Probability and PDE

August 30 - September 1, 2005

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Probability and PDE

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Program

Aug. 30 (Tues.)

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An identity in constructing approximate gradient flows for quasiconvex functionals

10:30-11:20 : Yoshihiro Tonegawa (Hokkaido Univ.)
The sharp-interface limit of the action functional for Allen-Cahn in one space dimension

13:00-13:50 : Shuya Kanagawa (Musashi Inst. Tech.)
Nonlinear Schrödinger Equation for Nearly Bichromatic Waves

14:00-14:50 : Atsushi Takeuchi (Osaka City Univ.)
Malliavin calculus for stochastic functional differential equations with jumps

15:20-16:10 : Kazuhiro Kuwae (Kumamoto Univ.)
Kato class measures of symmetric Markov processes under heat kernel estimates
(joint with Masayuki Takahashi)

16:20-17:00 : Masayoshi Takeda (Tohoku Univ.)
The principal eigenvalue for time-changed processes and applications

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Rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature

10:30-11:20 : Hitoshi Ishii (Waseda Univ.)
Periodic homogenization for nonlinear partially elliptic equations

13:00-13:50 : Naoyuki Ichihara (Osaka Univ.)
Stochastic representation for fully nonlinear PDEs and application to homogenization

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Risk-sensitive variational inequalities arising from optimal investment with transaction costs

15:10-16:00 : Hidehiro Kaise (Nagoya Univ.)
Min-max representation of critical value in ergodic type Bellman equation of first order

16:30-17:30 : Shuenn-Jyi Sheu (Academia Sinica, Taiwan) (also special department lecture)
Large time expectations for diffusion processes and an ergodic type Bellman equation

Sept. 1 (Thu.)

9:00-9:50 : Yasuhiro Fujita (Toyama Univ.)
On the principal eigenvalues of Kolmogorov operators on $\mathbb{R}^d$

10:00-10:50 : Masaaki Tsuchiya (Kanazawa Univ.)
An estimation problem for the shape of a domain via parabolic equations

11:00-11:40 : Masatoshi Fujisaki (Univ. Hyogo)
Common property resource and capital accumulation with random jump

11:50-12:30 : Toshio Mikami (Hokkaido Univ.)
Semimartingales from the Fokker-Planck equation.

An identity in constructing approximate gradient flows for quasiconvex functionals

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Probability and PDE, Aug. 30, 2005

1 Introduction

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with Lipschitz continuous boundary and let $F = F(p)$ be a function defined on the set of all N by n matrices with real elements, which is in this article simply denoted by $\boldsymbol{R}^{nN}$. Now let us consider the functional

$$J(u) = \int_\Omega F(\nabla u)dx,$$

where $\nabla u = (\dfrac{\partial u^i}{\partial x^\alpha})$. We suppose the following facts for the function F.

(A1) $F \in C^1(\boldsymbol{R}^{nN})$

(A2) F is *quasiconvex* with respect to p, that is,

$$\frac{1}{\mathcal{L}^n(D)}\int_D F(p_0 + \nabla\phi(x))dx \geq F(p_0)$$

for each bounded domain $D \subset \boldsymbol{R}^n$, for each $p_0 \in \boldsymbol{R}^{nN}$, and for each $\phi \in W_0^{1,\infty}(D; \boldsymbol{R}^N)$

(A3) There exist positive constants μ, λ and a constant $q > 1$ such that

$$\begin{cases} \lambda|p|^q \leq F(p) \leq \nu(1+|p|^q) \\ |F_p| \leq \mu(1+|p|^{q-1}) \end{cases}$$

The equation of gradient flow for J is given by

$$\frac{\partial u^i}{\partial t}(t,x) - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\{F_{p^i_\alpha}(\nabla u(x))\} = 0, \quad x \in \Omega. \tag{1.1}$$

We impose the initial and the boundary conditions

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(t,x) = w(x), \qquad x \in \partial\Omega. \tag{1.3}$$

We suppose that u_0 and w belong to $W^{1,q}(\Omega, \boldsymbol{R}^N) \cap L^2(\Omega)$ and that $\gamma u_0 = \gamma w$ on $\partial\Omega$ (γ is the trace operator to $\partial\Omega$).

In this article we say that a function u is a *weak solution* to (1.1)–(1.3) in $(0,\infty)\times\Omega$ if u satisfies i) $u\in L^\infty((0,\infty);W^{1,q}(\Omega)\cap L^2(\Omega))$, $u_t\in L^2((0,T)\times\Omega)$ for any $T>0$, ii) $\text{s-}\lim_{t\searrow 0} u(t,x)=u_0(x)$ in $L^2(\Omega)$, iii) $\gamma u(t)=\gamma w$ on Γ for $\mathcal{L}^1$-a.e. t, and iv) for any $\phi\in C_0^1((0,\infty)\times\Omega)$

$$\sum_{i=1}^N\int_0^\infty\int_\Omega\{u_t^i(t,x)\phi^i(t,x)+\sum_{\alpha=1}^n F_{p_\alpha^i}(\nabla u)\frac{\partial\phi^i}{\partial x^\alpha}(t,x)\}dxdt=0. \tag{1.4}$$

If u is a weak solution to (1.1), then $J(u(t))$ is absolutely continuous and it holds that $dJ(u(t))/dt=-(u_t,u_t)_{L^2(\Omega)}\le 0$ for $\mathcal{L}^1$-a.e. t. Thus this defines a gradient flow for J.

We construct an approximate solution to (1.1)–(1.3) by the method of discretization in time and minimizing variational functionals.

Let h be a positive number. A sequence $\{u_l\}$ in $W^{1,q}(\Omega,\boldsymbol{R}^N)$ is constructed as follows: we let u_0 be as in (1.2) and for $l\ge 1$ we define u_l as a minimizer of the functional

$$\mathcal{F}_l(v)=\frac{1}{2}\int_\Omega\frac{|v-u_{l-1}|^2}{h}dx+J(v)$$

in the class $w+W_0^{1,q}(\Omega,\boldsymbol{R}^N)$ (that is, among functions in $W^{1,q}(\Omega,\boldsymbol{R}^N)$ with $\gamma v=\gamma w$). The existence of a minimizer of $\mathcal{F}_l$ is assured by the quasiconvexity of F and (A3) (see, for example, [2, Chapter 4, Theorem 2.9]). Note also that (A3) assures $\mathcal{F}_l$ is Gâteaux differentiable. Approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ ($(t,x)\in(0,\infty)\times\Omega$) are defined as, for $(l-1)h<t\le lh$,

$$u^h(t,x)=\frac{t-(l-1)h}{h}u_l(x)+\frac{lh-t}{h}u_{l-1}(x)$$

and

$$\overline{u}^h(t,x)=u_l(x).$$

Then the following facts hold (see, for example, [1]).

Proposition 1.1 *We have*

1) $\{\|u_t^h\|_{L^2((0,\infty)\times\Omega)}\}$ *is uniformly bounded with respect to* h

2) $\{\|\overline{u}^h\|_{L^\infty((0,\infty);W^{1,q}(\Omega)\cap L^2(\Omega))}\}$ *is uniformly bounded with respect to* h

3) $\{\|u^h\|_{L^\infty((0,\infty);W^{1,q}(\Omega)\cap L^2(\Omega))}\}$ *is uniformly bounded with respect to* h

4) *for any* $T>0$, $\{\|u^h\|_{W^{1,\tilde q}((0,T)\times\Omega))}\}$ *is uniformly bounded with respect to* h, *where* $\tilde q=\min\{q,2\}$.

Then there exist a function u *such that, passing to a subsequence if necessary,*

5) $\overline{u}^h$ *converges to* u *as* $h\to 0$ *weakly star in* $L^\infty((0,\infty);W^{1,q}(\Omega))$

6) *for any* $T>0$, u^h *converges to* u *as* $h\to 0$ *weakly in* $W^{1,\tilde q}((0,T)\times\Omega))$

7) u^h *converges to* u *as* $h\to 0$ *strongly in* $L^{\tilde q}((0,T)\times\Omega)$

8) $\overline{u}^h$ *converges to* u *as* $h\to 0$ *strongly in* $L^{\tilde q}((0,T)\times\Omega)$

9) $s\text{-}\lim_{t\searrow 0}u(t)=u_0$ *in* $L^2(\Omega)$.

In [5] the limit u is called a *generalized minimizing movement* associated with J. Proposition 1.1 9) means that u satisfies (1.2) in a weak sense. Proposition 1.1 5) implies that u satisfies (1.3) in a weak sense since $\overline{u}^h-w\in L^\infty((0,\infty);W_0^{1,q}(\Omega))$ for each h (note that

$W_0^{1,q}(\Omega)$ is a closed subspace of $W^{1,q}(\Omega)$). Thus the problem is whether u satisfies (1.4). Since u_l is a minimizer of $\mathcal{F}_l(v)$, $d\mathcal{F}_l(u_l+\varepsilon\phi)/d\varepsilon|_{\varepsilon=0}=0$ for any $\phi\in W_0^{1,q}(\Omega)$, and noting that, for $(l-1)h<t<lh$, $u_t^h(t,x)=(u_l(x)-u_{l-1}(x))/h$, we have

$$\sum_{i=1}^{N}\int_{\Omega}\{(u_t^h)^i(x)\phi^i(x)+\sum_{\alpha=1}^{n}F_{p_\alpha^i}(\nabla\overline{u}^h)\frac{\partial\phi^i}{\partial x^\alpha}(x)\}dx=0 \tag{1.5}$$

for any $\phi\in W_0^{1,q}(\Omega)\cap L^2(\Omega)$ and any $t\in\bigcup_{\ell=0}^{\infty}((\ell-1)h,\ell h)$. This equality leads us to expect that the limit u is a weak solution to (1.1)–(1.3). In this article we show an identity, which would be a key in getting to our final destination. Our main result is mentioned in terms of geometric measure theory.

