An analogy of the theorem of Hector and Duminy

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words with Γ_0 as alphabet, that is, $W(\Gamma_0) = \coprod_{n=0}^{\infty}(\Gamma_0)^n$, where $(\Gamma_0)^n$ means n-direct products of Γ_0 and $(\Gamma_0)^0$ the empty word $(\ )$. This set $W(\Gamma_0)$ is useful to treat the pseudogroup $\langle\Gamma_0\rangle$, because

PROPOSITION 2.3. ([5], Proposition 2.6) *Define the map* $\Phi : W(\Gamma_0) \to \Gamma = \langle\Gamma_0\rangle$ *by* $\Phi((\ )) = \mathrm{id}_{\mathbf{R}^q}$ *for the empty word* $(\ )$ *and* $\Phi(w) = h_m \circ \cdots \circ h_1$ *for a word* $w = (h_m, \ldots, h_1)$. *Then this map* Φ *is surjective.*

For a word $w = (h_m, \ldots, h_1) \in W(\Gamma_0)$, we put $g_w = \Phi(w) = h_m \circ \cdots \circ h_1$. Note that for the *inverse word* $w^{-1} = (h_1^{-1}, \ldots, h_m^{-1})$ of w, $g_w^{-1} = g_{w^{-1}} = \Phi(w^{-1}) = h_1^{-1} \circ \cdots \circ h_m^{-1}$.

DEFINITION 2.4. Let $x_0 \in \mathbf{R}^q$. The Γ-*orbit of* x_0 is the set $\Gamma(x_0) = \{\ g(x) \mid g \in \Gamma,\ x \in D(g)\ \}$.

For every $x_0 \in \mathbf{R}^q$, the topological type of the Γ-orbit $\Gamma(x_0) \subset \mathbf{R}^q$ is classified into the following three types (for example see [**1**]):

(1) $\Gamma(x_0)$ is discrete; in this case $\Gamma(x_0)$ is called a *proper* orbit.

(2) The closure $\overline{\Gamma(x_0)}$ of $\Gamma(x_0)$ in $\mathbf{R}^q$ has non-empty interior; in this case $\Gamma(x_0)$ is called a *locally dense* orbit.

(3) Neither (1) nor (2), that is, the closure $\overline{\Gamma(x_0)}$ is a perfect set with empty interior; in this case $\Gamma(x_0)$ is called a *exceptional* orbit.

To investigate the structure of the closure of a Γ-orbit is an important problem.

For $q = 1$ (in our situation, Γ is a pseudogroup of local affine transformations of $\mathbf{R}$), but more generally, for pseudogroups of local diffeomorphisms of class C^r $(r \geq 0)$ of $\mathbf{R}$, we can get following notion.

DEFINITION 2.5. The orbit $\Gamma(x_0)$ of $x_0 \in \mathbf{R}$ is called *semiproper* if for every $x \in \Gamma(x_0)$, there exists an open interval $J \subset R$ such that x is a boundary point of J and $J \cap \Gamma(x_0) = \emptyset$. Therefore a semiproper orbit is either proper type or exceptional type.

PROPOSITION 2.6. *Suppose* $q = 1$. *Then the orbit* $\Gamma(x_0)$ $(x_0 \in \mathbf{R})$ *is exceptional type if* *and only if for some (and thus any) $x \in \Gamma(x_0)$, there exists a compact neighborhood I_x of x in $\mathbf{R}$ such that $\overline{\Gamma(x_0)} \cap I_x$ is a Cantor set. Furthermore, if $\Gamma(x_0)$ is a semiproper orbit of exceptional type, then for every $x \in \Gamma(x_0)$, x is a semi-isolated point of a Cantor set $\overline{\Gamma(x_0)} \cap I_x$.*

For semiproper orbits, following theorems are important (compare with theorems in introduction):

THEOREM 2.7.(Sacksteder [**6**]) *Suppose that $q = 1$ and $\Gamma(x_0)$ is a nonproper semiproper orbit. Then there exists $x \in \overline{\Gamma(x_0)}$ and $g \in \Gamma$ such that $x \in D(g)$, $g(x) = x$ and g is a contraction to x, that is, the derivative $g'(x)$ at x is less than 1.*

THEOREM 2.8.(Hector [**4**], Duminy (unpublished, but see Cantwell-Conlon [**3**])) *Suppose that $q = 1$ and $\Gamma(x_0)$ is a nonproper semiproper orbit. Then there exists $g \in \Gamma$ such that $x_0 \in D(g)$, $g(x_0) = x_0$ and g is a contraction to x_0.*

To consider analogies of these theorems for $q \geq 2$, Nishimori introduced the following notion as somewhat *semiproperness* of Γ-orbits.

