An Example Of A Regular Cantor Set Whose Difference Set Is A Cantor Set With Positive Measure

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An Example Of A Regular Cantor Set Whose Difference Set Is A Cantor Set With Positive Measure

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§.0 INTRODUCTION

In this paper, we give an example of a regular Cantor set whose self-difference set is a Cantor set and, at the same time, has a positive measure. This is a counter example of one of the questions posed by J.Palis related to homoclinic bifurcation of surface diffeomorphisms.

In [1], Palis–Takens investigated homoclinic bifurcation in the following context. Let M be a closed 2-dimensional manifold. We say a C^r-diffeomorphism $\phi : M \to M$ is *persistently hyperbolic* if there is a C^r-neighborhood $\mathcal{U}$ of ϕ and for every $\psi \in \mathcal{U}$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set (refer [2] for the definition and the notation of terminologies of dynamical systems). Let $\{\phi_\mu\}_{\mu\in\mathbf{R}}$ be a 1-parameter family of C^2-diffeomorphisms on M. We define $\{\phi_\mu\}_{\mu\in\mathbf{R}}$ has a *homoclinic Ω-explosion* at $\mu = 0$ if:

i) For $\mu < 0$, ϕ_μ is persistently hyperbolic;

ii) For $\mu = 0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\widetilde{\Omega}(\phi_0) = \lim_{\mu\uparrow 0}\Omega(\phi_\mu)$ together with a homoclinic orbit of tangency $\mathcal{O}$ associated with a fixed saddle point p, so that $\Omega(\phi_0) = \widetilde{\Omega}(\phi_0) \cup \mathcal{O}$; the product of the eigenvalues of $d\phi_0$ at p is different from one in norm;

iii) The separatrices have quadratic tangency along $\mathcal{O}$ unfolding generically; $\mathcal{O}$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of ϕ_0.

Let Λ be a basic set of a diffeomorphism ψ on M. $d^s(\Lambda)$ ($d^u(\Lambda)$) denotes the Hausdorff dimension in the transversal direction of the stable (unstable) foliation of the stable (unstable) manifold of Λ (refer [2] for the precise definition), and is called the stable (unstable) *limit capacity*. B denotes the set of valued $\mu > 0$ for which ϕ_μ is not persistently hyperbolic.

The result of Palis–Takens is;

THEOREM[1]. *Let $\{\phi_\mu;\ \mu \in \mathbf{R}\}$ be a family of diffeomorphisms of M with a homoclinic Ω-explosion at $\mu = 0$. Suppose that $d^s(\Lambda) + d^u(\Lambda) < 1$, where Λ is the basic set of ϕ_0 associated with the homoclinic tangency. Then*

$$\lim_{\delta \to 0} \frac{m(B \cap [0,\delta])}{\delta} = 0$$

where m denotes Lebesgue measure.

This result states that, in the case of $d^s(\Lambda) + d^u(\Lambda) < 1$, the measure of the parameters for which bifurcation occurs is relatively small.

For the next step, the case of $d^s(\Lambda) + d^u(\Lambda) > 1$ comes into question. In the proof of the theorem above, one of the essential point is a question of how two Cantor sets intersect each other when the one Cantor set is slided. In [3], Palis posed the following questions.

(Q.1) For affine Cantor sets X and Y in the line, is it true that $X - Y$ either has measure zero or contains intervals ?

(Q.2) Same for regular Cantor sets,

where for two subset X,Y of $\mathbf{R}$,

$$X - Y = \{\ x - y \mid x \in X,\ y \in Y\ \}\ .$$

This can be also written as;

$$X - Y = \{\ \mu \in \mathbf{R} \mid X \cap (\mu + Y) \neq \phi\ \}\ ,$$

namely, $X - Y$ is the set of parameters for which X and Y intersect when Y is slided.

Cantor set Λ in $\mathbf{R}$ is called affine (regular or C^r for $1 \leq r \leq \infty$) if Λ is defined with finite number of expanding affine (C^2 or C^r) maps, namely;

DEFINITION. Let Λ be a Cantor set on a closed interval I. For $1 \leq r \leq \infty$, Λ is called *C^r-Cantor set* if there are closed disjoint intervals $I_1, \cdots, I_k$ on I and onto C^r-maps $f_i : I_i \to I$ for all $1 \leq i \leq k$ such that;

(i) $|f'_i(x)| > 1 \quad \forall x \in I_i$,

(ii) $\Lambda = \bigcap_{n=0}^{\infty} \{ \bigcup_{\sigma \in \Sigma_n^k} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} \cdots f_{\sigma(n)}^{-1}(I) \}$,
where $\Sigma_n^k = \{\sigma : \{1, \cdots, n\} \to \{1, \cdots, k\}\}$.

