ON A REGULARITY CRITERION UP TO THE BOUNDARY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Shuji Takahashi

Series ♯113. May 1991

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 85: T. Morimoto, Théorème de Cartan-Kähler dans une classe de fonctions formelles Gevrey, 6 pages. 1990.

♯ 86: A. Arai, Diffusive behavior of an electron interacting with a quantized radiation field, 57 pages. 1990.

♯ 87: A. Inoue, The Alder-Wainwright effect for stationary processes with reflection positivity (II), 30 pages. 1990.

♯ 88: Y. Giga, S. Goto, H. Ishii and M.-H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, 32 pages. 1990.

♯ 89: A. Arai, Improper Bogoliubov transformations and instability of embedded eigenvalues, 47 pages. 1990.

♯ 90: K. Sugano, On bicommutators of modules over H-separable extension rings II, 9 pages. 1990.

♯ 91: T. Nakazi, Homogeneous polynomials and invariant subspaces in the polydiscs, 17 pages. 1990.

♯ 92: T. Nakazi, K. Takahashi, Homogoneous polynomials and invariant subspaces in the polydiscs II, 10 pages. 1990.

♯ 93: A. Arai, A theorem on essential self-adjointness with application to Hamiltonians in non-relativistic quantum field theory, 23 pages. 1990.

♯ 94: Y. Okabe, A. Inoue, On the exponential decay of the correlation functions for KMO-Langevin equations, 13 pages. 1990.

♯ 95: T. Sano, Y. Watatani, Angles between two subfactors, 62 pages. 1990.

♯ 96: S. Ninomiya, The Fourier-Sato transformation of pure sheaves, 22 pages. 1990.

♯ 97: Y. Okabe, A. Inoue, The theory of KM_2O-Langevin equations and its applications to data analysis (II): Causal analysis (1), 51 pages. 1990.

♯ 98: J. Ławrynowicz, S. Koshi and O. Suzuki, Dualities generated by the generalised Hurwitz problem and variation of the Yang-Mills field, 17 pages. 1991.

♯ 99: R. Agemi, K. Kubota and H. Takamura, On certain integral equations related to nonlinear wave equations, 52 pages. 1991.

♯ 100: S. Izumiya, Geometric singularities for Hamilton-Jacobi equation, 13 pages. 1991.

♯ 101: S. Izumiya, Legendrian singularities and first order differential equations, 16 pages. 1991.

♯ 102: A. Munemasa, Y. Watatani, Orthogonal pairs of $*$-subalgebras and association schemes, 11 pages. 1991.

♯ 103: A. Arai, O. Ogurisu, Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, 27 pages. 1991.

♯ 104: H. Takamura, Global existence of classical solutions to nonlinear wave equations with spherical symmetry for small data with noncompact support in three space dimensions, 14 pages. 1991.

♯ 105: R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, 11 pages. 1991.

♯ 106: T. Nakazi, Extremal problems in H^p, 13 pages. 1991.

♯ 107: T. Nakazi, ρ-dilations and hypo-Dirichlet algebras, 15 pages. 1991.

♯ 108: A. Arai, An abstract sum formula and its applications to special functions, 25 pages. 1991.

♯ 109: Y.-G. Chen, Y. Giga and S. Goto, Analysis toward snow crystal growth, 18 pages. 1991.

♯ 110: T. Hibi, M. Wakayama, A q-analogue of Capelli's identity for $GL(2)$, 7 pages. 1991.

♯ 111: T. Nishimori, A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem, 13 pages. 1991.

♯ 112: K. Matsuda, An analogy of the theorem of Hector and Duminy, 10 pages. 1991.

ON A REGULARITY CRITERION UP TO THE BOUNDARY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Shuji Takahashi

Department of Mathematics
Hokkaido University
Sapporo 060 Japan

Abstract. We are concerned with the behavior of weak solutions of the Navier-Stokes system around possible singularities on the boundary. We show that a weak solution locally belonging to some Lebesgue space cannot blowup there.

1. Introduction

We consider the Navier-Stokes equations:

$$(1.1) \qquad \begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla\phi = 0, & \text{in } Q = \Omega \times (-T, 0), \\ \nabla \cdot u = 0, & \text{in } Q, \\ u(x, -T) = u_0(x), & \text{on } \Omega, \\ u|_{\partial\Omega} = 0, & \end{cases}$$

where Ω is a domain in $\mathbb{R}^n (n \geqslant 2)$ with smooth boundary $\partial\Omega$, $0 < T < \infty$; $u = (u^i)_{i=1}^n$ and ϕ denote the unknown velocity and pressure, respectively, while $u_0 = (u_0^i)_{i=1}^n$ is a given initial velocity. Here external force is assumed to be zero for simplicity. Leray [Le] and Hopf [Ho] constructed global weak solutions in the class

$$u \in L^{2,\infty}(Q) \quad \text{and} \quad \nabla u \in L^{2,2}(Q) \tag{1.2}$$

for $u_0 \in L^2(\Omega)$ where $L^{p,q}(Q) := L^q(-T,0;L^p(\Omega))$. It is also known that a weak solution satisfying (1.2) moreover belongs to the class

$$\nabla u, \phi \in L^{r_0,r_0'}(Q) \quad \text{for all } 1 < r_0, r_0' < \infty \text{ with } n/r_0 + 2/r_0' = n \tag{1.3}$$

for a large class of domains including smoothly bounded domains provided that initial data is slightly regular (cf. Giga and Sohr [GS], Sohr and von Wahl [SW]). Serrin [Se] gave a local interior regularity criterion and Struwe [St] extended Serrin's result (cf. Takahashi [Ta]). They proved that the weak solution u of (1.1) in the class (1.2) is in $L^{\infty,\infty}(Q')$ and regular in the space variables provided that $u \in L^{p,q}(Q)$ for some p,q such that

$$n/p + 2/q \leqslant 1, \quad n < p \leqslant \infty, \tag{1.4}$$

where $Q' = \Omega' \times (-T',0)$, Ω' is relatively compact in Ω and $0 < T' < T$. When $\Omega = \mathbb{R}^n$, this was proved by Fabes, Jones and Riviere [FJR] (See also von Wahl [Wa1]).

Although global versions of Serrin-Struwe's results are available (cf. Giga [Gi], Sohr [So]), there seems no literature on a local version *up to the boundary*. Our goal is to give a local regularity criterion *up to the boundary* of Serrin-Struwe type. To avoid technical complexity in this paper we assume that the boundary $\partial\Omega$ is flat near a possible blowup point $x_0 \in \partial\Omega$. By translating variables we may assume that $x_0 = 0$. We take R so small that $\partial\Omega \cap B_R(0)$ is flat. Here $B_R(0)$ denotes the ball centered at 0 with redius R. We prove among other results in this paper that if the weak solution u of (1.1) in the class (1.2) and (1.3) in $Q \cap Q_R$ satisfies $u \in L^{p,q}(Q \cap Q_R)$ with

$$n/p + 2/q = 1, \quad n < p \leqslant \infty, \tag{1.5}$$

then

$$u \in L^{\infty,\infty}(Q \cap Q_{R'}),$$

where $Q_R = B_R(0) \times (-R^2, 0)$, $R^2 \leqslant T$ and $R' < R$. However, we are not sure whether the boundedness of u in space-time would imply the smoothness of u up to the boundary in the space variables, while it is true on the interior probrem (cf. [Se]). Concerning the interior regularity problem, the vorticity equation has been fully used (cf. Serrin [Se], Struwe [St] and Takahashi [Ta]). Unfortunately, the vorticity equation dose not apply to the regularity problem up to the boundary, because we cannot specify the boundary value of the vorticity $\omega = \operatorname{curl} u$ locally. Hence we are forced to analize (1.1) directly. When we localize the velocity, there arises a problem since the localized velocity is no longer solenoidal. We overcome this difficulty by applying Bogovski's lemma which gives a solution of $\nabla \cdot v = f$ with zero boundary conditions (cf. Bogovski [Bo1],[Bo2] and Borchers and Sohr [BS]). To carry out this idea we refine his result on the support of v.

