Motion of hypersurfaces and geometric equations

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Motion of hypersurfaces and geometric equations

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

By Yoshikazu GIGA[1)] and Shun'ichi GOTO[2)]

1. Introduction.

We are concerned with the motion of a hypersurface whose speed locally depends on the normal vector field and its derivatives. To be specific let Γ_t denote the hypersurface expressed as the boundary of a bounded open set D_t in $\mathbb{R}^n$ $(n \geq 2)$ at time t. Let $\mathbf{n}$ denote the unit exterior normal vector field to $\Gamma_t = \partial D_t$. It is convenient to extend $\mathbf{n}$ to a vector field (still denoted by $\mathbf{n}$) on a tubular neighborhood of Γ_t such that $\mathbf{n}$ is constant in the normal direction of Γ_t. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The equation for Γ_t we consider here is of form

$$V = f(t, x, \mathbf{n}(x), \nabla \mathbf{n}(x)) \quad \text{on} \quad \Gamma_t, \tag{1.1}$$

where f is a given function and ∇ stands for spatial derivatives. Material science provides a lot of examples of (1.1) where Γ_t is an interface bounding two phases of materials (see [2, 11, 12] and references therein). For example if

$$V = -\operatorname{div}\ \mathbf{n}, \tag{1.2}$$

the hypersurface Γ_t moves by its mean curvature and (1.2) is known as the mean curvature flow equation. We note that this equation arises as a singular limit of some reaction-diffusion equations [3,17]. It is also important to consider anisotropic properties of materials. A typical model (cf. [11, 12]) is

$$V = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial p_i}(\mathbf{n})\right) + \beta(\mathbf{n}), \tag{1.3}$$

where H is convex on $\mathbb{R}^n$ and positively homogeneous of degree one and β is a function on a unit sphere S^{n-1} in $\mathbb{R}^n$. The equation (1.3) includes (1.2) as a particular example with $H(p) = |p|$ and $\beta = 0$. We remark that in general the right hand side of (1.3) is *not* expressed as a functions of curvatures $\kappa_1, \cdots, \kappa_{n-1}$ of Γ_t and $\mathbf{n}$. In other words

$$V = g(\kappa_1, \cdots, \kappa_{n-1}, \mathbf{n}) \tag{1.4}$$

exclude (1.3), although (1.4) itself is interesting.

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A fundamental analytic question to (1.1) is to construct a global-in-time unique solution $\{\Gamma_t\}_{t\geq 0}$ for a given initial data Γ_0 (allowing that Γ_t becomes empty in a finite time). There are a couple of approach depending on description of (hyper)surfaces. A classical approach appeals to a parametrization of Γ_t. For the mean curvature flow equation (1.2) Huisken [13] constructed a unique smooth solution Γ_t which shrinks to a point in a finite time provided that Γ_0 is uniformly convex and C^2 and that $n \geq 3$. A similar result is proved by Gage and Hamilton [8] when $n = 2$. Moreover, Grayson [10] proved that any embedded curve moved by (1.2) never becomes singular unless it shrinks to a point. However, for $n \geq 3$ even embedded surface may develop singularities before it shrinks to a point. Even when $n = 2$ such singularities may develop if we consider

$$V = -\operatorname{div}\ \mathbf{n} + c \tag{1.5}$$

with some constant instead of (1.2). Angenent [1] constructed a unique solution across singularities for a class of parabolic equation (1.1) including (1.5) provided that $n = 2$ (see also [2]). However, it seems difficult to track the evolution of Γ_t across singularities by a parametrization of Γ_t when $n \geq 3$.