2 Graphs of Sobolev functions

Let $U=\Omega\times\boldsymbol{R}^N$ and let π denote the projection $U\ni z=(x,y)\mapsto x\in\Omega$. For each N by n matrix A, let $M(A)$ denote the vector consists of all minor determinants of A including 0-th order determinant, i.e., 1. Then $\int_D|M(A)|dx$ is the area of the plane $y=Ax$ over the set $D\subset\boldsymbol{R}^n$.

Proposition 2.1 *Let v be a function in $W^{1,1}(\Omega,\boldsymbol{R}^N)$ and put $\theta_0(z)=1/|M(Dv(x))|$, where $x=\pi(z)$.*
1) *G_v is countably n-rectifiable*
2) $\int_{G_v}\theta_0(z)d\mathcal{H}^n=\mathcal{L}^n(\Omega)$, *where G_v is the graph of v*
3) *θ_0 is $\mathcal{H}^n$-integrable on G_v*
4) *for each nonnegative continuous function f on $\Omega\times\boldsymbol{R}^N\times\boldsymbol{R}^{nN}$,*

$$\int_{\Omega}f(x,v(x),Dv(x))dx=\int_{U\times\boldsymbol{R}^{nN}}f(z,p)dV(z,p), \tag{2.1}$$

where $V=[(\mathcal{H}^n\llcorner G_v)\llcorner\theta_0]\otimes\delta_{Dv(x)}$.

Proof. Assertion 1) is well-known (compare to, for example, Theorems 4 of [4, I Section 3.1.5.]).

For $\mathcal{H}^n$-a.e. $z\in G_v$, the approximate tangent space T_zG_v exists and is expressed by the equation $y=Dv(\pi(z))x$. Hereby $\theta_0(z)=1/|M(Dv(\pi(z)))|$ and hence

$$\int_{G_v}\theta_0(z)d\mathcal{H}^n=\int_{\Omega}\frac{1}{|M(Dv(x))|}|M(Dv(x))|dx=\int_{\Omega}dx.$$

Thus Assertion 2) follows and Assertion 3) is the immediate consequence of 2).

When spt f is compact, we have by the definition of V

$$\int_{U\times\boldsymbol{R}^{nN}}f(z,p)dV(z,p)=\int_{G_v}f(z,Dv(\pi(z)))\theta_0(z)d\mathcal{H}^n(z). \tag{2.2}$$

Replacing Ω with any open set in Assertion 2), we have $\pi_\sharp((\mathcal{H}^n\llcorner G_v)\llcorner\theta_0)=\mathcal{L}^n$. Thus the right hand side of (2.2) coincides with the left hand side of (2.1). Hereby we obtain the conclusion for a function f with a compact support.

Suppose that f is a general nonnegative continuous function. Then, approximating f with an increasing sequence of functions in $C_0^0(\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN})$, we obtain the conclusion by the monotone convergence theorem. Q.E.D.

In general, a pair of a countably n-rectifiable set $\mathcal{M}$ in an open set $U \subset \boldsymbol{R}^{n+N}$ and a locally $\mathcal{H}^n$ integrable function θ on $\mathcal{M}$ is called a rectifiable n-varifold and denoted by $\boldsymbol{v}(\mathcal{M}, \theta)$. Proposition 2.1 shows that the pair (G_v, θ_0) defines a rectifiable n-varifold $\boldsymbol{v}(G_v, \theta_0)$. In this talk we call this varifold a *graph varifold* of v.

Given a rectifiable n-varifold $\boldsymbol{v}(\mathcal{M}, \theta)$ in U, a Radon measure V on $U \times G$, where G is the collection of all n-dimensional vector subspaces of $\boldsymbol{R}^{n+N}$, is defined by $V = [(\mathcal{H}^n \llcorner \mathcal{M}) \llcorner \theta] \otimes \delta_{T_z \mathcal{M}}$. Usually a Radon measure on $U \times G$ is called a general n-varifold in U. When $\mathcal{M}$ is a graph of a Sobolev function v, then for $\mathcal{H}^n$-a.e. $z_0 \in G_v$ the approximate tangent space $T_{z_0}\mathcal{M}$ is a graph of a linear function $y = A_{z_0} x$, where $A_{z_0} = Dv(x_0)$, $x_0 = \pi(z_0)$. Hence, in this case we could define a Radon measure V in $U \times \boldsymbol{R}^{nN}$ by $V = [(\mathcal{H}^n \llcorner \mathcal{M}) \llcorner \theta] \otimes \delta_{A_z}$. Let V be a Radon measure in $U \times \boldsymbol{R}^{nN}$. Let us assume that $V(K \times \boldsymbol{R}^{nN}) < \infty$ for each compact set $K \subset \boldsymbol{R}^{nN}$. A measure μ_V in U is defined by $\mu_V(B) = V(B \times \boldsymbol{R}^{nN})$ for each Borel set $B \subset U$. Our assumption implies μ_V is a Radon measure in U. Then there is a disintegration decomposition $V = \mu_V \otimes \eta_V^{(z)}$, namely, for each $\beta \in C_0^0(U \times \boldsymbol{R}^{nN})$,

$$\int_{U \times \boldsymbol{R}^{nN}} \beta(z,p) dV(z,p) = \int_U \Big(\int_{\boldsymbol{R}^{nN}} \beta(z,p) d\eta_V^{(z)}(p) \Big) d\mu_V.$$

Let v be a Sobolev function and let V be a Radon measure in $U \times \boldsymbol{R}^{nN}$ such that $V(K \times \boldsymbol{R}^{nN}) < \infty$ for each compact set $K \subset \boldsymbol{R}^{nN}$. In this talk we call V a *generalized graph varifold* of v if μ_V and $\mathcal{H}^n \llcorner G_v$ are mutually absolutely continuous.

Let V be a generalized graph varifold of $u \in W^{1,q}(\Omega)$. Then the functional J could be extended to V as follows:

$$J[V] = \int_{U \times \boldsymbol{R}^{nN}} F(p) dV.$$

The first variation of J for V, which is denoted by $\delta J[V](\phi)$, is defined as

$$\delta J[V](\phi) = \int_{U \times \boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} \sum_{i=1}^{N} F_{p_\alpha^i}(p) \Big(\frac{\partial \phi^i}{\partial x^\alpha}(z) + \sum_{j=1}^{N} \frac{\partial \phi^i}{\partial y^j}(z) p_\alpha^j \Big) dV(z,p) \tag{2.3}$$

for $\phi = (\phi^1, \ldots, \phi^N) \in C_0^1(U; \boldsymbol{R}^N)$. We say that V has locally bounded first variation of J in Ω if for each $W \subset\subset \Omega$ and each $\phi = (\phi^1, \ldots, \phi^N) \in C_0^1(U; \boldsymbol{R}^N)$ with $\mathrm{spt}\phi \subset W$ there exists a constant $C > 0$ such that $|\delta J[V](\phi)| \leq C \sup |\phi|$. Note that $\delta J[V]$ defines an $\boldsymbol{R}^N$-valued Radon measure. Its total variation is denoted by $\|\delta J[V]\|$.

Proposition 2.2 *Let V be a generalized graph varifold of a function $u \in W^{1,r}(\Omega)$, $r > q$. Suppose that V has locally bounded first variation of J in Ω. Then V satisfies*

$$\int_{\boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} F_{p_\alpha^i}(p)(p_\alpha^j - p_{0\alpha}^j) d\eta_V^{(z)}(p) = 0 \qquad (i,\ j = 1, \ldots, N)$$

for $\mathcal{H}^n$-a.e. $z \in G_u$, where $p_0 = Du(z)$.

