^1 \cap \gamma_t^2$ is compact.*

(i) *The two curves γ_t^1 and γ_t^2 intersect transversally except a discrete set of times $\{0 < \cdots < t_3 < t_2 < t_1 < T\}$ (unless $\gamma_t^1 \equiv \gamma_t^2$).*

(ii) *At each time t_j two curves γ_t^1 and γ_t^2 touch nontransversally, the number of intersections decreases immediately afterwards.*

We next apply these results to describe the picture at the time of pinching off. Let Γ_0 be the smooth boundary of a bounded open set D_0

defined in (2). Since each connected component evolves independently, we may assume D_0 is connected.

Proposition 2.5. *Let $(0, T_*)$ be the maximal time interval such that the interface evolution $\{\Gamma(t)\}_{0 \le t < T_*}$ with initial data Γ_0 remains smooth.*

(i) *The surface $\Gamma(t)$ is expressible as*

$$r = u(x,t), \quad a_1(t) < x < a_2(t), \quad 0 \le t < T_*.$$

(ii) *One of local minimal values of $u(x,t)$ on $(a_1(t), a_2(t))$ converges to zero as $t \to T_*$ provided that $a_{1*} < a_{2*}$*

(iii) *If $a_{1*} = a_{2*}$, then $\Gamma(t)$ shrinks to a point as $t \to T_*$. Here $a_{i*} = \lim_{t \to T_*} a_i(t)$, $i = 1, 2$. (The existence of limits is guaranteed by moving cap lemma.)*

Part (i) is clear if we apply the intersection lemma 2.4 with vertical lines $x = x_0$ which are stationary solutions of the motion by mean curvature.

Part (ii) needs the Korevaar interior gradient estimate by Evans and Spruck [ES2] for graphs moving by their mean curvature as well as Theorem 2.2. If all local minima stay away from zero, Theorem 2.2 implies that $\Gamma(T_*)$ is smooth except on the x-axis while the Evans-Spruck estimate implies that $\Gamma(T_*)$ is smooth near the x-axis (by using higher regularity results [LUS]). This contradicts the definition of T_*. If $a_{1*} = a_{2*}$ and $\Gamma(t)$ does not shrink to a point as $t \to T_*$, Theorem 2.2 is contradicted so (iii) is proved.

Theorem 2.6 (Regularity except the x-axis). *Let D_0 and Γ_0 be as in Proposition 2.5.*

(i) *The function $r = u(x,t)$ can be extended continuously up to $t = T_*$.*

(ii) *The extended function u is smooth on the place where $u > 0$.*

(iii) *Let Γ and D be, respectively, the interface and inner evolution with initial data Γ_0 and D_0. Then*

$$D(T_*) = \{(x,y) : r < u(x, T_*)\}$$
$$\Gamma(T_*) = \{(x,y) : r = u(x, T_*),\ a_{1*} \le x \le a_{2*}\}$$

Part (i) follows from the attracting axis Theorem 2.1, the moving caps Lemma 2.3 and the interior gradient estimate Theorem 2.2.

Part (ii) follows from higher regularity results [LUS] once the gradient is estimated.

Part (iii) depends on (left) continuity of the interface evolution $\Gamma(t)$ in t (in the Hausdorff distance).

Proposition 2.7 (Nonincrease of necks and bulges). *Let $k(t)$ and $h(t)$ be, respectively the number of necks and bulges of $D(t)$.*

(i) *$k(t)$ and $h(t)$ are nonincreasing on $[0, T_*)$ and finite except when $t = 0$.*

(ii) *There is a T_0 such that $k(t)$ and $h(t)$ are constants and that $u_x(\cdot, t)$ has no multiple zero provided that $t \in (T_0, T_*)$. In particular $k+1 = h$.*

This is another application of the Sturmian theorem [A1]. For $t \in (T_0, T_*)$ let $\{\eta_j(t)\}_{j=0}^{k}$ and $\{\xi_j(t)\}_{j=1}^{k}$ be, respectively, local maxima and minima of $u(x,t)$. Adapting the reflection argument of Chen and Matano [CM] with Lemma 2.4, one observes that the limits

$$\lim_{t \uparrow T_*} \xi_j(t) = \xi_{j*} \quad \text{and} \quad \lim_{t \uparrow T_*} \eta_j(t) = \eta_{j*}$$

exist for all j. We may assume $a_{1*} \le \eta_{0*} \le \xi_{1*} \le \cdots \le \xi_{k*} \le \eta_{k*} \le a_{2*}$.