2. Main theorem

We denote $Q_R^+ = B_R^+ \times (-R^2, 0)$, $B_R^+ = \{x \in \mathbb{R}^n \,|\, |x| < R, x_n > 0\}$ and $L^{p,q}(Q_R^+) = L^q(-R^2, 0; L^p(B_R^+))$.

We say u in the class

$$u \in L^{2,\infty}(Q_R^+), \quad \nabla u \in L^{2,2}(Q_R^+) \tag{2.1}$$

is a *weak solution* of

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla \phi = 0, & \text{in } Q_R^+, \\ \nabla \cdot u = 0, & \text{in } Q_R^+, \\ u|_{x_n = 0} = 0, & \end{cases} \tag{2.2}$$

if it holds

$$\iint_{Q_R^+} (\varphi_t + \Delta \varphi) \cdot u \, dx\, dt = \iint_{Q_R^+} (u \cdot \nabla) u \cdot \varphi \, dx \, dt \tag{2.3}$$

for all $\varphi = (\varphi^i)_{i=1}^n \in C_0^\infty(Q_R^+)$ with $\nabla \cdot \varphi = 0$ as well as

$$\nabla \cdot u = 0,$$

$$u|_{x_n=0} = 0.$$

Here $C_0^\infty(Q)$ is the space of smooth functions with compact support in Q. There is a scalar distribution ϕ on Q_R^+ such that

$$\nabla\phi = -u_t + \Delta u - (u\cdot\nabla)u$$

holds in the sense of distribution. Such ϕ is uniquely determined by u up to additive functions of t and called the pressure associated with u.

We do not distinguish the spaces of vector and scalar valued functions unless it causes confusion. We now state our main result in this paper.

THEOREM 2.1. *Suppose that u is a weak solution of (2.2) in the class (2.1) and that ϕ is the pressure associated with u. Let (u,ϕ) be moreover in the class*

$$\nabla u,\ \phi \in L^{r_0,r_0'}(Q_R^+) \quad \text{for all } 1 < r_0, r_0' < \infty \text{ with } \frac{n}{r_0} + \frac{2}{r_0'} = n. \tag{2.4}$$

(a) *Assume that $1 \leqslant p, q \leqslant \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. If $u \in L^{p,q}(Q_R^+)$, then*

$$u \in L^{\infty,\infty}(Q_{R/8}^+).$$

(b) *There exists a positive constant $\varepsilon = \varepsilon(n) < 1$ such that $||u||_{L^{n,\infty}(Q_R^+)} < \varepsilon$ implies that*

$$u \in L^{\infty,\infty}(Q_{R/8}^+).$$

We shall prove Theorem 2.1 in Section 5.

3. Localization

In this section we localize the weak solution u of (2.2) and get the linearized Navier-Stokes system for the modified localized weak solution.

We denote $\widetilde{B_R^+} = \{x \in \mathbb{R}^n \mid |x| < R, x_n \geqslant 0\}$. We first assume that $R = 1$ for simplicity. We cut off the weak solution u of (2.2) and the pressure ϕ on $Q_{1/2}^+$ to obtain higher regularity in $Q_{1/2}^+$. We set

$$\tilde{u} = u\psi \quad \text{and} \quad \rho = \phi\psi$$

where $\psi \in C_0^\infty(\widetilde{B_1^+} \times (-1,0])$ satisfies

$$\psi = 1 \qquad \text{in} \quad \overline{B_{1/2}^+} \times (-1/4, 0].$$

Then $(\widetilde{u}, \rho)$ satisfies

$$(3.1) \qquad \begin{cases} \widetilde{u}_t - \Delta \widetilde{u} + (u \cdot \nabla)\widetilde{u} + \nabla \rho = \phi \nabla \psi + \zeta(u, \psi), & \text{in } Q_1^+, \\ \nabla \cdot \widetilde{u} = u \cdot \nabla \psi, & \text{in } Q_1^+, \\ \widetilde{u}(x, -1) = 0, & \text{on } B_1^+, \\ \widetilde{u}\big|_{x_n = 0} = 0, \end{cases}$$

where

$$\zeta(u, \psi) = \psi_t u + u \Delta \psi - 2\nabla(u \cdot \nabla \psi) + (u \cdot \nabla \psi) u.$$

However $\widetilde{u}$ may not satisfy the incompressibility condition $\nabla \cdot \widetilde{u} = 0$. We recover this condition with Bogovski's lemma. To state it we prepare some function spaces:

Let D be a bounded domain in $\mathbb{R}^n$. Let $H_0^{j,r}(D)$ be the completion of $C_0^\infty(D)$ with respect to the norm $|\cdot|_{j,r}$, where $|f|_{j,r}^r = \sum_{|\alpha| \leqslant j} ||\nabla^\alpha f||_r^r$. Here we denote

$$\nabla^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $||f||_r^r = \int_D |f|^r dx$. We also denote the support of f by $\operatorname{supp} f$. We write $\nabla_i = \dfrac{\partial}{\partial x_i}$ and also simply write $||\nabla^j f||_r^r = \sum_{|\alpha| = j} ||\nabla^\alpha f||_r^r$.

LEMMA 3.1. *Assume that D is a bounded Lipschitz domain in $\mathbb{R}^n$ and that Γ is an open subset on ∂D. For any $j = 0, 1, 2, \cdots$, and any $r \in (1, \infty)$, there exist a bounded linear operator $K = K_{j,r} : H_0^{j,r}(D) \longrightarrow H_0^{j+1,r}(D)^n$ and positive constants $C = C(n, j, r, D)$ and $C' = C'(n, r, D)$ with the following properties:*

(a) $\nabla \cdot Kf = f$ *for all* $f \in H_0^{j,r}(D)$ *with* $\int_D f\, dx = 0$,

(b) $||\nabla^{j+1} Kf||_r \leqslant C ||\nabla^j f||_r$ *for all* $f \in H_0^{j,r}(D)$,

(c) $\operatorname{supp} Kf \subset D \cup \Gamma$ *if* $\operatorname{supp} f \subset D \cup \Gamma$,

(d) *For* $f \in L^r(D)$, *we can define* $K_{0,r}(\nabla_i f) \in L^r(D)$ $(i = 1, \cdots, n)$ *such that* $\nabla \cdot K_{o,r}(\nabla_i f) = \nabla_i f$ *for* $f \in H_0^{1,r}(D)$ *and that*

$$||K_{0,r}(\nabla_i f)||_r \leqslant C'||f||_r \quad \text{for all } f \in L^r(D).$$

Remark:

(1) The restriction of K on $C_0^\infty(D)$ is independent of j and r.

(2) If D is starshaped with respect to some ball, i.e., if there is a ball $B \subset D$ such that

$$D = \{tx + (1-t)y |\ x \in D,\ y \in B,\ t \in [0,1]\},$$

then the positive constants C and C' depend on D through the diameter of D and B.

