A note on the classification of non-singular flows with transverse similarity structures

Toshiyuki Nishimori

Series ♯116. June 1991

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 88: Y. Giga, S. Goto, H. Ishii and M.-H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, 32 pages. 1990.

♯ 89: A. Arai, Improper Bogoliubov transformations and instability of embedded eigenvalues, 47 pages. 1990.

♯ 90: K. Sugano, On bicommutators of modules over H-separable extension rings II, 9 pages. 1990.

♯ 91: T. Nakazi, Homogeneous polynomials and invariant subspaces in the polydiscs, 17 pages. 1990.

♯ 92: T. Nakazi, K. Takahashi, Homogoneous polynomials and invariant subspaces in the polydiscs II, 10 pages. 1990.

♯ 93: A. Arai, A theorem on essential self-adjointness with application to Hamiltonians in non-relativistic quantum field theory, 23 pages. 1990.

♯ 94: Y. Okabe, A. Inoue, On the exponential decay of the correlation functions for KMO-Langevin equations, 13 pages. 1990.

♯ 95: T. Sano, Y. Watatani, Angles between two subfactors, 62 pages. 1990.

♯ 96: S. Ninomiya, The Fourier-Sato transformation of pure sheaves, 22 pages. 1990.

♯ 97: Y. Okabe, A. Inoue, The theory of KM_2O-Langevin equations and its applications to data analysis (II): Causal analysis (1), 51 pages. 1990.

♯ 98: J. Ławrynowicz, S. Koshi and O. Suzuki, Dualities generated by the generalised Hurwitz problem and variation of the Yang-Mills field, 17 pages. 1991.

♯ 99: R. Agemi, K. Kubota and H. Takamura, On certain integral equations related to nonlinear wave equations, 52 pages. 1991.

♯ 100: S. Izumiya, Geometric singularities for Hamilton-Jacobi equation, 13 pages. 1991.

♯ 101: S. Izumiya, Legendrian singularities and first order differential equations, 16 pages. 1991.

♯ 102: A. Munemasa, Y. Watatani, Orthogonal pairs of $*$-subalgebras and association schemes, 11 pages. 1991.

♯ 103: A. Arai, O. Ogurisu, Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, 27 pages. 1991.

♯ 104: H. Takamura, Global existence of classical solutions to nonlinear wave equations with spherical symmetry for small data with noncompact support in three space dimensions, 14 pages. 1991.

♯ 105: R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, 11 pages. 1991.

♯ 106: T. Nakazi, Extremal problems in H^p, 13 pages. 1991.

♯ 107: T. Nakazi, ρ-dilations and hypo-Dirichlet algebras, 15 pages. 1991.

♯ 108: A. Arai, An abstract sum formula and its applications to special functions, 25 pages. 1991.

♯ 109: Y.-G. Chen, Y. Giga and S. Goto, Analysis toward snow crystal growth, 18 pages. 1991.

♯ 110: T. Hibi, M. Wakayama, A q-analogue of Capelli's identity for $GL(2)$, 7 pages. 1991.

♯ 111: T. Nishimori, A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem, 13 pages. 1991.

♯ 112: K. Matsuda, An analogy of the theorem of Hector and Duminy, 10 pages. 1991.

♯ 113: S. Takahashi, On a regularity criterion uo to the boundary for weak solutions of the Navier-Stokes equations, 23 pages. 1991.

♯ 114: T. Nakazi, Sum of two inner functions and exposed points in H^1, 18 pages. 1991.

♯ 115: A. Arai, De Rham operators, Laplacians, and Dirac operators on topological vector spaces, 27 pages. 1991.

A note on the classification of non-singular flows with transverse similarity structures

TOSHIYUKI NISHIMORI

§1. Introduction

The purpose of this paper is to classify non-singular flows with transverse similarity structures satisfying certain auxiliary conditions. We consider such flows as foliations with transverse similarity structures and the classification is done in this view point. A motivation for this study is as follows. In Nishimori [7], the author investigated the qualitative properties of foliations with transverse similarity structures and gave an analogy of Sacksteder's theorem (in Sacksteder [8]) on codimension one foliations. Furthermore for such foliations, Matsuda [6] gave an analogy of a theorem of Hector and Duminy (in Hector [5] and in Cantwell and Conlon [1]) on codimension one foliations. So we are interested in concrete examples of foliations with transverse similarity structures.

Here we give the definition of foliations with transverse similarity structures. A codimension q C^∞ foliation $\mathcal{F}$ of a C^∞ manifold M has a *transverse similarity structure* if there exists an open covering $\{U_i\}_{i\in I}$ of M, a family $\{h_i : U_i \to \mathbf{R}^q\}_{i\in I}$ of C^∞ submersions such that (1) $\mathcal{F}|_{U_i} = \{h_i^{-1}(t)\}_{t\in h_i(U_i)}$, and (2) for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there exists a similarity transformation $\gamma_{ji} : \mathbf{R}^q \to \mathbf{R}^q$ satisfying

$$\gamma_{ji} \circ (h_i|_{U_i\cap U_j}) = h_j|_{U_i\cap U_j}.$$

We call $\theta = \{U_i, h_i, \gamma_{ji}\}$ a *transverse similarity structure* of $\mathcal{F}$.

By starting from one of such submersions h_i's, we can construct the *analytic continuation* and obtain a C^∞ submersion $D : \widetilde{M} \to \mathbf{R}^q$, where $\widetilde{M}$ is the universal covering of M. We call D a *developing map* of the foliation $\mathcal{F}$ with the similarity structure $\theta = \{U_i, h_i, \gamma_{ji}\}$. As is well known (see Godbillon [4] for example), there exists a homomorphism $\Phi : \pi_1(M) \to \mathrm{Sim}(q)$ such that $D \circ \gamma = \Phi(\gamma) \circ D$ for all $\gamma \in \pi_1(M)$, where

$\mathrm{Sim}(q)$ is the group of similarity transformations of $\mathbf{R}^q$. In this paper, we work in the *oriented category* for the simplicity. So we suppose that γ_{ji}'s are orientation preserving, that is, $\gamma_{ji} \in \mathrm{Sim}_+(q)$.