3 Main Theorem

Suppose that $u \in L^\infty((0,\infty); W^{1,q}(\Omega)) \cap \bigcup_{T>0} W^{1,\bar{q}}((0,T)\times\Omega)$ is a weak solution to (1.1)–(1.3). By (1.4) and Proposition 2.1 we have, for each $\phi(z) = \phi(x,y) \in C_0^1((0,\infty)\times U)$

$$(3.1) \qquad \sum_{i=1}^N \int_0^\infty \{ \int_\Omega u_t^i(t,x)\phi^i(t,x,u(t,x))dx$$
$$+ \int_{U\times \boldsymbol{R}^{nN}} \sum_{\alpha=1}^n \sum_{i=1}^N F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(t,z) + \sum_{j=1}^N \frac{\partial \phi^i}{\partial y^j}(t,z)p_\alpha^j)dV_t(z,p)\}dt = 0,$$

where $V_t = \boldsymbol{v}(G_{u(t,\cdot)},\theta_0)$. Conversely suppose that a function u and a one parameter family $\{V_t\}$ of generalized graph varifolds of u satisfy (3.1). Then u is a weak solution to (1.1) if

$$(3.2) \qquad V_t = \boldsymbol{v}(G_{u(t,\cdot)},\theta_0) \quad \text{for} \quad \mathcal{L}^1\text{-a.e. } t.$$

Let $u^h(t,x)$ and $\overline{u}^h(t,x)$ be approximate solutions constructed in Section 1. By Proposition 2.1 there exists a one parameter family of graph varifolds

$$V_t^h = \boldsymbol{v}(G_{\overline{u}^h(t,\cdot)},\theta_0).$$

Further by Proposition 2.1 we can rewrite (1.5) as (3.1): for each $\psi(t) \in C_0^\infty(0,\infty)$ and $\phi(z) \in C_0^\infty(U)$

$$(3.3) \qquad \sum_{i=1}^N \int_0^\infty \{ \int_\Omega (u_t^h)^i(t,x)\phi^i(t,x,\overline{u}^h(t,x))dx$$
$$+ \int_{U\times \boldsymbol{R}^{nN}} [\sum_{\alpha=1}^n \sum_{i=1}^N F_{p_\alpha^i}(p)(\frac{\partial \phi^i}{\partial x^\alpha}(t,z) + \sum_{j=1}^N \frac{\partial \phi^i}{\partial y^j}(t,z)p_\alpha^j)]dV_t^h(z,p)\}dt = 0.$$

We have by Proposition 2.1 2)

$$(3.4) \qquad \operatorname*{ess.sup}_{t>0} | \int_{U\times \boldsymbol{R}^{nN}} \beta(z,p)dV_t^h(z,p)| \le K \sup|\beta|$$

for any $\beta(z,p) \in C_0^0(U\times \boldsymbol{R}^{nN})$.

The following proposition can be obtained by the use of (3.4) in the standard compactness argument (compare to Proposition 4.3 of [3]).

Lemma 3.1 *There exists a subsequence of $\{V_t^h\}$ (still denoted by $\{V_t^h\}$) and a one parameter family of Radon measures V_t in $U\times \boldsymbol{R}^{nN}$, for $t \in (0,\infty)$, such that, for each $\psi(t) \in L^1(0,\infty)$ and $\beta(z,p) \in C_0^0(U\times \boldsymbol{R}^{nN})$,*

$$\lim_{h\to 0} \int_0^\infty \psi(t) \int_{U\times \boldsymbol{R}^{nN}} \beta(z,p)dV_t^h(z,p)dt = \int_0^T \psi(t) \int_{U\times \boldsymbol{R}^{nN}} \beta(z,p)dV_t(z,p)dt.$$

Furthermore, for $\mathcal{L}^1$-a.e. t, V_t is a generalized graph varifolds of $u(t,\cdot)$.

By the use of Gehring theory and estimates in Proposition 1.1 we obtain

Lemma 3.2 $\{\|\overline{u}^h\|_{L^2((0,\infty);W^{1,r}(\Omega))}\}$ *is uniformly bounded with respect to h for some $r > q$.*

By the use of (1.5) we have

Lemma 3.3 *V_t has locally finite first variation of J in Ω for $\mathcal{L}^1$-a.e. $t \in (0, \infty)$.*

Thus by Proposition 2.2 we have our main theorem:

Theorem 3.4 *For $\mathcal{L}^1$-a.e. t, and for $\mathcal{H}^n$-a.e. $z \in G_{u(t,\cdot)}$,*

$$\int_{\boldsymbol{R}^{nN}} \sum_{\alpha=1}^{n} F_{p^i_\alpha}(z,p)\left(\frac{\partial u^j(z)}{\partial x^\alpha} - p^j_\alpha\right) d\eta^{(z)}_V(p) = 0 \qquad (i,\ j = 1, \ldots, N).$$

Applications of this theorem will be mentioned in the talk.

References

[1] F. Bethuel, J. M. Coron, J. M. Ghidallia, and A. Soyeur, *Heat flow and relaxed energies for harmonic maps*, Nonlinear Diffusion Equations and Their Equilibrium States, 3.

[2] B. Dacorogna, *Direct method in the calculus of variations*, Springer-Verlag, 1988.

[3] D. Fujiwara and S. Takakuwa, *A varifold solution to the nonlinear equation of motion of a vibrating membrane*, Kodai Math. J. **9** (1986), 84–116, correction, ibid. 14 (1991), 310-311.

[4] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents in the calculus of variations I, II*, Springer, 1998.

[5] E. De Giorgi, *New problems on minimizing movements*, Boundary Value Problems for PDE and Applications, Masson, 1993, pp. 81–98.

The sharp-interface limit of the action functional for Allen-Cahn in one space dimension

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The stochastically perturbed Allen-Cahn equation allows switching between stable equilibriums which is not possible for deterministic Allen-Cahn equation. The asymptotic small probability of such switching for the small noise limit is formally given and sometimes proved by the large deviation theory. The probability is given by a deterministic minimization of the Allen-Cahn action, which gives the cost for the transitions between stable equilibriums. In this talk, I explain the basic background materials and heuristic pictures, and present the rigorous one dimensional results. This is a joint work with Robert Kohn and Maria Reznikoff.

Nonlinear Schrödinger Equation for Nearly Bichromatic Waves

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(joint work with B.T. Nohara, A. Arimoto and K. Tchizawa)
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Let $u_m(t,x)$ be a wave function defined by for $t \in [0,\infty)$, $x \in R^d$

$$u_m(t,x) = \int_{-\infty}^{\infty} S(k) e^{i\{kx-\omega(k)t\}} dk,$$

where $S(k)$ is a spectrum and $\omega(k)$ is an angular frequency. If $S(k)$ is a unimodal function and the support of $S(k)$ is sufficiently small, then $u_m(t,x)$ is called a nearly monochromatic wave. Furthermore if $S(k)$ is multimodal function with two peaks and the support of $S(k)$ is sufficiently small, then

$$u_b(t,x) = \int_{-\infty}^{\infty} S(k) e^{i\{kx-\omega(k)t\}} dk,$$

is called a nearly bichromatic wave.

When $S(k) = \delta_{k_0}(k)$, $u_m(t,x)$ is a monochromatic wave and put

$$u_1(t,x) = e^{i\{k_0 x - \omega(k_0)t\}}.$$

Similarly if $S(k) = \delta_{k_0}(k) + \delta_{k_1}(k)$, then $u_m(t,x)$ is called a bichromatic wave defined by

$$u_2(t,x) = e^{i\{k_0 x - \omega(k_0)t\}} + e^{i\{k_1 x - \omega(k_1)t\}}.$$

The envelopes of the nearly monochromatic wave $u_m(t,x)$ for $u_1(t,x)$ and the nearly bichromatic wave $u_b(t,x)$ for $u_2(t,x)$ are defined by

$$A_m(t,x) = \frac{u_m(t,x)}{u_1(t,x)} \quad \text{and} \quad A_b(t,x) = \frac{2u_b(t,x)}{u_2(t,x)},$$

respectively. Using the Taylor expansion $\omega(k) = \sum_{j=0}^{\infty} \frac{\omega^{(j)}(k_0)}{j!}(k-k_0)^j$, we approximate the above envelope functions by for $n \geq 1$

$$A_m^n(t,x) = \frac{\int_{-\infty}^{\infty} S(k) e^{i\left\{kx - \sum_{j=0}^{n} \frac{\omega^{(j)}(k_0)}{j!}(k-k_0)^j t\right\}} dk}{u_1(t,x)}$$

and

$$A_b^n(t,x) = \frac{\int_{-\infty}^{\infty} S(k) e^{i\left\{kx - \sum_{j=0}^{n} \frac{\omega^{(j)}(k_0)}{j!}(k-k_0)^j t\right\}} dk}{u_2(t,x)},$$

respectively. They are called the nth order profile of nearly monochromatic wave or bichromatic wave, respectively.

In this talk we show that the envelope function satisfies linear or nonlinear Schrödinger equation. The following result is a key theorem for our talk.

Theorem. The second order profile $A_m^2(t,x)$ satisfies the following Schrödinger equation,

$$i\left(\frac{\partial A_m^2(t,x)}{\partial t} + \omega'(k_0)\frac{\partial A_m^2(t,x)}{\partial x}\right)A + \frac{1}{2}\omega''(k_0)\frac{\partial^2 A_m^2(t,x)}{\partial x^2} = 0.$$

Malliavin calculus for stochastic functional differential equations with jumps

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Fix $T, r > 0$. Let Ω_1 be the set of $\mathbb{R}^m$-valued continuous functions on $[0,T]$ with $\omega_1(0) = 0$, and Ω_2 the set of $\mathbb{Z}_+$-valued measures on $[0,T] \times \mathbb{R}^m$ with $\omega_2([0,T] \times \{0\}) = 0$. Denote by $\mathcal{F}_k$ the associated σ-field on Ω_k for $k = 1, 2$. Put $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. Consider a probability measure P on $(\Omega, \mathcal{F})$ such that

(a) $W(t)(\omega) := \omega_1(t)$ $(t \in [0,T])$ is an m-dimensional Brownian motion starting at $0 \in \mathbb{R}^m$,

(b) $J(dt, dz)(\omega) := \omega_2(dt, dz)$ is a Poisson random measure on $[0,T] \times \mathbb{R}^m$ with the intensity $\hat{J}(dt, dz) := dt\, \mu(dz)$,

(c) W and J are independent

for $\omega = (\omega_1, \omega_2) \in \Omega$, where $\mu(dz) = |z|^{-m-\alpha}\, dz$ $(0 < \alpha < 2)$. The space $(\Omega, \mathcal{F}, P)$ is called the Wiener-Poisson space with the Lévy measure $\mu(dz)$.