For the proof of Lemma 3.1 (c) we prepare a lemma on a starshaped domain. For any subset U we denote by $\overline{U}$ the closure of U in $\mathbb{R}^n$.

Lemma 3.2. *Let D be a bounded domain starshaped with respect to a closed ball $B \subset D$. Suppose that F is a subset in D and that B' is a closed ball contained in the interior of B. Then there exists a subset $F_{B'}$ in D such that*

(i) $F_{B'} \supset F$,

(ii) $F_{B'}$ *is starshaped with respect to* B',

(iii) $\overline{F_{B'}} \cap \partial D = \overline{F} \cap \partial D$.

Proof of Lemma 3.2: We set

$$F_{B'} := \{sz + (1-s)y \in D \mid z \in B', y \in F, 0 \leqslant s \leqslant 1\}. \tag{3.2}$$

$F_{B'}$ satisfies (i) and (ii). We prove (iii). For any set $U \subset \mathbb{R}^n$ and any element $x \in \mathbb{R}^n$, we denote by $(U; x)$ a convex hull spanned by U and x:

$$(U; x) = \{sz + (1-s)x |\ z \in U,\ 0 \leq s \leq 1\}.$$

We see $\overline{F_{B'}} = \{sz + (1-s)y \mid z \in B', y \in \overline{F}, 0 \le s \le 1\} = \{(B'; y) | \ y \in \overline{F}\}$. We also see $(B'; y_0) \subset D$ for $y_0 \in D$ and $(B'; y_1) \cap \partial D = \{y_1\}$ for $y_1 \in \partial D$. It thus holds $\{(B'; y) | \ y \in \overline{F}\} \cap \partial D = \overline{F} \cap \partial D$. ∎

REMARK: We need to use B' instead of B to get (iii). Indeed $\overline{F_B} \cap \partial D$ may not be contained in $\overline{F} \cap \partial D$ where F_B is defined in a similar way to $F_{B'}$, since $(B; y) \cap (\partial D \backslash \{y\})$ may not be empty for some $y \in \partial D$. See FIG.1.

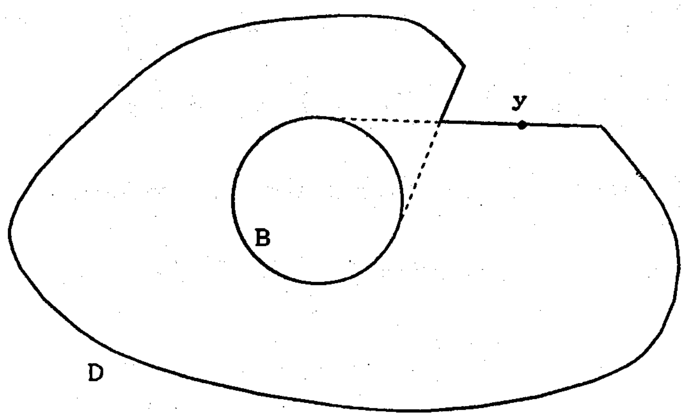

FIG. 1

PROOF OF LEMMA 3.1: Proofs of (a), (b) and (d) are found in [BS, Theorem 2.4]. We prove (c). We first assume that D is starshaped with respect to a closed ball $B \subset D$. As in Bogovski [Bo1], [Bo2] or Borchers and Sohr [BS], we define K on $C_0^\infty(D)$ by

$$Kf(x) = \int_D G(x,y) f(y)\, dy, \tag{3.3}$$

where

$$G(x,y) = (x-y) \int_1^\infty h(y + t(x-y)) t^{n-1} dt$$

and $h \in C_0^\infty(B')$ satisfies $\int_{B'} h\, dx = 1$. Here B' is a closed ball in the interior of B.

We see $\operatorname{supp} Kf$ is contained in the convex hull spanned by $\operatorname{supp} f$ and B'. Indeed, for a bounded set F we denote by $F_{B'}$ the convex hull spanned by F and B' such as in (3.2). Let $x \in D$, $y \in F$ and $t \geq 1$. Since $x = t^{-1}(y + t(x-y)) + (1-t^{-1})y$, we see $y + t(x-y) \in B'$ implies $x \in F_{B'}$ by the starshapedness of $F_{B'}$. We set $F = \operatorname{supp} f$ and $x \in \operatorname{supp} Kf$. The definition (3.3) yields $y + t(x-y) \in B'$. It thus holds $\operatorname{supp} Kf \subset F_{B'}$. Applying Lemma 3.2 implies $\operatorname{supp} Kf \cap \partial D \subset F_{B'} \cap \partial D = F \cap \partial D$.

When D is a general bounded Lipschitz domain, we can decompose D to finite star-shaped domains with respect to a ball by partition of unity (See [Ma, Section 1.1.9.]). Lemma 3.1 (c) is clear if we follow the decomposition in [BS, Theorem 2.4]. ∎

We apply Lemma 3.1 to (3.1) with $K = K_{1,2}$, $f = \nabla \cdot \tilde{u}$, $D = D_1$ and $\Gamma = \Gamma_1$, where $D_R := B_R^+ \backslash \overline{B_{R/2}^+}$ and $\Gamma_R := \{x \in \mathbb{R}^n; R/2 < |x| < R,\ x_n = 0\}$ for $R > 0$. We set

$$w = K(\nabla \cdot \tilde{u}) \quad \text{and} \quad v = \tilde{u} - w.$$

Since $u = 0$ on $\{x \in \mathbb{R}^n; x_n = 0\}$, using Stokes' theorem we have

$$\begin{aligned} \int_{D_1} \nabla \cdot \tilde{u}\, dx &= \int_{\partial B_{1/2}^+} \tilde{u} \cdot \overrightarrow{\mathbf{n}}\, d\sigma \\ &= \int_{B_{1/2}^+} \nabla \cdot u\, dx = 0. \end{aligned}$$

Here $\overrightarrow{\mathbf{n}}$ and σ denote the unit outer normal vector and the areal element of $\partial B_{1/2}^+$, respectively. Lemma 3.1 (a) yields $\nabla \cdot v = 0$. Since $\nabla \cdot \tilde{u}(\cdot, t) = u \cdot \nabla \psi(\cdot, t) \in H_0^{1,2}(D_1)$ for a.e. t, $w(\cdot, t) \in H_0^{2,2}(D_1)$. Integrating w and ∇w over x and t we see by Lemma 3.1 (b)

$$w \in L^{2,\infty}(D_1 \times (-1,0)) \quad \text{and} \quad \nabla w \in L^{2,2}(D_1 \times (-1,0)).$$

By the definition of ψ, $\operatorname{supp} \nabla \cdot \tilde{u}(\cdot, t) \subset D_1 \cup \Gamma_1$. Lemma 3.1 (c) implies $\operatorname{supp} w(\cdot, t) \subset D_1 \cup \Gamma_1$, which yields $w(\cdot, t) = 0$ on $\partial D_1 \backslash \Gamma_1$ with its all derivatives. Then we can extend w outside of D_1 by zero smoothly. We can also verify $w(x, -1) = 0$. Moreover, by the parallel argument on the class (2.4), we now conclude that v is in the class

$$(3.4) \qquad \begin{cases} v \in L^{2,\infty}(\mathbb{R}_+^n \times (-1,0)) \quad \text{and} \quad \nabla v \in L^{2,2}(\mathbb{R}_+^n \times (-1,0)), \\ \nabla v \in L^{r_0, r_0'}(\mathbb{R}_+^n \times (-1,0)) \text{ for all } r_0, r_0' \in (1, \infty) \text{ with } \dfrac{n}{r_0} + \dfrac{2}{r_0'} = n \end{cases}$$

and satisfies the linearized Navier-Stokes equations:

$$
(3.5) \qquad \begin{cases} v_t - \Delta v + (u \cdot \nabla)v + \nabla \rho = h, & \text{in } \mathbb{R}^n_+ \times (-1,0), \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^n_+ \times (-1,0), \\ v(x,-1) = 0, & \text{on } \mathbb{R}^n_+, \\ v\big|_{x_n=0} = 0, & \end{cases}
$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$ and

$$
(3.6) \qquad h = \phi \nabla \psi - w_t + \Delta w - (u \cdot \nabla) w + \zeta(u, \psi).
$$