Theorem 2.8 (Pinching). *Suppose that $a_{1*} < a_{2*}$.*

(i) *The interface evolution Γ is smooth in $[0, T_*] \times \mathbb{R}^{n+1}$ except at finitely many points*

$$(\xi_{j*}, 0, \cdots, 0) \in \mathbb{R}^{n+1} \quad \text{satisfying} \quad u(\xi_{j*}, T_*) = 0$$

at $t = T_*$. In particular $u(x, T_*)$ has only finitely many zeroes contained in $\{\xi_{j*}\}_{j=1}^{k}$ on the interval $[a_{1*}, a_{2*}]$.

(ii) *Let q be the number of ξ_{j*} which are zeros of $u(x, T_*)$. Then $k(T_*) \le k(T_* - 0) - q$.*

(iii) *$h(T_*) \le h(T_* - 0)$.*

We note that $k(T_*) < k(T_* - 0)$ since Proposition 2.5 (ii) implies $q \ge 1$.

The following lemma is a crucial step in the proof of Theorem 2.8.

Lemma 2.9 (Single point pinching). *Suppose that $u(\xi_{j*}, T_*) = 0$. If $\eta_{j*} > \xi_{j*}$ then $u(x, T_*) > 0$ for all $x \in (\xi_{j*}, \eta_{j*}]$. Likewise, if $\eta_j < \xi_{j+1}$ then $u(x, T_*) > 0$ for all $x \in [\eta_{j*}, \xi_{j+1*})$. Here j varies in $0 \le j \le k$ by interpreting $\xi_{0*} = a_{1*}$, $\xi_{k+1*} = a_{2*}$.*

We sketch the proof for the first case. Let $[a, b]$ be any compact interval in (ξ_{j*}, η_{j*}). There is a $t_1 < T_*$ such that $u_x > 0$ on $[a, b] \times (t_1, T_*)$. We set $\theta = \arctan u_x(x, t)$ and observe

$$\theta_t - \frac{\theta_{xx}}{1 + (u_x)^2} > 0.$$

We take a solution of the heat equation

$$\varphi(x,t) = \varepsilon e^{-\lambda^2 t} \sin \lambda(x-a), \quad \lambda = \pi/(b-a).$$

Since $\varphi_{xx} < 0$ on (a,b) we find

$$\varphi_t - \frac{\varphi_{xx}}{1+u_x^2} < 0.$$

Choosing ε small and applying the weak maximum principle yields $\theta(x,t) \geq \varphi(x,t)$ on $(a,b) \times (t_1, T_*)$. This gives the estimate

$$u_x(x,t) \geq \varepsilon e^{-\lambda^2 T_*} \sin(\lambda(x-a)), \quad a < x < b$$

by taking ε smaller. Integrating in x, we obtain

$$u(x,t) - u(\xi_j(t), t) \geq \int_a^b u_x(x,t)dt \geq \frac{\varepsilon}{\lambda} e^{-\lambda^2 T_*}(1 - \cos \lambda(x-a))$$
$$(a < x < b, t_1 < t < T_*)$$

since $u_x > 0$ on $(\xi_j(t), b)$. Letting $t \to T_*$ yields $u(x, T_*) > 0$ for $x \in (a,b)$. This implies $u(x,T) > 0$ on (ξ_{j*}, η_{j*}) since (a,b) is arbitrary. By monotonicity it follows that $u(\eta_{j*}, T) > 0$.

Remark 2.10. We learned that X.-Y. Chen [C] has also proved a stronger version of Lemma 2.9. Since our proof is very simple, we give it above.

Remark 2.11. The proof shows that u_x does not vanish, except at $\{\eta_{j*}\}$, $\{\xi_{j*}\}$.

§3. Instant smoothness of inner evolutions

The inner evolution may have many connected components at the time T_* when some necks pinch (cf. Theorem 2.6 (iii)). Fortunately the generalized evolutions of each connected component are mutually independent. Indeed, the theory of viscosity solutions for (1) yields:

Theorem 3.1 (Separation). *Let D_i be an inner evolution ($i = 1,2$). Let D be the inner evolution with initial data $D_1(0) \cup D_2(0)$. If $D_1(0)$ and $D_2(0)$ are disjoint, then $D = D_1 \cup D_2$.*

To track the inner evolution after a neck pinches off we study the evolution of each component. It is convenient to introduce the following terminology.