To overcome this difficulty one way would be to describe surfaces in a weak sense such as varifolds in geometric measure theory. For (1.2) Brakke [4] constructed a global varifold solution for arbitrary initial data. Unfortunately, the uniqueness of such a solution is not known. Another way is to describe a surface Γ_t as a level sets of a function u satisfying a second order evolution equation in $\mathbb{R}^n$:

$$\partial_t u + F(t, x, \nabla u, \nabla^2 u) = 0, \tag{1.6}$$

where $\partial_t = \partial/\partial t$ and $\nabla^2 u$ denotes the Hessian matrix of u in space variables. This idea is introduced by Osher and Sethian [18] for a numerical calculation of (1.5) and independently by Chen and the authors [5]. In [5] we introduced a weak motion for solution Γ_t of (1.1) through viscosity solutions of (1.6). We constructed a *unique* global weak solution $\{\Gamma_t\}_{t\geq 0}$ with arbitrary initial data for a certain class of (1.1) including (1.2), (1.3) and (1.5) (where H is C^2 outside the origin and β is continuous). Almost at the same time Evans and Spruck [7] constructed the same solution but only for (1.2). We note that Tso [19] applies a variant of a level surface approach to (1.4) when $-g$ is the Gauss-Kronecker curvature. He constructed smooth Γ_t shrinking to a point in a finite time provided that Γ_0 is uniformly convex and C^2. The corresponding result to (1.2) is proved by Huisken [13] as is explained in the second paragraph.

Our main goal is to clarify the class of equations of form (1.1) to which the level surface approach in [5] yields a unique global weak solution $\{\Gamma_t\}_{t\geq 0}$ with a given initial data. We first derive (1.6) from (1.1). Suppose that $u > 0$ in D_t and $u = 0$ on Γ_t. If u is C^2 and $\nabla u \neq 0$ near Γ_t, we see

$$\mathbf{n} = -\frac{\nabla u}{|\nabla u|}, \quad \nabla \mathbf{n} = -\frac{1}{|\nabla u|}(\nabla^2 u - \nabla^2 u(\frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|})), \tag{1.7}$$

where $\otimes$ denotes a tensor product of vectors in $\mathbb{R}^n$. It follows from (1.7) and $V = \partial_t u/|\nabla u|$

that (1.1) is formally equivalent to (1.6) on Γ_t with

$$F(t,x,p,X) = -|p| f(t,x,-\bar{p},-\frac{1}{|p|}(X - X\bar{p}\otimes\bar{p})), \quad \bar{p} = \frac{p}{|p|}. \tag{1.8}$$

Here p is a nonzero vector in $\mathbb{R}^n$ and X is an $n \times n$ real symmetric matrix. A direct calculation shows that F in (1.8) has the scaling invariance

$$\begin{aligned} F(t,x,\lambda p,\lambda X + \sigma p\otimes p) &= \lambda F(t,x,p,X) \\ \text{for all } \lambda > 0,\ \sigma \in \mathbb{R},\ & p \in \mathbb{R}^n \setminus \{0\},\ X \in \mathbf{S}_n, \end{aligned} \tag{1.9}$$

where $\mathbf{S}_n$ denotes the space of all $n \times n$ real symmetric matrices. In [5] F is called *geometric* if F satisfies (1.9). In this paper we shall show that every geometric F is of the form (1.8) with some f and f is (essentially) uniquely determined by F. This shows that the concept "geometric" is very natural to study the equation (1.1) by our level surface approach. It will turn out that the results in [5] yields a unique global weak solution $\{\Gamma_t\}_{t\geq 0}$ of (1.1) with an arbitrary initial data Γ_0 provided that $-f$ is degenerate elliptic, continuous and grows linearly in $\nabla\mathbf{n}$, where f is assumed to be independent of x. Our assumptions on f or F is equivalent to those in [5] when F is independent of t, x but simpler than in [5].

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2. Geometric equations.

The equation (1.6) is called *geometric* if F is geometric. We observe in this section that there is roughly an one-to-one correspondence from a geometric equation to (1.1). Indeed we shall show at least formally that every level surface of a function u moves by (1.1) for some f if and only if (1.6) is geometric. Moreover, f is uniquely determined by F.

