Comparison Principle and Convexity Preserving Properties for Singular Degenerate Parabolic Equations on Unbounded Domains

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Comparison Principle and Convexity Preserving Properties for Singular Degenerate Parabolic Equations on Unbounded Domains

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ABSTRACT: We prove comparison theorems for viscosity solutions of singular degenerate parabolic equations of general form in a domain not necessarily bounded.We also prove that the concavity of solutions is preserved as time develops under additional assumptions on the equations. Both results apply to various equations including the mean curvature flow equation where every level set of solutions moves by its mean curvature.

§1. **Introduction.** This paper is concerned with viscosity solutions of (possibly) degenerate parabolic equations of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega \tag{1.1}$$

or more general equations

$$u_t + F(t, x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega, \tag{1.2}$$

where Ω is a domain in $\mathbf{R}^n$ and $T > 0$. The equations are allowed to be singular in the sense that F has a singularity at $\nabla u = 0$. The unknown u will always be a real valued function on Q; $\partial_t u$, ∇u and $\nabla^2 u$ denote respectively the time derivative of u, the gradient of u and the Hessian of u in space variables. As explained in [1,6] a certain class of singular degenerate parabolic equations is important to study motion of a hypersurface whose speed locally depends on the normal and its derivatives. A typical example is

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \tag{1.3}$$

which says the level surface of u moves by its mean curvature provided that $\nabla u \neq 0$ on the surface. In [1] Y.-G. Chen and the first two authors introduced a weak notion of motion of a hypersurface and constructed a unique global evolution family moved by the given law including motion by mean curvature; see also [6]. Almost at the same time Evans and Spruck [4] obtained a similar result but only for motion by mean curvature. One of fundamental ingredients of both researches is a comparison principle of viscosity solutions for (1.1) or (1.3). However, they essentially consider the problem on a bounded domain Ω which excludes the possibility that initial hypersurface of the motion is noncompact.

In this paper we first establish a comparison principle for (1.1) and (1.2) on a domain Ω possibly unbounded. If the equation is regular and degenerate elliptic, Jensen [12] and the third author [10] showed that viscosity solutions enjoy a comparison principle; see also [19, 14, 13]. In[10,14] unbounded domain is also treated. For bounded domain Ω results in [10] were extended to singular degenerate parabolic equations (1.1) by [1, §4]. The crucial step in [10] for a comparison principle is a lemma on relation of 'second derivatives' of semicontinuous functions. It now becomes very explicit and clear in a recent paper of Crandall and the third author [3] through the study of [11, 2]. It turns out that their result is applicable to our problems although our equations (1.1), (1.2) are singular and Ω is not necessarily bounded. Our proof simplifies and clarifies that in [1] for a bounded domain. A corresponding results for singular degenerate elliptic equations are proved by the last author [20].

As another application of results in [3] we study a concavity of solution u of (1.1) when $\Omega = \mathbf{R}^n$. We will show that if the initial value u_0 of u is concave so is $u(t, \cdot)$ for all $t > 0$ provided that $F = F(p, X)$ is convex in X. In other words the concavity is preserved as time develops. Here we assume that u grows at most linearly near space infinity and that u_0 is globally Lipschitz. A similar technique is found in [11] where they prove the semi-concavity of solutions of Bellman equations. The technique goes back to the third author [9] where Hamilton-Jacobi equations are treated. Concavity of solutions is often studied for elliptic equations in a bounded convex domain [16, 17]. Their fundamental ingredient is the maximum principle for the concavity function. There are applications for parabolic equations in [16, 15, 18]. However, it seems that their result does not include ours because their equations are quasilinear (not fully nonlinear) and they do not consider the problem in $\mathbf{R}^n$.

If the equation (1.1) is linear, the concavity preserving is trivial although it is not written explicitly in the literature. Indeed we observe that any directional second derivative $\partial_\ell^2 u$ of the solution u solves the same linear equation (1.1). By the maximum principle $\partial_\ell^2 u$ is nonnegative for all time provided that it is initially nonnegative, so the concavity is preserved. We are grateful to Professor B. Kawohl who pointed out this fact as well as related references.

In [8] Huisken proved that the surface moved by its mean curvature remains convex if it is initially convex; strictly convexity is also proved (see also [5] for motion of curves). In [4] Evans and Spruck give another proof for the convexity preserving based on results of [16, 17]. We remark that our results, with some work, will extend the convexity preserving property to general motion of a hypersurface. The statement and details will be included in the second author's forthcoming paper [7].

We will prove our comparison principle for (1.1) in §2 and for (1.2) in §4. In §3 we will discuss the concavity preserving property. An application to motion of a hypersurface will be discussed in the second author's forthcoming paper [7].

§2. Comparison principle.

This section establishes a comparison principle for second-order degenerate parabolic equations with possible singularities on a domain not necessarily bounded. Our results essentially extend those in [1, §4] where the domain is assumed to be bounded. Moreover, our proof here simplifies their argument by using a recent result of Crandall and Ishii [3]. The equations we consider are fully nonlinear. To clarify the main idea of the proof we assume here that the equation does not depend on space and time variables explicitly. In §4 we will study more general equations depending space and time variables. As an application we will study the Lipschitz continuity of solutions.

Let Ω be a domain in $\mathbf{R}^n$ not necessarily bounded and let T be a positive number. We consider a degenerate parabolic equation of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega. \tag{2.1}$$

We first list assumptions on $F = F(p, X)$.

(F1) $F : (\mathbf{R}^n \setminus \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous, where $\mathbf{S}^n$ denotes the space of real $n \times n$ symmetric matrices.

(F2) F is degenerate elliptic, i.e., $F(p, X + Y) \leq F(p, X)$ for all $Y \geq 0$.

(F3) $-\infty < F_*(0, O) = F^*(0, O) < \infty$ where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $\mathbf{R}^n \times \mathbf{S}^n$, respectively, i.e.,

$$F_*(p, X) = \lim_{\varepsilon \downarrow 0} \inf\{F(q, Y); q \neq 0, |p - q| \leq \varepsilon, |X - Y| \leq \varepsilon\}$$

and $F^* = -(-F)_*$. Here $|X|$ denotes the operator norm of X as a self adjoint operator on $\mathbf{R}^n$.

(F4) For every $R > 0$

$$c_R = \sup\{|F(p, X)|; |p| \leq R, |X| \leq R, p \neq 0\} \quad \text{is finite.}$$

The assumption (F1) allows the possibility that (2.1) is singular at $\nabla u = 0$. The equation (2.1) is called degenerate parabolic if (F2) holds.

We next recall one of equivalent definitions of viscosity sub-and supersolutions of (2.1) (cf. [19]). A function $u : Q \to \mathbf{R}$ is called a *viscosity sub-(super) solution* of (2.1) in Q if $u^* < \infty$ (resp. $u_* > -\infty$) on $\bar{Q}$ and

$$\tau + F_*(p, X) \leq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}_Q^{2,+} u^*(t, x), (t, x) \in Q$$
$$(\text{resp.} \quad \tau + F^*(p, X) \geq 0 \quad \text{for all} \quad (\tau, p, X) \in \mathcal{P}_Q^{2,-} u_*(t, x), (t, x) \in Q).$$

Here $\mathcal{P}_Q^{2,+}$ denotes the *parabolic super 2-jet* in Q, i.e., $\mathcal{P}_Q^{2,+} u(t, x)$ is the set of $(\tau, p, X) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$ such that

$$\begin{aligned} u(s, y) \leq & u(t, x) + \tau(s - t) + \langle p, y - x \rangle + \frac{1}{2}\langle X(y - x), y - x \rangle \\ & + o(|s - t| + |y - x|^2) \quad \text{as} \quad (s, y) \to (t, x) \quad \text{in} \quad Q \end{aligned}$$

where $\langle , \rangle$ denotes the Euclidean inner product; similarly, $\mathcal{P}_Q^{2,-} u = -\mathcal{P}_Q^{2,+}(-u)$. In this paper we call a continuous function $m : [0, \infty) \to [0, \infty)$ a *modulus* if $m(0) = 0$ and it is nondecreasing. For $U = (0,T] \times D$, the set

$$\partial_p U = \{0\} \times D \cup [0,T] \times \partial D$$

is often called the *parabolic boundary* of U. We are now in position to state our main comparison theorem.

Theorem 2.1. *Suppose that F satisfies* (F1)-(F4). *Let u and v be, respectively, sub-and supersolutions of* (2.1) *in Q. Assume that*

(A1) *$u(t,x) \le K(|x|+1)$, $v(t,x) \ge -K(|x|+1)$ for some $K > 0$ independent of $(t,x) \in Q$;*

(A2) *there is a modulus m_T such that*

$$u^*(t,x) - v_*(t,y) \le m_T(|x-y|) \quad \text{for all} \quad (t,x,y) \in \partial_p U,$$

where $U = (0,T] \times D$ and $D = \Omega \times \Omega$;

(A3) *$u^*(t,x) - v_*(t,y) \le K(|x-y|+1)$ on $\partial_p U$ for some $K > 0$ independent of $(t,x,y) \in \partial_p U$.*

Then there is a modulus m such that

(2.2) $$u^*(t,x) - v_*(t,y) \le m(|x-y|) \quad \text{on} \quad U.$$

In particular $u^* \le v_*$ on Q.