Our result in this paper is that there is a counter example of (Q.2), namely;

THEOREM. *There exists a C^∞-Cantor set Λ such that*

(i) $m(\Lambda - \Lambda) > 0$,

(ii) $\Lambda - \Lambda$ *is a Cantor set.*

One will see in the proof of this theorem that Λ is constructed very artificially and cannot be defined as an *analytic* Cantor set. Therefore, this theorem may not give any clue to (Q.1), i.e. the affine case. In fact, the affine case seems to have an essential difficulty of these problems. In the case of $d^s(\Lambda) + d^u(\Lambda) > 1$, for the practical application to homoclinic bifurcation, the problems of "genericity " or "openness " may have more importance.

In the following sections 1 and 2, we give the proof of the theorem.

§.1 DEFINITION OF THE CANTOR SETS $\Lambda(s)$, $\Gamma(s)$

First of all, we define two cantor set depending on a sequence of real numbers.

DEFINITION 1. Let $I = [x_1, x_2]$ be a closed interval and λ a real number with $0 < \lambda < \frac{1}{2}$. We define,

$$I_0(\lambda; I) = [x_1, x_1 + \lambda(x_2 - x_1)]$$

$$I_1(\lambda; I) = [x_2 - \lambda(x_2 - x_1), x_2] \ .$$

Definition 2 (Cantor set $\Lambda(s)$). Let $I^0 = [0,1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$ for all $i \geq 1$. We define the Cantor set $\Lambda(s)$ as follows.

Let $I_0^1 = I_0(\lambda_1; I^0)$, $I_1^1 = I_1(\lambda_1; I^0)$ and $I^1 = I_0^1 \cup I_1^1$. Δ_n denotes the set of all sequences of 0 and 1 of length n. When I_β^{n-1}'s are defined for all $\beta \in \Delta_{n-1}$, we define;

$$I_{\beta 0}^n = I_0(\lambda_n; I_\beta^{n-1})$$
$$I_{\beta 1}^n = I_1(\lambda_n; I_\beta^{n-1}) \ .$$

Inductively, we can define I_α^n for all $\alpha \in \Delta_n$ and for all $n \geq 0$. Define

$$I^n = \bigcup_{\alpha \in \Delta_n} I_\alpha^n$$

and

$$\Lambda(s) = \bigcap_{n \geq 0} I^n \ .$$

This is clearly a Cantor set by the definition.

Next, we define another Cantor set $\Gamma(s)$.

Definition 3. Let $J = [x_1, x_2]$ and $0 < \lambda < \frac{1}{3}$. We define,

$$J_0(\lambda; J) = [x_1, x_1 + \lambda(x_2 - x_1)]$$
$$J_1(\lambda; J) = [\frac{x_1 + x_2}{2} - \frac{\lambda}{2}(x_2 - x_1), \frac{x_1 + x_2}{2} + \frac{\lambda}{2}(x_2 - x_1)]$$
$$J_2(\lambda; J) = [x_2 - \lambda(x_2 - x_1), x_2] \ .$$

Definition 4. Let $J^0 = [-1,1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Let

$$J_0^1 = J_0(\lambda_1; J^0)$$
$$J_1^1 = J_1(\lambda_1; J^0)$$
$$J_2^1 = J_2(\lambda_1; J^0)$$

and Π_n denote the set of all sequences of $0, 1, 2$ of length n. When J_δ^{n-1}'s are defined for all $\delta \in \Pi_{n-1}$, we define;

$$J_{\delta 0}^n = J_0(\lambda_n; J_\delta^{n-1})$$
$$J_{\delta 1}^n = J_1(\lambda_n; J_\delta^{n-1})$$
$$J_{\delta 2}^n = J_2(\lambda_n; J_\delta^{n-1}) .$$

Inductively, we can define J_γ^n for all $\gamma \in \Pi_n$ and for all $n \geq 0$. Define

$$J^n = \bigcup_{\gamma \in \Pi_n} J_\gamma^n$$

and

$$\Gamma(s) = \bigcap_{n \geq 0} J^n .$$

THEOREM 1. *Let* $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ *be a sequence of real numbers with* $0 < \lambda_i < \frac{1}{3}$ *for all* $i \geq 1$. *Then,*

$$\Lambda(s) - \Lambda(s) = \Gamma(s) .$$

PROOF:

$$\Lambda(s) - \Lambda(s) = (\bigcap_{n \geq 0} I^n) - (\bigcap_{n \geq 0} I^n) .$$

By a straightforward argument, it can be seen that,

$$(\bigcap_{n \geq 0} I^n) - (\bigcap_{n \geq 0} I^n) = \bigcap_{n \geq 0} (I^n - I^n) .$$

Therefore, it is enough to show that

$$I^n - I^n = J^n \qquad \forall n \geq 0 .$$

That is easily obtained by the following lemma 1.

LEMMA 1. *For all* $n \geq 0$,

(i) for all $\alpha, \beta \in \Delta_n$, *there exists a* $\gamma \in \Pi_n$ *such that*

$$I_\alpha^n - I_\beta^n = J_\gamma^n$$

(ii) for all $\gamma \in \Pi_n$, *there exist* $\alpha, \beta \in \Delta$ *such that*

$$J_\gamma^n = I_\alpha^n - I_\beta^n \ .$$

The following lemma 2 is trivial.