Let $a_0(t,f), a_1(t,f), \cdots, a_m(t,f)$ be $\mathbb{R}^d$-valued functions defined on $[0,T] \times D([-r,0];\mathbb{R}^d)$, and $b_z(t,f)$ an $\mathbb{R}^d$-valued function on $[0,T] \times D([-r,0];\mathbb{R}^d) \times \mathbb{R}^m$, which satisfy good conditions on the boundedness and the regularity. Given a deterministic path $\eta \in D([-r,0];\mathbb{R}^d)$, consider the stochastic functional differential equation

$$(1) \quad \begin{cases} x(t) = \eta(t) & (t \in [-r,0]), \\ dx(t) = \displaystyle\sum_{i=0}^{m} a_i(t, x_t)\, dW^i(t) + \int_{\mathbb{R}^m} b_z(t, x_{t-})\, \overline{J}(dt, dz) & (t \in (0,T]), \end{cases}$$

where $dW^0(t) = dt$, $\tilde{J}(dt,dz) := J(dt,dz) - \hat{J}(dt,dz)$ is a compensated Poisson random measure, $\overline{J}(dt,dz) = I_{\{|z|\le 1\}}\tilde{J}(dt,dz) + I_{\{|z|>1\}}J(dt,dz)$, and $x_t : [-r,0] \to \mathbb{R}^d$ is defined by $x_t(s) = x(t+s)$ for $s \in [-r,0]$. The main purpose in this talk is to study the existence and the smoothness of the density for the probability law of the random variable $x(T)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ via the Malliavin calculus on the Wiener-Poisson space $(\Omega, \mathcal{F}, P)$.

Put $\overline{b}(t,f) = \partial_\theta b_\theta(t,f)\big|_{\theta=0}$ and $B = c_{m,\alpha} \int_{|z|\leq 1} z\, z^* \mu(dz)$, where $c_{m,\alpha}$ is the positive constant depending only on m and α.

Theorem 1 *If the $d \times d$-matrix*

$$F(t,f) := \sum_{i=1}^{m} a_i(t,f)\, a_i(t,f)^* + \overline{b}(t,f)\, B\, \overline{b}(t,f)^* \tag{2}$$

is positive definite for any $t \in [0,T]$ and $f \in D([-r,0];\mathbb{R}^d)$, then the law of $x(T)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Fix $0 < q \leq r$. Consider the case where the coefficient $b_z(t,f)$ of the jump term in the equation (1) has the following expression

$$b_z(t,f) = b_z(t,f^q),$$

where $f^q = \{f(s)\,;\, s \in [-r,-q]\}$ for $f \in D([-r,0];\mathbb{R}^d)$. Then we have

Theorem 2 *If the $d \times d$-matrix*

$$F^q(t,f) := \sum_{i=1}^{m} a_i(t,f)\, a_i(t,f)^* + \overline{b}(t,f^q)\, B\, \overline{b}(t,f^q) \tag{3}$$

is positive definite for any $t \in [0,T]$ and $f \in D([-r,0];\mathbb{R}^d)$, then the law of $x(T)$ has a smooth density with respect to the Lebesgue measure on $\mathbb{R}^d$.

References

[1] Ishikawa, Y. and Kunita, H., *Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps*, preprint (2004).

[2] Picard, J., *On the existence of smooth densities for jump processes*, Probab. Theory Related Fields **105**, pp.481–511 (1996).

[3] Takeuchi, A., *The Malliavin calculus for stochastic functional differential equations with jumps*, in preparation.

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KATO CLASS MEASURES OF SYMMETRIC MARKOV PROCESSES UNDER HEAT KERNEL ESTIMATES

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1. Framework

Let (X, d) be a locally compact separable metric space and m a positive Radon measure with full support. Let $X_\Delta := X \cup \{\Delta\}$ be a one point compactification of X. For each $x \in X$ and $r > 0$, denote by $B_r(x) := \{y \in X \mid d(x,y) < r\}$ the open ball with center x and radius r. We consider and fix a symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X;m)$. Then there exists a Hunt process $\mathbf{M} = (\Omega, X_t, \zeta, P_x)$ such that for each Borel $u \in L^2(X;m)$, $T_t u(x) = E_x[u(X_t)]$ m-a.e. $x \in X$ for all $t > 0$, where $(T_t)_{t>0}$ is the semigroup associated with $(\mathcal{E}, \mathcal{F})$. Here $\zeta := \inf\{t \geq 0 \mid X_t = \Delta\}$ denotes the life time of $\mathbf{M}$. Further, we assume that there exists a kernel $p_t(x,y)$ defined for all $(t,x,y) \in]0,\infty[\times X \times X$ such that $E_x[u(X_t)] = P_t u(x) := \int_X p_t(x,y)u(y)m(dy)$ for any $x \in X$, bounded Borel function u and $t > 0$. $p_t(x,y)$ is said to be a *semigroup kernel*, or sometimes called a *heat kernel* of $\mathbf{M}$ on the analogy of heat kernel of diffusions. Then P_t can be extended to contractive semigroups on $L^p(X;m)$ for $p \geq 1$. The following are well-known:

(1) $p_{t+s}(x,y) = \displaystyle\int_X p_s(x,z)p_t(z,y)m(dz), \quad {}^\forall x, y \in X, {}^\forall t, s > 0.$

(2) $p_t(x,dy) = p_t(x,y)m(dy), \quad {}^\forall x \in X, {}^\forall t > 0.$

(3) $\displaystyle\int_X p_t(x,y)m(dy) \leq 1, \quad {}^\forall x \in X, {}^\forall t > 0.$

Throughout this paper, we fix $\nu, \beta \in]0,\infty[$ and $t_0 \in]0,\infty]$.

Assumption 1.1 (Life time condition). $\mathbf{M}$ has the following property that

$$\lim_{t\to 0} \sup_{x\in X} P_x(\zeta \leq t) =: \gamma \in [0,1[.$$

In particular, if $\mathbf{M}$ is stochastically complete, that is, $\mathbf{M}$ is conservative, then this condition is satisfied with $\gamma = 0$.

Assumption 1.2 (Bishop type inequality). There exists an increasing function $V(r)$ on $]0,\infty[$ such that $r \mapsto V(r)/r^\nu$ is increasing or bounded, and $\sup_{x\in X} m(B_r(x)) \le V(r)$ for all $r > 0$.

Assumption 1.3 (Upper and lower estimates of heat kernel). Let Φ_i $(i = 1,2)$ be positive decreasing functions defined on $[0,\infty[$ which may depend on t_0 if $t_0 < \infty$ and assume that Φ_2 satisfies the following condition $H(\Phi_2)$:

$$\int_1^\infty \frac{(V(t)\vee t^\nu)\Phi_2(t)}{t}dt < \infty$$

and $(\Phi E_{\nu,\beta})$: for any $x, y \in X$, $t \in]0, t_0[$

$$\frac{1}{t^{\nu/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \le p_t(x,y) \le \frac{1}{t^{\nu/\beta}}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right).$$

Remark 1.1. Grigor'yan [9] or Grigor'yan-Hu-Lau [10] proved that the lower estimate in $(\Phi E_{\nu,\beta})$ yields that there exist $C > 0$ and $r_0 \in]0,\infty]$ such that $m(B_r(x)) \le Cr^\nu$ for all $x \in X$ and $r \in]0, r_0[$. In that case, $r_0 = \infty$ if $t_0 = \infty$. If the stochastic completeness of $\mathbf{M}$ and Assumption 1.3 with $t_0 = \infty$ hold, then they proved the *Ahlfors regularity*, that is, there exists $C > 0$ such that $C^{-1}r^\nu \le m(B_r(x)) \le Cr^\nu$ for all $x \in X$ and $r \in]0,\infty[$ hence, Assumption 1.2 holds by taking $V(r) = Cr^\nu$ in this case.

Lemma 1.1. *Suppose that there exist $r_0 \in]0, t_0^{1/\beta}[$ and $c > 0$ such that $m(B_{r_0}(x)) \ge cr_0^\nu$ for all $x \in X$. Then Assumption 1.3 implies Assumption 1.1.*

2. Kato class measures

Definition 2.1 (Kato class S_K^0, Dynkin class S_D^0). For a positive Borel measure μ on X, μ is said to be *of Kato class relative to* $p_t(x,y)$ (write $\mu \in S_K^0$) if

$$\lim_{t\to 0}\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy) = 0 \tag{1}$$

and μ is said to be of *Dynkin class relative to* $p_t(x,y)$ (write $\mu \in S_D^0$) if

$$\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy) < \infty \quad {}^\exists t > 0. \tag{2}$$

Clearly, $S_K^0 \subset S_D^0$. For a positive Borel measure μ on X, μ is said to be *of local Kato class relative to* $p_t(x,y)$ (write $\mu \in S_{K,loc}^0$) if $I_G\mu \in S_K^0$ for any relatively compact open set G.

We set $r_\alpha(x,y) := \int_0^\infty e^{-\alpha t} p_t(x,y)dt$. The following are showed in [16].

Lemma 2.1 (Lemma 3.1 in [16]). *$\mu \in S_K^0$ is equivalent to*

$$\lim_{\alpha\to\infty} \sup_{x\in X} \int_X r_\alpha(x,y)\mu(dy) = 0 \tag{3}$$

and $\mu \in S_D^0$ is equivalent to

$$\sup_{x\in X} \int_X r_\alpha(x,y)\mu(dy) < \infty, \quad {}^\exists\alpha > 0. \tag{4}$$

Lemma 2.2 (Lemma 3.2 in [16]). *The following are equivalent to each other.*

(1) $\mu \in S_D^0$.