Definition 3.2 (α-domain). *Let $\alpha > 0$ be given and let $U \subset \mathbb{R}^{n+1}$ be an open set of the form*

$$U = \{(x, y) \in \mathbb{R}^{n+1} : r < u(x)\}$$

for some $0 \le u \in C(\mathbb{R})$. We say that U is an α-domain if
(i) *$I = \{x \in \mathbb{R} : u(x) > 0\}$ is a (connected) bounded interval,*
(ii) *u is smooth on I*
(iii) *In the $x - r$ plane ∂U intersects each cylinder ∂C_ρ with $0 < \rho \le \alpha$ exactly twice and these intersections are transverse. Here*

$$C_\rho = \{(x, y) : r < \rho\}.$$

Proposition 3.3. *Each connected component of $D(T_*)$ is an α-domain for some $\alpha > 0$.*

This follows from Theorems 2.6, 2.8 and Remark 2.11.

We next study the regularity of an inner evolution starting from an α-domain.

Theorem 3.4 (Instant smoothness). *Let U be an α-domain and let $D \subset [0, \infty) \times \mathbb{R}^{n+1}$ be the inner evolution with initial data U. Then there is a $T > 0$ such that $\partial D(t)$ is a smooth hypersurface for $0 < t < T$. Moreover D is smooth in $[0, T) \times \mathbb{R}^{n+1}$ except on the x-axis at $t = 0$.*

To prove the instant smoothness theorem, we first approximate U by a nondecreasing sequence $\{U_j\}_{j=1}^{\infty}$ of smoothly bounded α-domains from inside. This can be obtained by smoothing U near the axis. If the D_j are the inner solutions with initial data U_j, the theory of viscosity solutions for (1) implies that the nondecreasing sequence D_j approximates D from the inside.

We then show that for some $t_0 > 0$ and $0 < \beta < \alpha$ the $D_j(t)$ are smooth β-domains for $t \in (0, t_0)$. The choice of t_0 and β is independent of j. This is a crucial step in the proof and we rely on the Sturmian theorem to establish it.

We next use the interior gradient estimates in [ES2] for graphs moving by their mean curvature, and prove that the $\partial D_j(t)$ are uniformly smooth for $0 < t < t_0$ near the axis of rotation. This leads to the smoothness of $\partial D(t)$ near the axis for $0 < t < t_0$. The regularity away from the axis is easier to prove (even up to $t = 0$). Indeed, one obtains the (interior) gradient estimate (Theorem 2.2) and higher derivative estimates [LUS] for (3) up to initial data if $u \ge \delta > 0$.

Let us state the crucial lemma for our argument. Let V be a smoothly bounded domain in $\mathbb{R}^{n+1}$. Let $\{\Gamma(t)\}$ be the interface evolution with initial data ∂V. Let $(0, T_V)$ be the maximal interval during which the solution $\Gamma(t)$ remains smooth.

Lemma 3.5. *Let V be a smoothly bounded α-domain and let $D \subset [0,\infty) \times \mathbb{R}^{n+1}$ be its corresponding inner evolution. Then there exists a $t_V^\alpha \in (0, T_V]$ such that $D(t)$ is α-domain for all $0 < t < t_V^\alpha$, where*

$$\alpha(t) = \sqrt{\alpha^2 - 2(n-1)t}.$$

If $W \supset V$ is another α-domain, then we can choose $t_W^\alpha \geq t_V^\alpha$. In particular, it follows that $T_W \geq t_V^\alpha$.

For the proof we examine the intersections of the cylinder ∂C_ρ and $\partial D(t)$ in the $x - r$ plane.

Remark 3.6. Once smoothness of $\partial D(t)$ is established in Theorem 3.4 it is easy to discuss the disappearance at a time of singularity of necks and bulges. Actually the statements of Proposition 2.7 hold even if initial data $D(0)$ is an α-domain with singularities on the x-axis.

Combining the main results in §2 and §3 one can prove Theorems 1.1, 1.2 and Corollary 1.3 with Γ replaced by ∂D, the boundary of D in $[0,\infty) \times \mathbb{R}^{n+1}$. At the first time $t_1 = T_*$ when ∂D loses its smoothness, $D(T_*)$ consists of disjoint α-domains by Proposition 3.3. By the instant smoothness and separation we see $D(t)$ consists of disjoint smooth domains for $t \in (t_1, T + t_1)$ with some $T > 0$. By the moving cap lemma $\partial(D(t))$ consists of disjoint smooth surfaces for $t \in (t_1, T + t_1)$. Counting bulges and necks one concludes Theorem 1.2 (ii). The statement of Theorem 1.2 (iii) is essentially included in Theorem 2.8 (i) provided that Γ is replaced by ∂D. By Theorem 1.2 (ii), one concludes that the number ℓ of singular time is finite in Theorem 1.1 if $\Gamma = \partial D$. We will not give the detailed proof of Theorems 1.1, 1.2 and Corollary 1.3.