4. A priori estimates

This section establishes a priori estimates for the Stokes equations :

$$
(4.1) \qquad \begin{cases} v_t - \Delta v + \nabla \rho = f, & \text{in } \mathbb{R}^n_+ \times (-T,0), \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^n_+ \times (-T,0), \\ v(x,-T) = 0, & \text{on } \mathbb{R}^n_+, \\ v\big|_{x_n=0} = 0, & \end{cases}
$$

with function f in some Lebesgue space. We apply the estimates to the modified localized solution v of the linearized Navier-Stokes system (3.5), where $0 < T < \infty$.

Let $1 < r, r', \dot{r} < \infty$. We say v in the class

$$
(4.2) \qquad v \in L^{\dot{r}, r'}(\mathbb{R}^n_+ \times (-T,0)) \quad \text{with} \quad \nabla v \in L^{r,r'}(\mathbb{R}^n_+ \times (-T,0))
$$

is a *weak solution* of (4.1) if it holds

$$
\int_{-T}^{0} \int_{\mathbb{R}^n_+} (-\varphi_t - \Delta \varphi) \cdot v \, dx dt = \int_{-T}^{0} \int_{\mathbb{R}^n_+} f \cdot \varphi \, dx dt
$$

for all $\varphi = (\varphi^i)_{i=1}^n \in C_0^\infty(\mathbb{R}^n_+ \times [-T,0))$ with $\nabla \cdot \varphi = 0$ as well as

$$
\nabla \cdot v = 0,
$$

$$
v|_{x_n=0} = 0.
$$

Note that for the weak solution v defined as above, there is a scalar distribution ρ on $\mathbb{R}^n_+ \times (-T, 0)$ such that

$$\nabla \rho = f - v_t + \Delta v$$

holds in the sense of distribution. Such ρ is uniquely determined by v up to additive functions of t and called the pressure associated with v.

In what follows, we write

$$\xi(p,q) := \frac{n}{p} + \frac{2}{q}.$$

In this section we suppress $\mathbb{R}^n_+ \times (-T, 0)$ in norms and function spaces, which are simply written as $L^{p,q}$, C^∞ and so on.

Lemma 4.1. *(Giga and Sohr, [GS]) Let $1 < \ell, \ell' < \infty$. Then for every $f \in L^{\ell,\ell'}(\mathbb{R}^n_+ \times (-T, 0))$ there exists a unique solution $(v, \nabla\rho)$ of (4.1) satisfying*

$$||v_t||_{\ell,\ell'} + ||\nabla^2 v||_{\ell,\ell'} + ||\nabla \rho||_{\ell,\ell'} \leqslant C||f||_{\ell,\ell'}$$

with $C = C(n, \ell, \ell')$.

We note that weak solutions of (4.1) are unique in the class (4.2). The proof is, as usual, based on the existence of solutions of the dual problem (Lemma 4.1). We state next lemma without the proof:

Lemma 4.2. *Let $1 < r, r', \dot{r} < \infty$. Suppose that v is a weak solution of (4.1) with $f = 0$ in the class (4.2). Then $v \equiv 0$.*

We need another type a priori estimates.

Lemma 4.3. *Suppose that $1 < \beta, \beta', \alpha < \infty$. Let γ, γ' satisfy*

$$\begin{cases} \beta \leqslant \gamma < \infty, \quad \beta' \leqslant \gamma' < \infty, \\ \xi(\beta, \beta') = \xi(\gamma, \gamma') + 1. \end{cases} \tag{4.3}$$

Then for every $f \in L^{\beta,\beta'}$ *there exists a unique solution* v *of (4.1) in the class* $v \in L^{\alpha,\gamma'}$ *with* $\nabla v \in L^{\gamma,\gamma'}$ *satisfying*

$$||\nabla v||_{\gamma,\gamma'} \leqslant C||f||_{\beta,\beta'}$$

with $C = C(n, \beta, \beta', \gamma, \gamma')$.

PROOF: Since the uniqueness of weak solutions of (4.1) in the class (4.2) is given by Lemma 4.2, the weak solution is represented uniquely with the Stokes operator. Ukai [Uk, Theorem 3.1] showed that for $1 < q < p < \infty$

$$||\nabla u(\cdot, t)||_p \leqslant Ct^{-\alpha-1/2}||u_0||_q$$

with $\alpha = 2^{-1}n(q^{-1} - p^{-1})$ for the solution u of the Stokes system with zero external force and initial value u_0 on a halfspace (In fact this estimate holds for $1 < q \leqslant p < \infty$). We can prove Lemma 4.3 by applying the Hardy-Littlewood-Sobolev inequality in the time variable (cf. Takahashi [Ta, Section 4]). ∎

We next apply the above lemmas to the linearized Navier-Stokes equations:

$$(4.4) \qquad \begin{cases} v_t - \Delta v + (b \cdot \nabla)v + \nabla \rho = h, & \text{in } \mathbb{R}^n_+ \times (-T, 0), \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^n_+ \times (-T, 0), \\ v(x, -T) = 0, & \text{on } \mathbb{R}^n_+, \\ v\big|_{x_n=0} = 0, & \end{cases}$$

with irregular coefficient $b \in L^{p,q}$ for $p, q \geq 1$ satisfying $\xi(p, q) = 1$ and function h in some Lebesgue space.

Let $1 < \dot{r} < \infty$ and let $1 < \ell, \ell', r, r' < \infty$ satisfy

$$(4.5) \qquad 1/r = 1/\ell - 1/p, \qquad 1/r' = 1/\ell' - 1/q.$$

For $h \in L^{\ell,\ell'}$ and $b \in L^{p,q}$, we say v in the class

$$(4.6) \qquad v \in L^{\dot{r},r'} \quad \text{with} \quad \nabla v \in L^{r,r'}$$

is a *weak solution* of (4.4) if it holds

$$\int_{-T}^{0}\int_{\mathbb{R}^n_+}(-\varphi_t-\Delta\varphi)\cdot v\,dxdt=\int_{-T}^{0}\int_{\mathbb{R}^n_+}(h-(b\cdot\nabla)v)\cdot\varphi\,dxdt,$$

for any $\varphi=(\varphi^i)_{i=1}^n\in C_0^\infty(\mathbb{R}^n_+\times[-T,0))$ with $\nabla\cdot\varphi=0$ as well as

$$\nabla\cdot v=0,$$

$$v|_{x_n=0}=0.$$

In the same way as in (4.1), we define in (4.4) the pressure ρ associated with v:

$$\nabla\rho=h-v_t+\Delta v-(b\cdot\nabla)v.$$

PROPOSITION 4.1. *Assume that $1\leqslant p,q\leqslant\infty$ satisfies $n/p+2/q=1$. Let $1<r,r',\ell,\ell'<\infty$ satisfy (4.5) and let $1<\dot r<\infty$. Suppose that $h\in L^{\ell,\ell'}$ and $b\in L^{p,q}$. Assume that v is a weak solution of (4.4) in the class (4.6) and that ρ is the pressure associated with v. Then there exist positive constants $\varepsilon=\varepsilon(n,\ell,\ell',p)<1$ and $C_1=C_1(n,\ell,\ell',p)$ such that $||b||_{p,q}<\varepsilon$ implies*