DEFINITION 2.9. ([**5**], Definition 3.2) Let $x_0 \in \mathbf{R}^q$. We say that the Γ-orbit $\Gamma(x_0)$ of x_0 is *with bubbles* if for each $x \in \Gamma(x_0)$, there exists a non-empty, bounded, convex open subset B_x (called a *bubble at* x) of $\mathbf{R}^q$ satisfying the following three properties:

(a) $x \in \partial B_x$, where ∂B_x denotes the boundary of B_x.

(b) If $x_1, x_2 \in \Gamma(x_0)$ and $x_1 \neq x_2$, then $B_{x_1} \cap B_{x_2} = \emptyset$.

(c) If $h \in \Gamma_0$ and $x \in D(h) \cap \Gamma(x_0)$ satisfying $h(x) \neq x$, then $\overline{h}(B_x) = B_{h(x)}$, where $\overline{h}$ is the extension of h.

EXAMPLE. Let D^q be the unit disk in $\mathbf{R}^q$, $x_0 \in \partial D^q = S^{q-1}$ and $D_1, \ldots, D_n$ mutually disjoint disks contained in D^q and $\partial D_1 \ni x_0$. Let h_i $(i = 1, \ldots, n)$ be a unique similarity transformation which maps the unit disk D^q to the disk D_i and after suitable restriction of the domain of h_i to a bounded, convex open neighborhood of D^q, the ranges of h_i are mutually disjoint. (Clearly each h_i is a contraction.) And for special choice, $h_1(x_0) = x_0$. Now we obtain a pseudogroup $\Gamma = \langle \Gamma_0 \rangle \subset \Gamma^{\text{sim}}_{q,+}$, where $\Gamma_0 = \{h_1, \dots, h_n, h_1^{-1}, \dots, h_n^{-1}\}$. Then the Γ-orbit $\Gamma(x_0)$ is with bubbles and the closure $\overline{\Gamma(x_0)}$ is a Cantor set in $\mathbf{R}^q$. Furthermore h_1 is a contraction to $x_0 \in \Gamma(x_0)$. This construction is closely related to that of exceptional minimal sets of Markov type for $q = 1$ (see Cantwell-Conlon [**2**]).

Hereafter, we consider the following situation.

Let $\Gamma_0 \subset \Gamma^{\text{sim},*}_{q,+}$ be a finite, symmetric subset, $\Gamma = \langle \Gamma_0 \rangle$ and $x_0 \in \mathbf{R}^q$ satisfying the following two properties:

(S1) There exists a constant $\varepsilon > 0$ such that the distance $\text{dist}(\Gamma(x_0), \bigcup_{h \in \Gamma_0} \partial D(h))$ is greater than ε.

(S2) The Γ-orbit $\Gamma(x_0)$ of x_0 is nonproper and with bubbles $\{B_x\}_{x \in \Gamma(x_0)}$.

Here, an orbit $\Gamma(x_0)$ is *nonproper* if for every $x \in \Gamma(x_0)$, the closure $\overline{\Gamma(x_0) \setminus \{x\}}$ of $\Gamma(x_0) \setminus \{x\}$ contains x.

Remark that if $x \in \Gamma(x_0) \cap D(h)$ for some $h \in \Gamma_0$, then by (S1), $U(x; \varepsilon) \subset D(h)$, where $U(x; \varepsilon)$ denotes the ε-neighborhood of x.

Then an analogy of Sacksteder's theorem is as following.

