ON INTERIOR REGULARITY CRITERIA FOR WEAK SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

Shuji Takahashi

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ON INTERIOR REGULARITY CRITERIA FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Shuji TAKAHASHI

Department of Mathematics
Hokkaido University
Sapporo 060 Japan

Abstract. We are concerned with the behavior of weak solutions of the Navier-Stokes equations near possible singularities. We shall show that if a weak solution is in some Lebesgue space or small in some Lorentz space locally, it does not blowup there. Our basic idea is to estimate integral formulas for vorticity which satisfies parabolic equations.

1. Introduction

This paper studies local interior regularity criteria for weak solutions of the Navier-Stokes equations:

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla \phi = 0 \quad \text{in } Q \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } Q \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \; u(x,0) = u_0, \tag{1.3}$$

where $Q = \Omega \times (0,T)$, Ω is a domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary, $0 < T < \infty$; $u = (u^i)_{i=1}^n$ and ϕ denote, respectively, unknown velocity and pressure, while $u_0 = (u_0^i)_{i=1}^n$ is a given initial velocity. Here external force is assumed to be zero for simplicity. For every $u_0 \in L^2(\Omega)$ satisfying compatibility conditions, a global weak solution was constructed by Leray [9] (when $\Omega = \mathbb{R}^3$) and Hopf [6]. Their solutions are known to satisfy

$$u \in L^{2,\infty}(Q) \;\text{ and }\; \nabla u \in L^{2,2}(Q) \tag{1.4}$$

where

$$L^{p,q}(Q) = L^q(0,T;L^p(\Omega)).$$

However, the regularity of their weak solutions is not known unless $n = 2$ although some partial regularity is proved for $n = 3$ (see [2] and references therein).

Serrin [14] gave a nice local interior regularity criterion (cf. [12]). Let us recall his result. He proved among other results, that a weak solution u satisfying (1.4) is in $L^{\infty,\infty}(Q_{R/2})$ and regular in space variables provided that u satisfies $u \in L^{p,q}(Q_R)$ with

$$n/p + 2/q < 1, \ n < p < \infty. \tag{1.5}$$

Here $Q_R = Q_R(x_0, t_0)$ is a parabolic ball centered at $(x_0, t_0) \in Q$:

$$Q_R(x_0,t_0) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}; \ x \in B_R(x_0), -R^2 < t - t_0 < 0\}$$

such that $Q_R \subset Q$ where $B_R(x_0) = \{x \in \mathbb{R}^n; \ |x - x_0| < R\}$.

Recently Struwe [16] refined Serrin's result allowing the case

$$n/p + 2/q = 1, \ n < p \leq \infty. \tag{1.6}$$

The global version is known by Sohr [15] or Giga [4] when $p < \infty$. Indeed, if $u \in L^{p,q}(Q)$ solves the initial-boundary problem of the Navier-Stokes equations (1.1)-(1.3) with (1.5) or (1.6), u is regular in space-time up to boundary.

Our goal is to give a new interior regularity criterion for (1.1)-(1.2). We prove among other results, that there is $\varepsilon > 0$ such that

$$\sup_{x \in B_R(x_0)} |u(x,t)| \leq \varepsilon (t_0 - t)^{-1/2} \quad \text{for } -R^2 + t_0 < t < t_0 \tag{1.7}$$

implies $u \in L^{\infty,\infty}(Q_{R/2})$. Here ε is independent of u, R and (x_0, t_0). In other words (x_0, t_0) can not be a blowup point if (1.7) holds. Similar result is known for a semilinear heat equation

$$u_t - \Delta u - |u|^{p-1} u = 0 \quad \text{for} \quad p > 1$$

by Giga-Kohn [5]. Our basic idea is estimating integral formulas for vorticity $\omega = \text{curl } u$. This idea goes back to Serrin [14]. Struwe's proof is based on an energy method. However, the uniqueness of the limit of approximate solutions is not clearly explained in Struwe [16]. We verify the uniqueness. Our approximation argument avoids to use traces of functions and is simpler than that of Struwe [16]. We here recover his results by Serrin's method. This is indicated in Struwe [16] but is not carried out there.

The crucial part of our argument is regularity of solutions of a parabolic system

$$\omega_t - \Delta\omega + \nabla b\omega = 0 \qquad \text{in } Q \tag{1.8}$$

with nonregular coefficient b. We state our main results on (1.8) in Section 2 and results on Navier-Stokes equations in Section 3 including (1.7) where we use Lorentz spaces.

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2. Interior Regularity for Parabolic Equations

We consider a parabolic system

$$\omega_t - \Delta\omega + \nabla b\omega = 0 \tag{2.1}$$

in $Q = \Omega \times (0,T)$, where Ω is a domain in $\mathbb{R}^n$ with smooth boundary and $0 < T < \infty$. Here

$$\begin{aligned} &\omega = (\omega^1, \dots, \omega^d) \text{ with } \omega^i = \omega^i(x,t) \quad (i = 1, \dots, d), \\ &b(x,t) = (b^i_{jk}(x,t)) \;\text{ for } 1 \le i,k \le d \text{ and } 1 \le j \le n, \text{and} \\ &\nabla b\omega = \left(\sum_{j=1}^{n}\sum_{k=1}^{d} \frac{\partial}{\partial x_j} b^i_{jk}(x,t)\omega^k(x,t) \right)_{i=1}^{d} . \end{aligned} \tag{2.2}$$

We shall study a regularity of ω under minimal regularity assumptions on b. Let $L^{p,q}(Q)$ denote the space of $L^p(\Omega)$-valued L^q functions on $(0,T)$. The space $L^{p,q}(Q)$ is equipped with the norm

$$\|u\|_{L^{p,q}(Q)} = [\ \|u\|_{L^p(\Omega)}(t)\ ]_{L^q(0,T)} = \left\{ \int_0^T \left(\int_\Omega |u(x,t)|^p dx \right)^{q/p} dt \right\}^{1/q} .$$

Here $\|\cdot\|_{L^p(\Omega)}$ denotes the space L^p- norm, and $[\,\cdot\,]_{L^q(0,T)}$ denotes the time L^q- norm. We do not distinguish the spaces of vector and scalar valued functions.

We say $\omega \in L^{2,2}(Q)$ is a *weak solution* of (2.1) in Q, if it holds

$$\iint_Q (\varphi_t + \Delta\varphi + b\nabla\varphi)\omega \, dxdt = 0$$

for any $\varphi \in C_0^\infty(Q)$ where $C_0^\infty(Q)$ is the space of smooth functions with compact support in Q. Here $\varphi = (\varphi^i)_{i=1}^d$ and

$$b\nabla\varphi = \left(\sum_{j=1}^n \sum_{i=1}^d b_{jk}^i \frac{\partial}{\partial x_j} \varphi^i \right)_{k=1}^d .$$

We now state our main results on interior regularity of weak solutions of (2.1).