(2) $\sup_{x\in X} \int_X \left(\int_0^t p_s(x,y)ds\right)\mu(dy) < \infty$ *for* ${}^\forall t > 0$.

(3) $\sup_{x\in X} \int_X r_\alpha(x,y)\mu(dy) < \infty$ *for* ${}^\forall\alpha > 0$.

Definition 2.2 (Kato class $K_{\nu,\beta}$). Fix $\nu > 0$ and $\beta > 0$. For a positive Borel measure μ on X, μ is said to be *of Kato class relative to Green kernel* (write $\mu \in K_{\nu,\beta}$) if

$$\begin{aligned} \lim_{r\to 0} \sup_{x\in X} \int_{d(x,y)<r} G(x,y)\mu(dy) &= 0 \quad \text{for } \nu \geq \beta, \\ \sup_{x\in X} \int_{d(x,y)\leq 1} \mu(dy) &< \infty \quad \text{for } \nu < \beta, \end{aligned}$$

where $G(x,y) := G(d(x,y))$ with

$$G(r) := \begin{cases} r^{\beta-\nu} & \nu \neq \beta, \\ \log(r^{-1}) & \nu = \beta. \end{cases}$$

For a positive Borel measure μ on X, μ is said to be *of local Kato class relative to Green kernel* (write $\mu \in K_{\nu,\beta}^{loc}$) if $I_G\mu \in K_{\nu,\beta}$ for any relatively compact open set G. Clearly $K_{\nu,\beta} \subset K_{\nu,\beta}^{loc}$.

Lemma 2.3. *If $\mu \in K_{\nu,\beta}$, then $\sup_{x\in X} \mu(B_r(x)) < \infty$ for small $r \in]0, e^{-1}[$. In particular, every $\mu \in K_{\nu,\beta}^{loc}$ is a Radon measure.*

3. Main Theorems

Theorem 3.1. *Suppose that Assumption 1.3 and $\nu \geq \beta$ hold. Then the following are equivalent:*

(1) $\mu \in K_{\nu,\beta}$.

(2) $\displaystyle\limsup_{r\to 0} \sup_{x\in X} \int_{B_r(x)} r_\alpha(x,y)\mu(dy) = 0$ *for any* $\alpha > 0$.

(3) $\displaystyle\limsup_{r\to 0} \sup_{x\in X} \int_{B_r(x)} r_\alpha(x,y)\mu(dy) = 0$ *for some* $\alpha > 0$.

Remark 3.1. The assertion in Theorem 3.1 does not hold for $\nu < \beta$ in general. In fact, for 1-dimensional Brownian motion $\mathbf{M}^{\mathrm{w}}$, we see that $\mu = \delta_0 \in K_1$ does not satisfies the conditions (2), (3) in Theorem 3.1 because of $r_\alpha(x,y) = e^{-\sqrt{2\alpha}|x-y|}/\sqrt{2\alpha}$.

Theorem 3.2. *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then we have $S_K^0 = K_{\nu,\beta}$. Moreover, $\mu \in K_{\nu,\beta}$ implies that $\sup_{x\in X}\mu(B_R(x)) < \infty$ for all $R > 0$. For $\nu < \beta$, we have $S_D^0 = K_{\nu,\beta}$ and $\mu \in K_{\nu,\beta}$ is equivalent to one (hence all) of the following:*

(1) $\sup_{x\in X}\mu(B_R(x)) < \infty$ *for all* $R > 0$.

(2) $\sup_{x\in X}\mu(B_R(x)) < \infty$ *for some* $R > 0$.

(3) $\lim_{r\to 0}\sup_{x\in X}\int_{B_r(x)} d(x,y)^{\beta-\nu}\mu(dy) = 0$.

Theorem 3.3. *Suppose that Assumptions 1.2 and 1.3 hold. Then for any $f \in L^p_{\mathrm{unif}}(X;m)$, we have $|f|dm \in K_{\nu,\beta}$ if $p > \nu/\beta$ with $\nu \geq \beta$, or if $p \geq 1$ with $\nu < \beta$. Here $f \in L^p_{\mathrm{unif}}(X;m)$ means*

$$\sup_{x\in X} \int_{d(x,y)\leq 1} |f|^p dm < \infty.$$

4. Examples

Many examples satisfy Assumptions 1.1, 1.2 and 1.3. Symmetric α-stable process, relativistic free Hamiltonian process, relativistic α-stable process, jump type process on d-sets, Brownian motion on complete smooth Riemannian manifold with lower Ricci curvature bound and with positive injectivity radius, diffusion processes on nested fractals, Brownian motion on Sierpinski Carpet and so on.

References

[1] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure. Appl. Math.* **35** (1982), 209-273.

[2] S. Albeverio and Z.M. Ma, Additive functionals, nowhere Radon and Kato class smooth measures associated with Dirichlet forms. *Osaka J. Math.* **29** (1992), no. 2, 247-265.

[3] M.T. Barlow and R.F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets. *Canad. J. Math.* **51** (1999), 673-744.

[4] A. Bendikov, Asymtotic formulas for symmetric stable semigroups, *Expo. Math.*, **12** (1994) no. 4, 381–384.

[5] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process Appl.* **108** (2003), 27-62.

[6] K.L. Chung and Z. Zhao, *From Brownian motion to Schrödinger's Equation.* Springer, Berlin, 1995.

[7] C.B. Croke, Some isoperimetric inequalities and eigenvalue estimates, *Ann. Sci. École Norm. Sup.* (4) **13** (1980), no. 4, 419–435.

[8] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes.* de Gruyter Studies in Mathematics, **19** Walter de Gruyter & Co., Berlin, 1994.

[9] A. Grigor'yan, Heat kernels and function theory on metric measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 143–172, *Contemp. Math.*, **338**, Amer. Math. Soc., Providence, RI, 2003.

[10] A. Grigor'yan, J. Hu and K.-S. Lau, Heat kernels on metric measure spaces and an application to semilinear elliptic equations. *Trans. Amer. Math. Soc.* **355** (2003), no. 5, 2065–2095.

[11] A. Grigor'yan, Heat kernel upper bounds on fractal spaces, preprint, 2004.

[12] A. Grigor'yan, Heat kernels on weighted manifolds and applications, preprint 2005, to appear in Cont. Math.

[13] J. Hu and T. Kumagai, Nash-type inequalities and heat kernels for non-local Dirichlet forms, preprint, 2004.

[14] T. Kato, Schrödinger operators with singular potentials, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972). *Israel J. Math.* **13** (1972), 135–148 (1973).

[15] T. Kumagai, Estimates of transition densities for Brownian motion on nested fractals, *Probab. Theory Relat. Fields*, **96** (1993), 205–224.

[16] K. Kuwae and M. Takahashi, Kato class functions of Markov processes under ultracontractivity, to appear in ASPM XX, Potential Theory in Matsue, 2005.

[17] M.Ł. Ryznar, Estimates of Green function for relativistic α-stable process, *Potential Anal.*, **17** (2002), no. 1, 1–23.

[18] B. Simon, Schrödinger operators in the twentieth century, *J. Math. Phys.* **41** (2000), no. 6, 3523–3555.

[19] Z. Zhao, A probabilistic principle and generalized Schrödinger perturbation. *J. Funct. Anal.* **101** (1991), 162-176.

The principal eigenvalue for time-changed processes and applications

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Let M be a non-compact, complete Riemannian manifold. Let $\mathbb{M} = (\mathbb{P}_x, B_t)$ be the Brownian motion on M and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ the Dirichlet form generated by $\mathbb{M}$:

$$\mathcal{E}(u,v) = (-\frac{1}{2}\Delta u, v)_m = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm,$$
$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(M;m) : \int_M (\nabla u, \nabla u)\, dm < \infty \right\}$$

For a domain $D \subset M$, let $\mathbb{M}^D = (\mathbb{P}_x^D, B_t^D)$ be the absorbing Brownian motion on D and $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ its Dirichlet form. Assume that $\mathbb{M}^D$ is transient and denote by $G^D(x,y)$ the Green function.

Definition 1. Let μ be a Radon measure on D in the Kato class. The measure μ is said to be in the class $\mathcal{S}_\infty^D$, if for any $\epsilon > 0$, there exist a compact set $K \subset D$ and $\delta > 0$ such that

$$\sup_{(x,z)\in D\times D\setminus d} \int_{K^c} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \mu(dy) \le \epsilon,$$

and for any Borel set $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x,z)\in D\times D\setminus d} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \mu(dy) \le \epsilon.$$

For $\mu \in \mathcal{S}_\infty^D$, denote by A_t^μ the positive continuous additive functional in the Revus correspondence. Let $p_t^{\mu,D}(x,y)$ be the integral kernel of **Feynman-Kac semigroup**:

$$p_t^{\mu,D} f(x) := \mathbb{E}_x[\exp(A_t^\mu) f(X_t); t < \tau_D] = \int_D p_t^{\mu,D}(x,y) f(y) dy$$

($\tau_D = \inf\{t > 0 : B_t \notin D\}$). Let $G^{\mu,D}(x,y) = \int_0^\infty p_t^{\mu,D}(x,y)dt$.