The condition $\Gamma = \partial D$ is not always fulfilled. Indeed Evans and Spruck [ES1] constructed a counter example by giving an initial data like "figure eight" in the plane. In their example ∂D is strictly smaller than Γ. However, if the motion is monotone, then $\Gamma = \partial D$ always holds. For example if the initial (outward) means curvature is negative, then we see $\Gamma = \partial D$.

§4. Regular evolution

We say the inner evolution D is *regular* if its closure $\bar{D}$ is an outer evolution. If D is regular, it is easy to see that $\Gamma = \partial D$ where Γ is the interface evolution with initial data $\Gamma(0) = \partial D(0)$. The following result implies that Theorems 1.1, 1.2 and Corollary 1.3 hold for Γ instead of ∂D.

Theorem 4.1 (Regularity). *Let D_0 be a bounded open set defined in (2) with smooth boundary. Then the inner evolution D with initial data D_0 is regular.*

Since ∂D is smooth except at finitely many times, we only need to prove that $\bar{D}$ is the outer evolution across any singular time. By Theorems 2.6 and 2.9 we observe $\bar{D}$ is the outer evolution until the singular time (denoted T_*). Theorem 4.1 is now reduced to:

Theorem 4.2. *Let $D(T_*)$ be as in Theorem 2.6. Let D be the inner evolution with initial data $D(T_*)$. Then for some $t_0 > 0$, $\bar{D}$ is the outer evolution in $[0, t_0) \times \mathbb{R}^{n+1}$ with initial data $\bar{D}(T_*)$.*

We give a general criterion so that $\bar{D}$ is the outer evolution.

Lemma 4.3 (Regularity criterion). *Let U be a bounded open set in $\mathbb{R}^{n+1}$ which may be written as the union of a finite number of disjoint open sets $U^1, \cdots, U^k$. Denote the inner evolutions with initial data U and U^i by D and D^i respectively ($1 \le i \le k$) and let E^i be the outer evolutions with initial data $\bar{U}^i$. Suppose that*

$$(4) \qquad E^i = \bar{D}^i \quad \text{in } [0, t_0) \times \mathbb{R}^{n+1} \quad \text{for some } t_0 > 0.$$

Suppose that we are given a sequence of open covers $\{U^1_\alpha, \cdots, U^k_\alpha\}_{\alpha \ge 1}$ of U which satisfies

$$(5) \qquad U^i_\alpha \supset U^i_{\alpha+1} \text{ and } \bar{U}^i = \bigcap_{\alpha \ge 1} \bar{U}^i_\alpha \quad \text{for} \quad i = 1, 2, \cdots, k.$$

$$(6) \qquad \text{The } U^1_\alpha, \cdots, U^k_\alpha \quad \text{are pairwise disjoint for } i = 1, 2, \cdots, k.$$

Finally, let E be an outer evolution for which a double sequence $\{t_{\alpha,\ell}\}_{\alpha,\ell \ge 1}$ exists such that $t_{\alpha,\ell} \downarrow 0$ as $\ell \to \infty$ and

$$(7) \qquad E(t_{\alpha,\ell}) \subset U^1_\alpha \cup \cdots \cup U^k_\alpha.$$

Then $\bar{D} = E$ in $(0, t_0) \times \mathbb{R}^{n+1}$.

This can be proved by a general theory of generalized evolutions. We apply Lemma 4.3 with $U = D(T_*)$ to prove Theorem 4.2. Let U^i ($1 \le i \le k$) denote the connected components of U. Then property (4) follows from the following lemma because each U^i is an α-domain by Proposition 3.3.

Lemma 4.4. *Let D_0 be an α-domain. Then the inner evolution with initial data D_0 is regular in $[0, t_0) \times \mathbb{R}^{n+1}$ for some $t_0 > 0$.*

As in Theorem 3.4 one can prove that the boundary of the outer evolution E with initial data $\bar{D}_0$ is smooth for a short time $(0, t_0)$. Applying the maximum principle we observe $\partial E = \partial D$ in $(0, t_0)$ which yields Lemma 4.4.

We now go back to sketch the proof of Theorem 4.2. It is easy to find a sequence $\{U^1_\alpha, \cdots, U^k_\alpha\}_{\alpha \geq 1}$ satisfying (5), (6) such that U^i_α is of the form (2) with smooth boundary. One may arrange $U^i_\alpha \supset \bar{U}^i$ outside x-axis.