For $\bar{p} \in S^{n-1}$ we introduce a linear operator $Q_{\bar{p}}$ from $\mathbf{M}_n$ into itself defined by

$$Q_{\bar{p}}(X) = X - X(\bar{p}\otimes\bar{p}), \quad X \in \mathbf{M}_n, \tag{2.1}$$

where $\mathbf{M}_n$ denotes the space of all $n \times n$ real matrices. We note that the right hand side of (2.1) appears in (1.8).

LEMMA 2.1. *(i) The operator $Q_{\bar{p}}$ is a projection, i.e., $Q_{\bar{p}}^2 = Q_{\bar{p}}$.*
(ii) Let $L_{\bar{p}}$ denote a vector subspace of $\mathbf{S}_n$ defined by

$$L_{\bar{p}} = \{\sigma\bar{p}\otimes\bar{p};\ \sigma \in \mathbb{R}\}.$$

It holds

$$\mathbf{S}_n \cap \ker\ Q_{\bar{p}} = L_{\bar{p}}. \tag{2.2}$$

PROOF: (i) This follows directly from (2.1) if we observe

$$(\bar{p} \otimes \bar{p})(\bar{p} \otimes \bar{p}) = \bar{p} \otimes \bar{p}. \tag{2.3}$$

(ii) By (2.3) it is clear that $L_{\bar{p}}$ is contained in the kernel of $Q_{\bar{p}}$. It remains to prove that $Q_{\bar{p}}(X) = O$ for $X \in \mathbf{S}_n$ implies $X \in L_{\bar{p}}$. For an orthogonal matrix U it follows from the definition (2.1) that

$$U^{-1}Q_{\bar{p}}(X)U = Q_{\bar{q}}(Y), \quad \bar{q} = \bar{p}U, \quad Y = U^{-1}XU, \quad X \in \mathbf{M}_n.$$

We take U so that $\bar{q} = (1, 0, \cdots, 0)$ and observe that $Q_{\bar{q}}(Y) = O$ implies

$$Y = \begin{pmatrix} y_1 & \\ \vdots & O \\ y_n & \end{pmatrix} \quad \text{with} \quad y_j \in \mathbb{R}.$$

If Y is symmetric, we see $y_j = 0$ for $j \geq 2$. Since $X \in \mathbf{S}_n$ implies $Y \in \mathbf{S}_n$ we now conclude that for $X \in \mathbf{S}_n$ the condition $Q_{\bar{p}}(X) = O$ implies $Y = y_1 \bar{q} \otimes \bar{q}$ which is the same as $X \in L_{\bar{p}}$. □

We next introduce a (smooth) vector bundle E over S^{n-1} of the form

$$E = \{(\bar{p}, Q_{\bar{p}}(X));\ \bar{p} \in S^{n-1},\ X \in \mathbf{S}_n\}. \tag{2.4}$$

The bundle E is a subbundle of a trivial bundle $S^{n-1} \times \mathbf{M}_n$ (but not of $S^{n-1} \times \mathbf{S}_n$) and its fibre dimension equals $n(n+1)/2 - 1$. Let Q be a bundle map

$$Q : S^{n-1} \times \mathbf{S}_n \longrightarrow E$$

defined by

$$Q(\bar{p}, X) = (\bar{p}, Q_{\bar{p}}(X)).$$

Let L be a line bundle over S^{n-1} of form

$$L = \{(\bar{p}, X);\ \bar{p} \in S^{n-1},\ X \in L_{\bar{p}}\}. \tag{2.5}$$

The bundle L is a subbundle of $S^{n-1} \times \mathbf{S}_n$. Since Q is surjective, Lemma 2.1 provides a characterization of E as a quotient bundle.