Theorem 1. ([1],[2]) For $\mu \in \mathcal{S}_\infty^D$, following statements are equivalent:

(i) (**gaugeability**) $\sup_{x\in D} \mathbb{E}_x[e^{A_{\tau_D}^\mu}] < \infty$

(ii) (**subcriticality**) $G^{\mu,D}(x,y) < \infty$ for $x, y \in D$, $x \neq y$;

(iii) $\lambda(\mu; D) := \inf\left\{\mathcal{E}(u,u) : \ u \in C_0^\infty(D),\ \int_D u^2 d\mu = 1\right\} > 1.$

$\lambda(\mu; D)$ is the principal eigenvalue for the time-changed process by A_t^μ.

<u>**Applications of Theorem 1**</u> (i) Let $\bar{\mathbb{B}} = (\bar{B}_t, \bar{\mathbb{P}}_x)$ be a branching Brownian motion with **branching rate** k and **branching mechanism** $\{p_n(x)\}_{n\geq 2}$: $\bar{\mathbb{P}}_x[T > t|\sigma(X)] = \exp(-A_t^k)$ (T is **first splitting time**), $\sum_{n=2}^{\infty} p_n(x) = 1$. Set $Q(x) = \sum_{n\geq 2} np_n(x)$ and $\mu(dx) = (Q(x) - 1)k(dx)$. We assume that $\sup_{x\in\mathbb{R}^d} Q(x) < \infty$.

Theorem 2. For a closed set K with $\mathrm{Cap}(K) > 0$ and $\mu \in \mathcal{S}_\infty^{M\setminus K}$,

$$\lambda(\mu; M \setminus K) > 1 \Longleftrightarrow \bar{\mathbb{E}}_x[N_K] < \infty.$$

Here N_K is the number of branches hitting K.

To prove Theorem 2, we show that for $\mu \in \mathcal{S}_\infty^D$

$$\sup_{x\in D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu)] < \infty \Longleftrightarrow \sup_{x\in D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu); \tau_D < \infty] < \infty.$$

Then the identity

$$\bar{\mathbb{E}}_x[N_K] = \mathbb{E}_x[\exp(A_{\tau_D}^\mu); \tau_D < \infty]$$

leads us to Theorem 2.

(ii) Let $d(x, y)$ be the distance and m the volume. Let $p(t, x, y)$ be the heat kernel and assume that it satisfies the Gaussian lower and upper bounds (**Li-Yau estimate**): For any $x, y \in M$ and $t > 0$,

$$\frac{C_1 \exp\left(-c_1 \frac{d^2(x,y)}{t}\right)}{m(B(x, \sqrt{t}))} \leq p(t, x, y) \leq \frac{C_2 \exp\left(-c_2 \frac{d^2(x,y)}{t}\right)}{m(B(x, \sqrt{t}))}. \tag{1}$$

(C_1, c_1, C_2, c_2 are positive constants. $B(x, r) = \{y \in M : d(x, y) < r\}$). For $\mu \in \mathcal{S}_\infty^M$, Let $p^\mu(t, x, y)$ be the heat kernel associtated with the Schrödinger operator, $\frac{1}{2}\Delta + \mu$.

Theorem 3. Let $\mu \in \mathcal{S}_\infty^M$. Then $p^\mu(t, x, y)$ satisfies the Li-Yau estimate if and only if $\lambda(\mu; D) > 1$.

References

[1] Z.-Q. Chen, Gaugeability and conditional gaugeability, Trans. Amer. Math. Soc. 354 (2002), 4639-4679.

[2] M. Takeda, Subcriticality and conditional gaugeability of generalized Schrödinger operators, J. Funct. Anal. 191 (2002), 343-376.

[3] M. Takeda, Gaugeability for Feynman-Kac functionals with applications to symmetric α-Stable processes, to appear in Proc. Amer. Math. Soc.

[4] M. Takeda, Gaussian bounds of heat kernels for Schrödinger operators on Riemannian manifolds, preprint.

Rate of convergence of the Bence-Merriman-Osher algorithm for motion by mean curvature

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In 1992, Bence, Merriman and Osher proposed an algorithm for computing the motion of a hypersurface by its mean curvature. It is described as follows.

Let $C_0 \subset \mathbb{R}^N$ be a closed set and let $u = u(t,x)$ be the solution of

$$(1) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } (0,+\infty) \times \mathbb{R}^N, \\ u(0,x) = \begin{cases} 1 & x \in C_0, \\ -1 & x \in \mathbb{R}^N \backslash C_0. \end{cases} \end{cases}$$

Fix a time step $h > 0$ and set

$$C_1 = \{x \in \mathbb{R}^N \mid u(h,x) \geq 0\}.$$

Next we solve (1) with C_0 replacing C_1 and define a new set C_2 with u replaced by the solution of (1) with the new initial data. Repeating this procedure, we have a sequence $\{C_k\}_{k=0,1,\ldots}$ of closed sets in $\mathbb{R}^N$. Then we define

$$C_t^h = C_k \text{ if } kh \leq t < (k+1)h, \ k = 0,1,\ldots$$

for $t \geq 0$. Letting $h \to 0$, we obtain in the limit a flow $\{D_t\}_{t\geq 0}$ of closed subsets in $\mathbb{R}^N$ with $D_0 = C_0$ and then ∂D_t moves by its ($(N-1)$-times) mean curvature.

The convergence and the generalizations of the Bence-Merriman-Osher (BMO in short) algorithm were considered by many people. However, to my knowledge, there are a few results on the rate of convergence of the BMO algorithm. In 1996 Ruuth gave a time-local error estimate of the BMO algorithm in $\mathbb{R}^2$.

The purpose of this talk is to present the optimal rate of convergence of the BMO algorithm, valid before the onset of singularities, for the Hausdorff distance between Γ_t and $\Gamma_t^h (= \partial C_t^h)$.

Assume that $\{\Gamma_t\}_{0\leq t<T}$ is a motion of a smooth and compact hypersurface by mean curvature. Let $\{\Gamma_t^h\}_{t,h>0}$ be a flow constructed by the BMO algorithm satisfying $\Gamma_0^h = \Gamma_0$. First, we can show the following.

Theorem 0.1 *For each $T' < T$, there exist $h_0 > 0$ and $L > 0$ such that*

$$(2) \qquad \sup_{t\in[0,T']} d_H\left(\Gamma_t^h, \Gamma_t\right) \leq Lh \quad \text{for all } h \in (0, h_0).$$

Here $d_H(A,B)$ denotes the Hausdorff distance between A, $B \subset \mathbb{R}^n$.

To show the optimality of the estimate in this theorem, we consider the special case of a circle evolving by curvature. Let $\Gamma_t = \{x \in \mathbb{R}^2 \mid |x| = \phi(t)\}$ $(\phi(t) = \sqrt{1-2t})$. Then Γ_t moves by its curvature and shrinks to a point at $T = 1/2$. Let $\{\Gamma_t^h\}_{t,h>0}$ be a flow constructed by the BMO algorithm satisfying $\Gamma_0^h = \Gamma_0$. Then, for each t, $h > 0$, Γ_t^h is a circle and we can define its radius $R_h(t)$. In this situation, we have

Theorem 0.2 *For each $T' < 1/2$, there exist $h_0 > 0$ and $L > 0$ such that*

$$\sup_{t\in[0,T']} |R_h(t) - (\phi(t) - h\psi(t))| \leq Lh^{3/2}$$

for all $h \in (0, h_0)$, where $\psi(t) = -\log\phi(t)/3\phi(t)$.

This theorem yields the optimality of (2) with respect to the order of h.

To prove Theorem 0.1 and 0.2, we construct suitable sub- and super-solutions of (1) and use the comparison principle for (1). As to the constructions of sub- and super-solutions of (1), we apply the asymptotic analysis of solutions of Allen-Cahn equation.

Periodic homogenization for nonlinear partially degenerate elliptic equations

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In this talk I describe some of results on the homogenization of fully nonlinear partially degenerate elliptic equations in the frame work of periodic homogenization, which have been obtained in a joint work with K. Shimano and P. E. Souganidis.

The degenerate elliptic equation treated is as follow: Let $\Omega \subset \mathbf{R}^N$ be a bounded open set. Here $N = n + m$, with $n, m \in \mathbf{N}$, $\mathbf{R}^N = \mathbf{R}^n \times \mathbf{R}^m$, and a generic point $z \in \mathbf{R}^N$ will be denoted as $z = (x, y)$, with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. With this notation, the problem is:

$$\begin{cases} F_0\left(D_x^2 u^\varepsilon, x, y\right) + F_1\Big(D_y u^\varepsilon, x, y, \dfrac{x}{\varepsilon}, \dfrac{y}{\varepsilon}\Big) = 0 \quad \text{in } \Omega, \\ u^\varepsilon = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where F_0, F_1 are a real-valued continuous functions on $\mathbf{S}^n \times \Omega$ and $\mathbf{R}^m \times \Omega \times \mathbf{R}^n \times \mathbf{R}^m$, respectively, $\mathbf{S}^n$ denotes the space of $n \times n$ real symmetric matrices, $u^\varepsilon = u^\varepsilon(x, y)$ represents the unknown function, and $\varepsilon > 0$ is a parameter to be sent to zero. Moreover, I assume that the function F_0 is (partially) uniformly elliptic (or, more precisely, there are constants $0 < \lambda \leq \Lambda < \infty$ such that for all $X, P \in \mathbf{S}^n$ and $z \in \overline{\Omega}$, if $P \geq 0$, then

$$-\Lambda \operatorname{tr} P \leq F_0(X + P, z) - F_0(X, z) \leq -\lambda \operatorname{tr} P \quad),$$

the function F_1 is are coercive (or, more precisely, there are constants $C_0 > 0$ and $\kappa > 0$ such that for all $q \in \mathbf{R}^m$, $z \in \overline{\Omega}$, and $\zeta \in \mathbf{R}^N$,

$$C_0^{-1}|q|^\kappa - C_0 \leq F_1(q, z, \zeta) \leq C_0(|q|^\kappa + 1) \quad)$$

and the functions $\zeta \mapsto F_1(q, z, \zeta)$ are $\mathbf{Z}^N$-periodic. Under some additional assumptions, one of the results states that the solution u^ε of the above Dirichlet problem converges, as $\varepsilon \to 0$, to the solution of the homogenized problem.