To apply Lemma 4.3 it remains to show (7) where E is the outer evolution with initial data $\bar{U}$. We say ξ is a *pinching point* if $u(\xi, T_*) = 0$ and $a_{1*} < \xi < a_{2*}$ where a_{i*} $(i = 1, 2)$ is defined in Proposition 2.5. Since $t \mapsto E(t)$ is left continuous and the motion is monotone near α-axis (by the attracting axis Theorem 2.1), (7) is fulfilled provided that no pinching point belong to $E(t)$ for $t > T_*$. Fortunately, we have:

Lemma 4.5. *Let E be the outer evolution with initial data $\bar{U} = \overline{D(T_*)}$. Then $E(t)$ contains no pinching points (of $\bar{U}$) for $t > 0$.*

Our remaining task is to prove Lemma 4.5. We just mention a crucial lemma on the shape of $D(T_*)$ near a pinching point.

Lemma 4.6 (Near a pinching point). *Let ξ be a pinching point. Then for $\varepsilon > 0$ there is $\delta > 0$ such that $|x - \xi| < \delta$ implies that*

$$u(x, T_*) \leq \varepsilon |x - \xi|.$$

This can be proved from the asymptotic behavior of u near ξ obtained by Huisken [H2], and a comparison with a cylinder suggested by Ilmanen. In [H2] Huisken proved that the interface asymptotically looks like a cylinder near a pinching point under the assumption that the mean curvature H of the initial surface is everywhere negative and that $n = 2$. However, it turns out this assumption is unnecessary if we use the attracting axis Theorem 2.1.

After this work was completed, we were informed of recent work by H.M. Soner and P.E. Souganidis [SS] closely related to §4 in this paper. They extended the above mentioned result of Huisken without assuming $H < 0$ for the initial data under a symmetry assumption which forces the neck to not move at all. Although not stated explicitly, they apparently proved a statement similar to Theorem 4.1 but also under more assumptions as well as a symmetry assumption on the shape of the initial surface.

References

[Alt] S. Altschuler, Singularities of the curve shortening flow for space curve, J. Differential Geometry, to appear.

[AAG] S. Altschuler, S.B. Angenent and Y. Giga, Motion by mean curvature through singularities for surfaces of rotation, preprint.

[AGr] S. Altschuler and M. Grayson, Shortening space curves and flow through singularities, J. Differential Geometry, to appear.

[A1] S.B. Angenent, The zero set of a solution of a parabolic equation, Journal für die reine and angewandte Mathematik, 390 (1988), 79-96.

[A2] ———, On the formation of singularities in the curve shortening problem, J. Differential Geometry, 33 (1991), 601-633.

[A3] ———, Parabolic equations for curves on surfaces, II, Ann. of Math., 133 (1991), 171-215.

[A4] ———, Solutions of the $1-D$ porus medium equation are determined by their free boundary, Journal of the London Maths. Soc., 42 (1990), 339-353.

[C] X.-Y. Chen, in preparation.

[CM] X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity and finite-point blowup in one-dimensional semilinear heat equations, J. Differential Equations, 78 (1989), 160-190.

[CGG] Y.-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geometry, 33 (1991), 749-786.

[ES1] L.C. Evans and J. Spruck, Motion of level sets by mean curvature I, J. Differential Geometry, 33 (1991), 635-681.

[ES2] ———, Motion of level sets by mean curvature III, preprint.

[Gr] M. Grayson, A short note on the evolution of a surface by its mean curvature, Duke Math. J., 58 (1988), 555-558.

[H1] G. Huisken, Flow by mean curvature of convex surfaces into sphere, J. Differential Geometry, 20 (1984), 237-266.

[H2] ———, Asymptotic behaviour for singularities of the mean curvature flow, J. Differential Geometry, 31 (1990), 285-299.

[LUS] O.A. Ladyzhenskaya, V. Solonnikov and N. Ural'ceva, Linear and quasilinear equations of parabolic type, Translations of Mathematical Monographs, vol. 23, AMS (1968).

[SS] H.M. Soner and P.E. Souganidis, Uniqueness and singularities of cylindically symmetric surfaces moving by mean curvature, preprint.

Stephen Altschuler
Institute for Mathematics
and its Applications
University of Minnesota
514 Vincent Hall
206 Church St. SE.
Minneapolis, MN55455

(Current Address)
Centre for Mathematical Analysis
Australian National University
GPO Box 4
Canberra ACT 2601
Australia

Sigurd Angenent
Department of Mathematics, Van Vleck Hall
University of Wisconsin, Madison
Madison, Wisconsin 53706

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060, Japan