THEOREM 2.1. *Assume that $1 \le p, q \le \infty$ satisfies $n/p + 2/q = 1$.*

(i) Suppose that $b \in L^{p,q}(Q_R)$ where Q_R is given in Section 1. Assume that $\omega \in L^{2,2}(Q_R)$ is a weak solution of (2.1) in Q_R. Then there is a positive constant $\varepsilon < 1$ such that $\|b\|_{L^{p,q}(Q_R)} < \varepsilon$ implies

(a) $\omega \in L^{\infty,\beta}(Q_{R/2})$ for all $2 \le \beta < \infty$ when $p > n$.

(b) $\omega \in L^{\alpha,\beta}(Q_{R/2})$ for all $2 \le \alpha, \beta < \infty$ when $p = n$.

Here $\varepsilon = \varepsilon(n, p, \beta)$ if $p > n$ and $\varepsilon = \varepsilon(n, \alpha, \beta)$ if $p = n$.

(ii) Let $\omega \in L^{2,2}(Q)$ be a weak solution of (2.1) in Q.

(a) If $p > n$ and $b \in L^{p,q}(Q)$, then $\omega \in L^{\infty,\beta}(Q')$ for all $\beta \ge 2$ with $Q' = \Omega' \times (\sigma, T)$,where $\overline{\Omega'}$ is compact in Ω and $\sigma > 0$.

(b) If $b \in L^{n,\infty}(Q)$ and $\|b\|_{L^{n,\infty}(Q)}$ is sufficiently small, then $\omega \in L^{\alpha,\beta}(Q')$ for all $2 \le \alpha, \beta < \infty$.

REMARK: If $n/p + 2/q < 1$, Ladyzenskaya, Ural'ceva and Solonnikov [8] showed $\omega \in L^{\infty,\infty}$ under more regularity assumptions on ω than those in Theorem 2.1, where we only need $\omega \in L^{2,2}(Q_R)$. See the remark to the proof in Section 7 (cf. [8] Chap.5, §2).

We recall *Lorentz spaces* $L^{(q)}$ for $1 < q < \infty$:

$$L^{(q)}(0,T) = \{f \in L^1(0,T); [f]_{L^{(q)}(0,T)} < \infty\},$$

where

$$[f]_{L^{(q)}(0,T)} = \sup_{s>0} s(\mu\{t \in (0,T); |f(t)| > s\})^{1/q}.$$

Here μ denotes the Lebesgue measure on $\mathbb{R}$. Although $[f]_{L^{(q)}(0,T)}$ is not a norm (the triangle inequality fails to satisfy), there is an equivalent "norm" in $L^{(q)}(0,T)$ provided that $1 < q < \infty$ (cf. [1]). It thus holds

$$(2.3) \qquad [f+g]_{L^{(q)}(0,T)} \le C([f]_{L^{(q)}(0,T)} + [g]_{L^{(q)}(0,T)}).$$

When $0 < T < \infty$, we see

$$(2.4) \qquad [f]_{L^{p-\varepsilon}(0,T)} \le [f]_{L^{(p)}(0,T)} \le [f]_{L^{p}(0,T)}$$

for any $\varepsilon > 0$, and that $t^{-1/p} \in L^{(p)}(0,T)$. We now write

$$f(x,t) \in L^{p,(q)}(Q) \qquad \text{if} \quad ||f||_{L^{p,(q)}(Q)} = [\ ||f||_{L^p(\Omega)}(t)\ ]_{L^{(q)}(0,T)} < \infty.$$

THEOREM 2.2. *Assume that $1 \le p,q \le \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. Suppose that $\omega \in L^{2,2}(Q_R)$ is a weak solution of (2.1) in Q_R such that for any $0 < \delta < R^2$*

$$(2.5) \qquad \omega \in L^{\infty,\beta}(B_R(x_0) \times (-R^2 + t_0, -\delta + t_0)) \ \textit{ for any } 2 < \beta < \infty.$$

Then there exists a positive constant $\varepsilon < 1$ such that

$$||b||_{L^{p,(q)}(Q_R)} < \varepsilon$$

implies

$$\omega \in L^{\infty,\beta'}(Q_{R/2}) \ \textit{ for all } \beta' > 2.$$

Here $\varepsilon = \varepsilon(n,p,\beta')$.

We shall prove Theorems 2.1 and 2.2 in Section 7.

3. Interior Regularity for the Navier-Stokes Equations

As applications of Theorems 2.1 and 2.2, we derive some interior regularity results for weak solutions of the Navier-Stokes equations. Our results extend those of Serrin [14] and Struwe [16].

In this paper we say $u \in L^{2,\infty}(Q)$ with $\nabla u \in L^{2,2}(Q)$ is a *weak solution* of

$$(3.1) \qquad \begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla \phi = 0 \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } Q$$

if

$$(3.2) \qquad \begin{cases} \iint_Q (\varphi_t + \Delta \varphi + (u \cdot \nabla)\varphi) u \, dxdt = 0 \\ \iint_Q (u \cdot \nabla)\eta \, dxdt = 0, \end{cases}$$

for any $\varphi = (\varphi^i)_{i=1}^n \in C_0^\infty(Q)$ with $\nabla \cdot \varphi = 0$ and $\eta \in C_0^\infty(Q)$.

REMARK: If u is a weak solution of (3.1), we see the vorticity $\omega = \text{curl } u$ is a weak solution of (2.1) with $d = n(n-1)/2$ where b^i_{jk} is a linear combination of u^i. For example, if $n = 3$, applying the operator "curl" to (3.1) yields

$$(3.3) \qquad \omega_t - \Delta \omega + \nabla b \omega = 0 \quad \text{with } b^i_{jk} = u^j \delta_{ik} - u^i \delta_{jk}.$$

THEOREM 3.1. *If u is a weak solution of (3.1) in Q with*

$$u \in L^{2,\infty}(Q), \ \nabla u \in L^{2,2}(Q) \text{ and}$$

$$\begin{cases} \|u\|_{L^{p,q}(Q)} < \infty \text{ for some } p, q \text{ such that } n/p + 2/q = 1, n < p \le \infty \\ \text{or } \|u\|_{L^{n,\infty}(Q)} \text{ is sufficiently small,} \end{cases}$$

then

$$u \in L^{\infty,\infty}(Q') \quad \text{and} \quad \text{curl}\, u \in L^{\infty,\infty}(Q')$$

where Q' is as in Theorem 2.1.

(By Serrin's results in [14], this theorem yields that u is C^∞ in space variables.)