If the dimension of such $\mathcal{F}$ is zero, the triple $(M, \mathcal{F}, \{U_i, h_i, \gamma_{ji}\})$ is a similarity manifold, which are classified by Fried [**2**] in the case where M is closed. If the dimension of such $\mathcal{F}$ is one, the triple can be considered as a non-singular flow with a transverse similarity structure. In a sense, our result may be considered as a generalization of that of Fried [**2**].

The plan of this paper is as follows. In §2, we quote the results of Fried [**2**] on closed similarity manifold in a formulation which is convenient for our purpose. In §3, we give examples of non-singular flows with transverse similarity structures, which will make up the classification list. In §4, we describe the auxiliary conditions and state our main result (Theorem 4.5) on the classification. In §5, we prove Theorem 4.5. We work in the C^∞ category, and hereafter we omit the term C^∞.

§2. The classification of closed oriented similarity manifolds due to Fried

In this section, we quote the results in Fried [2] in a somewhat modified form which is convenient for the later application.

Let N be a connected closed oriented similarity manifold of dimension $q \geqq 1$, and $D : \widetilde{N} \to \mathbf{R}^q$ an orientation preserving developing map of N, where $\widetilde{N}$ and $\mathbf{R}^q$ are naturally oriented. The following theorem is the crucial result of Fried, which makes the classification possible.

THEOREM 2.1. *(Fried [2]) (1) If $q = 1$, then D is a diffeomorphism onto its image $D(\widetilde{N})$ and one of the following cases occurs:*

(i) $D(\widetilde{N}) = \mathbf{R}$,

(ii) $D(\widetilde{N}) =]a, \infty[$ *or* $]-\infty, a[$ *for some* $a \in \mathbf{R}$.

(2) If $q \geqq 2$, then D is a covering map onto its image $D(\widetilde{N})$ and one of the following cases occurs:

(i) $D(\widetilde{N}) = \mathbf{R}^q$,

(ii) $D(\widetilde{N}) = \mathbf{R}^q - \{a\}$ *for some* $a \in \mathbf{R}^q$.

(Note that D is a diffeomorphism onto its image except the case (ii) with $q = 2$.)

We call N *Euclidean* if $D(\widetilde{N}) = \mathbf{R}^q$, and *radiant* otherwise. When N is radiant, we may suppose that a in Theorem 2.1 coincides with the origin 0 of $\mathbf{R}^q$ by modifying the developing map D.

In order to state the classification result, we define the isomorphisms between closed oriented similarity manifolds as follows.

DEFINITION 2.2: Let N_1 and N_2 be closed oriented similarity manifolds. We say that N_1 is *isomorphic* to N_2 if there is an orientation preserving homeomorphism $g : N_1 \to N_2$ such that, if $D_2 : \widetilde{N} \to \mathbf{R}^q$ is an orientation preserving developing map and $\widetilde{g} : \widetilde{N}_1 \to \widetilde{N}_2$ is a lift of g, then $D_2 \circ \widetilde{g} : \widetilde{N}_1 \to \mathbf{R}^q$ is an orientation preserving developing map. We call such $g : N_1 \to N_2$ an *isomorphism.* (Note that such g becomes automatically a real analytic diffeomorphism.)

We give examples of closed oriented similarity manifolds, which will form the classification list of Fried [**2**].

EXAMPLE 2.3: Let Γ be a lattice group of $\mathbf{R}^q$ and denote by $N^q(\Gamma)$ the Euclidean closed oriented similarity manifold obtained as the quotient $\mathbf{R}^q/\Gamma$. Clearly $N^q(\Gamma_1)$ and $N^q(\Gamma_2)$ are isomorphic if and only if there exists an orientation preserving similarity transformation $g : \mathbf{R}^q \to \mathbf{R}^q$ such that $g\Gamma_1 g^{-1} = \Gamma_2$.

EXAMPLE 2.4: Take $r \in]0,1[$ and define a map $m_r : \mathbf{R} \to \mathbf{R}$ by $m_r(x) = rx$ for all $x \in \mathbf{R}$. Denote by $N^1(r,+)$ (respectively $N^1(r,-1)$) the radiant closed oriented similarity manifold obtained as the quotient of the interval $]0,\infty[$ (respectively $]-\infty,0[$) by the cyclic group generated by m_r. Clearly $N^1(r,+)$ and $N^1(s,-)$ are not isomorphic for all $r,s \in]0,1[$. If $r \neq s$, then $N^1(r,+)$ and $N^1(s,+)$ (respectively $N^1(r,-)$ and $N^1(s,-)$) are not isomorphic.

EXAMPLE 2.5: Suppose that $q = 2$ and identify $\mathbf{R}^2$ with $\mathbf{C}$. Consider the exponential map $\exp : \mathbf{C} \to \mathbf{C} - 0$ and the oriented similarity structure $\theta = \exp^* \theta_0$ on $\mathbf{C}$ induced from the canonical oriented similarity structure θ_0 of $\mathbf{C} - \{0\}$ by the map exp. Then a homeomorphism $g : \mathbf{C} \to \mathbf{C}$ is an automorphism of oriented similarity manifold $(\mathbf{C},\theta)$ if and only if g is a translation (since $\exp(z+\alpha) = \exp(z)\cdot\exp(\alpha)$). Hence a lattice group Γ of $\mathbf{C} = \mathbf{R}^2$ acts on $(\mathbf{C},\theta)$ as an automorphism group, and determines a radiant closed oriented similarity manifold $N^2(\exp\Gamma)$ as the quotient. Note that $N^2(\exp\Gamma_1)$ and $N^2(\exp\Gamma_2)$ are isomorphic if and only if $\Gamma_1 = \Gamma_2$.