LEMMA 2. *Let* $I = [x_1, x_2]$ *and* $J = [y_1, y_2]$ *be two closed intervals. Then,* $I - J = [x_1 - y_2, x_2 - y_1]$.

PROOF OF LEMMA 1: We prove (i) and (ii) simultaneously by induction.

When $n = 0$, the statement holds, because $I^0 - I^0 = J^0$. Assume that the statement is valid for n.

In the case of (i): Let $\alpha, \beta \in \Delta_{n+1}$ and $\alpha = \tilde{\alpha}\alpha_{n+1}, \beta = \tilde{\beta}\beta_{n+1}$ for $\tilde{\alpha}, \tilde{\beta} \in \Delta_n$ and $\alpha_{n+1}, \beta_{n+1} = 0$ or 1 . Then, by the hypothesis of induction, there exists a $\tilde{\gamma} \in \Pi_n$ such that $I^n_{\underset{\alpha}{\sim}} - I^n_{\underset{\beta}{\sim}} = J^n_{\underset{\gamma}{\sim}}$.

In the case of (ii): Let $\gamma \in \Pi_{n+1}$ and $\gamma = \tilde{\gamma}\gamma_{n+1}$ for some $\tilde{\gamma} \in \Pi_n$ and $\gamma_{n+1} = 0$ or 1. Then, by the hypothesis of induction, there exist $\tilde{\alpha}, \tilde{\beta} \in \Delta_n$ such that $I^n_{\underset{\alpha}{\sim}} - I^n_{\underset{\beta}{\sim}} = J^n_{\underset{\gamma}{\sim}}$.

In both cases, it is clear that the statement of lemma 1 is obtained from the following proposition.

PROPOSITION. *Suppose that*

$$I^n_{\underset{\alpha}{\sim}} - I^n_{\underset{\beta}{\sim}} = J^n_{\underset{\gamma}{\sim}}$$

for $\tilde{\alpha}, \tilde{\beta} \in \Delta_n$ and $\tilde{\gamma} \in \Pi_n$. Then,

$$\begin{aligned} J^{n+1}_{\tilde{\gamma}0} &= I^{n+1}_{\tilde{\alpha}0} - I^{n+1}_{\tilde{\beta}1} \\ J^{n+1}_{\tilde{\gamma}1} &= I^{n+1}_{\tilde{\alpha}0} - I^{n+1}_{\tilde{\beta}0} = I^{n+1}_{\tilde{\alpha}1} - I^{n+1}_{\tilde{\beta}1} \\ J^{n+1}_{\tilde{\gamma}2} &= I^{n+1}_{\tilde{\alpha}1} - I^{n+1}_{\tilde{\beta}0} \ . \end{aligned}$$

PROOF: Let $I^n_{\tilde{\alpha}} = [x_1, x_2]$ and $I^n_{\tilde{\beta}} = [y_1, y_2]$. Then, $J^n_{\tilde{\gamma}} = [x_1 - y_2, x_2 - y_1]$.

Since the length of $I^n_{\tilde{\alpha}}$ and $I^n_{\tilde{\beta}}$ are the same, we denote $\ell_n = x_2 - x_1 = y_2 - y_1$.

By the definition,

$$\begin{aligned} I^{n+1}_{\tilde{\alpha}0} &= [x_1, x_1 + \lambda_{n+1}\ell_n] \\ I^{n+1}_{\tilde{\alpha}1} &= [x_2 - \lambda_{n+1}\ell_n, x_2] \\ I^{n+1}_{\tilde{\beta}0} &= [y_1, y_1 + \lambda_{n+1}\ell_n] \\ I^{n+1}_{\tilde{\beta}1} &= [y_2 - \lambda_{n+1}\ell_n, y_2] \ . \end{aligned}$$

By lemma 2,

$$\begin{aligned} I^{n+1}_{\tilde{\alpha}0} - I^{n+1}_{\tilde{\beta}0} &= [x_1 - y_1 - \lambda_{n+1}\ell_n, x_1 - y_1 + \lambda_{n+1}\ell_n] \\ I^{n+1}_{\tilde{\alpha}0} - I^{n+1}_{\tilde{\beta}1} &= [x_1 - y_2, x_1 - y_2 + 2\lambda_{n+1}\ell_n] \\ I^{n+1}_{\tilde{\alpha}1} - I^{n+1}_{\tilde{\beta}0} &= [x_2 - y_1 - 2\lambda_{n+1}\ell_n, x_2 - y_1] \\ I^{n+1}_{\tilde{\alpha}1} - I^{n+1}_{\tilde{\beta}1} &= [x_2 - y_2 - \lambda_{n+1}\ell_n, x_2 - y_2 + \lambda_{n+1}\ell_n] \ . \end{aligned}$$

On the other hand, by the definition and (1),

$$\begin{aligned} J^{n+1}_{\tilde{\gamma}0} &= [x_1 - y_2, x_1 - y_2 + 2\lambda_{n+1}\ell_n] \\ J^{n+1}_{\tilde{\gamma}1} &= [\frac{1}{2}(x_1 - y_2 + x_2 - y_1) - \lambda_{n+1}\ell_n, \frac{1}{2}(x_1 - y_2 + x_2 - y_1) + \lambda_{n+1}\ell_n] \\ J^{n+1}_{\tilde{\gamma}2} &= [x_2 - y_1 - 2\lambda_{n+1}\ell_n, x_2 - y_1] \ . \end{aligned}$$

Since $\frac{1}{2}(x_1 - y_2 + x_2 - y_1) = x_1 - y_1$, the statement is obtained. □

§.2 Regularity And Positivity

The combination of Theorem 1 and the following Theorem 2 yields our main Theorem.