THEOREM 2.10. (Nishimori [**5**], Theorem 3.3) *Let Γ be the pseudogroup generated by a finite, symmetric subset Γ_0 of $\Gamma^{\text{sim},*}_{q,+}$ and $x_0 \in \mathbf{R}^q$ satisfying the assumptions (S1) and (S2). Then there exist $g \in \Gamma$ and $z \in \overline{\Gamma(x_0)}$ such that $z \in D(g)$, $g(z) = z$ and g is a contraction, that is, the similitude ratio of g is less than 1.*

We prove, in the rest of this paper, the following result which is a weak version of an analogy of the theorem of Hector-Duminy.

THEOREM 2.11. *Let Γ be the pseudogroup generated by a finite, symmetric subset Γ_0 of $\Gamma^{\text{sim},*}_{q,+}$ and $x_0 \in \mathbf{R}^q$ satisfying the assumptions (S1) and (S2). Then there exists $g \in \Gamma$ such that $x_0 \in D(g)$, $g(x_0) = x_0$ and g is not the identity of $D(g)$.*

REMARK. Therefore, such g is possibly a rotation at x_0. We do not know whether there exists an example that all elements of Γ which fixes x_0 are rotation at x_0.

3. The proof of Theorem 2.11

Let Γ be the pseudogroup generated by a finite, symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and $x_0 \in \mathbf{R}^q$ satisfying the assumptions (S1) and (S2). Let $\{B_x\}_{x \in \Gamma(x_0)}$ be bubbles of $\Gamma(x_0)$.

At first, we prepare some notions which play an important role in the proof of theorem 2.11.

DEFINITION 3.1. (1) For a word $w \in W(\Gamma_0)$, $|w|$ denotes the word length of w, that is, $|w| = 0$ for the empty word $w = (\quad)$ and $|w| = m$ for $w = (h_m, \ldots, h_1)$.
(2) For $x, y \in \mathbf{R}^q$ with $y \in \Gamma(x)$, put

$$d_{\Gamma_0}(x, y) = \min\{\ |w| \mid w \in W(\Gamma_0),\ x \in D(g_w) \text{ and } \ g_w(x) = y\ \}.$$

Then d_{Γ_0} is a natural distance on the orbit $\Gamma(x)$.

DEFINITION 3.2. Let $x, y \in \mathbf{R}^q$. A word $w \in W(\Gamma_0)$ is called a *short-cut at x to y* if $x \in D(g_w)$, $g_w(x) = y$ and $|w| = d_{\Gamma_0}(x, y)$.

Remark that if $w = (h_m, \ldots, h_1) \in W(\Gamma_0)$ is a short-cut at x to y, then the inverse word $w^{-1} = (h_1^{-1}, \ldots, h_m^{-1})$ of w is a short-cut at y to x and for every $k = 1, \ldots, m-1$, the word $w_k = (h_k, \ldots, h_1)$ is a short-cut at x to $g_{w_k}(x) = h_k \circ \cdots \circ h_1(x)$.

Following three lemmas are fundamental and for the proofs, see Nishimori [5].

LEMMA 3.3. ([5], Lemma 4.3) *Let $x \in \Gamma(x_0)$ and $w = (h_m, \ldots, h_1) \in W(\Gamma_0)$ be a short-cut at x. Then $\overline{g}_w(B_x) = B_{g_w(x)}$, where $g_w = h_m \circ \cdots \circ h_1$ and $\overline{g}_w$ is the extension of g_w (in the sense of section 2). Therefore the similitude ratio of g_w is the ratio of the diameters of bubbles,* $\mathrm{diam}(B_{g_w(x)})/\mathrm{diam}(B_x)$. *In particular, if $D(g_w) \supset U(x; r)$, then*

$$g_w(U(x; r)) = U\left(g_w(x); r \cdot \frac{\mathrm{diam}(B_{g_w(x)})}{\mathrm{diam}(B_x)}\right).$$

LEMMA 3.4. ([5], Lemma 4.4, 4.5) (1) *The union* $\cup_{x\in\Gamma(x_0)} B_x$ *of bubbles is a bounded subset of* $\mathbf{R}^q$.