EXAMPLE 2.6: Suppose that $q \geqq 3$. Let K be a finite orientation preserving isometry group of the standard sphere S^{q-1} such that the quotient S^{q-1}/K is a manifold. (See Wolf [**9**] for the classification of such K.) Take an orientation preserving similarity transformation $\gamma : \mathbf{R}^q \to \mathbf{R}^q$ such that $\gamma(0) = 0$ and $\|\gamma(x)\| < \|x\|$ if $x \neq 0$. (We call such γ *contracting*.) Denote by $\langle\gamma\rangle$ the cyclic group generated by γ. Let G be the group generated by $K \cup \{\gamma\}$ (that is, $G = K \rtimes \langle\gamma\rangle$). Denote by $N^q(K \rtimes \langle\gamma\rangle)$ the radiant closed oriented similarity manifold obtained as the quotient of $\mathbf{R}^q - \{0\}$ by G. Note that $N^q(K_1 \rtimes \langle\gamma_1\rangle)$ and

$N^q(K_2 \rtimes \langle\gamma_2\rangle)$ are isomorphic if and only if there exists an orientation preserving similarity transformation $g : \mathbf{R}^q \to \mathbf{R}^q$ such that $g(0) = 0$ and $gG_1g^{-1} = G_2$, where $G_1 = K_1 \rtimes \langle\gamma_1\rangle$ and $G_2 = K_2 \rtimes \langle\gamma\rangle$.

Now we can state the classification result.

THEOREM 2.7. *(Fried [2]). Let N be a connected closed oriented similarity manifold.*
(I) If N is Euclidean, then N is isomorphic to $N^q(\Gamma)$ for some lattice group Γ of $\mathbf{R}^q$.
(II) Suppose that N is radiant.
(i) If $q = 1$, then N is isomorphic to $N^1(r, +)$ or $N^1(r, -)$ for some $r \in]0,1[$.
(ii) If $q = 2$, then N is isomorphic to $N^2(\exp\Gamma)$ for some lattice group Γ of $\mathbf{C}$.
(iii) If $q \geqq 3$, then N is isomorphic to $N^q(K \rtimes \langle\gamma\rangle)$ for some K and γ.

One can easily classify the automorphisms $g : N \to N$. But we omit the details.

§3. Examples of non-singular flows with transverse similarity structures

In this and the next sections, we are going to classify non-singular flows $\phi : M \times \mathbf{R} \to M$ with transverse similarity structures under certain conditions. We treat only oriented underlying manifolds M. Hence such flows have the natural transverse orientation. We begin by describing the isomorphism between such flows.

DEFINITION 3.1: Let M_1 and M_2 be closed oriented manifolds of dimension $n \geqq 2$ and ϕ_1 and ϕ_2 non-singular flows with transverse similarity structures. We say that ϕ_1 is *isomorphic* to ϕ_2 if there exists an orientation preserving diffeomorphism $f : M_1 \to M_2$ such that

(1) for all $x \in M$, there exists an orientation preserving homeomorphism $\alpha : \mathbf{R} \to \mathbf{R}$ with

$$f \circ \phi_1(x,t) = \phi_2(f(x), \alpha(t)) \quad \text{for all } t \in \mathbf{R},$$

(2) if U is an open subset of M_2 and a submersion $h : U \to \mathbf{R}^{n-1}$ is compatible with the transverse similarity structure of ϕ_2, then $h \circ f : f^{-1}(U) \to \mathbf{R}^{n-1}$ is compatible with the transverse similarity structure of ϕ_1.

We call such $f : M_1 \to M_2$ an *isomorphism* between ϕ_1 and ϕ_2.

Our intention is the classification of non-singular flows with transverse similarity structures up to the above isomorphisms. Now we give examples.

EXAMPLE 3.2: *Suspension flows.* Let N be a closed oriented similarity manifold of dimension $n-1$ and $g : N \to N$ an automorphism of N. Consider a $\mathbf{Z}$-action on $N \times \mathbf{R}$ defined by

$$n \cdot (x,t) = (g^n(x), t-n) \quad \text{for } n \in \mathbf{Z} \text{ and } (x,t) \in N \times \mathbf{R}.$$

Since the vector field $\partial/\partial t$ on $N \times \mathbf{R}$ is preserved by this action, it induces a vector field X on the quotient manifold M of $N \times \mathbf{R}$ by this action. Furthermore this action preserves the transverse similarity structure of $\partial/\partial t$ induced from the similarity structure of N. Hence the vector field X generates a non-singular flow $\phi : M \times \mathbf{R} \to M$ with a natural transverse similarity structure. We call $\phi_{(N,g)} := \phi$ the *suspension flow* of N by g. Note that $\phi_{(N_1,g_1)}$ and $\phi_{(N_2,g_2)}$ are isomorphic if and only if there exits an isomorphism $h : N_1 \to N_2$ such that the diagram

$$\begin{tikzcd} N_1 \arrow[r, "g_1"] \arrow[d, "h"'] & N_1 \arrow[d, "h"] \\ N_2 \arrow[r, "g_2"'] & N_2 \end{tikzcd}$$

commutes.