Remark 2.2. When Ω is bounded, Theorem 2.1 has been proved in [1, §4] without assuming (F4). Note that (A2) is equivalent to $u^* \le v_*$ on $\partial_p Q$ and that (A1) and (A3) are unnecessary if Ω is bounded, since we may assume that u and v are bounded (cf. [10,1]). Although our proof given below uses (F4), one can circumvent (F4) when Ω is bounded; see Remark 2.9.

We will prove Theorem 2.1 in several steps. We begin by deriving a rough growth estimate for $u(t,x) - v(t,y)$ on U.

Proposition 2.3. *Suppose that F satisfies* (F1) *and* (F4). *Let u and v be, respectively, viscosity sub-and supersolutions of* (2.1) *in Q. Assume that u and v satisfy* (A1) *and* (A3) *and that* u *and* $-v$ *are upper semicontinuous in* Q. *Then for* $K' > K$ *there is a constant* $M = M(K', F) > 0$ *such that*

$$u(t,x) - v(t,y) \le K'|x-y| + M(1+t) \quad \text{on} \quad U. \tag{2.3}$$

Proof. We set

$$\begin{aligned} w(s,t,x,y) &= u(t,x) - v(s,y) \\ \psi(t,x,y) &= K'(|x-y|^2+1)^{1/2} + M(1+t). \end{aligned}$$

We will prove

$$w(t,t,x,y) \le \psi(t,x,y) \quad \text{for} \quad (t,x,y) \in U \tag{2.4}$$

by choosing M large. Let $\{g_R\}_{R>0}$ be a family of C^2 functions satisfying

(2.5a) $\qquad g_R(x) = 0 \quad \text{for} \quad |x| < R$

(2.5b) $\qquad g_R(x)/|x| \to 1 \quad \text{as} \quad |x| \to \infty$

(2.5c) $\qquad G = \sup\{|\nabla g_R(x)| + |\nabla^2 g_R(x)|; \quad x \in \mathbf{R}^n, R > 0\}$ is finite.

Using this barrier g_R, we set $\phi = \psi + 2K' g_R$. By (A1) and (2.5b) we observe that for sufficiently large R_1 it holds

$$w(s,t,x,y) - \phi(t,x,y) < 0 \quad \text{if} \quad |x|^2 + |y|^2 \ge R_1^2 \quad \text{and} \quad 0 \le t, s \le T. \tag{2.6}$$

By (A3) if $M > K$, we see

$$w(t,t,x,y) - \phi(t,x,y) < 0 \quad \text{for} \quad (t,x,y) \in \partial_p U. \tag{2.7}$$

Since w is upper semicontinuous, (2.6) and (2.7) yield

$$w(s,t,x,y) - \frac{(t-s)^2}{\delta} - \phi(t,x,y) < 0 \quad \text{for} \quad (s,x,y) \in \partial_p U \quad \text{or} \quad (t,x,y) \in \partial_p U \tag{2.8}$$

with sufficiently small δ (independent of t, s, x, y). Suppose that (2.4) were false. Then from (2.5a) it would follow that

$$\sup_{\bar{V}}(w - \Psi) > 0 \tag{2.9}$$

with $\Psi = (t-s)^2/\delta + \phi$ and $V = (0,T] \times U$ if R is sufficiently large. By (2.6)-(2.9) we now observe that $w - \Psi$ attains a maximum over $\bar{V}$ at a point $(\hat{s}, \hat{t}, \hat{x}, \hat{y}) \in V$. This implies that

$$(\partial_t \Psi, \nabla_x \Psi, \nabla_x^2 \Psi)(\hat{s}, \hat{t}, \hat{x}, \hat{y}) \in \mathcal{P}_Q^{2,+} u(\hat{t}, \hat{x})$$
$$(-\partial_s \Psi, -\nabla_y \Psi, -\nabla_y^2 \Psi)(\hat{s}, \hat{t}, \hat{x}, \hat{y}) \in \mathcal{P}_Q^{2,-} v(\hat{s}, \hat{y}),$$

where ∇_x denotes spatial derivatives in x variables. Since u and v are, respectively, viscosity, sub-and supersolutions of (2.1), we see

$$\partial_t \Psi + F_*(\nabla_x \phi, \nabla_x^2 \phi) \le 0, \tag{2.10a}$$

$$-\partial_s \Psi + F^*(-\nabla_y \phi, -\nabla_y^2 \phi) \ge 0 \quad \text{at} \quad (\hat{s}, \hat{t}, \hat{x}, \hat{y}). \tag{2.10b}$$

By (2.5c) and definition of ψ it holds

$$|\nabla \phi|, \quad |\nabla^2 \phi| \le N, \quad \nabla = (\nabla_x, \nabla_y)$$

with $N = N(K', G)$. Subtracting (2.10b) from (2.10a) and noting (F4) yield

$$\partial_t \Psi + \partial_s \Psi \le 2c_N.$$

Since $\partial_t (t-s)^2 = -\partial_s (t-s)^2$, this implies $M \le 2c_N$. If M is taken larger than $2c_N$ and K, we have a contradiction. We thus prove (2.4) for

$$M > \max(2c_N, K).$$

The estimate (2.3), with M replaced by $M + K'$, follows from (2.4). ∎

The basic idea of the proof of Theorem 2.1 is similar to that of Proposition 2.3. Roughly speaking we will find a parabolic super 2-jet of

$$w(t, x, y) = u(t, x) - v(t, y)$$

at a point where (2.2) may not hold. This time, in order to obtain a contradiction, we should find a nice element of the closure $\bar{\mathcal{P}}^{2,+}u$, $\bar{\mathcal{P}}^{2,-}v$ so that "Hessian matrix" of both u and v is comparable. Here Ishii's idea [10] plays an important role. We will use its sophisticated version due to Crandall and Ishii [3].

For ε, δ, $\gamma > 0$ we set

$$\begin{aligned}\Phi(t,x,y) &= w(t,x,y) - \Psi(t,x,y), \quad w(t,x,y) = u(t,x) - v(t,y), \\ \Psi(t,x,y) &= \frac{|x-y|^4}{4\varepsilon} + B(t,x,y), \quad B(t,x,y) = \delta(|x|^2 + |y|^2) + \frac{\gamma}{T-t}.\end{aligned} \tag{2.11}$$

The function B plays the role of a barrier for space infinity and $t = T$.

Proposition 2.4. *Suppose that u and v satisfy* (2.3) *and that*

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, (t,x,y) \in \bar{U}\} > 0. \tag{2.12}$$

Then there are positive constants δ_0 and γ_0 such that

$$\sup_{\bar{U}} \Phi(t,x,y) > \frac{\alpha}{2} \tag{2.13}$$

holds for all $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$, $\varepsilon > 0$.

Proof. By (2.3) we see $\alpha < \infty$. By (2.12) there is a point (t_0, x_0, y_0) such that $w(t_0,x_0,y_0) > 3\alpha/4$ and $|x_0 - y_0|^4/4\varepsilon < \alpha/4$. We now observe that $\Phi(t_0,x_0,y_0) > \alpha/2$ if δ and γ is sufficiently small. ∎

Proposition 2.5. *Let u, v, δ_0, γ_0 be as in Proposition 2.4. Suppose that w is upper semicontinuous in $\bar{U}$.*

(i) *Φ attains a maximum over $\bar{U}$ at $(\hat{t}, \hat{x}, \hat{y}) \in \bar{U}$ with $\hat{t} < T$.*

(ii) *$|\hat{x} - \hat{y}|$ is bounded as a function of $0 < \varepsilon < 1$, $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$.*

(iii) *$\delta\hat{x}$ and $\delta\hat{y}$ tend to zero as $\delta \to 0$; the convergence is uniform in $0 < \varepsilon < 1$ and $0 < \gamma < \gamma_0$. In particular, for fixed $\delta > 0$, $\hat{x}$ and $\hat{y}$ are bounded on $0 < \varepsilon < 1$, $0 < \gamma < \gamma_0$.*

(iv) *$|\hat{x} - \hat{y}|$ tends to zero as $\varepsilon \to 0$; the convergence is uniform in $0 < \delta < \delta_0$ and $0 < \gamma < \gamma_0$.*

Proof.

(i) By (2.3) and the definition of B we see Φ is negative outside a compact set W in $[0,T) \times \bar{D}$, $D = \Omega \times \Omega$. Since Φ is upper semicontinuous and $\sup \Phi > 0$ by (2.13), Φ takes a maximum over $\bar{U}$ at a point of W.

(ii) From (2.13) it follows $\Phi(\hat{t}, \hat{x}, \hat{y}) > 0$ for $0 < \delta < \delta$, $0 < \gamma < \gamma_0$, $\varepsilon > 0$. This yields

$$w(\hat{t}, \hat{x}, \hat{y}) \geq \frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} + B(\hat{t}, \hat{x}, \hat{y}) \geq \frac{|\hat{x} - \hat{y}|^4}{4\varepsilon}. \tag{2.14}$$

Applying (2.3) we observe

$$\frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} \leq K'|\hat{x} - \hat{y}| + M(1 + T) \tag{2.15}$$

which yields (ii).