Theorem 2. *There exists a sequence of real numbers $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$ such that;*

(i) $\Lambda(s)$ is a C^∞–Cantor set,

(ii) $m(\Lambda(s) - \Lambda(s)) > 0$,

where $m(\ )$ denotes the Lebesgue measure.

In the following, we shall prove this theorem.

Let $\{r_n\}_{n\geq 0}$ be a sequence of positive real numbers such that

$$\sum_{n=0}^{\infty} r_n < 1 . \tag{1}$$

We define $\{\lambda_n\}_{n\geq 1}$ using this $\{r_n\}_{n\geq 0}$ as follows.

$$\begin{cases} \lambda_1 = \dfrac{1}{3}(1 - r_0) \\ \lambda_{n+1} = \dfrac{1}{3}\left(\dfrac{1 - \sum_{i=0}^{n} r_i}{1 - \sum_{i=0}^{n-1} r_i}\right) . \end{cases} \tag{2}$$

It is clear that

$$0 < \lambda_n < \frac{1}{3} \qquad \forall n \geq 1 . \tag{3}$$

Lemma 3.

$$\sum_{i=0}^{n} r_i = 1 - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j \qquad \forall n \geq 0 .$$

PROOF: We prove this lemma by induction. For $n = 0$, it is the definition of λ_1. Assume that the statement is valid for n.

$$\begin{aligned}
\sum_{i=0}^{n+1} r_i &= \sum_{i=0}^{n} r_i + r_{n+1} \\
&= \sum_{i=0}^{n} r_i + (1 - 3\lambda_{n+2})(1 - \sum_{i=0}^{n} r_i) \\
&= 1 - 3\lambda_{n+2} + 3\lambda_{n+2} \sum_{i=0}^{n} r_i \\
&= 1 - 3\lambda_{n+2} + 3\lambda_{n+2}(1 - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j) \\
&= 1 - 3^{n+2} \prod_{j=1}^{n+2} \lambda_j \ . \qquad \square
\end{aligned}$$

LEMMA 4.

$$r_n = 3^n(1 - 3\lambda_{n+1}) \prod_{j=0}^{n} \lambda_j \qquad \forall n \geq 0 \ .$$

where, we assume $\lambda_0 = 1$ for the simplicity of notation.

PROOF: For $n = 0$, it is the definition of λ_1. For $n \geq 1$,

$$\begin{aligned}
&3^n(1 - 3\lambda_{n+1}) \prod_{j=1}^{n} \lambda_j \\
=&3^n \prod_{j=1}^{n} \lambda_j - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j \\
=&(1 - \sum_{j=1}^{n-1} r_j) - (1 - \sum_{j=1}^{n} r_j) \\
=&r_n \ . \qquad \square
\end{aligned}$$

2.1 The positivity of the measure of $\Gamma(s)$.

LEMMA 5. *Let* $\{r_n\}_{n\geq 0}$ *be a sequence of positive real numbers such that* $\sum_{n=0}^{\infty} r_n < 1$, *and* $\{\lambda_n\}_{n\geq 1}$ *be the sequence defined by (2). Then,* $m(\Gamma(s)) > 0$.

This lemma is a consequence of the following lemma by applying Lemma 4.

LEMMA 6. *Let* $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ *be a sequence of real numbers such that* $0 < \lambda_n < \frac{1}{3}$ $\forall n \geq 1$. *Then,*

$$m(\Gamma(s)) = 2(1 - \sum_{n=0}^{\infty}(3^n(1-3\lambda_{n+1})\prod_{j=1}^{n}\lambda_j)) .$$

PROOF: Let w_n denotes the length of each interval of J^n. For example, $w_0 = 2$, $w_1 = 2\lambda_1$, $w_2 = \lambda_2 w_1 = 2\lambda_1\lambda_2$. In general, $w_n = \lambda_n w_{n-1}$, and,

$$w_n = 2\prod_{j=1}^{n}\lambda_j .$$

In each interval of J^{n-1}, there are three intervals of J^n and therefore, there are two gaps in it. The sum of the lengths of these gaps in J^{n-1} is

$$w_{n-1} - 3w_n .$$

Since there are 3^{n-1} intervals in J^{n-1} , the sum of the lengths of the open gaps of the n–th level is

$$3^{n-1}(w_{n-1} - 3w_n) .$$

Therefore, the sum of the lengths of the all open gaps is ,

$$\begin{aligned}
&\sum_{n=0}^{\infty} 3^n(w_n - 3w_{n+1}) \\
=&\sum_{n=0}^{\infty} 3^n w_n(1-3\lambda_{n+1}) \\
=&2\sum_{n=0}^{\infty} 3^n(1-3\lambda_{n+1})\prod_{j=1}^{n}\lambda_j . \qquad \square
\end{aligned}$$

2.2 The regularity of $\Lambda(s)$.

In the following, we define a sequence $\{r_n\}_n \geq 0$ (and so $\{\lambda_n\}_{n\geq 1}$), and prove that $\Lambda(s)$ is C^∞.