Stochastic representation for fully nonlinear PDEs and application to homogenization

Naoyuki Ichihara (2005/08/31)

We consider the following second-order partial differential equation of parabolic type with small parameter $\varepsilon > 0$:

$$\text{(1)} \qquad \begin{cases} -u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, u^\varepsilon, u_x^\varepsilon, u_{xx}^\varepsilon\right) = 0\,, & \text{in} \quad [0,T) \times \mathbb{R}^d\,, \\ u^\varepsilon(T,x) = h(x) \in C_b^3(\mathbb{R}^d)\,, & \text{on} \quad \mathbb{R}^d\,, \end{cases}$$

where $H = H(\eta, y, p, X) : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$ is assumed to be $\mathbb{Z}^d$-periodic in η and satisfies the following conditions for some ν and $K > 0$:

(A1) H is of C^2 class with respect to all variables and all second derivatives are bounded.

(A2) H is convex in X.

(A3) For every (η, y, p, X) and $\xi \in \mathbb{R}^d$,

$$\nu|\xi|^2 \le H(\eta, y, p, X) - H(\eta, y, p, X + \xi \otimes \xi) \le \nu^{-1}|\xi|^2\,,$$

where $\xi \otimes \xi$ stands for the $(d \times d)$-matrix defined by $(\xi \otimes \xi)_{ij} := \xi^i \xi^j$.

(A4) For every (y, p, X), (y', p', X') and η,

$$|H(\eta, y, p, X) - H(\eta, y', p', X')| \le K\{|y - y'| + |p - p'| + |X - X'|\}\,.$$

(A5) $|H(\eta, 0, 0, 0)| \le K$.

(A6) For every η, η' and (y, p, X),

$$|H(\eta, y, p, X) - H(\eta', y, p, X)| \le K(1 + |p| + |X|)|\eta - \eta'|\,.$$

We are interested in the asymptotic behavior of solutions $\{u^\varepsilon; \varepsilon > 0\}$ as ε tends to zero. Such kind of homogenization problems have been studied by the viscosity solution method (see [1], [2]).

Theorem 1. *[Evans ('92), Alvarez-Bardi ('03)] The family of solutions $\{u^\varepsilon(t,x); \varepsilon > 0\}$ to PDEs (1) converges uniformly on compacts to $u^0(t,x)$ as ε goes to zero, where $u^0(t,x)$ is a unique (classical) solution of the PDE*

$$\text{(2)} \qquad \begin{cases} -u_t^0 + \overline{H}(u^0, u_x^0, u_{xx}^0) = 0\,, & \textit{in} \quad [0,T) \times \mathbb{R}^d\,, \\ u^0(T,x) = h(x)\,, & \textit{on} \quad \mathbb{R}^d\,. \end{cases}$$

The effective Hamiltonian $\overline{H} = \overline{H}(y, p, X)$ is characterized as a unique constant of the following cell problem

$$\text{(3)} \qquad \overline{H} = H(\eta, y, p, X + v_{\eta\eta}(\eta))\,, \qquad (v(\cdot), \overline{H}) : \textit{unknown.}$$

In this talk, we consider homogenization of PDE (1) from a probabilistic viewpoint. For this purpose, we introduce a class of stochastic equations called backward stochastic differential equations that play a important role in our approach. The novelty of our result is that by virtue of probabilistic arguments, we can naturally calculate the rate of convergence of solutions that has not been obtained in previous works. More precisely, let $\delta \in (0,1)$ be the exponent of Hölder continuity for $u^0 \in C^{1+\delta/2,2+\delta}([0,T] \times \mathbb{R}^d)$. Remark that under our assumption there exists a unique solution to PDE (2) in the Hölder space stated above if $\delta \in (0,1)$ is sufficiently small. Our main result is the following.

Theorem 2. *For every compact subset Q of $[0,T] \times \mathbb{R}^d$, there exists $C > 0$ such that*

$$\sup_{(t,x)\in Q} |u^\varepsilon(t,x) - u^0(t,x)| \leq C\, \varepsilon^{\frac{2\delta}{2+\delta}}\,.$$

References

[1] Alvarez O, Bardi M (2003) Singular perturbations of nonlinear degenerate parabolic PDEs: a general convergence result. Arch Ration Mech Anal 170(1):17–61

[2] Evans LC (1992) Periodic homogenization of certain fully nonlinear partial differential equations. Proc Roy Soc Edinburgh Sect A 120:245-265

Risk-sensitive variational inequalities arising from optimal investment with transaction costs

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Risk-sensitive variational inequalities are studied. The inequalities arise from power utility maximization problems on infinite time horizon with transaction costs in mathematical finance. We can reduce them to studying a family of stopping problems of certain multiplicative functionals. The stopping problems are solved by showing unique existence of the solutions of relevant risk-sensitive variational inequalities. Optimal investment strategies of utility maximization with transaction costs are constructed by using the solutions.

Min-max representation of critical value in ergodic type Bellman equation of first order

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It is known that the critical value of second order linear differential operators has the min-max type representation. This can be understood as a generalization of Rayleigh-Ritz variational formula for the principal eigenvalue of Schrödinger operators. For general linear differential operators of second order, we could derive the min-max representation by convex duality analysis because the critical value is convex on perturbations (0th order terms).

On the other hand, ergodic type Bellman equation of second order with a particular quadratic Hamiltonian can be obtained by the eigenvalue problem of Schrödinger operators through logarithmic transformation. This indicates that the structure of solutions for Bellman equation of second order with general quadratic Hamiltonian may be similar to positive solutions for linear equation of second order. In our previous work, we specified the structure of classical solutions of ergodic Bellman equation of second order and had the min-max representation of the critical value, which generalizes the representation result of the critical value in second order linear operators.

In this talk, motivated by the results on second order case, we will discuss the min-max representation of the critical value for ergodic Bellman equations of first order with quadratic Hamiltonian. We already proved that the structure of viscosity solutions of first order equation is the same as the second order case and there exists the critical value in the solutions. By noting that the critical value is convex on 0th order term, we will show that the min-max representation can be obtained by convex duality argument.

Large time expectations for diffusion processes and an ergodic type Bellman equation

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We consider a general diffusion process $X(t)$ satisfying the SDE,

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t).$$

We assume only the smoothness of the coefficients $b(\cdot), \sigma(\cdot)$ and the non-degeneracy of the diffusion coefficient. We study the expectations

$$E_x[f(X(T)exp(\int_0^T V(X(t))dt)]$$

with large T for various functions $f(\cdot), V(\cdot)$. The asymptotics is given by $\exp(\Lambda(V)T)$. Our main concern is to determine $\Lambda(V)$ and also other finer asymptotics of the expectation. We shall see such problems closely relates to the equation,

$$\frac{1}{2}a_{ij}(x)D_{ij}W(x) + b(x) \cdot \nabla W(x) + \frac{1}{2}a_{ij}(x)D_iW(x)D_jW(x) + V(x) = \Lambda,$$

where Λ is a real number and W is a smooth function. A solution is given by a pair, Λ and W. One may notice that this equation has another familiar form,

$$\frac{1}{2}a_{ij}(x)D_{ij}\phi(x) + b(x) \cdot \nabla\phi(x) + V(x)\phi(x) = \Lambda\phi(x),$$

with $\phi = \exp(W)$. This is a equation for eigenvalue and eigenfunction. An equation with two different forms actually suggests very different approaches. This has been revealed in our previous study, where a more general class of nonlinear equations were discussed and in the analysis the function-space argument, as is normally used in PDE literature, was totally avoided. In this talk, we shall briefly review some results for this type of nonlinear equations that will be used in the study of our problem. The asymptotics of large time expectaions for diffusion process indeed suggest interesting questions for such nonlinear equations. We shall mention these in our discussion. Aiming for the solution, we are able to obtain some results that in the end will give some satisfactory answers for the large time asymptotics of the expectations described above.