(iii) From (2.14) it follows

$$\delta(|\hat{x}|^2 + |\hat{y}|^2) \leq u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}).$$

Let c be a bound on $|\hat{x} - \hat{y}|$ given in (ii). Applying (2.3) yields

$$\delta(|\hat{x}|^2 + |\hat{y}|^2) \leq C = K'c + M(1 + T)$$

or

$$(\delta|\hat{x}|)^2 + (\delta|\hat{y}|)^2 \leq \delta C.$$

Since C is independent of $0 < \varepsilon < 1$, $0 < \delta < \delta_0$, $0 < \gamma < \gamma_0$, letting $\delta \to 0$ completes the proof of (iii).

(iv) Similarly, from (2.15) it follows

$$\frac{|\hat{x} - \hat{y}|^4}{4\varepsilon} \leq C,$$

which yields (iv) as $\varepsilon \to 0$. ∎

Proposition 2.6. *Assume the hypotheses of Proposition 2.5. Suppose that* (A2) *holds for u and v. Then there is $\varepsilon_0 > 0$ such that Φ attains a maximum over $\bar{U}$ at an* *interior point* $(\hat{t}, \hat{x}, \hat{y})$ *of* U*, i.e,.* $(\hat{t}, \hat{x}, \hat{y}) \in (0,T) \times \Omega \times \Omega$ *for all* $0 < \varepsilon < \varepsilon_0$*,* $0 < \delta < \delta_0$ *and* $0 < \gamma < \gamma_0$*.*

Proof. Suppose that the conclusion were false. Since $\hat{t} < T$ by Proposition 2.5, there would exist sequences $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, $\{\delta_j\} \subset (0, \delta_0)$ and $\{\gamma_j\} \subset (0, \gamma_0)$ such that $\partial_p U$ contains a maximizer $(\hat{t}_j, \hat{x}_j, \hat{y}_j)$ of Φ for the value $\varepsilon = \varepsilon_j$, $\delta = \delta_j$, $\gamma = \gamma_j$. By (2.13) and (A2) we see

$$\frac{\alpha}{2} \leq \Phi(\hat{t}_j, \hat{x}_j, \hat{y}_j) \leq w(\hat{t}_j, \hat{x}_j, \hat{y}_j) \leq m_T(|\hat{x}_j - \hat{y}_j|).$$

Since $\varepsilon_j \to 0$, applying Proposition 2.5 (iv) yields $|\hat{x}_j - \hat{y}_j| \to 0$ which leads a contradiction $0 < \alpha/2 \leq 0$. ∎

Lemma 2.7 ([3]). *Let* u_i *be an upper semicontinuous function with* $u_i < \infty$ *in* $(0,T) \times \mathbf{R}^{N_i}$ *for* $i = 1, 2, \cdots, k$*. Let* w *be a function in* $(0,T) \times \mathbf{R}^N$ *given by*

$$w(t,x) = u_1(t, x_1) + \cdots + u_k(t, x_k) \quad \text{for} \quad x = (x_1, \cdots, x_k) \in \mathbf{R}^N,$$

where $N = N_1 + \cdots + N_k$*. For* $s \in (0,T)$*,* $z \in \mathbf{R}^N$ *suppose that*

$$(\tau, p, A) \in \mathcal{P}^{2,+} w(s,z) \subset \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N.$$

Assume that there is an $\omega > 0$ *such that for every* $M > 0$

$$\begin{aligned} &\sigma_i \leq C \quad \textit{whenever} \quad (\sigma_i, q_i, Y_i) \in \mathcal{P}^{2,+} u(t, x_i), \\ &|x_i - z_i| + |s - t| < \omega \quad \textit{and} \quad |u_i(t, x_i)| + |q_i| + |Y_i| \leq M \quad (i = 1, \cdots, k), \end{aligned} \tag{2.16}$$

with some $C = C(M)$*. Then for each* $\lambda > 0$ *there exists* $(\tau_i, X_i) \in \mathbf{R} \times \mathbf{S}^{N_i}$ *such that*

$$(\tau_i, p_i, X_i) \in \bar{\mathcal{P}}^{2,+} u_i(s, z_i) \quad \textit{for} \quad i = 1, \cdots k$$

and

$$-\left(\frac{1}{\lambda} + |A|\right) I \leq \begin{pmatrix} X_1 & \cdots & O \\ \vdots & & \vdots \\ O & \cdots & X_k \end{pmatrix} \leq A + \lambda A^2 \quad \textit{and} \quad \tau_1 + \cdots + \tau_k = \tau,$$

where I denotes the identity matrix and $p = (p_1, \cdots, p_k)$.

Remark 2.8. This lemma is Theorem 6 in [3]. Here and hereafter the subscript of $\mathcal{P}^{2,+}$ is suppressed. The bar over $\mathcal{P}^{2,+}$ means the closure. Although the domain considered here is $\mathbf{R}^{N_i}$, it is easily seen that the result is local and that $\mathbf{R}^{N_i}$ may be replaced by a neighborhood of $z_i \in \mathbf{R}^{N_i}$.

Proof of Theorem 2.1. We may assume that u and v are, respectively, upper and lower semicontinuous so that

$$w(t,x,y) = u(t,x) - v(t,y)$$

is upper semicontinuous in $\bar{U}$. Suppose that (2.2) were false. Then we would have (2.12), i.e.,

$$\alpha = \lim_{\theta \downarrow 0} \sup\{w(t,x,y); |x-y| < \theta, \quad (t,x,y) \in \bar{U}\} > 0.$$

By Proposition 2.3 and (2.12) we see all conclusions in Propositions 2.4-2.6 would hold for Φ defined in (2.11). Proposition 2.6 says that Φ attains a maximum over $\bar{U}$ at $(\hat{t}, \hat{x}, \hat{y}) \in (0,T) \times \Omega \times \Omega$ for small ε, δ, γ. In particular

$$w(t,x,y) \le w(\hat{t},\hat{x},\hat{y}) + \Psi(t,x,y) - \Psi(\hat{t},\hat{x},\hat{y}) \quad \text{in} \quad U.$$

Expanding Ψ at $(\hat{t},\hat{x},\hat{y})$ yields

$$(2.17) \qquad (\hat{\Psi}_t, \hat{\Psi}_{x,y}, A)(\hat{t},\hat{x},\hat{y}) \in \mathcal{P}^{2,+} w(\hat{t},\hat{x},\hat{y}) \quad \text{with} \quad \nabla^2 \Psi(\hat{t},\hat{x},\hat{y}) \le A$$

where $\hat{\Psi}_t = \partial_t \Psi(\hat{t},\hat{x},\hat{y})$, $\hat{\Psi}_{x,y} = \nabla\Psi(\hat{t},\hat{x},\hat{y})$ and $\nabla = (\nabla_x, \nabla_y)$.

We will apply Lemma 2.7 with $k = 2$, $u_1 = u$, $u_2 = -v$, $s = \hat{t}$, $z = (\hat{x},\hat{y})$. Since u and v are, respectively, sub-and supersolution of (2.1) with F satisfying (F4), we easily see the assumption (2.16) holds. Since $(\hat{t},\hat{x},\hat{y})$ is an interior point of U, by Remark 2.8 we now apply Lemma 2.7 and conclude that for each $\lambda > 0$ there are (τ_1, X) and $(\tau_2, Y) \in \mathbf{R} \times \mathbf{S}^n$ such that

$$(2.18) \qquad (\tau_1, \hat{\Psi}_x, X) \in \bar{\mathcal{P}}^{2,+} u(\hat{t},\hat{x}), \quad (-\tau_2, -\hat{\Psi}_y, -Y) \in \bar{\mathcal{P}}^{2,-} v(\hat{t},\hat{y}), \quad \hat{\Psi}_t = \tau_1 + \tau_2$$

$$(2.19)\qquad -\left(\frac{1}{\lambda}+|A|\right) I \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le A+\lambda A^2,$$

where $\hat{\Psi}_t = \partial_t \Psi(\hat{t},\hat{x},\hat{y})$, $\hat{\Psi}_x = \nabla_x \Psi(\hat{t},\hat{x},\hat{y})$, etc. Since u and v are, respectively, sub-and supersolution of (2.1) it follows from (2.18) that

$$\tau_1 + F_*(\hat{\Psi}_x, X) \le 0, \quad -\tau_2 + F^*(-\hat{\Psi}_y, -Y) \ge 0,$$

which yields

$$(2.20)\qquad 0 \ge \hat{\Psi}_t + F_*(\hat{\Psi}_x, X) - F^*(-\hat{\Psi}_y, -Y).$$

We next take a special A. Differentiating Ψ in (2.11) yields

$$(2.21)\qquad \hat{\Psi}_x = |\eta|^2\eta/\varepsilon + 2\delta\hat{x}, \quad \hat{\Psi}_y = -|\eta|^2\eta/\varepsilon + 2\delta\hat{y}, \qquad (\eta = \hat{x}-\hat{y})$$

$$\begin{aligned}\begin{pmatrix} \hat{\Psi}_{xx} & \hat{\Psi}_{xy} \\ \hat{\Psi}_{yx} & \hat{\Psi}_{yy} \end{pmatrix} &= \frac{1}{\varepsilon}(|\eta|^2 + 2\eta\otimes\eta)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\delta \begin{pmatrix} I & O \\ O & I \end{pmatrix} \\ &\le \frac{3}{\varepsilon}|\eta|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\delta \begin{pmatrix} I & O \\ O & I \end{pmatrix} = A.\end{aligned}$$

With this A the estimate (2.19) becomes

$$(2.22)\qquad \begin{aligned} &-\mu \begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & O \\ O & I \end{pmatrix}, \\ &\mu = \lambda^{-1} + 6|\eta|^2/\varepsilon + 2\delta, \quad \nu = (18|\eta|^2\lambda + 3\varepsilon + 12\delta\varepsilon\lambda)|\eta|^2/\varepsilon^2, \\ &\omega = 4\delta^2\lambda + 2\delta. \end{aligned}$$

We will study (2.20). We take $\lambda = 1$ in (2.22) and fix ε, γ such that $0<\varepsilon<\varepsilon_0$, $0<\gamma<\gamma_0$, where ε_0 and γ_0 are as in Propositions 2.6 and 2.4. We let $\delta \to 0$ in (2.20). We divide the situation in two cases depending on the behavior of $\eta = \hat{x}-\hat{y}$ as $\delta \to 0$.