EXAMPLE 3.3: *Circle bundle flows.* Let N be a closed oriented similarity manifold, and $\xi = (M, \pi, N)$ an oriented circle bundle. Let $\phi : M \times \mathbf{R} \to M$ be a non-singular flow such that the orbits of ϕ are the fibers of ξ, and the natural orientation of an orbit of ϕ coincides with that as a fiber of ξ. Then the similarity structure of N determines a transverse similarity structure of ϕ. We call $\phi_\xi := \phi$ a *circle bundle flow* over N. Note that the circle bundle flows ϕ_{ξ_1} and ϕ_{ξ_2} are isomorphic if and only if there exists an orientation preserving bundle isomorphism

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ N_1 & \xrightarrow[h]{} & N_2 \end{array}$$

such that $h : N_1 \to N_2$ is an isomorphism. If ξ is a trivial bundle, then the circle bundle flow ϕ_ξ is isomorphic to the suspension flow $\phi_{(N,\mathrm{id})}$.

We can generalize the circle bundle flows as follows. Let N be a closed oriented similarity manifold of dimension 2. Then N is diffeomorphic to T^2 and any Seifert bundle M over N has a non-singular flow $\phi : M \times \mathbf{R} \to M$ with a transverse similarity structure in the similar way as circle bundles.

EXAMPLE 3.4: *Contraction flows.* For $n \geqq 2$, take an orientation preserving contracting similarity automorphism $g : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ with $g(0) = 0$, and define a map $f : \mathbf{R}^n \to \mathbf{R}^n$ by

$$f(x,t) = (g(x), rt) \qquad \text{for } \ x \in \mathbf{R}^{n-1} \ \text{ and } \ t \in \mathbf{R},$$

where $r > 0$ means the similitude ratio of g. Then f is an orientation preserving similarity transformation of $\mathbf{R}^n$. Consider a vector field $\overline{X}$ on $\overline{M} := \mathbf{R}^n - \{0\}$ defined by

$$\overline{X}(x,t) = \|(x,t)\| \cdot \frac{\partial}{\partial t} \qquad \text{for } \ (x,t) \in \mathbf{R}^{n-1} \times \mathbf{R} - \{(0,0)\} = \overline{M}.$$

Note that $\overline{X}$ is a non-singular vector field with the transverse similarity structure induced from the canonical similarity structure of $\mathbf{R}^{n-1}$ by the submersion $h : \overline{M} \to \mathbf{R}^{n-1}$ defined by

$$h(x,t) = x \qquad \text{for } \ (x,t) \in \mathbf{R}^{n-1} \times \mathbf{R} - \{(0,0)\} = \overline{M}.$$

Since $f|\overline{M}$ is an automorphism of $\overline{X}$, we have a vector field X on the quotient manifold M of $\overline{M}$ by the cyclic group generated by $f|\overline{M}$. Then X generates a non-singular flow $\phi : M \times \mathbf{R} \to M$ with the transverse similarity structure induced from that of $\overline{X}$. We call $\phi_g := \phi$ a *contraction flow.* Clearly ϕ_g has exactly two closed orbits and M is diffeomorphic to $S^1 \times S^{n-1}$. The subset $\overline{N} := (\mathbf{R}^{n-1} - \{0\}) \times \{0\}$ of $\overline{M}$ is invariant by $f|\overline{M}$, and submersed onto a submanifold N of M. Since N is transverse to ϕ_g, it is a closed similarity manifold. If $n = 3$, then N is isomorphic to $N^2(\exp \Gamma)$, where $\Gamma = \{pz + q \cdot 2\pi\sqrt{-1} : p, q \in \mathbf{Z}\}$ and z is any complex number such that

$$g(x) = (\exp z) \cdot x \qquad \text{for } \ x \in \mathbf{C} = \mathbf{R}^2.$$

If $n \geqq 4$, then N is isomorphic to $N^{n-1}(\{\mathrm{id}\} \rtimes \langle g \rangle)$. Note that ϕ_{g_1} and ϕ_{g_1} are isomorphic if and only if there exists an orientation preserving similarity automorphism $h : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ such that $h \circ g_1 \circ h^{-1} = g_2$.

EXAMPLE 3.5: *Generalized contraction flows.* We consider the case $n = 2$ in Example 3.4. Then there exists uniquely $r \in]0, 1[$ such that $g(x) = rx$ for all $x \in \mathbf{R}$, that is, $g = m_r$. We see that N is isomorphic to the disjoint union $N^1(r, +) \cup N^1(r, -)$. Take $\nu \in \mathbf{N}$ and consider the ν-fold covering $\pi : \widehat{M} \to M$ such that $\pi^{-1}(N)$ has 2ν connected components. The lift $\phi_{(r;\nu)}$ of ϕ_g by π is naturally a non-singular flow with a transverse similarity structure. We call $\phi_{(r;\nu)}$ a *generalized contraction flow.* Note that $\phi_{(r;\nu)}$ has exactly 2ν closed orbits and M is diffeomorphic to T^2. Clearly $\phi_{(r;\nu)}$ and $\phi_{(s;\mu)}$ are isomorphic if and only if $r = s$ and $\nu = \mu$.

§4. Statement of the results

In this section, we give a classification of non-singular C^∞ flows with transverse similarity structures satisfying certain conditions. We begin by giving definitions, which are needed in order to describe the conditions. Let M be a closed oriented manifold of dimension $\geqq 2$ and $\phi : M \times \mathbf{R} \to M$ a non-singular flow with a transverse similarity structure.

DEFINITION 4.1: A submanifold N of M is a *closed transversal* to ϕ if N is a codimension one closed submanifold and N is transverse to ϕ.

REMARK 4.2: Since a closed transversal N to ϕ has the canonical similarity structure induced from the transverse similarity structure of ϕ, we can find N in the classification table of the closed similarity manifolds (due to Fried [**2**]) described in §2. This is the starting point of our research.