THEOREM 3.2. *Assume that u is a weak solution of (3.1) in Q_R such that*

$$u \in L^{2,\infty}(Q_R) \text{ and } \nabla u \in L^{2,2}(Q_R).$$

Suppose that $1 \le p, q \le \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. Then there exists a positive constant $\varepsilon = \varepsilon(n,p) < 1$ such that

$$\|u(t)\|_{L^p(B_R(x_0))} \le \frac{\varepsilon}{(t_0 - t)^{1/q}} \quad \text{for} \quad t \in (-R^2 + t_0, t_0) \tag{3.4}$$

implies

$$u \in L^{\infty,\infty}(Q_{R/4}) \quad \text{and} \quad \operatorname{curl} u \in L^{\infty,\infty}(Q_{R/4}).$$

PROOF THAT THEOREM 2.1 IMPLIES THEOREM 3.1: Applying Theorem 2.1(ii) to (3.3) we see $\omega \in L^{\infty,\beta}(Q')$ for any $\beta > 2$. Since $u \in L^{2,\infty}(Q)$ and $-\Delta u = \operatorname{curl} u$ in Q, we obtain $u \in L^{\infty,\beta}(Q^2)$ for any $\beta > 2$ by a standard argument (cf. Serrin [14], P193, Step II). As in Serrin [14], the remark of Theorem 2.1 yields $\omega \in L^{\infty,\infty}(Q^3)$, which implies $u \in L^{\infty,\infty}(Q^4)$. Here $Q^i = \Omega^i \times (\sigma_i, T)$, $\Omega^{i+1} \Subset \Omega^i$, $\sigma_{i+1} > \sigma_i$ for $1 \le i \le 4$ and $Q^1 = Q'$. ∎

PROOF THAT THEOREMS 2.1 AND 2.2 IMPLY THEOREM 3.2: The inequality (3.4) yields for any $0 < \delta < R^2$

$$u \in L^{p,q}(B_R(x_0) \times (-R^2 + t_0, -\delta + t_0)) \text{ and}$$

$$u \in L^{p,(q)}(Q_R) \quad \text{with} \quad \|u\|_{L^{p,(q)}(Q_R)} < \varepsilon.$$

If ε is sufficiently small, applying Theorems 2.1 and 2.2 with $\omega \equiv \operatorname{curl} u$ yields

$$\omega \in L^{\infty,\beta}(Q_{R/2}) \quad \text{for any } \beta > 2.$$

The proof of Theorem 3.1 now yields

$$u \in L^{\infty,\infty}(Q_{R/4}) \text{ and } \operatorname{curl} u \in L^{\infty,\infty}(Q_{R/4}). \quad ∎$$

4. A priori Estimates for Weak Solutions of Linear Parabolic Equations

To prove Theorems 2.1 and 2.2, through Sections 4-6 we shall prepare a priori estimates for weak solutions of linear parabolic equations with nonregular coefficients in $\mathbb{R}^n \times (0,T)$. In these sections we suppress $\mathbb{R}^n \times (0,T)$ in norms and function spaces, which are simply written as $L^{p,q}$, C^∞ and so on.

This section establishes a priori estimates for weak solutions of

$$(4.1) \qquad \begin{cases} v_t - \Delta v + \nabla b v = F & \text{in } \mathbb{R}^n \times (0,T) \\ v(x,0) = 0 \end{cases}$$

with regularity assumptions of v. Here $v = (v^i)_{i=1}^d$ and $F = \nabla g + h$ with $\nabla g = (\sum_{j=1}^n \frac{\partial}{\partial x_j} g^{ij})_{i=1}^d$ and $h = (h^i)_{i=1}^d$; b and $\nabla b v$ are as in (2.2).

We say $v \in L^{2,2}$ is a *weak solution* of (4.1) if it holds

$$-\int_0^T \int_{\mathbb{R}^n} (\varphi_t + \Delta\varphi + b\nabla\varphi) v \, dx dt = \int_0^T \int_{\mathbb{R}^n} F\varphi \, dx dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$ with $\varphi = (\varphi^i)_{i=1}^d$.

We begin with well known properties of $L^{(p)}$. We here also suppress $(0,T)$ in norms.

Lemma 4.1. *Suppose that $\theta \geq 0$ and $0 < T < \infty$. It holds*

$$t^{-\theta} \in L^{(p)}(0,T) \quad \text{if and only if} \quad \theta p \leq 1.$$

Moreover $[t^{-\theta}]_{(p)} = 1$ if $\theta p = 1$.

This follows from a direct calculation:

$$\begin{aligned} [t^{-\theta}]_{(p)} &= \sup_{s>0} s(\mu\{t \in (0,T) |\, t^{-\theta} > s\})^{1/p} \\ &= \begin{cases} T^{1/p} & \text{if } \theta = 0, \\ \sup_{s > T^{-\theta}} s^{1-1/\theta p} & \text{if } \theta > 0. \end{cases} \end{aligned}$$

Lemma 4.2. *Suppose that $f \in L^{(p)}(0,T)$ and $g \in L^{(q)}(0,T)$. It holds*

(i)

$$[fg]_{(r)} \leq 2[f]_{(p)}[g]_{(q)} \quad \text{if} \quad 1/r = 1/p + 1/q \quad \text{and} \quad 1 < p,q,r < \infty$$

(ii) (weak Young)

$$[f*g]_{(r)} \le C_{pq}[f]_{(p)}[g]_{(q)} \quad \textit{if} \;\; 1/r+1 = 1/p+1/q \;\; \textit{and} \;\; 1<p,q,r<\infty.$$

If $g \in L^q(0,T)$, then it holds

(iii) (Hardy-Littlewood-Sobolev)

$$[f*g]_r \le C_{pq}[f]_{(p)}[g]_q \quad \textit{if} \;\; 1+1/r = 1/p+1/q \;\; \textit{and} \;\; 1<p,q,r<\infty.$$

Here $f*g$ denotes the convolution

$$(f*g)(t) = \int_0^T f(t-s)g(s)\,ds.$$

PROOF: The inequalities (ii) and (iii) are now standard and found for example in [13] Chap.9, §4. We now prove (i). We may assume that $[f]_{(p)}[g]_{(q)} > 0$. Set $0<\theta<1$ as $r = \theta p$, and set $\lambda := [f]_{(p)}^{\theta-1}[g]_{(q)}^{\theta}$. Since

$$\begin{aligned}
&\mu\{t \in (0,T); |fg(t)| > s\} \\
&\le \mu\{t \in (0,T); |f(t)| > \frac{s^\theta}{\lambda}\} + \mu\{t\in(0,T); |g(t)| > \lambda s^{1-\theta}\},
\end{aligned}$$

it holds

$$\begin{aligned}
[fg]_{(r)} &\le \lambda^{1/\theta}[f]_{(\frac{r}{\theta})}^{1/\theta} + \lambda^{-1/(1-\theta)}[g]_{(\frac{r}{1-\theta})}^{1/(1-\theta)} \\
&= 2[f]_{(\frac{r}{\theta})}[g]_{(\frac{r}{1-\theta})}.
\end{aligned}$$

This is the same as (i). ∎

We write

$$(e^{t\Delta}f)(x) = \int_{\mathbb{R}^n} G(x-y,t)f(y)\,dy,$$

where

$$G(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/4t).$$

It is easy to see for $1 \le l \le r \le \infty$

$$(4.2) \qquad ||e^{t\Delta} f||_{L^r(\mathbb{R}^n)} \le Ct^{-n(1/l-1/r)/2}||f||_{L^l(\mathbb{R}^n)}$$

$$(4.3) \qquad ||e^{t\Delta} \nabla f||_{L^r(\mathbb{R}^n)} \le Ct^{-(1/2+n(1/l-1/r)/2)}||f||_{L^l(\mathbb{R}^n)}$$

where $C = C(n)$ and $\nabla f = (\frac{\partial}{\partial x_j} f)_{j=1}^n$. We now estimate

$$(V(f))(x,t) = \int_0^t (e^{(t-s)\Delta} f)(x,s)\, ds.$$

LEMMA 4.3.