Case 1. $\quad \eta = \hat{x}-\hat{y} \to 0$ *as* $\delta \to 0$. From (2.22) it follows that

$$\begin{aligned}\begin{pmatrix} X & O \\ O & Y \end{pmatrix} &\le \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & O \\ O & I \end{pmatrix} \\ &\le \theta \begin{pmatrix} I & O \\ O & I \end{pmatrix} \quad \text{with} \quad \theta = 2\nu + \omega.\end{aligned}$$

This implies $X \le \theta I$ and $-Y \ge -\theta I$. By the degenerate ellipticity (F2) we have

$$F_*(\hat{\Psi}_x, X) \ge F_*(\hat{\Psi}_x, \theta I), \quad F^*(-\hat{\Psi}_y, -Y) \le F^*(-\hat{\Psi}_y, -\theta I), \tag{2.23}$$

where $\hat{\Psi}_x$, $\hat{\Psi}_y$ is defined by (2.21). If $\delta \to 0$, we see $\hat{\Psi}_x$ and $\hat{\Psi}_y$ converge to zero since $\eta \to 0$ and $\delta\hat{x}$, $\delta\hat{y} \to 0$ by Proposition 2.5. Letting $\delta \to 0$ in (2.23) yields

$$\varliminf_{\delta \to 0} F_*(\hat{\Psi}_x, X) \ge F_*(0, O), \quad \varlimsup_{\delta \to 0} F^*(-\hat{\Psi}_y, -Y) \le F^*(0, O)$$

since $\theta \to 0$. Applying this estimate to (2.20) and noting that

$$\Psi_t = \gamma(T-t)^{-2} \ge \gamma T^{-2},$$

we obtain

$$0 \ge \gamma T^{-2} + F_*(0, O) - F^*(0, O).$$

By (F3) this yields $0 \ge \gamma T^{-2}$, which contradicts $\gamma > 0$.

Case 2. $\hat{x} - \hat{y} \to a \ne 0$ *for some subsequence* $\delta_j \to 0$. Since the singularity of F is not important in this case our argument is essentially the same as in [10]. From (2.22) it follows that

$$\langle Xp, p \rangle + \langle Yq, q \rangle \le \nu(|p|^2 + |q|^2) - 2\nu\langle p, q \rangle + \omega(|p|^2 + |q|^2).$$

Taking $p = q$ yields

$$X + Y \le 2\omega I.$$

By (F2) we see

$$F^*(-\hat{\Psi}_y, -Y) \le F^*(-\hat{\Psi}_y, X - 2\omega I) \tag{2.24}$$

Since X and Y are bounded as $\delta \to 0$ by (2.22) there are a subsequence $X_j = X(\delta_j)$ and $\bar{X} \in \mathbf{S}^n$ such that $X_j \to \bar{X}$ as $\delta_j \to 0$ (see e.g. [10, Lemma 5.3]). Applying (2.24) to (2.20) and letting $\delta_j \to 0$ now yield

$$0 \ge \gamma T^{-2} + F_*(|a|^2 a/\varepsilon, \bar{X}) - F^*(|a|^2 a/\varepsilon, \bar{X}).$$

Since F is continuous at $(|a|^2 a/\varepsilon, \bar{X})$ for $a \neq 0$, this again contradicts $\gamma > 0$. We thus prove (2.2). ∎

Remark 2.9. The assumption (F4) in Theorem 2.1 is unnecessary if we assume that u and v satisfy the rough growth estimate (2.3). In particular, if u and v are bounded (F4) is unnecessary. Indeed, other than in Proposition 2.3 we use (F4) only to prove (2.16) in Lemma 2.7 so that we derive (2.18)-(2.20). However to carry out the proof of Theorem 2.1 we only need (2.19) and (2.20). Without showing (2.18) one can circumvent (F4) to derive (2.19) and (2.20) by applying the following lemma, which can be proved similarly as Lemma 2.7.

Lemma 2.10. *Assume the hypotheses of Lemma* 2.7 *except* (2.16). *Suppose that* u_i *is a viscosity subsolution of*

$$u_t + F_i(\nabla u, \nabla^2 u) = 0 \tag{2.25}$$

in a neighborhood of $(s, z_i) \in (0,T) \times \mathbf{R}^{N_i}$ *for* $i = 1, 2, \cdots k$, *where* $F_i : \mathbf{R}^{N_i} \times \mathbf{S}^{N_i} \to \mathbf{R} \cup \{\pm\infty\}$ *is lower semicontinuous. Let*

$$(\tau, p, A) \in \mathcal{P}^{2,+} w(s,z), \quad w(s,z) = \sum_{i=1}^{k} u_i(s, z_i),$$

where $p = (p_1, \cdots, p_k)$, $z = (z_1, \cdots, z_k)$. *Then for each* $\lambda > 0$ *there exists* $X_i \in \mathbf{S}^{N_i}$ *such that*

$$\tau + \sum_{i=1}^{k} F_i(p_i, X_i) \leq 0$$

and

$$-\left(\frac{1}{\lambda} + |A|\right) I \leq \begin{pmatrix} X_1 & & O \\ & \ddots & \\ O & & X_k \end{pmatrix} \leq A + \lambda A^2.$$

We conclude this section with a simple application of Theorem 2.1. We will prove that the Lipschitz continuity in x is preserved for $t > 0$.

Corollary 2.11. *Suppose that F satisfies* (F1)-(F4). *Let u be a continuous viscosity solution of* (2.1) *in* $[0,T] \times \mathbf{R}^n$. *Assume that*

$$|u(t,x)| \leq K(|x|+1) \quad \textit{on} \quad [0,T] \times \mathbf{R}^n$$

with $K > 0$ independent of t and x and that $u(0,x)$ is Lipschitz, i.e.,

$$|u(0,x) - u(0,y)| \leq L|x-y|$$

with $L > 0$ independent of x and y. Then it holds

$$|u(t,x) - u(t,y)| \leq L|x-y| \quad \textit{on} \quad [0,T] \times \mathbf{R}^n. \tag{2.26}$$

Proof. Since F is independent of x and u, we see

$$v(t,x) = u(t,x+h) + L|h|$$

is also a continuous viscosity solution of (2.1) in $[0,T] \times \mathbf{R}^n$. By assumptions on u we see u and v satisfy (A1)-(A3) for $Q = (0,T] \times \mathbf{R}^n$. Applying Theorem 2.1 yields

$$u(t,x) - u(t,x+h) \leq L|h|.$$

We next set

$$u(t,x) := u(t,x+h) - L|h| \quad \text{and} \quad v(t,x) := u(t,x),$$

and observe that Theorem 2.1 now yields

$$u(t,x) - u(t,x+h) \geq -L|h|.$$

We thus prove (2.26). ∎

§3. Convexity preserving.

We consider the Cauchy problem

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \mathbf{R}^n \tag{3.1}$$

$$u(0,x) = u_0(x). \tag{3.2}$$

We will show that the concavity of u in x is preserved as time develops provided that $F(p,X)$ is convex in X and that u grows at most linearly near space infinity. For this purpose we apply Lemma 2.7 to

$$w(t,\xi) = u(t,x) + u(t,y) - 2u(t,z), \quad \xi = (x,y,z).$$

and conclude that

$$w(t,\xi) \le L|x+y-2z|$$

with some constant L. Similar technique is found in [11], where it is applied to the semi-concavity of solutions of Bellman equations.

Theorem 3.1. *Suppose that F satisfies* (F1)-(F4) *and*

(F5) $$X \mapsto F(p,X) \text{ is convex on } \mathbf{S}^n \text{ for all } p \in \mathbf{R}^n \backslash \{0\}.$$

Let u be a viscosity solution of (3.1) *with* (3.2). *Assume that u is continuous in $[0,T] \times \mathbf{R}^n$ and that*

(3.3) $$|u(t,x)| \le K(|x|+1) \quad \text{with} \quad K \quad \text{independent of} \quad (t,x) \in Q.$$

If the initial data u_0 is concave and globally Lipschitz with constant L, then it holds

(3.4) $$u(t,x) + u(t,y) - 2u(t,z) \le L|x+y-2z|, \quad x,y,z \in \mathbf{R}^n, \quad 0 \le t \le T.$$

In particular $x \mapsto u(t,x)$ is concave for all $t \in [0,T]$.