REMARK 4.3: All the suspension flows $\phi_{(N,g)}$ in Example 3.2, all the contraction flows ϕ_g in Example 3.4, and all the generalized contraction flows $\phi_{(\tau;\nu)}$ in Example 3.5 have closed transversals. On the other hand, some circle bundle flows ϕ_ξ in Example 3.3 have no closed transversal. For example, take a circle bundle $\xi = (M, \pi, N_0)$ such that N_0 is a closed oriented similarity manifold of dimension 2 (which implies that N_0 is diffeomorphic to the 2-torus T^2), and the Euler number $\mathrm{eu}(\xi)$ of ξ is not zero. Then the circle bundle flow ϕ_ξ has no closed transversal. For, otherwise, the restriction $\pi|_N : N \to N_0$ is a covering map, where N is a closed transversal to ϕ. Since the induced bundle $(\pi|_N)^*\xi$ has a section, its Euler number $\mathrm{eu}((\pi|_N)^*\xi)$ is zero. This implies that $\mathrm{eu}(\xi) = 0$, a contradiction.

DEFINITION 4.4: Suppose that N has a closed transversal N. We say that the pair (ϕ, N) has the *lifting property* if, for any continuous map $c : [a,b] \to N$, any number $\tau \in \mathbf{R} - \{0\}$ and any compact codimension one submanifold T of M such that $\phi(c(a), \tau) \in \mathrm{Int}\, T$ and T is transverse to ϕ, there exists $b^* \in]a,b]$ and a continuous map $H : [a,b^*] \times J \to M$ (where $J = [0,\tau]$ if $\tau > 0$ and $J = [\tau, 0]$ if $\tau < 0$) such that

(1) $\quad H(s,0) = c(s) \quad$ for all $s \in [a,b^*]$,

(2) $\quad H(a,t) = \phi(c(0),t) \quad$ for all $t \in J$,

(3) $\quad H(\{s\} \times J) \subset \phi(\{c(s)\} \times \mathbf{R}) \qquad$ for all $s \in [a, b^*]$,

(4) $\quad H([a, b^*] \times \{\tau\}) \subset T$,

(5) if $b^* < b$, then $H(b^*, \tau) \in \partial T$.

We call H a *lift-homotopy* of $\phi|_{\{c(a)\} \times [0,\tau]}$ along c and T.

Now we can state the result of classification, which is the main result of this paper.

THEOREM 4.5. *Let M be a closed oriented C^∞ manifold of dimension $n \geqq 2$ and ϕ a non-singular C^∞ flow with a transverse structure. Suppose that ϕ has a closed transversal N and the pair (ϕ, N) has the lifting property. Then ϕ is isomorphic to (1) the suspension flow $\phi_{(N,g)}$ for some closed similarity manifold N and some C^∞ automorphism g of N, (2) the contraction flow ϕ_g for some contracting similarity transformation $g : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$, or (3) the generalized contraction flow $\phi_{(r;\nu)}$ for some $r > 0$ and $\nu \in \mathbf{N}$ (in the case $n = 2$).*

REMARK 4.6: When $n = 3$, the concept of a transverse similarity structure to a non-singular flow coincides with that of a transverse complex affine structure. Ghys [**3**] treated this case, independently and without our auxiliary conditions.

§5. The proof of Theorem 4.5

Let M be a connected closed oriented manifold of dimension $n \geqq 2$ and $\phi : M \times \mathbf{R} \to M$ a non-singular flow with a transverse similarity structure. Suppose that ϕ has a closed transversal N and the pair (ϕ, N) has the lifting property. We may suppose that N is connected.

Here we recall the usual notations. Let $x \in M$. We call $O(x) := \{\phi(x,t) : t \in \mathbf{R}\}$ the *orbit* of x and $O^+(x) := \{\phi(x,t) : t > 0\}$ (respectively $O^-(x) := \{\phi(x,t) : t < 0\}$) the *positive* (respectively *negative*) *semi-orbit* of x. We call $L^+(x) := \cap_{\tau>0}\overline{\{\phi(x,t) : t \geqq \tau\}}$ (respectively $L^-(x) := \cap_{\tau<0}\overline{\{\phi(x,t) : t \leqq \tau\}}$) the *$\omega$-limit set* (respectively *α-limit set*) of x. For a subset A of M, we call $\mathrm{Sat}(A) := \{\phi(x,t) : x \in A, t \in \mathbf{R}\}$ the *saturation* of A. It is well known that $L^+(x), L^-(x)$ and $\mathrm{Sat}(A)$ are *invariant* (that is, the union of a family of orbits of ϕ).

Now we divide the situation into the following two cases:

Case I. $O^+(x) \cap N \neq \emptyset$ for all $x \in N$.

Case II. $O^+(x_0) \cap N = \emptyset$ for some $x_0 \in N$.

First consider Case I. For $x \in N$, let $g(x)$ be the first intersecting point of $O^+(x)$ with N. Then the obtained map $g : N \to N$ is an immersion. Hence the image $g(N)$ is a compact open subset of N. Since N is Hausdorff and connected, it follows that $g(N) = N$, which implies that $\mathrm{Sat}(N) = M$. Furthermore we see that $g : N \to N$ is an automorphism of the closed oriented similarity manifold N, and that ϕ is isomorphic to the suspension flow $\phi_{(N,g)}$.

Hereafter we consider Case II. Note that the ω-limit set $L^+(x_0)$ of x_0 is compact, connected, non-empty and invariant, and that the saturation $\mathrm{Sat}(N)$ of N is an open subset of M.

Lemma 5.1. $L^+(x_0) \subset \partial\,\mathrm{Sat}(N) := \overline{\mathrm{Sat}(N)} - \mathrm{Sat}(N)$.