LEMMA 2.2. *The vector bundle E is isomorphic to the quotient bundle*

$$S^{n-1} \times \mathbf{S}_n / L = \{(\bar{p}, [X]);\ \bar{p} \in S^{n-1},\ [X] \in \mathbf{S}_n / L_{\bar{p}}\}.$$

We now turn to study relation (1.8) of f and F. Since our argument is pointwise in t and x we suppress t, x-dependence of f and F in this section. The expression (1.7) of $\nabla \mathbf{n}$

shows that our f in (1.1) needs to be defined only on E not whole $S^{n-1} \times \mathbf{M}_n$. We thus consider the space $\mathcal{F}$ of all real valued functions f defined on E. To each f we correspond a function F on $(\mathbb{R}^n \setminus \{0\}) \times \mathbf{S}_n$ by (1.8), i.e.,

$$F(p, X) = -|p| f(-\bar{p}, -\frac{1}{|p|} Q_{\bar{p}}(X)), \quad \bar{p} = \frac{p}{|p|}.$$

Let $\mathcal{G}$ denote the set of all geometric real valued function F defined on $(\mathbb{R}^n \setminus \{0\}) \times \mathbf{S}_n$. Lemma 2.2 now shows that the concept "geometric" is very natural.

THEOREM 2.3. *The mapping $f \mapsto F$ is a bijection from $\mathcal{F}$ to $\mathcal{G}$.*

PROOF: Let $\mathcal{G}'$ be the set of all real valued functions F' on $S^{n-1} \times \mathbf{S}_n$ satisfying

$$(2.6) \qquad F'(\bar{p}, X + \sigma \bar{p} \otimes \bar{p}) = F'(\bar{p}, X) \quad \text{for all} \quad \sigma \in \mathbb{R}, \quad (\bar{p}, X) \in S^{n-1} \times \mathbf{S}_n.$$

By (1.9) we see the mapping $F' \mapsto F$ defined by

$$F(p, X) = |p| F'(\bar{p}, \frac{X}{|p|}), \quad \bar{p} = \frac{p}{|p|}$$

gives a bijection from $\mathcal{G}'$ to $\mathcal{G}$. By the definition (2.5) of L and (2.6) one may identify $F' \in \mathcal{G}'$ with a function on $S^{n-1} \times \mathbf{S}_n / L$. By Lemma 2.2 the mapping $f \mapsto F'$ defined by

$$F'(\bar{p}, X) = -f(-\bar{p}, -Q_{\bar{p}}(X))$$

gives a bijection from $\mathcal{F}$ to $\mathcal{G}'$ since $Q_{\bar{p}} = Q_{-\bar{p}}$. Since the mapping $f \mapsto F$ is a composition of $f \mapsto F'$ and $F' \mapsto F$, it gives a bijection from $\mathcal{F}$ to $\mathcal{G}$. □

By Theorem 2.3 we see every level surface of a function u moves by (1.1) for some f if and only if (1.6) is geometric at least formally, where F is uniquely determined from f by (1.8).

3. Existence and uniqueness of weak solutions.

We shall clarify the class of hypersurface evolution equations (1.1) to which our theory of geometric parabolic equations developed in [5] yields a unique global weak solution for a given initial data. We shall also simplify the assumptions of [5]. We first define a weak solution $\{(\Gamma_t, D_t)\}_{t \geq 0}$ of (1.1) through a viscosity solution of (1.6) similarly to [5]. As in [5] we discuss the case when Γ_t is compact.

DEFINITION 3.1: Let D_0 be a bounded open set and $\Gamma_0 (\subset \mathbb{R}^n \setminus D_0)$ be a compact set containing ∂D_0. Let $\{(\Gamma_t, D_t)\}_{t \geq 0}$ be a family of compact sets and bounded open sets in $\mathbb{R}^n$. Suppose that for some $\alpha < 0$ there is a viscosity solution $u \in C_\alpha([0, T] \times \mathbb{R}^n)$ for (1.6) with (1.8) in $(0, \infty) \times \mathbb{R}^n$ such that zero level surface of $u(t, \cdot)$ at time $t \geq 0$ equals Γ_t and that the set D_t where $u > 0$ is bounded open. If $(\Gamma_t, D_t)|_{t=0} = (\Gamma_0, D_0)$, we say

$\{(\Gamma_t, D_t)\}_{t\geq 0}$ is a *weak solution* of (1.1) with initial data (Γ_0, D_0). Here $T > 0$ is arbitrary and $v \in C_\alpha(A)$ means $v - \alpha$ is continuous and has compact support in A.