(i) Suppose that $1 \le l \le r \le \infty$ *and* $1 < l' \le r' < \infty$ *such that*

$$(4.4) \qquad \frac{n}{l} + \frac{2}{l'} \le \frac{n}{r} + \frac{2}{r'} + 2.$$

There is a positive constant C *such that*

(a) $||V(f)||_{r,r'} \le C||f||_{l,l'}$

(b) $||V(f)||_{r,(r')} \le C||f||_{l,(l')}$.

(ii) Suppose that $1 \le m \le r \le \infty$ *and* $1 < m' \le r' < \infty$ *such that*

$$(4.5) \qquad \frac{n}{m} + \frac{2}{m'} \le \frac{n}{r} + \frac{2}{r'} + 1.$$

There is a positive constant C' *such that*

(a) $||V(\nabla f)||_{r,r'} \le C'||f||_{m,m'}$

(b) $||V(\nabla f)||_{r,(r')} \le C'||f||_{m,(m')}$.

REMARK: The constant C (resp. C') depends on n. If the strict inequality holds in (4.4) (resp. (4.5)), it also depends on a bound on T. Other than these dependence, C (resp. C') depends only on exponents through $1/l - 1/r, r', l'$ (resp. $1/m - 1/r, r', m'$) if $l' < r'$ (resp. $m' < r'$) or (4.4) (resp. (4.5)) holds with strict inequality. Otherwise C and C' depends only on n, r and r', and r should be $1 < r < \infty$.

PROOF:

(i)(*a*) We divide situations in several cases.

Case 1. $l' < r'$.

Since

$$\| \int_0^t g(x,s)\, ds \|_{L^r(\mathbb{R}^n_x)} \leq \int_0^t \|g(x,s)\|_{L^r(\mathbb{R}^n_x)}\, ds,$$

(4.2) yields

$$\begin{aligned} \|V(f)\|_{r,r'} &\leq C[\int_0^t (t-s)^{-n(1/l-1/r)/2} \|f\|_{L^l(\mathbb{R}^n)}(s)\, ds]_{r'} \\ &= C[\int_0^T H(t-s)(t-s)^{-n(1/l-1/r)/2} \|f\|_{L^l(\mathbb{R}^n)}(s)\, ds]_{r'} \end{aligned}$$

with $C = C(n)$ where H is the Heaviside function. Applying Lemma 4.2(iii) yields

$$\|V(f)\|_{r,r'} \leq C'[H(t)t^{-n(1/l-1/r)/2}]_{(a)} \|f\|_{l,l'}$$

for $1/r' + 1 = 1/a + 1/l'$ with $C' = C'(n, r', l')$. Note that one may take $1 < a < \infty$ since $l' < r'$. Since $an(1/l - 1/r)/2 \leq 1$, Lemma 4.1 implies that $[t^{-n(1/l-1/r)/2}]_{(a)}$ is finite. We thus obtain (i)(a).

Case 2. $l' = r'$ and $1/l < 1/r + 2/n$.

Applying Young's inequality instead of Lemma 4.2(iii), the parallel argument to case 1 yields

$$\|V(f)\|_{r,r'} \leq C'' A \|f\|_{l,l'} \tag{4.6}$$

with $C'' = C''(n)$ and $A = [H(t)t^{-n(1/l-1/r)/2}]_1 < \infty$ since $1/l < 1/r + 2/n$.

Case 3. $1 < r < \infty$, $l' = r'$ and $1/l = 1/r + 2/n$.

We first assume that $1/r' - 1/r = 2/n$. By Sobolev's inequality, it holds

$$\|V(f)\|_{r,r'} \leq C \|\nabla^2 V(f)\|_{r',r'} \tag{4.7}$$

with $C = C(n, r)$ where

$$\|\nabla^2 g\|^p_{L^p(\mathbb{R}^n)} = \sum_{i,j=1}^n \| \frac{\partial^2}{\partial x_j \partial x_i} g \|^p_{L^p(\mathbb{R}^n)}.$$

Applying Calderón-Zygmund's inequality yields

$$\begin{aligned}||\nabla^2 V(f)||_{r,r'} &= ||\int_0^t \nabla^2 e^{(t-s)\Delta} f\, ds||_{r',r'} \\ &\leq C'||f||_{r',r'}\end{aligned}$$

with $C' = C'(n, r')$. Since $r' = l = l'$ we now obtain (i)(a). Although so far we assume that $1/r' - 1/r = 2/n$, this restriction can be removed if we use $L^{r,r'}$ estimate for singular integral operators (cf. [3]).

The proof of (ii)(a) parallels that of (i)(a) if we use (4.3) instead of (4.2) and Sobolev's inequality

$$||V(f)||_{r,r'} \leq C||\nabla V(f)||_{r',r'} \text{ if } 1/r' - 1/r = 1/n$$

instead of (4.7).

In the case 1 the proof of (i)(b) essentially parallels that of (i)(a) if we use Lemma 4.2(ii) instead of Lemma 4.2(iii). In the cases 2 and 3, applying Hunt's interpolation to

$$T : ||f||_l(t) \mapsto ||V(f)||_r(t),$$

we see (i)(a) yields (i)(b) (cf. Hunt [7]). The proof of (ii)(b) parallels that of (i)(b). ∎

We next recall uniqueness of solutions of the heat equation. The proof is based on existence of solutions of the dual problem. A more general version will be proved in Section 5.

LEMMA 4.4. *Let $v \in L^{2,2}(\mathbb{R}^n \times (0,T))$ a weak solution of*

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ v(x,0) = 0. \end{cases}$$

Then $v \equiv 0$.

We now state our main result in this section.

PROPOSITION 4.1. *Let $0 < T_0 < \infty$ and $T \le T_0$. Assume that $1 \le p, q \le \infty$ and $n/p + 2/q = 1$. Assume that l and l' satisfy $1/l = 1/p + 1/m$ and $1/l' = 1/q + 1/m'$ where $2 \le m \le \infty$ and $2 \le m' < \infty$. Let $m \le r \le \infty$ and $m' \le r' < \infty$ sastisfy (4.5). Suppose that $v \in L^{2,2}$ is a weak solution of (4.1) with $F = \nabla g + h$ and $g, h \in L^{2,2}$. Then there exists a positive constant ε such that*

(i) $||b||_{p,q} < \varepsilon$ implies

$$||v||_{r,r'} \le C(||g||_{m,m'} + ||h||_{l,l'})$$

provided that $v \in L^{r,r'}$;

(ii) $||b||_{p,(q)} < \varepsilon$ implies

$$||v||_{r,(r)} \le C(||g||_{m,(m')} + ||h||_{l,(l')})$$

provided that $v \in L^{2,(2+\varrho)} \cap L^{r,(r')}$ for some $\varrho > 0$ and that $m' > 2$, where

$$C = C(n, T_0, r, r', m, m').$$

Here $\varepsilon = \varepsilon(n, p, r')$ if $p > n$ and $\varepsilon = \varepsilon(n, r, r')$ if $p = n$. The exponent r should be $1 < r < \infty$ if (1) $(p, q) = (n, \infty)$ or (2) $m' = r'$ and $1/m = 1/r + 1/n$.