Proof: It is clear that $L^+(x_0) \subset \overline{\mathrm{Sat}(N)}$. Suppose that there exists a point $y \in L^+(x_0) \cap \mathrm{Sat}(N)$. Since $y \in \mathrm{Sat}(N)$, there exists $x \in N$ and $t \in \mathbf{R}$ such that $y = \phi(x,t)$. Since $L^+(x_0)$ is invariant and $y \in L^+(x_0)$, it follows that $x \in L^+(x_0)$. Hence the positive semi-orbit $O^+(x_0)$ must intersect N infinitely many times, which contradicts the assumption $O^+(x_0) \cap N = \emptyset$. Hence $L^+(x_0) \cap \mathrm{Sat}(N) = \emptyset$. $\square$

Let $\iota : N \to M$ the inclusion map, and $\pi : \widetilde{M} \to M$ and $\pi_N : \widetilde{N} \to N$ the universal covering maps. We can construct a developing map $D : \widetilde{M} \to \mathbf{R}^{n-1}$ of the non-singular flow ϕ with the transverse similarity structure in a natural way. We have the following diagram:

$$\begin{tikzcd} \widetilde{N} \arrow[d, "\pi_N"'] & \widetilde{M} \arrow[r, "D"] \arrow[d, "\pi"'] & \mathbf{R}^{n-1} \\ N \arrow[r, "\iota"'] & M & \end{tikzcd}$$

We are going to take three compact disks $T \subset M$, $\widetilde{T} \subset \widetilde{M}$ and $\widehat{T} \subset \mathbf{R}^{n-1}$, and six points $y, z \in \mathrm{Int}\, T$, $\widetilde{y}, \widetilde{z} \in \mathrm{Int}\, \widetilde{T}$ and $\widehat{y}, \widehat{z} \in \mathrm{Int}\, \widehat{T}$. First take a point $z \in L^+(x_0) \subset M$. Choose a point $\widetilde{z} \in \pi^{-1}(z) \subset \widetilde{M}$ and put $\widehat{z} = D(\widetilde{z}) \in \mathbf{R}^{n-1}$. Second take a sufficiently small compact $(n-1)$-disk $\widetilde{T}$ in $\widetilde{M}$ such that $\widetilde{z} \in \mathrm{Int}\, \widetilde{T}$ and $\widetilde{T}$ is transverse to the lift $\widetilde{\phi} : \widetilde{M} \times \mathbf{R} \to \widetilde{M}$ of ϕ. Put $T = \pi(\widetilde{T})$ $(\subset M)$ and $\widehat{T} = D(\widetilde{T})$ $(\subset \mathbf{R}^{n-1})$. We may suppose that $\pi|_{\widetilde{T}} : \widetilde{T} \to T$ and $D|_{\widetilde{T}} : \widetilde{T} \to \widehat{T}$ are diffeomorphisms. Then T and $\widehat{T}$ are compact disks. Furthermore $z \in \mathrm{Int}\, T$ and $\widehat{z} \in \mathrm{Int}\, \widehat{T}$. Last we take the points $y, \widetilde{y}$ and $\widehat{y}$ as follows. Since $z \in L^+(x_0)$, there exists $\tau > 0$ with $y := \phi(x_0, \tau) \in \mathrm{Int}\, T$. Put $\widetilde{y} = (\pi|_{\widetilde{T}})^{-1}(y) \in \mathrm{Int}\, \widetilde{T}$ and $\widehat{y} = D(\widetilde{y}) \in \mathrm{Int}\, \widehat{T}$.

Define a curve $\omega : [0, \tau] \to M$ by $\omega(t) = \phi(x_0, t)$ for all $t \in [0, \tau]$, and let $\widetilde{\omega} : [0, \tau] \to \widetilde{M}$ be the lift of ω with $\widetilde{\omega}(\tau) = \widetilde{y}$. Put $\widetilde{x}_0 = \widetilde{\omega}(0)$. Note that $D(\widetilde{x}_0) = D(\widetilde{y}) = \widehat{y}$ because the points $\widetilde{x}_0$ and $\widetilde{y}$ are on the same orbit of the flow ϕ. Since $\widetilde{x}_0 \in \pi^{-1}(x_0)$, there exists an immersion $\widetilde{\iota} : \widetilde{N} \to \widetilde{M}$ such that $\widetilde{x}_0 \in \widetilde{\iota}(\widetilde{N})$ and the diagram

$$\begin{tikzcd} \widetilde{N} \arrow[r, "\widetilde{\iota}"] \arrow[d, "\pi_N"'] & \widetilde{M} \arrow[d, "\pi"] \\ N \arrow[r, "\iota"'] & M \end{tikzcd}$$

commutes. Choose a point $\widetilde{x}_N \in \widetilde{\iota}^{-1}(\widetilde{x}_0)$.

Since the similarity structure of N comes from the transverse similarity structure of ϕ, the composition $D_N := D\widetilde{\iota} : \widetilde{N} \to \mathbf{R}^{n-1}$ is a developing map of N. By Theorem 2.1, the map $D_N : \widetilde{N} \to D_N(\widetilde{N})$ is a covering map and the image $D_N(\widetilde{N})$ is one of the following:

(a) the whole $\mathbf{R}^{n-1}$,

(b) $\mathbf{R}^{n-1}$ deleted a point $(n-1 \geqq 2)$,

(c) a connected component of $\mathbf{R}$ deleted a point.

This implies that the intersection $\mathrm{Int}\,\widehat{T} \cap D_N(\widetilde{N})$ is diffeomorphic to (a) an open disk, (b) a punctured open disk of dimension $n-1 \geqq 2$, or (c) an open interval. Note that $\mathrm{Int}\,\widehat{T} \cap D_N(\widetilde{N})$ is path-connected.

LEMMA 5.2. $\quad \widehat{z} = D(\widetilde{z}) \notin D_N(\widetilde{N})$.