Instead of giving a definition of a viscosity solution we just remark that a viscosity solution is a kind of weak solutions satisfying the comparison principle for nonlinear degenerate elliptic equations. A fundamental theory is established by Jensen [16] and Ishii [14] (see also [15] and [6]). Since our F in (1.8) is not continuous at $p = 0$ even if f is continuous, we were forced to extend their theory. We here reproduce results on geometric parabolic equations in [5]. We consider (1.6) in $(0,\infty) \times \mathbb{R}^n$ with F independent of x. The function F is assumed to satisfy the following conditions.

(F0) $F : J = (0,\infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$ is geometric, i.e., F satisfies (1.9).

(F1) $F : J \to \mathbb{R}$ is continuous.

(F2) F is degenerate elliptic, i.e.,

$$F(t,p,X) \leq F(t,p,Y) \quad \text{if} \quad X \geq Y.$$

(F3) $-\infty < F_*(t,0,O) = F^*(t,0,O) < \infty$.

(F4) Let T be a positive number. It holds

$$\begin{aligned} &(-) \quad F_*(t,p,-I) \leq c_-(|p|) \\ &(+) \quad F^*(t,p,I) \geq -c_+(|p|) \end{aligned}$$

for all $0 < t < T$ with some $c_\pm(\sigma) \in C^1[0,\infty)$ and $c_0 > 0$ (depending only on T) such that $c_\pm(\sigma) \geq c_0$ for all $\sigma \geq 0$.

Here I denotes the identity matrix and $F_* : \bar{J} \to \mathbb{R} \cup \{\pm\infty\}$ is the lower semicontinuous relaxation of $F : J \to \mathbb{R}$, i.e.,

$$F_*(z) = \lim_{\epsilon \downarrow 0} \inf_{\substack{|w-z|<\epsilon \\ w \in J}} F(w), \qquad z = (t,p,X) \in \bar{J}.$$

The function F^* is defined by $F^* = -(-F)_*$.

PROPOSITION 3.2([5, Theorem 6.8 and 7.1]). *Assume that (F0)-(F4).*

(i) Let $\alpha < 0$. For $a \in C_\alpha(\mathbb{R}^n)$ there is a unique global viscosity solution u_a of (1.6) such that $u_a(0,x) = a(x)$ and that u_a is in $C_\alpha([0,T] \times \mathbb{R}^n)$ for every $T > 0$.

(ii) Let Γ_t denote the zero level surface of $u_a(t,\cdot)$ and D_t denote the set where $u_a(t,\cdot) > 0$. The family $\{(\Gamma_t, D_t)\}_{t\geq 0}$ is uniquely determined by (Γ_0, D_0) and independent of α and a.

By Theorem 2.3 (F0) is equivalent to the condition that F is expressed as in (1.8) with $f : (0,\infty) \times E \to \mathbb{R}$ where E is the bundle defined by (2.4). Proposition 3.2 yields a unique global solution of (1.1) (cf.[5, Theorem 7.3]).

PROPOSITION 3.3. *Assume that F defined in (1.8) satisfies (F1)-(F4). Suppose that D_0 is a bounded open set and Γ_0 ($\subset \mathbb{R}^n \setminus D_0$) is a compact set containg ∂D_0. Then there is a unique global weak solution $\{(\Gamma_t, D_t)\}_{t\geq 0}$ of (1.1) with initial data (Γ_0, D_0).*

REMARK 3.4: Proposition 3.2 is based on the comparison principle for viscosity solutions in a bounded domain. It turns out that the proof in [5] of the comparison principle can be simplified if we appeals to a maximum principle of Crandall and Ishii [6]. We shall give the simplified proof in our forthcoming paper with Ishii and Sato [9] as well as extensions to the case when F depends on x and the domain is unbounded.