PROOF:

(i) Lemma 4.3-(i)(a) and -(ii)(a) yield $\bar{v} = V(-\nabla bv + \nabla g + h) \in L^{2,2}$. By Lemma 4.4 we see $\bar{v}$ is the unique weak solution v of (4.1) with $F = \nabla g + h$. We easily see Lemma 4.3-(i)(a) and -(ii)(a) also yield

$$||v||_{r,r'} \le C_1||b||_{p,q}||v||_{r,r'} + C_2(||g||_{m,m'} + ||h||_{l,l'})$$

where $C_1 = C_1(n, p, r')$ if $p > n$ and $C_1 = C_1(n, r, r')$ if $p = n$ and

$$C_2 = C_2(n, T_0, r, r', m, m').$$

Since $v \in L^{r,r'}$, setting $\varepsilon = (2C_1)^{-1}$ we obtain (i).

(ii) Lemmas 4.2(i) and 4.3(ii)(b) yield $V(\nabla bv) \in L^{2,(2+\varrho)}$. Since $L^{2,2}$ is contained in $L^{2,(2+\varrho)}$, as in (i) we see $v = V(-\nabla bv + \nabla g + h) \in L^{2,2}$ is the unique weak solution of

(4.1) with $F = \nabla g + h$. The proof parallels (i) if we use Lemma 4.3-(i)(b) and -(ii)(b) instead of Lemma 4.3-(i)(a) and -(ii)(a), and use Lemma 4.2(i) and (2.3). ∎

5. Existence and Uniqueness

This section proves the uniqueness of weak solutions of (4.1) with $F = 0$. Let $W^{2,1}$ denote the Sobolev space:

$$W^{2,1} = \{\varphi \in L^{2,2}(\mathbb{R}^n \times (0,T));\ ||\varphi||_{W^{2,1}} < \infty\}$$

where

$$||\varphi||^2_{W^{2,1}} = ||\varphi||^2_{2,2} + ||\varphi_t||^2_{2,2} + ||\nabla\varphi||^2_{2,2} + ||\nabla^2\varphi||^2_{2,2}.$$

Throughout this section we assume that $1 \le p, q \le \infty$ satisfies $n/p + 2/q = 1$. We begin with $W^{2,1}$ estimates for solutions of the heat equations.

LEMMA 5.1. *Let $0 < T_0 < \infty$, $T \le T_0$. Let $V(f)$ be as in Lemma 4.3. It holds*

$$||V(f)||_{W^{2,1}} \le C||f||_{2,2} \quad \text{with } C = C(n, T_0)$$

for all $f \in L^{2,2}$.

Since the proof is now standard and given in [8] Chap. 4 §3, we don't prove here.

LEMMA 5.2. *There is a positive constant $C = C(n, p, T_0)$ such that*

$$||b\nabla\varphi||_{2,2} \le C||b||_{p,q}||\varphi||_{W^{2,1}}$$

provided that $\varphi \in W^{2,1}$ satisfies $\varphi(x, T) = 0$.

PROOF: By Hölder's inequality we get

$$||b\nabla\varphi||_{2,2} \le ||b||_{p,q}||\nabla\varphi||_{\gamma,\delta}$$

for

$$1/2 = 1/p + 1/\gamma \quad \text{and} \quad 1/2 = 1/q + 1/\delta. \tag{5.1}$$

Applying Gagliardo-Nirenberg's inequality (cf. [11]) yields

$$||\nabla\varphi||_\gamma \le C||\nabla^2\varphi||_2^\theta ||\nabla\varphi||_2^{1-\theta} \quad \text{with} \quad C = C(n,\theta)$$

for $0 \le \theta \le 1$ such that

$$\begin{aligned} n/\gamma &= \theta(-1+n/2) + (1-\theta)n/2 \\ &= -\theta + n/2. \end{aligned} \tag{5.2}$$

The assumption, restrictons (5.1) and (5.2) on exponents imply

$$\theta = n/p \quad \text{and} \quad 1/\delta = \theta/2.$$

Applying the inequality

$$[f^\vartheta g^{1-\vartheta}]_k \le [f]_l^\vartheta [g]_m^{1-\vartheta} \quad \text{for} \quad 1/k = \vartheta/l + (1-\vartheta)/m$$

yields

$$\begin{aligned} ||\nabla\varphi||_{\gamma,\delta} &\le C||\nabla^2\varphi||_{2,2}^\theta ||\nabla\varphi||_{2,\infty}^{1-\theta} \\ &\le C(||\nabla^2\varphi||_{2,2} + ||\nabla\varphi||_{2,\infty}). \end{aligned}$$

This now yields

$$||b\nabla\varphi||_{2,2} \le C||b||_{p,q}(||\nabla^2\varphi||_{2,2} + ||\nabla\varphi||_{2,\infty}).$$

The proof is now complete since

$$||\nabla\varphi||_{2,\infty} \le C||\varphi||_{W^{2,1}}$$

with $C = C(n, T_0)$ for $\varphi(x,T) = 0$. The last estimate is found in [10]. However it is easily proved by multiplying $\Delta\varphi$ with the equation

$$\varphi_t + \Delta\varphi = f$$

and integrating by parts. ∎

We now state our main result in this section.

THEOREM 5.1. *(Uniqueness) Suppose that $v \in L^{2,2}$ satisfies (4.1) with $F = 0$ and $0 < T \leq \infty$. Then there exists a positive constant ϵ depending only on n and p such that $||b||_{p,q} < \epsilon$ implies $v \equiv 0$. If $p > n$, $b \in L^{p,q}$ implies $v \equiv 0$ without smallness of b.*

REMARK: Since the corresponding assertion to Lemma 5.2 is not verified for $b \in L^{p,(q)}$, we don't obtain the uniqueness when $b \in L^{p,(q)}$.

We shall prove the existence of solutions of the dual problem to get the uniqueness.