We will prove Theorem 3.1 in several steps.

Lemma 3.2. *Suppose that u_0 is concave and globally Lipschitz with constant L in $\mathbf{R}^n$. Then it holds*

(3.5) $$u_0(x) + u_0(y) - 2u_0(z) \le L|x+y-2z| \quad \text{for all} \quad x,y,z \in \mathbf{R}^n.$$

Proof. Since u_0 is concave, it follows that

$$\begin{aligned}&u_0(x)+u_0(y)-2u_0(z)\\ =&u_0(x)+u_0(y)-2u_0((x+y)/2)+2(u_0((x+y)/2)-u_0(z))\\ \leq&2(u_0((x+y)/2)-u_0(z)).\end{aligned}$$

The right hand side is dominated by $2L|(x+y)/2-z|$ so (3.5) follows. ∎

Proposition 3.3. *Suppose that F satisfies* (F1) *and* (F4). *Assume that the hypotheses of Theorem* 3.1 *concerning u hold. Then for $K' > L$ there is a constant $M = M(K', F) > 0$ such that*

$$u(t,x)+u(t,y)-2u(t,z)\leq K'|x+y-2z|+M(1+t) \tag{3.6}$$

for all $\xi=(x,y,z)\in\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n,\quad 0\leq t\leq T$.

Proof. This is a variant of Proposition 2.3 so we just indicate the proof. By (2.26) we may assume that $K = L$. We set

$$\begin{aligned}w(s_1,s_2,t,\xi)&=u(t,x)+u(s_1,y)-2u(s_2,z),\\ \psi(t,\xi)&=K'(|x+y-2z|^2+1)^{1/2}+M(1+t),\end{aligned}$$

and will prove, for large M,

$$w(t,t,t,\xi)\leq\psi(t,\xi)\quad\text{for}\quad\xi\in\mathbf{R}^{3n},\quad 0\leq t\leq T. \tag{3.7}$$

Let $\{g_R\}_{R>0}$ be as in (2.5a-c) and set

$$\phi=\psi+\frac{3K'}{2}(g_R(x)+g_R(y)).$$

By (3.3) g_R plays the role of barrier at space infinity so that $w-\phi<0$ for large ξ. By (3.5) we see $w<\phi$ at $t=s_i=0$ $(i=1,2)$. We now suppose that (3.7) were false and study a positive maximum of $w-\Psi$ with

$$\Psi=((t-s_1)^2+(t-s_2)^2+(s_1-s_2)^2)/\delta+\phi$$

for small $\delta > 0$. The above observation shows that the maximum is attained at $(\hat{s}_1, \hat{s}_2, \hat{t}, \hat{\xi})$ with $(\hat{s}_1, \hat{s}_2, \hat{t}) \neq (0,0,0)$. A parallel argument to Proposition 2.3 yields a contradiction for large M. We thus prove (3.7) yielding (3.6). ∎

Proof of Theorem 3.1. We may assume $L = 1$. Our goal is to prove

$$w(t,\xi) = u(t,x) + u(t,y) - 2u(t,z) \leq |x+y-2z| \tag{3.8}$$

for all $\xi = (x,y,z) \in \mathbf{R}^{3n}$, $0 \leq t \leq T$. For γ, δ, $\kappa > 0$ and $K' > 1$ we set

$$\begin{aligned} &\Phi(t,\xi) = w(t,\xi) - \Psi(t,\xi), \quad \Psi(t,\xi) = K'b(\xi) + B(t,\xi) \\ &\text{with} \quad b(\xi) = \frac{1}{4\kappa}|x+y-2z|^4 + \frac{3}{4}\kappa^{1/3}, \quad B(t,\xi) = \delta|\xi|^2 + \gamma(T-t)^{-1}. \end{aligned} \tag{3.9}$$

We will show that for every κ, $\gamma > 0$ and $K' > 1$ there is $\delta_0 = \delta_0(\kappa,\gamma,K') > 0$ such that $0 < \delta < \delta_0$ implies

$$\Phi(t,\xi) \leq 0 \quad \text{on} \quad U = (0,T] \times \mathbf{R}^{3n}. \tag{3.10}$$

By Young's inequality it holds

$$|x+y-2z| \leq b(\xi) \tag{3.11}$$

and the equality holds if $\kappa = |x+y-2z|^3$. Taking this κ and letting γ, $\delta \to 0$, $K' \to 1$ in (3.10) yields (3.8).

It suffices to prove (3.10). Suppose that (3.10) were false. There would exist κ_0, $\gamma_0 > 0$ and $K_0' > 1$ such that

$$\sup_{U} \Phi(t,\xi) > 0 \quad \text{with} \quad \kappa = \kappa_0, \quad \gamma = \gamma_0, \quad K' = K_0' \tag{3.12}$$

holds for a subsequence $\delta_j \to 0$. By (3.6), (3.9) and (3.11) we see $\Phi < 0$ for sufficiently large ξ. By (3.5) and (3.11) we also see $\Phi \leq 0$ at $t = 0$. Clearly $\Phi = -\infty$ at $t = T$; so (3.12) now implies that Φ takes its positive maximum over $\bar{U}$ at $(\hat{t}, \hat{\xi})$ with $0 < \hat{t} < T$. As in the proof of Proposition 2.5 we first study the behavior of $\hat{\xi} = (\hat{x}, \hat{y}, \hat{z})$ as parameter $\delta = \delta_j \to 0$. Since $\Phi(\hat{t},\hat{\xi}) > 0$ by (3.12), it follows from (3.6) that

$$K_0'b(\hat{\xi}) + \delta|\hat{\xi}|^2 \leq w(\hat{t},\hat{\xi}) \leq K_0'|\hat{x}+\hat{y}-2\hat{z}| + M(1+T). \tag{3.13}$$

with $\kappa = \kappa_0$, $\gamma = \gamma_0$. By (3.11) this yields

$$c = \sup_{\delta=\delta_j} |\hat{x} + \hat{y} - 2\hat{z}| < \infty \tag{3.14}$$

The estimate (3.13) in turn yields

$$\delta_j |\hat{\xi}|^2 \leq C = K_0' c + M(1+T)$$

or

$$(\delta_j |\hat{\xi}|)^2 \leq C\delta_j.$$

This yields

$$\delta_j \hat{\xi} \to 0 \quad \text{as} \quad \delta_j \to 0. \tag{3.15}$$

Since Φ attains its maximum over $\bar{U}$ at $(\hat{t}, \hat{x}, \hat{y})$ it holds

$$w(t,\xi) \leq w(\hat{t},\hat{\xi}) + \Psi(t,\xi) - \Psi(\hat{t},\hat{\xi}) \quad \text{in} \quad U.$$

Expanding Ψ at $(\hat{t},\hat{\xi})$ yields

$$(\partial_t \Psi(\hat{t},\hat{\xi}), \nabla\Psi(\hat{t},\hat{\xi}), A) \in \mathcal{P}^{2,+} w(\hat{t},\hat{\xi}) \quad \text{with} \quad \nabla^2 \Psi \leq A \tag{3.16}$$

where $\nabla = (\nabla_x, \nabla_y, \nabla_z)$. We apply Lemma 2.7 with

$$k = 3, \quad u_1 = u(t,x), \quad u_2 = u(t,y), \quad u_3 = -2u(t,z), \quad s = \hat{t}, \quad (z_1, z_2, z_3) = (\hat{x}, \hat{y}, \hat{z}).$$

Since u is a viscosity solution of (3.1) with F satisfying (F4), we easily see the assumption (2.16) holds. Since $(\hat{t},\hat{\xi})$ is an interior point of U, we now apply Lemma 2.7 with (3.16) to conclude that for each $\lambda > 0$ there are (τ_1, X), (τ_2, Y), $(\tau_3, Z) \in \mathbf{R} \times \mathbf{S}^n$ such that

$$\begin{aligned} &(\tau_1, \hat{\Psi}_x, X) \in \bar{\mathcal{P}}^{2,+} u(\hat{t},\hat{x}), \quad (\tau_2, \hat{\Psi}_y, Y) \in \bar{\mathcal{P}}^{2,+} u(\hat{t},\hat{y}), \\ &(\tau_3, \hat{\Psi}_z, Z) \in \bar{\mathcal{P}}^{2,+}(-2u(\hat{t},\hat{z})) \quad \text{with} \quad \hat{\Psi}_t = \tau_1 + \tau_2 + \tau_3, \end{aligned} \tag{3.17}$$

$$-\left(\frac{1}{\lambda} + |A|\right) I \leq \begin{pmatrix} X & O & O \\ O & Y & O \\ O & O & Z \end{pmatrix} \leq A + \lambda A^2. \tag{3.18}$$