(a) $||\nabla v||_{r,r'}\leqslant C_1||h||_{\ell,\ell'}$,

(b) $||\nabla\rho||_{\ell,\ell'}\leqslant C_1||h||_{\ell,\ell'}$:

if moreover $1<\ell<n$, it also holds

(c) $||\nabla v||_{m,\ell'}\leqslant C_1||h||_{\ell,\ell'}$ *for $1<m<\infty$ with $1/m=1/\ell-1/n$.*

PROOF: We first note that $\xi(r,r')=\xi(m,\ell'),r\leqslant m,r'\geqslant\ell'$, in particular, $(r,r')=(m,\ell')$ if $(p,q)=(n,\infty)$.

(a) Since $(\beta,\beta',\gamma,\gamma')=(\ell,\ell',r,r')$ satisfies (4.3), applying Lemma 4.3 and Hölder's inequality with (4.5) yields

$$||\nabla v||_{r,r'}\leqslant C(||b||_{p,q}||\nabla v||_{r,r'}+||h||_{\ell,\ell'}).$$

Since $||\nabla v||_{r,r'}<\infty$, setting $\varepsilon=(2C)^{-1}$ implies (a).

(b) Lemma 4.1 and Hölder's inequality yield

$$||\nabla\rho||_{\ell,\ell'}\leq C(||b||_{p,q}||\nabla v||_{r,r'}+||h||_{\ell,\ell'}).$$

The estimate (a) yields (b).

(c) Setting $(\beta, \beta', \gamma, \gamma') = (\ell, \ell', m, \ell')$, as in (a), we have

$$||\nabla v||_{m,\ell'} \leq C(||b||_{p,q}||\nabla v||_{r,r'} + ||h||_{\ell,\ell'}).$$

The estimate (a) yields (c). ∎

We next get parallel results to Proposition 4.1 without the regularity assumptions on v in (4.6) by an approximate argument.

THEOREM 4.1. *Assume that $n \geq 3$ or $p > n$. Let $1 \leq p, q \leq \infty$ satisfy $n/p + 2/q = 1$. Let $1 < r, r', \ell, \ell' < \infty$ satisfy (4.5). Let $1 < \theta, \theta', \dot{\theta} < \infty$ satisfy*

$$(4.7) \qquad \begin{cases} \dfrac{n}{\theta} + \dfrac{2}{\theta'} = n, \quad \dfrac{p}{p-1} < \theta < n, \quad \theta' > 2, \\ \dfrac{1}{\dot{\theta}} = \dfrac{1}{\theta} - \dfrac{1}{n}. \end{cases}$$

Suppose that $h \in L^{\ell,\ell'}$, $b \in L^{p,q}$ and that v is a weak solution of (4.4) in the class

$$(4.8) \qquad \begin{cases} v \in L^{2,\infty} \quad \text{and} \quad \nabla v \in L^{2,2}, \\ v \in L^{\dot{\theta},\theta'} \quad \text{and} \quad \nabla v \in L^{\theta,\theta'} \end{cases}$$

with the pressure ρ. If

$$r \geq \theta \quad \text{and} \quad r' \geq \theta',$$

then v is in the class

$$\nabla v \in L^{r,r'}$$

and the conclusions of Proposition 4.1 hold.

PROOF: We prove only the estimate (a) in Proposition 4.1 since (a) automatically yields (b) and (c). Since the mapping $h \longmapsto v$ is linear we may assume that $h \in C_0^\infty(\mathbb{R}^n_+ \times (-T, 0))$. We approximate b by $b_k \in C_0^\infty$ such that

$$b_k \to b \quad \text{in} \quad L^{p,q},$$

$$||b_k||_{p,q} \leq M||b||_{p,q} \qquad (M > 0).$$

There exists a smooth solution v_k of the approximate equations:

$$(4.9) \qquad \begin{cases} \partial_t v_k - \Delta v_k + (b_k \cdot \nabla) v_k + \nabla \rho_k = h, & \text{in } \mathbb{R}^n_+ \times (-T, 0), \\ \nabla \cdot v_k = 0, & \text{in } \mathbb{R}^n_+ \times (-T, 0), \\ v_k(x, -T) = 0, & \text{on } \mathbb{R}^n_+, \\ v_k|_{x_n=0} = 0 \end{cases}$$

in the class (4.8) with the pressure ρ_k (The existence of solutions is shown by Solonnikov [Sol, Theorem 4.1 and Theorem 9.1] when $n = 3$. His method applies to an arbitrary dimension $n \geq 2$ (cf. von Wahl [Wa2])).

We first show that there exists a weak limit of some subsequence $v_{k'}$ such that $\nabla v_{k'}$ weakly converges in $L^{\theta,\theta'}$. Let $1 < \ell_\theta, \ell'_\theta < \infty$ satisfy $1/\ell_\theta = 1/\theta + 1/p$ and $1/\ell'_\theta = 1/\theta' + 1/q$. The assumptions $r \geq \theta$, $r' \geq \theta'$ and (4.5) implies $\ell \geq \ell_\theta$ and $\ell' \geq \ell'_\theta$. Since $h \in L^{\ell,\ell'}$ is compactly supported, we see $h \in L^{\ell_\theta,\ell'_\theta}$. Proposition 4.1 (a) yields

$$(4.10) \qquad \|\nabla v_k\|_{\theta,\theta'} \leq C\|h\|_{\ell_\theta,\ell'_\theta}.$$

By Sobolev's inequality we see

$$(4.11) \qquad \|v_k\|_{\dot\theta,\theta'} \leqslant C'\|h\|_{\ell_\theta,\ell'_\theta}.$$

Since $1 < \theta, \theta', \dot\theta < \infty$, $L^{\theta,\theta'}$ and $L^{\dot\theta,\theta'}$ are respectively the dual spaces of L^{θ^*,θ'^*} and $L^{\dot\theta^*,\theta'^*}$, where θ^*, θ'^* and $\dot\theta^*$ are the dual numbers, respectively. By the estimates (4.10)-(4.11) there exist a subsequence $\{k'\}$ and $\widetilde{v} \in L^{\dot\theta,\theta'}$ with $\nabla\widetilde{v} \in L^{\theta,\theta'}$ such that

$$\nabla v_{k'} \to \nabla \widetilde{v} \quad \text{weakly in } L^{\theta,\theta'},$$
$$v_{k'} \to \widetilde{v} \quad \text{weakly in } L^{\dot\theta,\theta'}.$$