First of all, we fix a C^∞–function $h(t)$ on $[0,1]$ with the following properties.

(i) $h(t) \geq 0$,

(ii) $\int_0^1 h(t)dt = 1$,

(iii) for all $n \geq 0$,

$$\begin{cases} \lim_{t\downarrow 0} h^{(n)}(t) = 0 , \\ \lim_{t\uparrow 1} h^{(n)}(t) = 0 , \end{cases}$$

where $h^{(n)}$ denotes the n-th derivative of h.

(For example,

$$\begin{cases} h(t) = \dfrac{e^{-\frac{1}{t(1-t)}}}{\int_0^1 e^{-\frac{1}{s(1-s)}} ds} & 0 < t < 1 \\ 0 & t = 0,1 . \end{cases}$$

is a such function.)

To define $\{r_n\}_{n\geq 0}$, we define the following sequences. For each integers $n \geq 0$, let

$$q_n = \max\{q_0, q_1, \cdots, q_{n-1}, 1, \sup_{t\in[0,1]} |h^{(n)}(t)| \} . \tag{4}$$

Clearly,

$$1 \leq q_0 \leq q_1 \leq q_2 \leq \cdots .$$

For $n \geq 0$, we define,

$$r_n = \frac{4^{-(n^2+2)}}{q_n} \tag{5}$$

Clearly,

$$\frac{1}{16} \geq r_0 > r_1 > r_2 > \cdots . \tag{6}$$

Since $r_n \leq 4^{-(n^2+2)} \leq 4^{-(n+2)}$, we have,

$$\sum_{n=0}^{\infty} r_n < \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{12} . \tag{7}$$

Therefore, $\{r_n\}_{n\geq 0}$ satisfy (1). Let $\{\lambda_n\}_{n\geq 1}$ be a sequence defined by (2) from this $\{r_n\}_{n\geq 0}$.

By (6) and (7), we can easily see that $\lambda_n > \frac{1}{4}$ for all $n \geq 1$. So, together with (3), we have,

$$\frac{1}{4} < \lambda_n < \frac{1}{3} \qquad \forall n \geq 1 \ . \tag{8}$$

We define another sequence of positive real numbers;

$$m_n = \frac{3(3r_{n-1} - 2r_n)}{1 - \sum_{i=0}^{n-1} r_i} \qquad \forall n \geq 1 \ .$$

Since $\{r_n\}_{n\geq 0}$ is monotonically decreasing and by (7), $m_n > 0$ for all $n \geq 1$. Moreover,

$$\begin{aligned} m_n &< \frac{9r_{n-1}}{1 - \sum_{i=0}^{n-1} r_i} \\ &< 10 \cdot r_{n-1} \ . \end{aligned} \tag{9}$$

U^0 denotes the open interval between I_0^1 and I_1^1, namely;

$$U^0 = I^0 \backslash (I_0^1 \cup I_1^1) \ .$$

In general, U_α^{n-1} $(\alpha \in \Delta_{n-1})$ denotes the open interval between $I_{\alpha 0}^n$ and $I_{\alpha 1}^n$ in I_α^{n-1}, namely;

$$U_\alpha^{n-1} = I_\alpha^{n-1} \backslash (I_{\alpha 0}^n \cup I_{\alpha 1}^n) \ .$$

Let $\ell_n = \ell(I_\alpha^n)$. Then, by the definition,

$$\ell_n = \lambda_n \ell_{n-1} \ , \tag{10}$$

$$\ell_n = \prod_{j=1}^{n} \lambda_j \ . \tag{11}$$

Let $u_n = \ell(\overline{U_\alpha^n})$, and $\overline{U_\alpha^n} = [x_\alpha, y_\alpha]$. Then,

$$u_n = \ell_n - 2\ell_{n+1} \ ,$$

and

$$u_n = y_\alpha - x_\alpha \ .$$

We prove the smoothness of $\Lambda(s)$ as follows. We define a non-negative C^∞–function $f(t)$ on $[0, \lambda_1]$ and define;

$$g(t) = \int_0^t (f(s) + 3) ds .$$

We put;

$$\begin{cases} g_0(t) = g(t) & on\ [0, \lambda_1] \\ g_1(t) = g(t - 1 + \lambda_1) & on\ [1 - \lambda_1, 1] . \end{cases}$$

and prove that these g_0 and g_1 define $\Lambda(s)$.

DEFINITION OF $f(t)$. Recall that we have already defined a C^∞–function $h(t)$ on $[0, 1]$. We define $f(t)$ using this $h(t)$ as follows.