We seek simple conditions on f so that Proposition 3.3 is applicable to (1.1). For this purpose we first study conditions (F1)-(F4). It is convenient to introduce

$$M(s) = \sup_{\substack{|p|\leq 1 \\ p\neq 0}} F(s,p,-I), \quad m(s) = \inf_{\substack{|p|\leq 1 \\ p\neq 0}} F(s,p,I). \tag{3.1}$$

LEMMA 3.5. *Assume that F satisfies (F0) and (F2).*
(i) For $t \geq 0$ it holds

$$F^*(t,0,O) = \lim_{\varepsilon\downarrow 0}(\varepsilon \sup_{\substack{|t-s|<\varepsilon \\ s>0}} M(s)), \quad F_*(t,0,O) = \lim_{\varepsilon\downarrow 0}(\varepsilon \inf_{\substack{|t-s|<\varepsilon \\ s>0}} m(s)).$$

(ii) If $M^*(t) < \infty$ (resp. $m_*(t) > -\infty$), then $F^*(t,0,O) \leq 0$ ($F_*(t,0,O) \geq 0$).
(iii) If F is independent of t, the following three conditions are equivalent.

$$\begin{array}{lll}
\text{(a)} & F^*(0,O) < \infty & (\text{resp. } F_*(0,O) > -\infty) \\
\text{(b)} & M < \infty & (m > -\infty) \\
\text{(c)} & F^*(0,O) \leq 0 & (F_*(0,O) \geq 0).
\end{array}$$

PROOF: (i) If $|X|$ denotes the operator norm of $X \in \mathbf{S}_n$, the estimate $|X| \leq \varepsilon$ implies

$$-\varepsilon I \leq X \leq \varepsilon I.$$

Since F is degenerate elliptic by (F2), we see

$$\sup_{|X|\leq\varepsilon} F(s,p,X) \leq F(s,p,-\varepsilon I), \quad (s,p,X) \in J.$$

The converse inequality is trivial since $|-\varepsilon I| = \varepsilon$. We thus observe that

$$\sup_{\substack{|p|\leq\varepsilon \\ p\neq 0}} \sup_{|X|\leq\varepsilon} F(s,p,X) = \sup_{\substack{|p|\leq\varepsilon \\ p\neq 0}} F(s,p,-\varepsilon I) = \varepsilon \sup_{\substack{|p|\leq\varepsilon \\ p\neq 0}} F(s,p/\varepsilon,-I) = \varepsilon M(s)$$

since F is geometric by (F0). This yields the first identity of (i). The second identity is parallelly proved.

(ii) This follows immediately from (i).

(iii) By (i) the condition (b) follows from (a). By (ii) the condition (b) implies (c). Clearly (c) implies (a) and the proof is now complete. □

We consider a slightly stronger condition than (F1) on the continuity of F in t.

(F1′) $F : [0,\infty) \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

LEMMA 3.6. *Assume that F satisfies (F1′). Let M and m be as in (3.1). The condition (F4−) (resp. (F4+)) is equivalent to*

(3.2−) $\quad M^*(t) < \infty \quad$ for $\quad t \geq 0$.

((3.2+) $\quad m_*(t) > -\infty \quad$ for $\quad t \geq 0$.)

PROOF: We only prove that (F4−) is equivalent to (3.2−) since the other equivalence is parallelly proved. The condition (F4−) implies

$$M(t) \leq \sup_{|p| \leq 1} c_-(|p|) \quad \text{for} \quad 0 \leq t \leq T$$

which yields (3.2−). Since $M^*(t)$ is upper semicontinuous, (3.2−) implies that

$$\sup_{0 \leq t \leq T} M(t) = c_T < \infty.$$

This yields (F4−) since $F(t,p,-I)$ is bounded on

$$[0,T] \times \{p \in \mathbb{R}^n;\ 1 \leq |p| \leq R\}$$

for every $R > 1$ by (F1′). □

LEMMA 3.7. *Assume that F satisfies (F0), (F1′) and (F2).*

(i) The conditions (3.2±) imply (F3)-(F4).

(ii) If F is independent of t, then

(3.3) $$M < \infty \quad \text{and} \quad m > -\infty$$

is equivalent to (F3)-(F4). Here M and m are defined by (3.1).