We next show that $\widetilde{v}$ is a weak solution of (4.4). It is clear that $\nabla \cdot \widetilde{v} = 0$. By the definition of (4.9) it holds

$$< -\varphi_t - \Delta\varphi, v_{k'} > = < h - (b_{k'} \cdot \nabla) v_{k'}, \varphi >$$

for any $\varphi \in C_0^\infty(\mathbb{R}_+^n \times [-T,0))$ with $\nabla \cdot \varphi = 0$. Here

$$< f, g > := \int_{-T}^{0} \int_{\mathbb{R}_+^n} f \cdot g \; dxdt.$$

Letting $k' \to \infty$ yields

$$< -\varphi_t - \Delta\varphi, v_{k'} > \;\to < -\varphi_t - \Delta\varphi, \widetilde{v} >,$$
$$< (b_{k'} \cdot \nabla) v_{k'}, \varphi > \;\to < (b \cdot \nabla)\widetilde{v}, \varphi > .$$

We next prove that $\widetilde{v}|_{x_n=0} = 0$. We consider the Banach space

$$W = \{v \in L^{\dot{\theta},\theta'}; \nabla v \in L^{\theta,\theta'}\}$$

equipped with the norm $||v||_{\dot{\theta},\theta'} + ||\nabla v||_{\theta,\theta'}$. We set

$$W_0 = \{v \in W; v|_{x_n=0} = 0\}.$$

Since W is reflexive and W_0 is strongly closed in W, Mazur's theorem (cf. Yosida [Yo, Theorem 11 in Sect.1, Chap.5]) implies that W_0 is weakly closed in W. Since $v_{k'} \to \widetilde{v}$ weakly in W and $v_{k'} \in W_0$, we conclude $\widetilde{v} \in W_0$. Therefore $\widetilde{v}$ satisfies (4.4) in the sense of distribution.

We now obtain the desired estimate for $\widetilde{v}$. Applying Lemma 4.3 to (4.9) yields

$$||\nabla v_k||_{\gamma,\gamma'} \leqslant C(||(b_k \cdot \nabla) v_k||_{\beta,\beta'} + ||h||_{\beta,\beta'})$$

with (4.3). Since $b_k, h \in C_0^\infty$, we see $||\nabla v_k||_{\beta,\beta'} < \infty$ implies $||\nabla v_k||_{\gamma,\gamma'} < \infty$. Repeating this argument with $\nabla v_k \in L^{\theta,\theta'}$ yields $\nabla v_k \in L^{\alpha,\alpha'}$ for all $\theta \leqslant \alpha < \infty$ and $\theta' \leq \alpha' < \infty$. Applying Proposition 4.1 (a) we see

$$||\nabla v_{k'}||_{r,r'} \leq C_1 ||h||_{\ell,\ell'}. \tag{4.12}$$

Since $\nabla v_{k'} \to \nabla \widetilde{v}$ weakly in $L^{\theta,\theta'}$ and since the norm in $L^{r,r'}$ is lower semicontinuous with respect to this topology, sending $k' \to \infty$ yields

$$||\nabla \widetilde{v}||_{r,r'} \leq C_1 ||h||_{\ell,\ell'}.$$

We observe that $\widetilde{v} = v$ by the uniqueness for solutions of (4.4) in the class (4.8) (by arranging $||b||_{p,q}$ smaller if necessary). Let v_1 and v_2 be weak solutions of (4.4) in the class (4.8). We set $V = v_1 - v_2$. Then Lemma 4.3 yields

$$||\nabla V||_{\theta,\theta'} \leqslant C||b||_{p,q}||\nabla V||_{\theta,\theta'}.$$

This implies $V \equiv 0$ if $||b||_{p,q} \leqslant (2C)^{-1}$. The proof is now complete. ∎

We next estimate the minor terms h. We denote $||f||_{p,q,Q} := ||f||_{L^{p,q}(Q)}$ and $||f||_{p,q} := ||f||_{L^{p,q}(\mathbb{R}^n_+ \times (-1,0))}$.

PROPOSITION 4.2. *Suppose that u in the class (2.1) is a weak solution of (2.2) in Q_1^+ with the pressure ϕ and that (u,ϕ) is moreover in the class (2.4). Let $1 \leq p, q \leq \infty$ satisfy $n/p + 2/q = 1$. Let $1 < \ell, \ell', r, r' < \infty$ satisfy (4.5). Assume that h is as in (3.6), $u \in L^{p,q}(Q_1^+) \cap L^{r,r'}(Q_1^+)$, $\nabla u \in L^{\ell,\ell'}(Q_1^+)$ and $\phi \in L^{\ell,\ell'}(Q_1^+)$. Then it holds*

$$||h||_{\ell,\ell'} \leqslant C(||u||_{r,r',Q_1^+} + ||\nabla u||_{\ell,\ell',Q_1^+} + ||\phi||_{\ell,\ell',Q_1^+})$$

with $C = C(n, r, r', p, Q_1^+)$.

PROOF: We recall

$$\begin{aligned}
&h = \phi\nabla\psi - w_t + \Delta w - (u\cdot\nabla)w + \zeta(u,\psi),\\
&w = K(\nabla\cdot u\psi),\\
&\zeta(u,\psi) = \psi_t u + u\Delta\psi - 2\nabla(\nabla\psi\cdot u) + (u\cdot\nabla\psi)u,\\
&\psi \in C_0^\infty(\widetilde{B_1^+ \times (-1,0]}).
\end{aligned}$$

We here suppress the subscripts j and r since $K_{j,r}$ is independent of j and r on $C_0^\infty(D_1)$. We first calculate w_t. Since $\nabla\cdot u = 0$, we see $w = K(u\cdot\nabla\psi)$. It holds

$$\begin{aligned}
w_t &= \partial_t K(u\cdot\nabla\psi)\\
&= K(u\cdot\nabla\psi_t) + K(u_t\cdot\nabla\psi).
\end{aligned}$$

The equations (2.2) yield

$$K(u_t\cdot\nabla\psi)=K(\nabla\psi\cdot(\Delta u-(u\cdot\nabla)u-\nabla\phi)).$$

Here the above two equalities are justified by Lemma 3.1 (d) and

$$u_t\in L^1(-1,0;H^{-1,\gamma}(B_1^+))\quad\text{for some } 1<\gamma<\infty,$$

which is given by

$$\begin{aligned}(u\cdot\nabla)u&\in L^1(-1,0;H^{-1,\gamma_1}(B_1^+))\quad\text{for some } \gamma_1=\gamma_1(r_0),\\ \Delta u&\in L^2(-1,0;H^{-1,2}(B_1^+)),\\ \nabla\phi&\in L^1(-1,0;H^{-1,\gamma_2}(B_1^+))\quad\text{for some } \gamma_2=\gamma_2(r_0)\end{aligned}$$

(See the assumption (2.4). We have applied Sobolev's inequality to ∇u in (2.4) for some $r_0\in(1,n)$ since $u|_{x_n=0}=0$ (cf. [Zi, Corollary 4.5.3])). Here $H^{-1,\gamma}(B_1^+)$ is the dual space of $H_0^{1,\gamma^*}(B_1^+)$ for γ^* satisfying $1/\gamma+1/\gamma^*=1$. Since K is linear, we thus rewrite h as follows:

$$h=h_1+h_2+h_3+h_4$$

where

$$\begin{aligned}h_1=&\sum_{i=1}^n K(\nabla_i(\phi\nabla_i\psi))-K(\phi\Delta\psi)+\phi\nabla\psi,\\ h_2=&\sum_{i,j=1}^n K(\nabla_j(u^ju^i\nabla_i\psi))-\sum_{i,j=1}^n K(u^ju^i\nabla_i\nabla_j\psi)+(u\cdot\nabla\psi)u,\\ h_3=&-(u\cdot\nabla)K(u\cdot\nabla\psi),\\ h_4=&-K(u\cdot\nabla\psi_t)-\sum_{i,j=1}^n K(\nabla_j(\nabla_i\psi\nabla_ju^i))+\sum_{i,j=1}^n K(\nabla_ju^i\nabla_j\nabla_i\psi)\\ &-\Delta K(u\cdot\nabla\psi)+\psi_tu+u\Delta\psi-2\nabla(u\cdot\nabla\psi).\end{aligned}$$