First, note that

$$u_n > \frac{\ell_n}{3}$$

because

$$\begin{aligned} u_n &= \ell_n - 2\ell_{n+1} \\ &= \ell_n(1 - 2\lambda_{n+1}) \\ &> \ell_n(1 - 2 \cdot \frac{1}{3}) = \frac{\ell_n}{3} . \end{aligned}$$

Let $[x'_\alpha, y'_\alpha]$ be the interval of length $\frac{\ell_n}{3}$ in the middle of U^n_α such that

$$[x'_\alpha, y'_\alpha] = [x_\alpha + \frac{1}{2}(u_n - \frac{\ell_n}{3}), y_\alpha - \frac{1}{2}(u_n - \frac{\ell_n}{3})] .$$

We define $f(t)$ on $[0, \lambda_1]$ as follows.

(i) On $U^n_\alpha \cap [0, \lambda_1]$ $(n \neq 1)$,

$$\begin{cases} f(t) = m_n h(\frac{t - x'_\alpha}{\frac{\ell_n}{3}}) & t \in [x'_\alpha, y'_\alpha] \\ f(t) = 0 & \text{otherwise} . \end{cases}$$

(ii) On $\Lambda(s) \cap [0, \lambda_1]$, $f(t) = 0$.

In the following, we show that;

(I) $f(t)$ is a C^∞–function on $[0, \lambda_1]$.

(II) g_0 and g_1 define $\Lambda(s)$.

2.3 The smoothness of $f(t)$.

For any $p \geq 0$, we define a function $f_p(t)$ as follows. Let $\Delta_n^0 = \{\alpha = \alpha_1 \cdots \alpha_n \in \Delta_n | \alpha_1 = 0\}$ and $U = \bigcup_{n \geq 1, \alpha \in \Delta_n^0} U_\alpha^n$. Note that $U = [0, \lambda_1] \backslash \Lambda(s)$. Since $f(t)$ is C^∞ on U, $f^{(p)}(t)$ exists for all $p \geq 0$ on U. We define,

$$\begin{cases} f_p(t) = f^{(p)}(t) & \text{for} \quad t \in U \\ f_p(t) = 0 & \text{otherwise (} i.e. \quad t \in \Lambda(s) \text{) .} \end{cases}$$

In order to show the smoothness of $f(t)$, we prove that;

LEMMA 7. *For any $p \geq 0$, f_p is differentiable at any $t \in [0, \lambda_1]$ and $f_p'(t) = f_{p+1}(t)$.*

Since $f_0 = f$, this lemma says that f is C^∞.

We fix a $p \geq 0$. Since f_p is differentiable at any $t \in U$, it is enough to show that f_p is differentiable at any $t \in \Lambda(s) \cap [0, \lambda_1]$. Clearly, Lemma 7 follows from the following lemma.

LEMMA 8. *At any $t \in \Lambda(s) \cap [0, \lambda_1]$, f_p is differentiable and $f_p'(t) = 0$.*

PROOF: Let $t_0 \in \Lambda(s) \cap [0, \lambda_1]$. Since $f_p(t_0) = 0$, it is enough to show that

$$\lim_{t \to t_0} \frac{f_p(t)}{t - t_0} = 0 \quad . \tag{12}$$

There are following two cases.

(i) t_0 is an end point of a $\overline{U_\alpha^n}$ and t approaches to t_0 in U_α^n.

(ii) otherwise.

In the case (i), (12) is clear because $f(t) \equiv 0$ in an neighbourhood of the end points of $\overline{U_\alpha^n}$. Therefore, we consider the case (ii). Assume that $t_0 < t$, namely t approaches to t_0 from the above. The similar argument gives the converse.

We shall show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $t - t_0 < \delta$, then

$$\frac{f_p(t)}{t - t_0} < \epsilon \, .$$

Now let $\epsilon > 0$ be given. Let $n_0 \geq p+2$ be an integer such that, if $n \geq n_0$ then

$$10 \cdot 3^{p+1} \cdot 4^{(n(p+1)-(n-1)^2-2)} < \epsilon .$$

Clearly such n_0 exists.

By the definition of U_α^n's, it can be seen that taking $\delta > 0$ sufficiently small, if $t - t_0 < \delta$ then for any U_α^n with $[t_0, t] \cap U_\alpha^n \neq \emptyset$, it holds that $n \geq n_0$. Because, if it is not the case, t_0 must be an end point of a $\overline{U_\alpha^n}$ with $n < n_0$ and this contradicts with the assumption.

If $t \in \Lambda(s)$, then by the definition of f_p, $f_p(t) = 0$. Therefore, we assume that $t \in U_\alpha^n$ for some $n \geq n_0$ and $\alpha \in \Delta_n$.