PROOF: This follows from a combination of Lemmas 3.5 and 3.6. □

We now rewrite our conditions in terms of f when F is of the form (1.8). The condition (F1′) is clearly equivalent to

(f1′) $f : [0,\infty) \times E \to \mathbb{R}$ is continuous, where E is the bundle defined by (2.4).

The condition (F2) is clearly equivalent to
(f2) $f(t,-\bar{p},-Q_{\bar{p}}(X)) \geq f(t,-\bar{p},-Q_{\bar{p}}(Y))$ for $X \geq Y$, $\bar{p} \in S^{n-1}$ and $t \geq 0$.
This condition means that $-f$ is degenerate elliptic. By (1.8) and (3.1) we observe that

$$\begin{aligned} M(s) &= -\inf_{0<\rho<1} \rho \inf_{|\bar{p}|=1} f(s,-\bar{p},\frac{I-\bar{p}\otimes\bar{p}}{\rho}) \\ m(s) &= -\sup_{0<\rho<1} \rho \sup_{|\bar{p}|=1} f(s,-\bar{p},\frac{-I+\bar{p}\otimes\bar{p}}{\rho}). \end{aligned} \tag{3.4}$$

It is easy to see that (3.3) is equivalent to

$$\begin{aligned} &\liminf_{\rho\downarrow 0} \rho \inf_{|\bar{p}|=1} f(-\bar{p},\frac{I-\bar{p}\otimes\bar{p}}{\rho}) > -\infty \\ &\limsup_{\rho\downarrow 0} \rho \sup_{|\bar{p}|=1} f(-\bar{p},\frac{-I+\bar{p}\otimes\bar{p}}{\rho}) < \infty. \end{aligned} \tag{3.5}$$

This condition (and also (3.3)) is fulfilled if $f = f(\bar{p}, Z)$ is positively homogeneous of degree one in Z, where $(\bar{p}, Z) \in E$, i.e.

$$f(\bar{p}, \lambda Z) = \lambda f(\bar{p}, Z) \quad \text{for all} \quad \lambda > 0. \tag{3.6}$$

By Lemma 3.7 Proposition 3.3 deduces the unique existence of global weak solutions under conditions easier to check.

THEOREM 3.8. *Assume that f is independent of x and satisfies (f1′) and (f2). Assume that f satisfies (3.2±) with (3.4) or that f is independent of t and satisfies (3.5). Let D_0 be a bounded open set in $\mathbb{R}^n$ and let Γ_0 ($\subset \mathbb{R}^n \setminus D_0$) be a compact set containing ∂D_0. Then there is a unique global weak solution $\{(\Gamma_t, D_t)\}_{t\geq 0}$ of (1.1) with initial data (Γ_0, D_0).*

REMARK 3.9: The examples (1.2), (1.3) and (1.5) fulfill all the assumptions of Theorem 3.8; here we assume that $H \in C^2(\mathbb{R}^n \setminus \{0\})$ is convex and positively homogeneous of degree one and that β is continuous. Indeed, it is easy to check (f1′) and (f2) directly. In these examples f is independent of t and satisfies (3.6). Since (3.6) implies (3.5), our f satisfies all assumptions of Theorem 3.8.

REMARK 3.10: For the mean curvature flow equation (1.2) Evans and Spruck [7] proved that the family $\{\Gamma_t\}_{t\geq 0}$ of the weak solution $\{(\Gamma_t, D_t)\}_{t\geq 0}$ is determined only by Γ_0 and is independent of D_0. In other words there is no need to distinguish interior and exterior bounded by Γ_t. This property holds for more general equation

$$V = f(t, \mathbf{n}, \nabla \mathbf{n})$$

with f in Theorem 3.8 provided that

$$f(t, -\bar{p}, -\bar{Z}) = -f(t, \bar{p}, \bar{Z}), \quad (\bar{p}, Z) \in E.$$

Instead of giving a proof we remark that this fact is easily proved by combining arguments in [7, 9].

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