Since u is a solution of (3.1), (3.17) implies

$$\tau_1 + F_*(\hat{\Psi}_x, X) \leq 0, \quad \tau_2 + F_*(\hat{\Psi}_y, Y) \leq 0,$$
$$\frac{-\tau_3}{2} + F^*(-\hat{\Psi}_z/2, -Z/2) \geq 0.$$

Adding first two inequalities and subtracting the last one twice yield

$$0 \geq \hat{\Psi}_t + F_*(\hat{\Psi}_x, X) + F_*(\hat{\Psi}_y, Y) - 2F^*(-\hat{\Psi}_z/2, -Z/2). \tag{3.19}$$

Differentiating Ψ in (3.9) yields

$$\begin{aligned} \hat{\Psi}_x &= K'|\eta|^2\eta/\kappa + 2\delta\hat{x}, \quad \hat{\Psi}_y = K'|\eta|^2\eta/\kappa + 2\delta\hat{y}, \\ \hat{\Psi}_z &= -2K'|\eta|^2\eta/\kappa + 2\delta\hat{z}, \quad \eta = \hat{x} + \hat{y} - 2\hat{z}, \end{aligned} \tag{3.20}$$

with $K' = K_0'$, $\kappa = \kappa_0$ and $\delta = \delta_j$. Since

$$\nabla^2\Psi(\hat{t}, \xi) \leq \zeta S + 2\delta I, \quad \zeta = \frac{3K'}{\kappa}|\eta|^2 \quad \text{with} \quad S = \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix},$$

we take $A = \zeta S + 2\delta I$ as in the proof of Theorem 2.1. The estimate (3.18) becomes

$$-\mu I \leq \begin{pmatrix} X & O & O \\ O & Y & O \\ O & O & Z \end{pmatrix} \leq \nu S + \omega I, \tag{3.21}$$
$$\mu = \lambda^{-1} + 6\zeta + 2\delta, \quad \nu = \zeta + \lambda(6\zeta^2 + 2\delta\zeta), \quad \omega = 2\delta + 4\delta^2,$$

since $S^2 = 6S$ and the largest eigenvalue of S is 6. We will study (3.19). We take $\lambda = 1$ in (3.21) and set $K' = K_0'$, $\kappa = \kappa_0$, $\gamma = \gamma_0$ and $\delta = \delta_j$. We let $\delta_j \to 0$ in (3.19). We divide the situation in two cases depending on the behavior of $\eta = \hat{x} + \hat{y} - 2\hat{z}$ as $\delta_j \to 0$.

Case 1. $\eta = \hat{x} + \hat{y} - 2\hat{z} \to 0$ *as* $\delta_j \to 0$. Since $S \leq 6I$, (3.21) yields X, Y, $Z \leq \theta I$ with $\theta = 6\nu + \omega$. By (F2) we see

$$F_*(\hat{\Psi}_x, X) \geq F_*(\hat{\Psi}_x, \theta I), \quad F_*(\hat{\Psi}_y, Y) \geq F_*(\hat{\Psi}_y, \theta I),$$
$$F^*(-\hat{\Psi}_z/2, -Z/2) \leq F^*(-\hat{\Psi}_z/2, -\theta I/2).$$

Since $\eta \to 0$, $\theta \to 0$ as $\delta_j \to 0$, letting $\delta_j \to 0$ yields

$$\varliminf_{j\to\infty} F_*(\hat{\Psi}_x, X) \geq F_*(0,O), \quad \varliminf_{j\to\infty} F_*(\hat{\Psi}_y, Y) \geq F_*(0,O)$$
$$\varlimsup_{j\to\infty} F^*(-\hat{\Psi}_z/2, -Z/2) \leq F^*(0,O).$$

Since $\hat{\Psi}_t \geq \gamma_0 T^{-2}$, letting $j \to \infty$ (or $\delta_j \to 0$) in (3.19) we obtain

$$0 \geq \gamma_0 T^{-2} + 2F_*(0,O) - 2F^*(0,O).$$

By (F3) this means $0 \geq \gamma_0 T^{-2}$ which contradicts $\gamma_0 > 0$.

Case 2. $\eta \to a \neq 0$ *for a subsequence of* $\{\delta_j\}$ *(still denoted* $\{\delta_j\}$*) as* $\delta_j \to 0$. As in the proof of Theorem 2.1, (3.21) implies that there is a further subsequence of $\{\delta_j\}$ (still denoted $\{\delta_j\}$) and $\bar{X}$, $\bar{Y}$, $\bar{Z} \in \mathbf{S}^n$ such that

$$X_j \to \bar{X}, \quad Y_j \to \bar{Y}, \quad Z_j \to \bar{Z},$$

where $X_j = X(\delta_j)$ and so on. Letting $\delta_j \to 0$ in (3.21) we see

$$\begin{pmatrix} \bar{X} & O & O \\ O & \bar{Y} & O \\ O & O & \bar{Z} \end{pmatrix} \leq \bar{\nu} S \quad \text{with} \quad \bar{\nu} = \bar{\zeta} + 6\bar{\zeta}^2, \quad \bar{\zeta} = \frac{3K'}{\kappa}|a|^2,$$

which yields $\bar{X} + \bar{Y} + \bar{Z} \leq 0$ since

$$\langle p, Sp \rangle = 0 \quad \text{for} \quad p = (q,q,q), \quad q \in \mathbf{R}^n.$$

As $\delta_j \to 0$ it follows from (3.19) that

$$0 \geq \gamma_0 T^{-2} + F(p, \bar{X}) + F(p, \bar{Y}) - 2F(p, (\bar{X} + \bar{Y})/2), \quad p = \frac{|a|^2 a}{\kappa_0}$$

since $F(p, X)$ is continuous for $p \neq 0$. We now invoke the convexity assumption (F5) to get $0 \geq \gamma_0 T^{-2}$ which contradicts $\gamma_0 > 0$. We thus prove (3.10) and obtain (3.8).
The proof of (3.4) is now complete.

Taking $x + y = 2z$ yields

$$(u(t,x) + u(t,y))/2 \le u(t,(x+y)/2)$$

which says that $u(t,x)$ is mid-concave in x. Successive use of this inequality yields

$$\lambda u(t,x) + (1-\lambda)u(t,y) \le u(t,z) \quad z = \lambda x + (1-\lambda)y \tag{3.22}$$

for all $0 < \lambda < 1$ of form $\lambda = k2^{-h}$ with positive integers k, h. Since u is continuous, we now conclude that (3.22) holds for all $0 \le \lambda \le 1$ so u is concave in x. ∎

§4 **General comparison results.** This section extends the comparison principle in §2 to a general equation of form

$$u_t + F(t,x,u,\nabla u,\nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T] \times \Omega, \tag{4.1}$$

where $T > 0$ and Ω is a domain in $\mathbf{R}^n$. Our approach is basically the same as in §2. However, since F depends on x, we are forced to let $\varepsilon \to 0$ in our test function Ψ of (2.11) at the end of the proof. The crucial step is to establish that $|\hat{x} - \hat{y}|^4/4\varepsilon$ converges to zero as $\varepsilon \to 0$ after we let $\delta \to 0$, $\gamma \to 0$.

We consider F satisfying

(F1) $F : J_0 = Q \times \mathbf{R} \times (\mathbf{R}^n \backslash \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous.

We continue to assume (F2) and (F3) i.e.,

(F2) F is degenerate elliptic,i.e., $F(t,x,r,p,X+Y) \le F(t,x,r,p,X)$ in J_0 if $Y \ge 0$.

(F3) $-\infty < F_*(t,x,r,0,O) = F^*(t,x,r,0,O) < \infty$ for all $(t,x,r) \in Q \times \mathbf{R}$.

For boundedness of F we also impose uniformity in t, x and r.

(F4) For every $R > 0$, $c_R = \sup\{|F(t,x,r,p,X)|; |p|, |X| \le R, (t,x,r,p,X) \in J_0\} < \infty$;

this, of course, is the same as (F4) in §2 when F is independent of t, x and r. We assume a kind of monotonicity in r.

(F5) For every $H > 0$, there is a constant $c_0 = c_0(n,T,H)$ such that $r \mapsto F(t,x,r,p,X) + c_0 r$ is nondecreasing for all $(t,x,r,p,X) \in J_0$ with $|r| \le H$.

Outside singularities we assume uniform continuity in (p, X).

(F6) For every $R > \rho > 0$ there is a modulus $\sigma = \sigma_{R\rho}$ such that

$$|F(t,x,r,p,X) - F(t,x,r,q,Y)| \leq \sigma_{R\rho}(|p-q| + |X-Y|)$$

for all $(t,x,r) \in Q \times \mathbf{R}$, $\rho \leq |p|$, $|q| \leq R$, $|X|$, $|Y| \leq R$.

The behavior near $(p, X) = (0, O)$ is assumed to be uniform in t, x and r.

(F7) There are $\rho_0 > 0$ and a modulus σ_1 such that

$$F^*(t,x,r,p,X) - F^*(t,x,r,0,O) \leq \sigma_1(|p| + |X|)$$
$$F_*(t,x,r,p,X) - F_*(t,x,r,0,O) \geq -\sigma_1(|p| + |X|)$$

provided that $(t,x,r) \in Q \times \mathbf{R}$ and $|p|$, $|X| \leq \rho_0$.

We further assume some equicontinuity in x.