Let t_1 be the left end point of $\overline{U_\alpha^n}$. Clearly, $t_0 < t_1 < t$ and

$$\frac{f_p(t)}{t-t_0} < \frac{f_p(t)}{t-t_1} .$$

Since f_p is differentiable on $\overline{U_\alpha^n}$, by the mean value theorem,

$$\begin{aligned} \frac{f_p(t)}{t-t_1} &\leq \sup_{t \in U_\alpha^n} |f_p'(t)| \\ &= \sup_{t \in [x_\alpha', y_\alpha']} |f^{(p+1)}(t)| \\ &= m_n (\frac{3}{\ell_n})^{p+1} \sup_{t \in [0,1]} h^{(p+1)}(t) \end{aligned} \tag{13}$$

By (8),(9) and (11), we have, $\ell_n = \prod_{j=1}^n \lambda_j > (\frac{1}{4})^n$ and $m_n < 10 \cdot r_{n-1}$. Therefore,

$$\begin{aligned} (13) &< 10 \cdot r_{n-1} \cdot 3^{p+1} \cdot (4^n)^{p+1} \cdot \{ \sup_{t \in [0,1]} h^{(p+1)}(t) \} \\ &= 10 \cdot \frac{4^{-((n-1)^2+2)}}{q_{n-1}} \cdot 3^{p+1} \cdot (4^n)^{p+1} \cdot \{ \sup_{t \in [0,1]} h^{(p+1)}(t) \} \end{aligned} \tag{14}$$

Since $n - 1 \geq p+1$, by the definition of q_{n-1},

$$\begin{aligned} (14) &\leq 10 \cdot 3^{p+1} \cdot 4^{n(p+1)} \cdot 4^{-((n-1)^2+2)} \\ &= 10 \cdot 3^{p+1} \cdot 4^{(n(p+1)-(n-1)^2-2)} < \epsilon . \end{aligned}$$

□

2.4 g_0 and g_1 define $\Lambda(s)$.

For the proof, we need some lemmas.

LEMMA 9. *For any $n \geq 1$ and $\alpha \in \Delta_n^0$,*

$$\int_{U_\alpha^n} f(t)dt = \frac{1}{3} m_n \ell_n \ .$$

PROOF: By the definition of $f(t)$,

$$\begin{aligned}\int_{U_\alpha^n} f(t)dt &= \int_{x'_\alpha}^{y'_\alpha} m_n h(\frac{t - x'_\alpha}{\frac{\ell_n}{3}})dt \\ &= m_n \int_0^{\frac{\ell_n}{3}} h(\frac{t}{\frac{\ell_n}{3}})dt \\ &= m_n \int_0^1 h(s)\frac{\ell_n}{3} ds \\ &= \frac{1}{3} m_n \ell_n \ . \qquad \square\end{aligned}$$

LEMMA 10. *For all $n \geq 1$,*

$$\ell_{n-1} = g_0(\ell_n) \ .$$

PROOF:

$$\begin{aligned}g_0(\ell_n) &= \int_0^{\ell_n} (f(t) + 3)dt \\ &= 3\ell_n + \int_0^{\ell_n} f(t)dt \ . \qquad (15)\end{aligned}$$

In $[0, \ell_n]$, $f(t)$ has positive value only on countable number of open intervals U_α^k such that $U_\alpha^k \subset [0, \ell_n]$. Note that $U_\alpha^k \subset [0, \ell_n]$ for $\alpha = \alpha_1 \cdots \alpha_k$ if and only if $k \geq n$ and $\alpha_1 \cdots \alpha_n = 0 \cdots 0$. Therefore, for $k \geq n$, there exist 2^{k-n} open intervals of U_α^k's in $[0, \ell_n]$.

By lemma 9,

$$\int_0^{\ell_n} f(t)dt = \sum_{U_\alpha^k \subset [0,\ell_n]} \left(\int_{U_\alpha^k} f(t)dt \right)$$
$$= \sum_{i=n}^{\infty} 2^{i-n} \cdot \frac{1}{3} m_i \ell_i$$
$$= \sum_{i=n}^{\infty} 2^{i-n} \frac{1}{3} \ell_i \frac{3(r_{i-1} - 2r_i)}{1 - \sum_{j=1}^{i-1} r_j} \tag{16}$$

By lemma 3, $\ell_i = \prod_{j=1}^{i} \lambda_j = \frac{1}{3^i}(1 - \sum_{j=1}^{i-1} r_j)$. Therefore, by lemma 4,

$$\begin{aligned}
(16) &= \sum_{i=n}^{\infty} 2^{i-n} \frac{1}{3^i}(3r_{i-1} - 2r_i) \\
&= 2^{-(n-1)} \sum_{i=n}^{\infty} \left\{ \left(\frac{2}{3}\right)^{i-1} r_{i-1} - \left(\frac{2}{3}\right)^{i} r_i \right\} \\
&= 2^{-(n-1)} \lim_{k \to \infty} \left\{ \left(\frac{2}{3}\right)^{n-1} r_{n-1} - \left(\frac{2}{3}\right)^{k} r_k \right\} \\
&= 2^{-(n-1)} \left(\frac{2}{3}\right)^{n-1} r_{n-1} \\
&= \frac{r_{n-1}}{3^{n-1}} \\
&= \frac{1}{3^{n-1}} 3^{n-1}(1 - 3\lambda_n)\ell_{n-1} \\
&= \ell_{n-1} - 3\lambda_n \ell_{n-1} \\
&= \ell_{n-1} - 3\ell_n \ .
\end{aligned}$$

Hence by (15), we have,

$$g_0(\ell_n) = \ell_{n-1} \ .$$

□

Let $I_\alpha^n = [r_\alpha^n, s_\alpha^n]$.