(2) *Total volume* $\sum_{x\in\Gamma(x_0)} \mathrm{vol}(B_x)$ *of bubbles is bounded. So* $\sum_{x\in\Gamma(x_0)}(\mathrm{diam}(B_x))^q$ *is also bounded.*

LEMMA 3.5. (The short-cut theorem. [5], Lemma 4.7) *Let* $w \in W(\Gamma_0)$ *be a short-cut at* x_0. *Then*

$$U\left(x_0; \varepsilon \cdot \frac{\mathrm{diam}(B_{x_0})}{\delta}\right) \subset D(g_w),$$

where $\delta = \sup\{\ \mathrm{diam}(B_y) \mid y \in \Gamma(x_0)\ \}$.

For the proof of our theorem, following argument is essentially due to Hector [4, Théorème CIII 1] in the case of $q = 1$.

Put $\Delta = \{\ y \in \Gamma(x_0) \mid \mathrm{diam}(B_y) \geqq \mathrm{diam}(B_{x_0})\ \}$, then by lemma 3.4, it is a non-empty, finite subset of $\Gamma(x_0)$ which contains x_0. Since the pseudogroup Γ is finitely generated and Δ is finite, so there exists a non-negative integer $N = \sup\{\ d_{\Gamma_0}(x, y) \mid x, y \in \Delta\ \}$.

LEMMA 3.6. *There exists* $\varepsilon' > 0$ *such that*

(1) $\varepsilon/3 \geqq \varepsilon' > 0$,

(2) $d_{\Gamma_0}(x_0, z) > N$ *for each* $z \in U(x_0; \varepsilon' \cdot \mathrm{diam}(B_{x_0})/\delta)$ *with* $z \in \Gamma(x_0) \setminus \{x_0\}$. *Therefore* $z \notin \Delta$.

PROOF. Since Γ is finitely generated, the set $\{\ y \in \Gamma(x_0) \mid d_{\Gamma_0}(x_0, y) \leqq N\ \}$ is finite. By assumption, the orbit $\Gamma(x_0)$ is nonproper, so we can take $\varepsilon' > 0$ satisfying (1) and (2). □

Hereafter we assume that

$$(\sharp) \qquad \text{for each } g \in \Gamma \text{ which fixes } x_0,\ g \text{ is the identity on } D(g)$$

and deduce a contradiction.

Lemma 3.7. *Let $\varepsilon' > 0$ be a constant as in lemma 3.6 and $z \in U\left(x_0; \varepsilon' \cdot \mathrm{diam}(B_{x_0})/\delta\right)$ with $z \in \Gamma(x_0) \setminus \{x_0\}$. Let $w \in W(\Gamma_0)$ be a short-cut at x_0 to z. Then $x_0 \in D(g_w^{-1})$ and w^{-1} is a short-cut at x_0 to $g_w^{-1}(x_0)$.*

Proof. Note that the word length $|w| = d_{\Gamma_0}(z, x_0) > N$. By assumption, $w^{-1} \in W(\Gamma_0)$ is a short-cut at z to x_0.

We write $w^{-1} = (h_m, \ldots, h_1)$ ($m \geqq 1$, $h_i \in \Gamma_0$), and put $w_k^{-1} = (h_k, \ldots, h_1)$ and $g_k = g_{w_k^{-1}} = g_{w_k}^{-1} = h_k \circ \cdots \circ h_1$ for $k = 1, 2, \ldots, m$. And, for convention, $w_0^{-1} = (\ )$ (the empty word) and $g_0 = g_{w_0^{-1}} = \mathrm{id}_{\mathbf{R}^q}$. Then w_k^{-1} is a short-cut at z to $g_k(z)$ for $k = 0, 1, \ldots, m$.

We prove the following assertions by induction on $k = 0, 1, \ldots, m$:

$$(\mathrm{A})_k: \qquad U\left(x_0; \varepsilon' \cdot \frac{\mathrm{diam}(B_{x_0})}{\delta}\right) \subset D(g_k).$$

$(\mathrm{B})_k$: The word w_k^{-1} is a short-cut at x_0 to $g_k(x_0)$.

For $k = 0$, all assertions are trivial.