PROOF: We suppose that $\widehat{z} \in D_N(\widetilde{N})$ and will bring out a contradiction. Since $\mathrm{Int}\,\widehat{T} \cap D_N(\widetilde{N})$ is path-connected and the points $\widehat{y}$ and $\widehat{z}$ belong to $\mathrm{Int}\,\widehat{T} \cap D_N(\widetilde{N})$, we can take a curve $\widehat{c} : [0,1] \to \mathrm{Int}\,\widehat{T} \cap D_N(\widetilde{N})$ such that $\widehat{c}(0) = \widehat{y}$ and $\widehat{c}(1) = \widehat{z}$. Since $D_N : \widetilde{N} \to D_N(\widetilde{N})$ is a covering map and

$$D_N(\widetilde{x}_N) = D\widetilde{\iota}(\widetilde{x}_N) = D(\widetilde{x}_0) = D(\widetilde{y}) = \widehat{y} = \widehat{c}(0),$$

there exists uniquely a continuous map $\widetilde{c}_N : [0,1] \to \widetilde{N}$ such that $D_N \widetilde{c}_N = \widehat{c}$ and $\widetilde{c}_N(0) = \widetilde{x}_N$. Put $c = \pi_N \widetilde{c}_N : [0,1] \to N$. Then it follows that

$$c(0) = \pi_N \widetilde{c}_N(0) = \pi_N(\widetilde{x}_N) = \pi\widetilde{\iota}(\widetilde{x}_N) = \pi(\widetilde{x}_0) = x_0 = \phi(x_0, 0) = \omega(0).$$

Since the pair (ϕ, N) has the lifting property, there exists $b^* \in]0,1]$ and a lift-homotopy $H : [0,b^*] \times [0,\tau] \to M$ of $\phi|_{\{x_0\} \times [0,\tau]}$ along c and T. Then $H(0,\tau) = \omega(\tau) = \phi(x_0,\tau) = y$. Let $\widetilde{H} : [0,b^*] \times [0,\tau] \to \widetilde{M}$ be the lift of H with $\widetilde{H}(0,\tau) = \widetilde{y}$.

Since $\widetilde{H}(0,\tau) = \widetilde{\omega}(\tau)$ and for all $t \in [0,\tau]$,

$$\pi\widetilde{H}(0,t) = H(0,t) = \omega(t) = \pi\widetilde{\omega}(t),$$

the unique path lifting property of covering maps implies that $\widetilde{H}(0,t) = \widetilde{\omega}(t)$ for all $t \in [0,\tau]$. Hence $\widetilde{H}(0,0) = \widetilde{\omega}(0) = \widetilde{x}_0 = \widetilde{\iota}\widetilde{c}_N(0)$. Next note that $\pi\widetilde{H}(s,0) = H(s,0) =$

$c(s) = \pi_N \widetilde{c}_N(s) = \pi \widetilde{\iota} \widetilde{c}_N(s)$ for all $s \in [0, b^*]$. By the unique path lifting property, it follows that $\widetilde{H}(s,0) = \widetilde{\iota}\widetilde{c}_N(s)$ for all $s \in [0, b^*]$. Note that $\widetilde{H}(\{s\} \times [0,\tau])$ is contained in some orbit of $\widetilde{\phi}$. This implies that

$$D\widetilde{H}(s,\tau) = D\widetilde{H}(s,0) = D\widetilde{\iota}\widetilde{c}_N(s) = D_N\widetilde{c}_N(s) = \widehat{c}(s) \in \operatorname{Int}\widehat{T}.$$

Hence $\widetilde{H}(s,\tau) \in \operatorname{Int}\widetilde{T}$ and $H(s,\tau) \in \operatorname{Int} T$ for all $s \in [0, b^*]$. Therefore $b^* = 1$ and $H(1,\tau) = z$. It follows that $z = H(1,\tau) \in O^+(c(1)) \subset \operatorname{Sat}(N)$, which contradicts Lemma 5.1. $\square$

By Lemma 5.2 and Theorem 2.1, we conclude that N is radiant, and that if $n-1 \geqq 2$ then $D_N(\widetilde{N}) = \mathbf{R}^{n-1}$, and if $n-1=1$ then $D_N(\widetilde{N}) =]-\infty, \widehat{z}[$ or $]\widehat{z}, +\infty[$.

Lemma 5.3. *If $n-1 \geqq 2$, then $\operatorname{Sat}(N) \cap T = T - \{z\}$. If $n-1=1$, then $\operatorname{Sat}(N) \cap T$ is a connected component of $T - \{z\}$.*

Proof: We treat only the case $n-1 \geqq 2$ and omit the proof for the case $n-1=1$. It is clear that $\operatorname{Sat}(N) \cap T \subset T - \{z\}$. Conversely take $w \in T - \{z\}$ and put $\widetilde{w} = (\pi|_{\widetilde{T}})^{-1}(w) \in \widetilde{T}$ and $\widehat{w} = D(\widetilde{w}) \in \widehat{T}$. Choose a curve $\widehat{c} : [0,1] \to \operatorname{Int}\widehat{T} - \{\widehat{z}\}$ such that $\widehat{c}(0) = \widehat{y}$ and $\widehat{c}(1) = \widehat{w}$. By the similar argument as in Lemma 5.2, we see that $w \in \operatorname{Sat}(N)$. Therefore $T - \{z\} \subset \operatorname{Sat}(N)$. $\square$

By Lemma 5.3, we see that the orbit $O(z)$ of z is proper, which implies that $L^+(z) \subset \overline{O(z)} - O(z)$. Furthermore, we have the following.