LEMMA 11. *For all* $\alpha, \alpha' \in \Delta_n$,

$$\int_{I_\alpha^n} f(t)dt = \int_{I_{\alpha'}^n} f(t)dt \ .$$

PROOF: Since $\ell(I_\alpha^n) = \ell(I_{\alpha'}^n)$, by the definition of $f(t)$, the statement is clear because $f(t + r_\alpha^n) = f(t + r_{\alpha'}^n)$ for $0 \leq t \leq s_\alpha^n - r_\alpha^n = s_{\alpha'}^n - r_{\alpha'}^n$ □

Finally, we prove that,

$$\Lambda(s) = \bigcap_{n \geq 0} \{ \bigcup_{\sigma \in \Sigma_n^2} g_{\sigma(1)}^{-1} g_{\sigma(2)}^{-1} \cdots g_{\sigma(n)}^{-1}(I^0) \} \ .$$

This is obtained from the following lemma 12.

LEMMA 12. *For all* $n \geq 0$ *and* $\alpha \in \Delta_n$,

$$g_0(I_{0\alpha}^{n+1}) = I_\alpha^n \ , \qquad g_1(I_{1\alpha}^{n+1}) = I_\alpha^n \ .$$

PROOF: We prove this lemma by induction on n.

When $n = 0$, it suffices to show that;

$$g_0(I_0^1) = I^0, \qquad g_1(I_1^1) = I^0. \tag{16}$$

Since g_0, g_1 are monotonically increasing and $I_0^1 = [0, \ell_1]$, $I_1^1 = [1 - \ell_1, 1]$, by the definition of g_0, g_1 and lemma 10, we have

$$\begin{cases} g_0(0) = 0 \ , & g_0(\ell_1) = 1 \\ g_1(1 - \ell_1) = 0 \ , & g_1(1) = 1 \ . \end{cases}$$

This means (16).

Assume that the statement be true for $n - 1$. What we have to show is that;

$$\begin{aligned} (i) \quad & g_0(r_{0\alpha}^{n+1}) = r_\alpha^n \\ (ii) \quad & g_0(s_{0\alpha}^{n+1}) = s_\alpha^n \\ (iii) \quad & g_1(r_{1\alpha}^{n+1}) = r_\alpha^n \\ (iv) \quad & g_1(s_{1\alpha}^{n+1}) = s_\alpha^n \ . \end{aligned}$$

Let $\alpha = \alpha'\alpha_n$. Then, by the hypothesis of induction,

$$g_0(I^n_{0\alpha'}) = I^{n-1}_{\alpha'} , \qquad g_1(I^n_{1\alpha'}) = I^{n-1}_{\alpha'} .$$

Namely;

$$g_0(r^n_{0\alpha'}) = r^{n-1}_{\alpha'} , \qquad g_0(s^n_{0\alpha'}) = s^{n-1}_{\alpha'}$$
$$g_1(r^n_{1\alpha'}) = r^{n-1}_{\alpha'} , \qquad g_1(s^n_{1\alpha'}) = s^{n-1}_{\alpha'} .$$

When $\alpha_n = 0$, since $r^{n+1}_{0\alpha} = r^n_{0\alpha'}$ and $r^n_\alpha = r^{n-1}_{\alpha'}$, (i) is clear. As to (ii), by lemma 10 and 11, we have,

$$\begin{aligned} g_0(s^{n+1}_{0\alpha}) &= \int_0^{s^{n+1}_{0\alpha}} (f(t)+3)dt \\ &= \int_0^{r^{n+1}_{0\alpha}} (f(t)+3)dt + \int_{I^{n+1}_{0\alpha}} (f(t)+3)dt \\ &= r^n_\alpha + \ell_n \\ &= r^n_\alpha + (s^n_\alpha - r^n_\alpha) \\ &= s^n_\alpha . \end{aligned}$$

This proves (ii). The similar argument with g_1 gives (iii) and (iv).

When $\alpha_n = 1$, since $s^{n+1}_{0\alpha} = s^n_{0\alpha'}$ and $s^n_\alpha = s^{n-1}_{\alpha'}$, (i) is clear. As to (ii), we have,

$$\begin{aligned} g_0(r^{n+1}_{0\alpha}) &= \int_0^{r^{n+1}_{0\alpha}} (f(t)+3)dt \\ &= \int_0^{s^{n+1}_{0\alpha}} (f(t)+3)dt - \int_{r^{n+1}_{0\alpha}}^{s^{n+1}_{0\alpha}} (f(t)+3)dt \\ &= s^n_\alpha - \ell_n \\ &= s^n_\alpha - (s^n_\alpha - r^n_\alpha) \\ &= r^n_\alpha . \end{aligned}$$

This proves (ii). The similar argument with g_1 gives (iii) and (iv). □

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