Assume that the assertions $(\mathrm{A})_k$ and $(\mathrm{B})_k$ hold true for $k \geqq 0$. By the special choice of $z \in U\left(x_0; \varepsilon' \cdot \mathrm{diam}(B_{x_0})/\delta\right)$ and $(\mathrm{A})_k$,

$$\begin{aligned} g_k(z) &\in g_k\left(U\left(x_0; \varepsilon' \cdot \frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\right) \\ &= U\left(g_k(x_0); \varepsilon' \cdot \left(\frac{\mathrm{diam}(B_{x_0})}{\delta}\right) \cdot \left(\frac{\mathrm{diam}(B_{g_k(x_0)})}{\mathrm{diam}(B_{x_0})}\right)\right) \\ &= U\left(g_k(x_0); \varepsilon' \cdot \frac{\mathrm{diam}(B_{g_k(x_0)})}{\delta}\right) \\ &\subset U\left(g_k(x_0); \varepsilon'\right). \end{aligned}$$

Since $g_k(z) \in D(h_{k+1}) \cap \Gamma(x_0)$, $U\left(g_k(z); \varepsilon\right) \subset D(h_{k+1})$ by (S1). Therefore

$$\begin{aligned} g_k\left(U\left(x_0; \varepsilon' \cdot \frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\right) &\subset U\left(g_k(x_0); \varepsilon'\right) \\ &\subset U\left(g_k(z); \varepsilon\right) \\ &\subset D(h_{k+1}) \end{aligned}$$

Then $U(x_0;\varepsilon'\cdot\mathrm{diam}(B_{x_0})/\delta)\subset D(h_{k+1}\circ g_k)=D(g_{k+1})$. This establishes the assertion $(\mathrm{A})_{k+1}$.

For next, we take a short-cut $\xi\in W(\Gamma_0)$ at x_0 to $g_{k+1}(x_0)$. Then $g_\xi^{-1}\circ g_{k+1}(x_0)=x_0$, so $g_\xi=g_{k+1}$ on $D(g_\xi)\cap D(g_{k+1})$ by assumption ($\sharp$).

Since w_{k+1}^{-1} is a short-cut at z, then $z\in D(g_{k+1})$ and by lemma 3.5 and the choice of ε', $z\in U(x_0;\varepsilon'\cdot\mathrm{diam}(B_{x_0})/\delta)\subset D(g_\xi)$. Therefore $z\in D(g_\xi)\cap D(g_{k+1})$.

By the definition of a short-cut,

$$|w_{k+1}^{-1}|=d_{\Gamma_0}(z,g_{k+1}(z))\leqq|\xi|=d_{\Gamma_0}(x_0,g_{k+1}(x_0))\leqq|w_{k+1}^{-1}|,$$

so $|w_{k+1}^{-1}|=d_{\Gamma_0}(x_0,g_{k+1}(x_0))$, that is, w_{k+1}^{-1} is a short-cut at x_0 to $g_{k+1}(x_0)$. This establishes the assertion $(\mathrm{B})_{k+1}$.

Now consider $k=m$, this completes the proof. □

Remark that $g_w^{-1}(x_0)\notin\Delta$. This is because $d_{\Gamma_0}(x_0,g_w^{-1}(x_0))=|w^{-1}|=d_{\Gamma_0}(x_0,z)>N$.

By lemma 3.7, the word w^{-1} is a short-cut at $z\notin\Delta$ to $x_0\in\Delta$, furthermore, that at $x_0\in\Delta$ to $g_w^{-1}(x_0)\notin\Delta$. Then, by lemma 3.3, the similitude ratio of g_w^{-1} is

$$\frac{\mathrm{diam}(B_{x_0})}{\mathrm{diam}(B_z)}=\frac{\mathrm{diam}(B_{g_w^{-1}(x_0)})}{\mathrm{diam}(B_{x_0})}.$$

But the definition of the set Δ yields

$$1<\frac{\mathrm{diam}(B_{x_0})}{\mathrm{diam}(B_z)}=\frac{\mathrm{diam}(B_{g_w^{-1}(x_0)})}{\mathrm{diam}(B_{x_0})}<1,$$

a contradiction. This completes the proof of theorem 2.11.

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