Lemma 5.4. *The orbit $O(z)$ is a closed orbit.*

Proof: Consider the case $n-1 \geqq 2$. Suppose that $O(z)$ is not a closed orbit. Then there exists a point $z_1 \in L^+(z) (\subset \overline{O(z)} - O(z) \subset L^+(x_0))$. Take a sufficiently small $(n-1)$-disk T_1 in M such that $z_1 \in \operatorname{Int} T_1$ and T_1 is transverse to ϕ. Since Lemma 5.3 is valid for z_1, it follows that $\operatorname{Sat}(N) \cap T_1 = T_1 - \{z_1\}$. This implies that $O(z)$ cannot approach z_1, a contradiction. This argument is valid for the case $n-1=1$, too. $\square$

LEMMA 5.5. $L^+(x_0) = O(z)$.

PROOF: By Lemmas 5.3 and 5.4, we can conclude that $L^+(x_0)$ is the union of some isolated closed orbits. Since $L^+(x_0)$ is connected, it follows that $L^+(x_0) = O(z)$. □

Now consider the holonomy g along the closed orbit $O(z)$. Since $L^+(x_0) = O(z)$, the similarity transformation g of $\mathbf{R}^{n-1}$ must be contracting. We can take a compact tubular neighborhood W of $O(z)$ such that ∂W is transverse to ϕ. Then the closed oriented similarity manifold ∂W is isomorphic to $N(g) := (\mathbf{R}^{n-1} - \{0\})/\langle g \rangle$, where $\langle g \rangle$ is the cyclic group generated by g. Note that if $n - 1 = 1$ then $N(g) = N^1(r, -) \cup N^1(r, +)$ (where $r \in \,]0, 1[$ is the similitude ratio of g), that if $n - 1 = 2$ then $N(g) = N^2(\exp \Gamma)$ (where we take $\alpha_0 \in \mathbf{C}$ with $g(x) = \exp(\alpha_0)x$ and let $\Gamma = \{\ \mu \cdot \alpha_0 + \nu \cdot 2\pi\sqrt{-1}\ : \mu, \nu \in \mathbf{Z}\}$), and that if $n - 1 \geqq 3$ then $N(g) = N^{n-1}(\{\mathrm{id}\} \rtimes \langle g \rangle)$.

LEMMA 5.6. *For all $x \in N$, the positive semi-orbit $O^+(x)$ intersects ∂W exactly once.*

PROOF: Clearly any orbit of ϕ intersects ∂W at most once. Since $L^+(x_0) = O(z)$, the positive semi-orbit $O^+(x_0)$ intersects ∂W. We may suppose that $y = \phi(x_0, \tau) \in \partial W$. Take a point $x \in N$ arbitrarily and consider a curve $c : [0, 1] \to N$ such that $c(0) = x_0$ and $c(1) = x$. By the lifting property of (ϕ, N), we can lift $\phi|_{\{x_0\} \times [0,\tau]}$ along c and ∂W. This implies that $O^+(x) \cap \partial W \neq \emptyset$. □

For $x \in N$, denote by $f(x)$ the intersecting point of $O^+(x)$ and ∂W. Since the subset $f(N) := \{f(x) : x \in N\}$ of ∂W is connected, open and closed, it is one of the connected component of ∂W. If $n - 1 \geqq 2$, then $f(N) = \partial W$. Furthermore the obtained map $f : N \to f(N) (\subset \partial W)$ is an isomorphism between the closed oriented similarity manifolds N and $f(N)$. Therefore N is isomorphic to $N(g)$.

Now repeat the same argument as above for the negative side of N. Then $L^-(x_0)$ is a closed orbit of ϕ and its holonomy h is expanding in this case. Furthermore the contracting similarity transformation $h^{-1} : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ determines the transverse similarity structure of the boundary ∂V of a small compact tubular neighborhood V of $L^-(x_0)$. Since one of the connected components of ∂V is isomorphic to N, it follows that $h^{-1} = g$. If $n-1 \geqq 2$, then we have the following decomposition of M:

$$M = V \cup (N \times [-1,1]) \cup W.$$

By using this decomposition, we see that ϕ is isomorphic to the contraction flow ϕ_g.

If $n-1 = 1$, then we see that ϕ has exactly an even number of closed orbits, whose holonomies are g or g^{-1}. Furthermore we have the similar decomposition as in the case $n-1 \geqq 2$. Thus we see that ϕ is isomorphic to the generalized contraction flow $\phi_{(r;\nu)}$ for some $r > 0$ and $\nu \in \mathbf{Z}$. This completes the proof of Theorem 4.5.

References

1. Cantwell J. and Conlon L., *Endsets of exceptional leaves: a theorem of Duminy, an account*, informal notes.
2. Fried D., *Closed similarity manifolds*, Comment. Math. Helv. **55** (1980), 576–582.
3. Ghys E., *Flots transversalement affines et tissus feuilletés*, Prépubl. Math. de l'Ecole Normal Supérieure de Lyon no. 20 (1990).
4. Godbillon C., "Feuilletages; Études Géométriques I, II," Publ. Inst. de Recherche Math. Université Louis Pasteur, Strasbourg, 1985–1986.
5. Hector G., *Sur un théorem de structure des feuilletages de codimension un*, thesis, Université Louis Pasteur, Strasbourg.
6. Matsuda K., *An analogy of the theorem of Hector and Duminy*, Hokkaido Univ. Preprint Series in Math. no. 112 (1991).
7. Nishimori T., *A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem*, Hokkaido Univ. Preprint Series in Math. no. 111 (1991).
8. Sacksteder R., *Foliations and pseudogroups*, Amer. J. Math. **87** (1965), 79–102.
9. Wolf J., "Spaces of Constant Curvature," McGraw-Hill, 1967.

Department of Mathematics
Faculty of Science
Hokkaido University
060 Sapporo, Japan