Here $\nabla_i=\dfrac{\partial}{\partial x_i}$. We note that $\operatorname{supp}\nabla\psi\subset\overline{D_1}$. Applying Lemma 3.1 (d) on $D=D_1$ yields

$$\begin{aligned}||h_1||_{L^\ell(\mathbb{R}^n_+)}&=||h_1||_{L^\ell(D_1)}\leqslant C||\phi||_{L^\ell(B_1^+)}\\ ||h_2||_{L^\ell(\mathbb{R}^n_+)}&=||h_2||_{L^\ell(D_1)}\leqslant C||\,|u|^2||_{L^\ell(B_1^+)}.\end{aligned}$$

Applying Lemma 3.1 -(b) and -(d) yields

$$||h_4||_{L^\ell(\mathbb{R}^n_+)} = ||h_4||_{L^\ell(D_1)} + ||\psi_t u||_{L^\ell(B_1^+)} \leqslant C||\nabla u||_{L^\ell(B_1^+)}.$$

Here $C = C(\ell, \psi)$ and estimates of lower derivatives are given by Poincaré's inequality. Hölder's inequality yields

$$||h_2||_{\ell,\ell'} \leqslant C||u||_{p,q,Q_1^+}||u||_{r,r',Q_1^+}.$$

The term h_3 can be also estimated as above. ∎

5. Proof of Theorem 2.1.

In this section we denote $||f||_{p,q} = ||f||_{L^{p,q}(\mathbb{R}^n_+\times(-R^2,0))}$ and $||f||_{p,q,Q} = ||f||_{L^{p,q}(Q)}$.

We first prove Theorem 2.1 when $n \geq 3$ or $p > n$.

We observe that the class (3.4) is included in the class (4.8). Let $1 < s < n$ and set r such that $1/r = 1/s - 1/n$. If $||u||_{p,q,Q_R^+} < \varepsilon$ for $\varepsilon = \varepsilon(n,p,\ell,\ell',R)$ given as in Proposition 4.1, applying Theorem 4.1 and Proposition 4.2 to (3.5), we obtain

$$\begin{aligned} &||\nabla v||_{r,r'} + ||\nabla v||_{m,\ell'} + ||\rho||_{m,\ell'} \\ &\leqslant C(||\nabla u||_{s,r',Q_R^+} + ||\nabla u||_{\ell,\ell',Q_R^+} + ||\phi||_{\ell,\ell',Q_R^+}) \end{aligned} \tag{5.1}$$

with $C = C(n,p,s,r',R)$ where the other exponents are the same as in Proposition 4.1. We have here applied Sobolev's inequality to $\nabla\rho$ and u since $\rho = \psi\phi$ has compact support and u satisfies $u|_{x_n=0} = 0$ (cf. [Zi, Corollary 4.5.3]). We note that $1 < \ell < n$ and $\xi(r,r') = \xi(m,\ell') = \xi(s,r') - 1 = \xi(\ell,\ell') - 1$. We use (5.1) inductively. In the similar way to the definition of (v,ρ) in Section 3, we set

$$v_j = u\psi_j - K(\nabla\cdot u\psi_j) \quad \text{and} \quad \rho_j = \phi\psi_j$$

where

$$\psi_j \in C_0^\infty(\widetilde{B_{R_j}^+} \times (-R_j^2, 0])$$

satisfies $\psi_j = 1$ in $\overline{B^+_{R_{j+1}}} \times (-R^2_{j+1}, 0]$. Here $R_j := (1/2 + 1/2^j)R$. We also set

$$1/r_{j+1} = 1/r_j - 1/n,$$
$$1/\ell_j = 1/r_{j+1} + 1/p,$$
$$1/\ell' = 1/r' + 1/q.$$

We apply Lemma 3.1 on $D = D_{R_j}$. We see $v_j = u$ and $\rho_j = \phi$ on $Q^+_{R_{j+1}}$ since $\psi_j = 1$ and $K(\nabla \cdot u\psi_j) = 0$ there. Then

$$||u||_{p,q,Q^+_{R_j}} < \varepsilon_j$$

for $\varepsilon_j = \varepsilon_j(n, p, r_j, r', R_j)$ implies

$$\begin{aligned} &||\nabla u||_{r_{j+1},r',Q^+_{R_{j+1}}} + ||\nabla u||_{\ell_{j+1},\ell',Q^+_{R_{j+1}}} + ||\phi||_{\ell_{j+1},\ell',Q^+_{R_{j+1}}} \\ &\leqslant ||\nabla v_j||_{r_{j+1},r'} + ||\nabla v_j||_{\ell_{j+1},\ell'} + ||\rho_j||_{\ell_{j+1},\ell'} \\ &\leqslant C_j \left(||\nabla u||_{r_j,r',Q^+_{R_j}} + ||\nabla u||_{\ell_j,\ell',Q^+_{R_j}} + ||\phi||_{\ell_j,\ell',Q^+_{R_j}}\right) \end{aligned}$$

where $C_j = C_j(n, p, r_j, r', R_j)$ and $1 < r_j, \ell_j < n$. For any $2 < \alpha' < \infty$, we set $r' = \alpha'$ and r_1 such that $n/r_1 + 2/\alpha' = n$. The assumption (2.4) implies

$$\nabla u \in L^{r_1,\alpha'}(Q^+_{R_1}).$$

Since $\xi(\ell_1, \ell') = \xi(r_1, \alpha') = n$, it also holds

$$\nabla u, \phi \in L^{\ell_1,\ell'}(Q^+_{R_1}).$$

We see $1 < r_{n-1} < n < r_n < \infty$. And setting $\alpha' > 2p/n$ we also see $1 < \ell_{n-1} < n < \ell_n < \infty$. We set $\varepsilon = \min(\varepsilon_1, \cdots, \varepsilon_{n-1})$. Here ε depends only on n, p, α' and R. Since $||u||_{p,q,Q^+_R} < \varepsilon$ implies $||u||_{p,q,Q^+_{R_j}} < \varepsilon_j$ for all $j = 1, \cdots, n-1$, the induction yields

$$\nabla u \in L^{r_n,\alpha'}(Q^+_{R/4}) \quad \text{and} \quad \nabla u,\ \phi \in L^{\ell_n,\ell'}(Q^+_{R/4}) \tag{5.2}$$

with

$$r_n = \frac{n\alpha'}{\alpha' - 2} \quad \text{and} \quad \frac{1}{\ell_n} = \frac{1}{n} + \frac{1}{p} - \frac{2}{n\alpha'},$$

provided that $||u||_{p,q,Q_R^+} < \varepsilon$. We see by Sobolev's inequality

$$u \in L^{\infty,\alpha'}(Q^+_{R/4}). \tag{5.3}$$

By the L^p-L^q estimates for the Stokes operator, it holds

$$||\, e^{-tA}\, a||_{L^\infty(\mathbb{R}^n_+)} \le C\, t^{-n/2r}||a||_{L^r(\mathbb{R}^n_+)} \quad \text{for } 1 \le r < \infty. \tag{5.4}$$

Here e^{-tA} denotes the semi-group of the Stokes operator A on the upper half space (See [BM, Proposition 4.1]). Applying (5.4) to (3.5) and using Young's inequality in the time variable, we obtain

$$||v||_{\infty,\infty} \le C(||u\nabla v||_{\delta,\delta'} + ||h||_{\delta,\delta'})$$

for $1 < \delta, \delta' < \infty$ such that $n/\delta + 2/\delta' < 2$. Proposition 4.2 and (5.2)-(5.3) with $(\delta, \delta') = (\ell_n, \ell')$ yield

$$u \in L^{\infty,\infty}(Q^+_{R/8}) \tag{5.5}$$

provided that $||u||_{p,q,Q_R^+} < \varepsilon$.