PROPOSITION 5.1. *(Existence) Let $0 < T \leq T_0 < \infty$. There is a positive constant ϵ depending only on n, p and T_0 such that $||b||_{p,q} < \epsilon$ implies that for any $f \in L^{2,2}$ there exists a weak solution $\varphi \in W^{2,1}$ of*

$$\begin{cases} \varphi_t - \Delta\varphi - b\nabla\varphi = f & \text{in } \mathbb{R}^n \times (0,T) \\ \varphi(x,0) = 0. \end{cases} \tag{5.3}$$

PROOF THAT PROPOSITION 5.1 IMPLIES THEOREM 5.1: We may assume that $T \leq 1$ by dividing the interval into subintervals and we first assume that $||b||_{p,q} < \epsilon$. The equation (5.3) can be transformed by $s = T - t$ into a backward parabolic equation with a terminal data:

$$\begin{cases} \varphi_s + \Delta\varphi + b\nabla\varphi = -f & \text{in } \mathbb{R}^n \times (0,T) \\ \varphi(x,T) = 0. \end{cases} \tag{5.4}$$

By the definition of weak solutions of (4.1) with $F = 0$, it holds

$$\iint_{\mathbb{R}^n \times (0,T)} (\varphi_t + \Delta\varphi + b\nabla\varphi)v \, dxdt = 0 \tag{5.5}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$. Since $C_0^\infty(\mathbb{R}^n \times [0,T))$ is dense in $W^{2,1} \cap \{\varphi \in L^{2,2}; \varphi(x,T) = 0\}$, by a density argument we see by Lemma 5.2 that (5.5) holds for any $\varphi \in W^{2,1}$ with $\varphi(x,T) = 0$. Hence applying Proposition 5.1 to (5.4) yields

$$\iint_{\mathbb{R}^n \times (0,T)} vf \, dxdt = 0$$

for any $f \in L^{2,2}$, which implies $v \equiv 0$.

If $p > n$ and $b \in L^{p,q}$, one can divide $(0,T)$ into small subintervals so that the $L^{p,q}$ norm of b is smaller than ϵ on each subinterval (T_i, T_{i+1}) with $T_1 = 0$. Applying the uniqueness for small b yields $v \equiv 0$ on $\mathbb{R}^n \times (0, T_2]$. We repeat this argument to get $v \equiv 0$ on $\mathbb{R}^n \times (T_2, T_3]$ and so on. We thus end up with $v \equiv 0$ on $\mathbb{R}^n \times (0,T)$.

Proof of Proposition 5.1: We show that there exists $\varphi \in W^{2,1}$ satisfying

$$\varphi(x,t) = \int_0^t (e^{(t-s)\Delta}\zeta(\varphi))(x,s)\, ds \tag{5.6}$$

where $\zeta(\varphi) = f + b\nabla\varphi$. As is usual we construct φ by a successive approximation. We set $\varphi_0 = 0$ and let φ_{j+1} $(j \geq 0)$ be

$$\varphi_{j+1} = V(\zeta(\varphi_j)) = \int_0^t e^{(t-s)\Delta}\zeta(\varphi_j)\, ds.$$

Suppose that $\varphi_j \in W^{2,1}$. Then by Lemma 5.2 we obtain

$$||\zeta(\varphi_j)||_{2,2} \leq ||f||_{2,2} + C||b||_{p,q}||\varphi_j||_{W^{2,1}}.$$

We now apply Lemma 5.1 to get $\varphi_{j+1} \in W^{2,1}$. Since $\varphi_0 = 0 \in W^{2,1}$ we obtain that $\varphi_j \in W^{2,1}$ for all $j \geq 0$. Since

$$\varphi_{j+1} - \varphi_j = \int_0^t e^{(t-s)\Delta} b\nabla(\varphi_j - \varphi_{j-1})\, ds,$$

Lemma 5.1 and Lemma 5.2 yield

$$\begin{aligned} ||\varphi_{j+1} - \varphi_j||_{W^{2,1}} &\leq C||b\nabla(\varphi_j - \varphi_{j-1})||_{2,2} \\ &\leq C'||b||_{p,q}||\varphi_j - \varphi_{j-1}||_{W^{2,1}} \quad \text{with } C' = C'(n,p,T_0). \end{aligned}$$

This shows that $\{\varphi_j\}$ is a Cauchy sequence in $W^{2,1}$ provided that $C'||b||_{p,q} < 1$. Since $W^{2,1}$ is a Banach space, there exists $\varphi \in W^{2,1}$ such that $\varphi_j \to \varphi$ in $W^{2,1}$ as $j \to \infty$. By Lemmas 5.1 and 5.2 the mapping $\varphi_j \mapsto \varphi_{j+1}$ is continuous in $W^{2,1}$, so we see the limit $\varphi \in W^{2,1}$ satisfies (5.6). This means φ solves (5.3). ∎

Remark: Existence and uniqueness results in this section are also valid if $\mathbb{R}^n \times (0,T)$ is replaced by $Q = \Omega \times (0,T)$ with the Dirichlet boundary condition where Ω is bounded. We state the uniqueness without the proof because we do not use it in our paper.

We say $v \in L^{2,2}(Q)$ is a weak solution of

$$\begin{cases} v_t - \Delta v + \nabla b v = F & \text{in } Q \\ v(x,0) = 0, \ v|_{\partial\Omega} = 0 \end{cases} \tag{5.7}$$

if

$$-\iint_Q (\varphi_t + \Delta\varphi + b\nabla\varphi)v \; dxdt = \iint_Q F\varphi \; dxdt$$

for any $\varphi \in C^\infty(\overline{\Omega} \times [0,T])$ with $\varphi(x,T) = 0$.

Theorem 5.2. *Suppose that $v \in L^{2,2}(Q)$ satisfies (5.7) with $F = 0$. Then there exists a positive constant ϵ depending only on n and p such that $||b||_{L^{p,q}(Q)} < \epsilon$ implies $v \equiv 0$. If $p > n$, $b \in L^{p,q}(Q)$ implies $v \equiv 0$ without smallness of b.*

6. Estimates for L^2 Weak Solutions

This section establishes a priori estimates for a weak solution $v \in L^{2,2}$ of (4.1) without assuming additional regularity on v such that $v \in L^{r,r'}$ as in Proposition 4.1. We first approximate (4.1) by the equations with smooth coefficients. We next apply a priori estimates in Proposition 4.1 to the approximated equations. Passing to the limit yields the desired a priori estimate for the solution $v \in L^{2,2}$ of (4.1) since the solution of (4.1) is unique by Theorem 5.1.

Theorem 6.1. *The assertion of Proposition 4.1(i) holds without assuming that $v \in L^{r,r'}$.*

Proof: We first assume that g, h are compactly supported in $\mathbb{R}^n \times (0,T)$. We introduce approximate equations for (4.1). We extend b, g and h by zero outside $\mathbb{R}^n \times (0,T)$ and set

$$b_k := \rho_{1/k} * *b, \ F_k := \rho_{1/k} * *F$$

where $\rho_{1/k}$ is Friedrich's mollifier, i.e.,

$$\rho_{1/k}(x,t) = k^{n+1}\rho(kx, kt)$$

with $\rho \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ supported in $|(x,t)| \leq 1$ such that

$$\rho \geq 0 \quad \text{and} \quad \iint_{\mathbb{R}^n \times \mathbb{R}} \rho \, dxdt = 1.$$

Here $**$ denotes the space-time convolution, i.e.,

$$(f * *g)(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} f(x-y, t-s) g(y,s) \, dyds.$$

Assume that $v_k \in L^{2,2}$ is a weak solution of

$$\text{(6.1)} \qquad \begin{cases} \partial_t v_k - \Delta v_k + \nabla b_k v_k = F_k & \text{in } \mathbb{R}^n \times (0,T) \\ v_k(x,0) = 0. \end{cases}$$