(F8) There is a modulus σ_2 such that

$$|F(t,x,r,p,X) - F(t,y,r,p,X)| \leq \sigma_2(|x-y|(|p|+1))$$

for $y \in \Omega$, $(t,x,r,p,X) \in J_0$.

Theorem 4.1. *Suppose that F satisfies* (F1)-(F8). *Let u and v be, respectively, sub-and supersolutions of* (4.1) *in Q. Assume that* (A1)-(A3) *holds for u and v. Then there is a modulus m such that*

$$u^*(t,x) - v_*(t,y) \leq m(|x-y|) \quad \text{on} \quad U. \tag{4.2}$$

The assumption (F8) has a disadvantage because it excludes variable coefficients in second order terms, even if the equation is linear. We will prove (4.2) under weaker assumptions.

(F6') For every $R > \rho > 0$ there is a modulus $\sigma = \sigma_{R\rho}$ such that

$$|F(t,x,r,p,X) - F(t,x,r,q,X)| \leq \sigma_{R\rho}(|p-q|)$$

for all $(t,x,r,p,X) \in J_0$, $\rho \leq |p|, |q| \leq R$, $|X| \leq R$.

(F9) There is a modulus σ_2 such that

$$F_*(t,x,r,0,O) - F^*(t,y,r,0,O) \geq -\sigma_2(|x-y|)$$

for all $(t,x,r) \in Q \times \mathbf{R}$, $y \in \Omega$.

(F10) Suppose that

$$-\mu \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \nu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \omega \begin{pmatrix} I & O \\ O & I \end{pmatrix} \tag{4.3}$$

with μ, ν, $\omega \geq 0$. Let R be taken so that $R \geq \max(\mu, \theta) + 2\omega$ with $\theta = 2\nu + \omega$. Let ρ be a positive number. Then it holds

$$\begin{aligned} &F_*(t,x,r,p,X) - F^*(t,y,r,p,-Y) \\ \geq &-\bar{\sigma}(|x-y|(|p|+1) + \nu|x-y|^2) - \bar{\sigma}(2\omega) \quad \text{for} \quad \rho \leq |p| \leq R. \end{aligned}$$

with some modulus $\bar{\sigma} = \bar{\sigma}_{R\rho}$ independent of t, x, y, r, X, Y, μ, ν, ω.

Theorem 4.2. *Suppose that F satisfies* (F1), (F3)-(F5), (F6'), (F7), (F9), (F10). *Let u and v be respectively, sub-and supersolutions of* (4.1) *in Q. Assume that* (A1)-(A3) *holds for u and v. Then there is a modulus m such that* (4.2) *holds.*

Here is a typical example to which Theorem 4.2 applies without fulfilling (F8). Let $\Sigma(x,p)$ be a bounded function on $\bar{\Omega} \times (\mathbf{R}^n \backslash \{0\})$ with value in the space of $n \times n$ real matrices. Suppose that Σ is Lipschitz on $\bar{\Omega} \times \{p \in \mathbf{R}^n; |p| \geq \rho\}$ for every $\rho > 0$. Since (4.3) implies

$$\langle q_1, Xq_1 \rangle + \langle q_2, Yq_2 \rangle \leq \mu|q_1 - q_2|^2 + \omega(|q_1|^2 + |q_2|^2),$$

we see

$$F(x,p,X) = -\operatorname{trace}\,(\Sigma(x,p)^t \Sigma(x,p) X)$$

fulfills (F10) by taking $q_1 = \Sigma_i(x,p)$, $q_2 = \Sigma_i(y,p)$, where i denotes the i-th column vector of Σ. It is easy to see F satisfies all other assumptions in Theorem 4.2 although F may not satisfy (F8). The following proposition shows that Theorem 4.1 is the special case of Theorem 4.2.

Proposition 4.3. (i) *The assumptions* (F3) *and* (F8) *imply* (F9).
(ii) *Assumptions* (F2),(F6),(F8) *imply* (F10).

Proof. (i) We suppress t and r to simplify notations. By (F8) we observe

$$\varliminf_{\substack{p\to 0\\ X\to O}} (F(x,p,X) - F(y,p,X)) \geq -\sigma_2(|x-y|).$$

The left hand side is dominated from above by

$$\lim_{\epsilon\to 0}\Big(\inf_{|p|+|X|\leq\epsilon} F(x,p,X) - \inf_{|p|+|X|\leq\epsilon} F(y,p,X)\Big) = F_*(x,0,O) - F_*(y,0,O).$$

The condition (F3) now yields (F9).

(ii) As is observed in Case 2 of the proof of Theorem 2.1, (4.3) yields $X+Y \leq 2\omega I$. From (F2) it follows that

$$\begin{aligned}
&F(x,p,X) - F(y,p,-Y)\\
\geq &F(x,p,X) - F(y,p,X-2\omega I)\\
\geq &-\tilde{\sigma}_{R\rho}(2\omega I) + F(x,p,X) - F(y,p,X) \quad \text{for} \quad \rho\leq|p|\leq R \quad \text{by(F6)}
\end{aligned}$$

since (4.3) yields $|X|$, $|Y| \leq \max(\mu,\theta)$ so that $|X|$, $|X-2\omega I| \leq R$. From (F8) it now follows (F10). ∎

The rest of this section is devoted to the proof of Theorem 4.2. We begin with a rough growth estimate for $u(t,x)-v(t,y)$ on $U=(0,T]\times D$ with $D=\Omega\times\Omega$. Since the proof is the same as that of Proposition 2.3 with trivial modifications, we omit its proof.

Proposition 2.3'. *Suppose that F satisfies* (F1) *and* (F4). *Let u and v be, respectively, viscosity sub-and supersolutions of* (4.1) *in Q and that u and $-v$ are upper* *semicontinuous in Q. Then for $K' > K$ there is a constant $M = M(K', F) > 0$ such that* (2.3) *holds.*

We now recall Φ and Ψ of (2.11) and let $(\hat{t}, \hat{x}, \hat{y})$ be a point attaining a maximum of Φ over $\bar{U}$ defined in Propositions 2.5 and 2.6. To carry out the proof of Theorem 4.2 we need to study $|\hat{x} - \hat{y}|^4/\varepsilon$ as $\varepsilon \to 0$.

Proposition 4.4. *Suppose that u and v satisfies* (2.3) *and that* (2.12) *holds. Let $(\hat{t}, \hat{x}, \hat{y})$ be as in Proposition* 2.5. *It holds*

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{\delta, \gamma \downarrow 0}} \frac{|\hat{x} - \hat{y}|^4}{\varepsilon} = 0. \tag{4.4}$$

Proof. By the definition of α in (2.12) there is a point $(t_0, x_0, y_0) \in \bar{U}$ such that

$$w(t_0, x_0, y_0) = u(t_0, x_0) - v(t_0, y_0) > \alpha - \frac{\varepsilon}{4} \quad \text{and} \quad |x_0 - y_0|^4 < \varepsilon^2.$$

This yields

$$\alpha - \frac{\varepsilon}{2} < w(t_0, x_0, y_0) - \frac{|x_0 - y_0|^4}{4\varepsilon}.$$

We now take δ, γ sufficiently small (say $\delta < \delta_0(\varepsilon)$, $\gamma < \gamma_0(\varepsilon)$) so that

$$\alpha - \varepsilon < \Phi(t_0, x_0, y_0),$$

which yields

$$\alpha - \varepsilon < \Phi(\hat{t}, \hat{x}, \hat{y}) \le u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) - \frac{|\hat{x} - \hat{y}|^4}{4\varepsilon}. \tag{4.5}$$

Applying (2.3) to (4.5) we see

$$\alpha - \varepsilon < K'|\hat{x} - \hat{y}| + M' - \frac{|\hat{x} - \hat{y}|^4}{4\varepsilon}, \quad M' = M(1 + T)$$

which yields

$$|\hat{x} - \hat{y}| \le \zeta(\varepsilon) \quad \text{for} \quad 0 < \delta < \delta_0(\varepsilon), \quad 0 < \gamma < \gamma_0(\varepsilon)$$

with some modulus ζ. From (4.5) it follows that

$$\frac{|\hat{x}-\hat{y}|^4}{4\varepsilon} \le w(\hat{t},\hat{x},\hat{y}) - \alpha + \varepsilon \le \beta(\varepsilon) - \alpha + \varepsilon$$
$$\text{with} \quad \beta(\varepsilon) = \sup\{w(t,x,y); |x-y| < \zeta(\varepsilon), \quad (t,x,y) \in U\}$$

for $0 < \delta < \delta_0(\varepsilon)$, $0 < \gamma < \gamma_0(\varepsilon)$. We thus obtain

$$\overline{\lim_{\delta,\gamma\downarrow 0}} \frac{|\hat{x}-\hat{y}|^4}{4\varepsilon} \le \beta(\varepsilon) - \alpha + \varepsilon.$$

Letting $\varepsilon \to 0$ yields (4.4) since $\beta(\varepsilon) \to \alpha$ by the definition of α. ∎

Proof of Theorem 4.2. We may assume that (4.1) has a form

$$u_t + u + F(t,x,u,\nabla u,\nabla^2 u) = 0 \tag{4.1'}$$

with

(F5') $\quad r \mapsto F(t,x,r,p,X)$ is nondecreasing for all $(t,x,r,p,X) \in J_0$

(stronger than (F5)) if we replace u (resp. v) by $e^{\lambda t}u$ (resp. $e^{\lambda t}v$) with sufficiently large λ. Indeed, other assumptions on F is unaltered by this transformation. The estimate (2.3) still holds for (4.1').