When $p = n = 2$ we can get (5.3) only with Proposition 4.1 and 4.2. Indeed, the class (3.4) implies

$$\nabla v,\ \rho \in L^{\ell,\ell'} \quad \text{for all } 1 < \ell, \ell' < \infty \text{ with } \frac{1}{\ell} + \frac{1}{\ell'} = 1. \tag{5.6}$$

For any $2 < \alpha' < \infty$ we set (ℓ_0, ℓ_0') such that $1/\ell_0 = 1 - 1/\alpha'$ and $1/\ell_0' = 1 - 1/\ell_0$. Then Proposition 4.1, 4.2 and (5.6) with $(\ell, \ell') = (\ell_0, \ell_0')$ yield (5.1) for (s, r, r', m) given by $s = \ell_0$, $1/r = 1/\ell_0 - 1/2$, $r' = \alpha'$ and $m = r$. We see (5.2) holds for $n = p = 2$. We now also get (5.5) when $n = p = 2$.

We can show that ε is independent of R. Indeed, we rescale the weak solution u of (2.2) in Q_R^+ and the pressure ϕ around $(0, 0)$ with

$$\begin{cases} u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\ \phi_\lambda(x,t) = \lambda^2 \phi(\lambda x, \lambda^2 t) \end{cases}$$

for $\lambda > 0$. Then $(u_\lambda, \phi_\lambda)$ satisfies (2.2) in $Q^+_{R/\lambda}$ and

$$\|u_\lambda\|_{p,q,Q^+_{R/\lambda}} = \|u\|_{p,q,Q^+_R}$$

for $n/p + 2/q = 1$. We see

(1) $\|u_R\|_{p,q,Q^+_1} < \varepsilon^1$ implies $u_R \in L^{\infty,\infty}(Q^+_{1/8})$

is equivalent to

(2) $\|u\|_{p,q,Q^+_R} < \varepsilon^1$ implies $u \in L^{\infty,\infty}(Q^+_{R/8})$.

If $p > n$, that is, if $q < \infty$, we can remove the smallness condition on u. Indeed, let $0 \leqslant t^0 \leqslant R^2/4$. Let x^0 be on the flat boundary and x^1 be in the interior of the upper half ball, that is,

$$x^0 \in \overline{B^+_{R/2}} \cap \{x \in \mathbb{R}^n | x_n = 0\} \quad \text{and} \quad x^1 \in B^+_{R/2}.$$

Then the assumption $u \in L^{p,q}(Q^+_R)$ with $q < \infty$ implies that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\|u\|_{p,q,Q^+_{R_\varepsilon}(x^0,t^0)} < \varepsilon, \tag{5.7}$$

$$\|u\|_{p,q,Q_{R_\varepsilon}(x^1,t^0)} < \varepsilon, \tag{5.8}$$

where

$$\begin{aligned} Q^+_R(x^0,t^0) &= B^+_R(x^0) \times (t^0 - R^2, t^0), \\ Q_R(x^1,t^0) &= B_R(x^1) \times (t^0 - R^2, t^0), \\ B^+_R(x^0) &= \{x \in \mathbb{R}^n | x_n > 0,\ |x - x^0| < R\}, \\ B_R(x^1) &= \{x \in \mathbb{R}^n |\ |x^1 - x| < R\}. \end{aligned}$$

It is known that (5.8) leads to $u \in L^{\infty,\infty}(Q_{R_\varepsilon/4}(x^1,t^0))$ (cf. Struwe [St] and Takahashi [Ta]). Since ε does not depend on radius R, (5.7) implies $u \in L^{\infty,\infty}(Q^+_{R_\varepsilon/8}(x^0,t^0))$. A covering argument yields $u \in L^{\infty,\infty}(Q^+_{R/8})$. ∎

REMARK: Derivative estimates (5.2) are also valid in Theorem 2.1.

Acknowledgments

The author is grateful to Professor Y. Giga who brought this problem to his attention. The author is also grateful to Professor H. Kozono for his useful comments.

References

[Bo1] M. E. Bogovski, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl. **20** (1979), 1094-1098.

[Bo2] ——, *Solution of some vector analysis problems connected with operators* div *and* grad, in Russian, Trudy Seminar N. L. Sobolev No.1 **80** (1980), 5-40. Akademia Nauk SSSR, Sibirskoe Otdelenie Mathematiki Nowosibirsk

[BM] W. Borchers and T. Miyakawa, *L^2 decay for the Navier-Stokes flow in halfspaces*, Math. Ann. **282** (1988), 139-155.

[BS] W. Borchers and H. Sohr, *On the equations* $\operatorname{rot} v = g$ *and* $\operatorname{div} u = f$ *with zero boundary conditions*, Hokkaido Math. J. **19** (1990), 67-87.

[FJR] E. B. Fabes, B. F. Jones and N. M. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p*, Arch. Rational Mech. Anal. **45** (1972), 222-240.

[Gi] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (1986), 186-212.

[GS] Y. Giga and H. Sohr, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, Hokkaido Univ. Preprint Series **60** (1989). (to appear in J. Funct. Anal.)

[Ho] E. Hopf, *Über die anfangswertaufgabe für die hydrodynamischen grundgleichungen*, Math. Nachr. **4** (1951), 213-231.

[Le] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193-248.

[Ma] V. G. Maz'ja, "Sobolev Spaces," English edn., Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.

[Se] J. Serrin, *On the interior regularity of weak solutions of the Navier- Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187-195.

[So] H. Sohr, *Zur regularitätstheorie der instantionären Gleichungen von Navier-Stokes*, Math. Z. **184** (1983), 359-375.

[SW] H. Sohr and W. von Wahl, *On the regularity of the pressure of weak solutions of Navier-Stokes equations*, Arch. Math. **46** (1986), 428-439.

[Sol] V. A. Solonnikov, *Estimates for solutions of nonstationary Navier-Stokes equations*, J. Soviet Math. **8** (1977), 467-529.

[St] M. Struwe, *On partial regularity results for the Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), 437-458.

[Ta] S. Takahashi, *On interior regularity criteria for weak solutions of the Navier-Stokes equations*, Manuscripta Math. **69** (1990), 237-254.

[Uk] S. Ukai, *A solution formula for the Stokes equation in* $\mathbb{R}^n_+$, Comm. Pure Appl. Math. **40** (1987), 611-621.

[Wa1] W. von Wahl, "The Equations of Navier-Stokes and Abstract Parabolic Equations," Vieweg, Braunschweig, 1985.

[Wa2] ——, *Regularitätsfragen für die instationären Navier-Stokesschen Gleichungen in höheren Dimensionen*, J. Math. Soc. Japan **32** (1980), 263-283.

[Yo] K. Yosida, "Functional Analysis," Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[Zi] W. P. Ziemer, "Weakly Differentiable Functions," Springer-Verlag, New York, 1989.