Since F_k is in C_0^∞ and $b_k \in C^\infty$ is bounded with its all derivatives, v_k uniquely exists and $v_k \in C^\infty \cap L^{r,r'}$ for any $r, r' \geq 2$ (cf. [8] Chap. 4 §5. Theorem 5.1 and §14.). Since F_k is of the form

$$F_k = \nabla g_k + h_k \quad \text{with} \quad g_k = \rho_{1/k} * *g \quad \text{and} \quad h_k = \rho_{1/k} * *h$$

and since

$$||g_k||_{m,m'} \leq ||g||_{m,m'}, \ ||h_k||_{l,l'} \leq ||h||_{l,l'} \text{ and } ||b_k||_{p,q} \leq ||b||_{p,q},$$

applying Proposition 4.1(i) yields

$$\text{(6.2)} \qquad ||v_k||_{r,r'} \leq C(||g||_{m,m'} + ||h||_{l,l'})$$

with $C = C(n, T_0, r, r', m, m')$. When $r = r' = m = m' = 2$, it holds

$$||v_k||_{2,2} \leq C'(||g||_{2,2} + ||h||_{l_1,l_1'})$$

with $1/l_1 = 1/p + 1/2$ and $1/l_1 = 1/q + 1/2$. This now yields

$$\text{(6.3)} \qquad ||v_k||_{2,2} \leq C^\circ(||g||_{m,m'} + ||h||_{l,l'})$$

where C^o depends in addition on bounds on areas of supp g and supp h, since g and h have compact support. Since $2 \leq r, r' \leq \infty$, $L^{r,r'}$ is the dual of Lebesgue space L^{r_*,r'_*} with $1/r + 1/r_* = 1,\ 1/r' + 1/r'_* = 1$. The estimates (6.2) and (6.3) yield that there exists a subsequence $\{k'\}$ and $\tilde{v} \in L^{r,r'} \cap L^{2,2}$ such that $v_k \to \tilde{v}$ $*$-weakly in $L^{r,r'}$ and weakly in $L^{2,2}$. Letting $k' \to \infty$ in (6.2) and (6.3) yields

$$||\tilde{v}||_{r,r'} \leq C(||g||_{m,m'} + ||h||_{l,l'}) \tag{6.4}$$

$$||\tilde{v}||_{2,2} \leq C^o(||g||_{m,m'} + ||h||_{l,l'}). \tag{6.5}$$

We shall show that $\tilde{v}$ is a weak solution of (4.1). The inequality (6.5) yields $\tilde{v} \in L^{2,2}$. By the definition of (6.1), it holds

$$< -\varphi_t - \Delta\varphi - b_{k'}\nabla\varphi,\ v_{k'} >=< F_{k'},\ \varphi >$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$. Here

$$< f,\ g >= \int_0^T \int_{\mathbb{R}^n} fg\ dxdt.$$

Letting $k' \to \infty$ yields

$$< \varphi_t + \Delta\varphi,\ v_{k'} > \to < \varphi_t + \Delta\varphi,\ \tilde{v} > .$$

By the definition of F_k, we see

$$< F_{k'},\ \varphi >=< \nabla g_{k'},\ \varphi > + < h_{k'},\ \varphi > .$$

Since $g_{k'} \to g$ in $L^{m,m'}$ and $h_{k'} \to h$ in $L^{l,l'}$, the right hand side converges to

$$\begin{aligned} &- < g,\ \nabla\varphi > + < h,\ \varphi > \\ &=< F,\ \varphi > . \end{aligned}$$

If $b \in L^{p,q}$ is compactly supported in $\mathbb{R}^n \times [0,T]$, $b_{k'} \to b$ in $L^{2,2}$. Since $v_{k'} \to \tilde{v}$ weakly in $L^{2,2}$ and $||v_k||_{2,2}$ are bounded,

$$< b_{k'}\nabla\varphi,\ v_{k'} > \to < b\nabla\varphi,\ \tilde{v} > \quad \text{as} \quad k' \to \infty.$$

This convergence is still valid without assuming that b is compactly supported. We do not give its proof because it is standard and we shall not apply the case when b doesn't have compact support in this paper. We now see $\tilde{v}$ is a weak solution of (4.1), which means $\tilde{v} = v$ by Theorem 5.1. (When $p = n$, we set $\varepsilon < \epsilon$.) The estimate (6.4) now yields

$$||v||_{r,r'} \leq C(||g||_{m,m'} + ||h||_{l,l'})$$

provided that g and h are compactly supported. Since C is independent of g and h, this estimate holds for every $g \in L^{m,m'}$ and $h \in L^{l,l'}$ by a standard density argument. ∎

7. Proofs of Theorems 2.1 and 2.2

PROOF OF THEOREM 2.1: We see (i) implies (ii). Indeed, $b \in L^{p,q}(Q)$ with $q < \infty$ implies that for any $\varepsilon > 0$ and $(x_0, t_0) \in Q$ there exists R_ε such that $||b||_{L^{p,q}(Q_{R_\varepsilon})} < \varepsilon$ where $Q_{R_\varepsilon} = Q_{R_\varepsilon}(x_0, t_0) \subset Q$. Since (i) yields $\omega \in L^{\infty,\beta}(Q_{R_\varepsilon/2})$ for all $2 \leq \beta < \infty$ and since Q' is relatively compact in $\Omega \times (0, T]$, by a covering argument it holds that $\omega \in L^{\infty,\beta}(Q')$ for all $2 \leq \beta < \infty$.

We now prove (i). We may assume that $(x_0, t_0) = (0, 0)$ and $R = 1$. Indeed, if we rescale a weak solution ω of (2.1) in Q_R around (0,0) with

$$\begin{cases} \omega_\lambda(x,t) = \omega(\lambda x, \lambda^2 t) \\ b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t), \end{cases}$$

we easily see $(\omega_\lambda, b_\lambda)$ satisfies (2.1) in $Q_{R/\lambda}$ and that

$$||b_\lambda||_{L^{p,q}(Q_{R/\lambda})} = ||b||_{L^{p,q}(Q_R)}.$$

(a) If we obtain that $||b_R||_{L^{p,q}(Q_1)} < \varepsilon_1$ implies $\omega_R \in L^{\infty,\beta}(Q_{1/2})$ for any $2 \leq \beta < \infty$ where $\varepsilon_1 = \varepsilon_1(n, p, \beta)$, it follows that $||b||_{L^{p,q}(Q_R)} < \varepsilon_1$ implies $\omega \in L^{\infty,\beta}(Q_{R/2})$ for any $2 \leq \beta < \infty$. The parallel argument works for (b).