We argue by a contradiction. Suppose that (4.2) were false. In other words α in (2.12) were positive. By the proof of Theorem 2.1 we obtain (2.18) and (2.22). Since u and v satisfies (4.1') instead of (2.1) we see, by (2.18),

$$0 \ge \hat{u} - \hat{v} + \hat{\Psi}_t + F_*(\hat{t},\hat{x},\hat{u},\hat{\Psi}_x,X) - F^*(\hat{t},\hat{y},\hat{v},-\hat{\Psi}_y,-Y)$$

instead of (2.20), where $\hat{u} = u(\hat{t},\hat{x})$, $\hat{v} = v(\hat{t},\hat{y})$. By the monotonicity (F5') this estimate yields

$$0 > \alpha/2 + F_*(\hat{t},\hat{x},\hat{u},\hat{\Psi}_x,X) - F^*(\hat{t},\hat{y},\hat{u},-\hat{\Psi}_y,-Y) \tag{4.6}$$

since Proposition 2.3 implies $\hat{u} - \hat{v} > \alpha/2$ and since $\hat{\Psi}_t \ge \gamma/T^2 > 0$.

We divide the situation in two cases depending on the behavior of $\eta = \hat{x} - \hat{y}$ as $\delta \to 0$, $\gamma \to 0$ being ε fixed in $0 < \varepsilon < \varepsilon_0$. Since the convergence as $\delta \to 0$ is always uniform in $0 < \gamma < \gamma_0$ we will often suppress $\gamma \to 0$.

Case 1. $\eta = \hat{x} - \hat{y} \to 0$ *as* $\delta \to 0$. We take

$$\lambda = \min\left(\frac{1}{|\eta|}, \frac{1}{\delta}\right)$$

in (2.22) and observe that μ, ν, $\omega \to 0$. From (2.22) it now follows that X, $Y \to O$. Since $\delta\hat{x}$, $\delta\hat{y} \to 0$ by Proposition 2.5 it follows that $\hat{\Psi}_x$, $\hat{\Psi}_y \to 0$. By (F9) and (F7) we see

$$\begin{aligned}
&F_*(\hat{t}, \hat{x}, \hat{u}, \hat{\Psi}_x, X) - F^*(\hat{t}, \hat{y}, \hat{u}, -\hat{\Psi}_y, -Y) \\
\geq &F_*(\hat{t}, \hat{x}, \hat{u}, \hat{\Psi}_x, X) - F_*(\hat{t}, \hat{x}, \hat{u}, 0, O) \\
&+F^*(\hat{t}, \hat{y}, \hat{u}, 0, O) - F^*(\hat{t}, \hat{y}, \hat{u}, -\hat{\Psi}_y, -Y) \\
&-\sigma_2(|x - y|) \\
\geq &-\sigma_1(|\hat{\Psi}_x| + |X|) - \sigma_1(|\hat{\Psi}_y| + |Y|) - \sigma_2(|x - y|).
\end{aligned}$$

The right hand side converges to zero since $\delta \to 0$. Letting $\delta \to 0$ in (4.6) now yields $0 > \alpha/2$ which is a contradiction.

Case 2. $\eta = \hat{x} - \hat{y} \to a \neq 0$ *for some subsequence* $\delta_j \to 0$. From (2.21) it follows that

$$\hat{\Psi}_x \quad \text{and} \quad -\hat{\Psi}_y \to |a|^2 a/\varepsilon = b \quad \text{as} \quad \delta = \delta_j \to 0$$

since $\delta\hat{x}$, $\delta\hat{y} \to 0$ by Proposition 2.5. We apply (F6') and (F10) with $\rho = 2|b|$ to get

(4.7)

$$\begin{aligned}
&F_*(\hat{t}, \hat{x}, \hat{u}, \hat{\Psi}_x, X) - F^*(\hat{t}, \hat{y}, \hat{u}, -\hat{\Psi}_y, -Y) \\
=&F(\hat{t}, \hat{x}, \hat{u}, \hat{\Psi}_x, X) - F(\hat{t}, \hat{x}, \hat{u}, \zeta, X) \\
&+F(\hat{t}, \hat{x}, \hat{u}, \zeta, X) - F(\hat{t}, \hat{y}, \hat{u}, \zeta, -Y) \\
&+F(\hat{t}, \hat{y}, \hat{u}, \zeta, -Y) - F(\hat{t}, \hat{y}, \hat{u}, -\hat{\Psi}_y, -Y) \qquad (\zeta = |\eta|^2\eta/\varepsilon) \\
\geq &-\sigma_{R\rho}(2|\delta\hat{x}|) - \sigma_{R\rho}(2|\delta\hat{y}|) - \bar{\sigma}_{R\rho}(|\hat{x} - \hat{y}|(|\zeta| + 1) + \nu|\hat{x} - \hat{y}|^2) - \bar{\sigma}_{R\rho}(2\omega).
\end{aligned}$$

We take $\lambda > 0$ such that $\mu = \nu$ and observe that

$$\mu = \nu \to 3|a|^2/\varepsilon, \quad \omega \to 0 \quad \text{as} \quad \delta_j \to 0. \tag{4.8}$$

By (4.8) we may take R independent of small δ, γ. Letting $\delta_j \to 0$ in (4.7) yields

$$\begin{aligned} &\varliminf_{\delta_j \to 0} (F_*(\hat{t}, \hat{x}, \hat{u}, \hat{\Psi}_x, X) - F^*(\hat{t}, \hat{y}, \hat{u}, -\hat{\Psi}_y, -Y)) \\ &\geq 0 + 0 - \bar{\sigma}_{R\rho}(|a|^4/\varepsilon + |a| + 3|a|^4/\varepsilon) - 0 \end{aligned} \tag{4.9}$$

by (4.8) and Proposition 2.5 (iii). Applying (4.9) to (4.6) yields

$$0 > \alpha/2 - \bar{\sigma}_{R\rho}(4|a|^4/\varepsilon + |a|). \tag{4.10}$$

After letting $\gamma \to 0$ we see

$$|a|^4/\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0$$

by Proposition 4.4. Since $a \to 0$ as $\varepsilon \to 0$ by Proposition 2.5 (iv), from (4.10) it follows that $0 > \alpha/2$ which yields a contradiction. We thus prove (4.2). ∎

Remark 4.5. As explained in Remark 2.9, (F4) in Theorems 4.1 and 4.2 is unnecessary if we assume that u and v satisfy the rough growth estimate (2.3). If (F4) is not assumed, we use Lemma 2.10 instead of Lemma 2.7. Note that Lemma 2.10 is still valid if (2.25) is replaced by more general equations (4.1) or (4.1').

Remark 4.6. When Ω is bounded, (F6), (F6'), (F7) and (A1), (A3) are unnecessary, because we may assume that u and v are bounded; (A2) may be replaced by $u^* \leq v_*$ on $\partial_p Q$. Moreover, we may take $\delta = 0$ in the definition of Φ in (2.11). If δ is taken as zero, we may take $\omega = 0$ in (F10). Since Theorems 4.1 and 4.2 are new for F depending on x even if Ω is bounded, we restate them for bounded Ω.

Theorem 4.7. *Let Ω be a bounded domain in $\mathbf{R}^n$. Suppose that F satisfies* (F1)-(F3), (F5), (F8) *or* (F1), (F3), (F5), (F9), (F10) *with $\omega = 0$. Let u and v be, respectively, sub-and supersolutions of* (4.1) *in Q. Assume that $u^* \leq v_*$ on $\partial_p Q$. Then $u^* \leq v_*$ on Q.*

Remark 4.8. By Theorem 4.7 all results in [1, §6, §7] extend to F depending on x. We state one of typical results on global existence of solutions.

Theorem 4.9. *Let $\Omega = \mathbf{R}^n$ and $\beta \in \mathbf{R}$. Assume the hypotheses of Theorem 4.7 concerning F. Suppose that F is geometric, i.e., F is independent of r and*

$$F(t, x, \lambda p, \lambda X + \sigma p \otimes p) = \lambda F(t, x, p, X)$$

for all $\lambda > 0$, $\sigma \in \mathbf{R}$, $(t, x) \in Q$, $(p, X) \in (\mathbf{R}^n \backslash \{0\}) \times \mathbf{S}^n$ and that

$$F_*(t, x, p, -I) \leq c(|p|), \quad F^*(t, x, p, I) \geq -c(|p|)$$

for some $c(q) \in C^1[0, \infty)$ and $c(q) \geq c_0 > 0$ with some constant c_0. Then for $a \in C_\beta(\mathbf{R}^n)$ there is a unique viscosity solution $u_a \in C_\beta([0, T] \times \mathbf{R}^n)$ of (4.1) with $u_a(0, x) = a(x)$.Here $C_\beta(K)$ denotes the space of continuous function u such that $u - \beta$ is compactly supported in K.

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