We first cut off ω on $Q_{1/2}$ to obtain higher regularity in $Q_{1/2}$. We set $v = \omega\psi$ where

$$\psi \in C_0^\infty(B_1(0) \times (-1, 0]) \text{ with } \psi = 1 \text{ in } B_{1/2}(0) \times (-1/4, 0].$$

Then v equals ω in $Q_{1/2}$ and $v \in L^{2,2}(\mathbb{R}^n \times (-1,0))$ solves

$$\begin{cases} v_t - \Delta v + \nabla b v = F(b,\omega) & \text{in } \mathbb{R}^n \times (-1,0) \\ v(x,-1) = 0 \end{cases} \tag{7.1}$$

with $F(b,\omega) = \psi_t \omega - 2\nabla(\omega\nabla\psi) + \omega\Delta\psi + (b\nabla\psi)\omega$. Here

$$\nabla(\omega\nabla\psi) = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\omega^i \frac{\partial}{\partial x_j}\psi)\right)_{i=1}^{d} \quad \text{and} \quad (b\nabla\psi)\omega = \left(\sum_{k=1}^{d} \omega^k \sum_{j=1}^{n} b^i_{jk} \frac{\partial}{\partial x_j}\psi\right)_{i=1}^{d}$$

and both b and ω are extended by zero outside $B_1(0)$. Applying Theorem 6.1 yields

$$||v||_{r,r'} \leq C(||g||_{m,m'} + ||h||_{l,l'})$$

with $C = C(n,r,r',m,m',\psi)$ provided that $||b||_{p,q}$ is sufficiently small. Here

$$g = -2\omega\nabla\psi \text{ and } h = \omega(\psi_t + \Delta\psi) + (b\nabla\psi)\omega \tag{7.2}$$

and all exponents are the same as in Theorem 6.1. Since

$$||g||_{m,m'} \leq C_1||\omega||_{m,m'} \quad \text{with} \quad C_1 = C_1(\psi) \quad \text{and}$$

$$||h||_{l,l'} \leq C_2||\omega||_{m,m'} \quad \text{with} \quad C_2 = C_2(\psi,p),$$

we now obtain

$$||v||_{r,r'} \leq C'||\omega||_{L^{m,m'}(Q_1)} \tag{7.3}$$

with $C' = C'(n,r,r',m,m',\psi)$. Since $\omega \in L^{2,2}(Q_1)$, we get $v \in L^{2,\beta}$ for all $2 \leq \beta < \infty$, which means $\omega \in L^{2,\beta}(Q_{1/2})$.

We now apply (7.3) inductively. Indeed, we set $v_j = \omega\psi_j$ $(j \geq 1)$ where

$$\psi_j \in C_0^\infty(B_{R_j}(0) \times (-R_j^2, 0]) \text{ with } \psi_j = 1 \text{ in } B_{R_{j+1}}(0) \times (-R_{j+1}^2, 0].$$

Here $R_j = 1/4 + 1/4^j$. We see

$$||\omega||_{L^{r_{j+1},\beta}(Q_{R_{j+1}})} \leq ||v_j||_{r_{j+1},\beta} \leq C^j||\omega||_{L^{r_j,\beta}(Q_{R_j})}$$

with $C^j = C^j(n, r_j, r_{j+1}, \beta, \psi_j)$ provided that $2 \le r_j < r_{j+1} \le \infty$ and $1/r_j < 1/r_{j+1} + 1/n$. Here r_{j+1} should be finite when $p = n$. Setting $r_1 = 2$ yields $1/r_{j_0} < 1/2 + (-1/n)(j_0 - 1)$. Since $j_0 > n/2 + 1$ implies $1/2 - (j_0 - 1)/n < 1/\alpha$ for all $2 \le \alpha \le \infty$, it holds $1/r_{j_0} < 1/\alpha$. We now obtain $\omega \in L^{\alpha,\beta}(Q_{1/4})$ for all $2 \le \alpha \le \infty$ and $2 \le \beta < \infty$. Here α should be finite when $p = n$. ∎

REMARK: When $n/p + 2/q < 1$ and $b \in L^{p,q}(Q_R)$, since it holds

$$||\omega||_{L^{r,r'}(Q_{R/2})} \le C(||b||_{L^{p,q}(Q_R)}||\omega||_{L^{s,s'}(Q_R)} + ||\omega||_{L^{m,m'}(Q_R)})$$

for $2 \le m \le s < r \le \infty$ and $2 \le m' \le s' < r' \le \infty$ such that

$$n/s + 2/s' \le n/r + 2/r' + 1 - (n/p + 2/q) \text{ and } n/m + 2/m' \le n/r + 2/r' + 1.$$

By a standard bootstrap argument, we obtain $\omega \in L^{\infty,\infty}(Q_{R/4})$ without smallness conditions on $||b||_{L^{p,q}(Q_R)}$.

PROOF OF THEOREM 2.2: Proof of Theorem 2.2 parallels that of Theorem 2.1. We may again assume that $(x_0, t_0) = (0, 0)$ and $R = 1$. We denote $Q_1^\delta = B_1(0) \times (-1, -\delta)$ and denote the norm of $f \in L^{r,(r')}(Q_1)$ (resp. $f \in L^{r,(r')}(Q_1^\delta)$) by $||f||_{r,(r')}$ (resp. ${}_\delta||f||_{r,(r')}$). We see

$$(7.4) \qquad {}_\delta||f||_{r,(r')} \le ||f||_{r,(r')}$$

for any $0 < \delta < 1$. It is easy to see that (2.4) and (2.5) yield

$$(7.5) \qquad \omega \in L^{\infty,(\beta)}(Q_1^\delta)$$

for all $2 < \beta < \infty$ and $0 < \delta < 1$. We get (7.1) in $\mathbb{R}^n \times (-1, -\delta)$. We see (7.5) yields

$$v \in L^{2,(2+\varrho)}(Q_1^\delta) \cap L^{r,(r')}(Q_1^\delta)$$

for all $2 \le r \le \infty$, $2 < r' < \infty$ and some $\varrho > 0$. Since $p > n$, Proposition 4.1(ii) yields that there exists a positive constant $\varepsilon = \varepsilon(n, p, r')$ such that $||b||_{p,(q)} < \varepsilon$ implies

$${}_\delta||v||_{r,(r')} \le C({}_\delta||g||_{m,(m')} + {}_\delta\,||h||_{l,(l')})$$

with $C = C(n, r, r', m, m')$. Recalling (7.2) we see (7.4) yields

$$\delta\|g\|_{m,(m')} \leq C_1\|\omega\|_{m,(m')},$$

and that applying Lemma 4.2(i) and (7.4) yields

$$\delta\|h\|_{l,(l')} \leq C_2\|\omega\|_{m,(m')}.$$

Here $C_1 = C_1(\psi)$ and $C_2 = C_2(\psi, p)$. We see

$$\delta\|v\|_{r,(r')} \leq M\|\omega\|_{m,(m')}.$$

Since M does not depend on δ, it holds

$$\|v\|_{r,(r')} \leq M\|\omega\|_{m,(m')}.$$

The assumption $\omega \in L^{2,2}(Q_1)$ yields $\omega \in L^{2,(2)}(Q_1)$. By the parallel induction to the previous proof and (2.4) we see

$$\omega \in L^{\infty,\beta}(Q_{1/2}) \text{ for all } 2 < \beta < \infty. \quad ∎$$

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