On the principal eigenvalues of Kolmogorov operators on $\mathbb{R}^d$

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Let L_0 be the Kolmogorov operator on $\mathbb{R}^d$ defined by

$$L_0 = \frac{1}{2}\Delta - DU(x) \cdot D.$$

Following to [1, Hypothesis 1.1], we make the following assumptions on the function U:

(A1) $U \in C^5(\mathbb{R}^d)$,

(A2) $^\exists m \geq 0$ such that $\sup \left\{ \frac{|D^{1+\ell}U(x)|}{(1+|x|)^{2m+1-\ell}} \,\middle|\, x \in \mathbb{R}^d \right\} < +\infty \quad \text{for } \ell = 0, 1, 2, 3, 4,$

(A3) $\alpha := \inf \left\{ \min \left[D^2U(x)y \cdot y \mid y \in \mathbb{R}^d, |y| = 1 \right] \,\middle|\, x \in \mathbb{R}^d \right\} > 0,$

(A4) $^\exists a, {}^\exists\gamma, {}^\exists c > 0$ such that

$$y \cdot (DU(x+y) - DU(x)) \geq a|y|^{2m+2} - c(|x|^\gamma + 1) \quad \text{for } x, y \in \mathbb{R}^d.$$

Let $d\nu$ be the unique invariant probability measure on $\mathbb{R}^d$ associated with L_0, and $H^1(\nu)$ the Sobolev space with the norm

$$\|\varphi\|_{H^1(\nu)} = \left[\|\varphi\|^2 + \|D\varphi\|^2 \right]^{1/2},$$

where

$$\|\varphi\| = \left(\int_{\mathbb{R}^d} \varphi^2 \, d\nu \right)^{1/2}, \qquad \|D\varphi\| = \left(\sum_{i=1}^d \|\partial\varphi/\partial x_i\|^2 \right)^{1/2}.$$

Let β be the infimum of the Rayleigh quotient on $H_0^1(\nu)$ defined by

$$\beta = \inf \left\{ \frac{\|D\varphi\|^2}{\|\varphi\|^2} \,\middle|\, \varphi \in H_0^1(\nu) \setminus \{0\} \right\},$$

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where

$$H^1_0(\nu) = \{ \varphi \in H^1(\nu) \ : \ \overline{\varphi} = 0 \} \quad \text{and} \quad \overline{\varphi} = \int_{\mathbb{R}^d} \varphi \, d\nu.$$

Then, the best constant of the Poincaré inequality in $L^2(\nu)$ is β^{-1}. In [2], it was shown that $2\alpha \le \beta$. It is easy to see that the equality holds when L_0 is the Ornstein-Uhlenbeck operator. Here and in what follows, we say that L_0 is the Ornstein-Uhlenbeck operator if there exist a $d \times d$ positive definite matrix A and a vector b such that $DU(x) = A(x-b)$ in $\mathbb{R}^d$.

Our goal of this talk is to show that $\beta = 2\alpha$ *only* when L_0 is the Ornstein-Uhlenbeck operator. Hence, if L_0 is not the Ornstein-Uhlenbeck operator, the principal eigenvalue $\beta/2$ of $-L_0$ is strictly greater than α.

Our main result is stated as follows: Define the constant γ by

$$\gamma = \inf \left\{ \frac{1}{\|\varphi\|^2} \int_0^\infty \|D^2 P_t \varphi\|^2 \, dt \,\middle|\, \varphi \in H^1_0(\nu) \setminus \{0\} \right\}.$$

Here we use the notation such that $\|D^2\psi\| = \left(\sum_{i,j=1}^{d} \|\partial^2\psi/\partial x_i \partial x_j\|^2 \right)^{1/2}$.

THEOREM ([6]). The following three statements are equivalent:

(a) $\beta = 2\alpha$.

(b) $\gamma = 0$.

(c) $DU(x) = A(x-b)$ in $\mathbb{R}^d$, where A is a $d \times d$ positive definite matrix with the minimum eigenvalue α, and $b = \int_{\mathbb{R}^d} y \, d\nu(y) \in \mathbb{R}^d$. $\square$

References

[1] S. Cerrai, Second Order PDE's in Finite and Infinite Dimension. A Probabilistic Approach, Lecture Notes in Math. 1762. Springer, Berlin, 2001.

[2] G. Da Prato and B. Goldys, *Elliptic operators on $\mathbb{R}^d$ with unbounded coefficients*, J. Differential Equations, 172 (2001), pp. 333–358, *Erratum*, J. Differential Equations, 184 (2002), p. 620.

[3] G. Da Prato and A. Lunardi, *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*, J. Functional Analysis, 131 (1995), pp. 94–114.

[4] G. Da Prato and J. Zabczyk, Second Order Partial Differential Equations in Hilbert spaces, London Mathematical Society Lecture Note Series 293. Cambridge University Press, Cambridge, 2002.

[5] Y. Fujita, *A refinement of the Poincaré inequality for the Kolmogorov operators on $\mathbb{R}^d$*, J. Inequalities and Applications, Vol.2005 (2005), 25-31.

[6] Y. Fujita, *On the best constants of the Poincaré inequality for Kolmogorov operators on $\mathbb{R}^d$*, preprint.

An Estimation Problem for the Shape of a Domain via Parabolic Equations

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1 Introduction

This work is concerned with the inverse problem of determining the shape of some unknown portion of the boundary of a domain based on a parabolic equation on the domain; the details will be given in [2]. Such problems have been already studied by several authors. Especially, Bryan and Caudill [1] treat the problem based on the usual heat equation in general dimensional cases. More precisely they mainly consider the case of small and smooth unknown deformation of the lower surface of a box, which is the unknown portion of the boundary of the deformed domain. In fact they examine a suitably linearized problem and determine the shape from measurements of the temperature on the upper surface of the box. Then they impose Neumann's condition on the boundary, in particular, zero-Neumann's on the unknown portion.

In this work, the same problem is investigated in a somewhat different situation: Dirichlet's condition is imposed on an unknown portion and Neumann's ones on the other portions; that is, a mixed boundary condition is imposed. This case is also necessary in practical applications. We also examine a linearized problem and present a new reconstruction procedure for the shape of the unknown portion and its stability property. In addition, the problem is treated in more general framework: the heat operator and the box are replaced by a second order parabolic operator of general type and a more general cylindrical domain, respectively, and further the shape of deforming unknown portion is allowed depending on time and assuming only Lipschitz continuity. We therefore have to use generalized solutions as in Ladyženskaja et al [3] and give an a priori estimate for generalized solutions. This ensures rigorous derivation of the linearized problem.

2 The inverse problem and its reduction

We start with defining a basic cylindrical domain in $\boldsymbol{R}^n$ $(n \geq 2)$. Let B be a bounded Lipschitz domain in $\boldsymbol{R}^{n-1}$ and set

$$\Omega := \left\{(x', x^n) \in \boldsymbol{R}^n : x' = (x^1, \ldots, x^{n-1}) \in B,\ 0 < x^n < 1\right\} = B \times (0, 1). \tag{2.1}$$

We decompose $\Gamma := \partial\Omega$ into a disjoint union $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_0 := \{(x', x^n) \in \partial\Omega : x^n = 0\}, \quad \Gamma_1 := \{(x', x^n) \in \partial\Omega : x^n = 1\}, \quad \Gamma_2 := \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1).$$

Let $S = S(t, y')$ be a function on $[0, \infty) \times \boldsymbol{R}^{n-1}$ such that

$$\operatorname{supp} S \subset [0, \infty) \times B, \qquad S < 1, \tag{2.2}$$

which gives the shape of the lower surface of the deformed domain. Then the deformed domain is given by

$$\Omega_{(t)} \equiv \Omega_{S(t,\,\cdot\,)} := \{y = (y', y^n) :\ y' \in B,\ y^n > S(t, y')\} \quad (t \geq 0). \tag{2.3}$$

For a given $T > 0$, we further define the space-time domain $\tilde{Q}_T$ with time dependent section $\Omega_{(t)}$ by

$$\tilde{Q}_T := \left\{(t, y) :\ 0 < t < T,\ y \in \Omega_{(t)}\right\}$$

and decompose $\partial\tilde{Q}_T$ into a disjoint union

$$\partial\tilde{Q}_T = R_{0,T} \cup R_{1,T} \cup R_{2,T} \cup (\{0\} \times \Omega_{(0)}) \cup (\{T\} \times \Omega_{(T)}),$$

where

$$\begin{aligned} R_{0,T} &:= \left\{(t, y) :\ 0 \leq t \leq T,\ y' \in B,\ y^n = S(t, y')\right\}, \\ R_{1,T} &:= \left\{(t, y) :\ 0 \leq t \leq T,\ y' \in B,\ y^n = 1\right\}, \\ R_{2,T} &:= \left\{(t, y) :\ 0 \leq t \leq T,\ y' \in \partial B,\ 0 \leq y^n \leq 1\right\}. \end{aligned}$$

Let $A \equiv A(t, y) = [a^{ij}(t, y)]$, $\boldsymbol{b} \equiv \boldsymbol{b}(t, y) = [b^1(t, y), \ldots, b^n(t, y)]^T$ and $a \equiv a(t, y)$ be an $n \times n$ real matrix-valued function, an n-dimensional real vector-valued function and a real-valued function defined on the closure of $\left\{(t, y) :\ 0 < t < T,\ y \in \cup_{0 \leq \tau \leq T}\Omega_{(\tau)} \cup \Omega\right\}$, respectively. Furthermore suppose that $A(t, y)$ is symmetric and positive definite. For such quantities A, $\boldsymbol{b}$ and a, define a second order differential operator $\mathcal{P}$:

$$\mathcal{P}v(t, y) := \nabla_y \bullet (A(t, y)\nabla_y v(t, y)) - \boldsymbol{b}(t, y) \bullet \nabla_y v(t, y) - a(t, y)v(t, y),$$

where $\nabla_y := [\partial/\partial y^1, \ldots, \partial/\partial y^n]^T$. Let $\nu = \nu(t, y) = [\nu_1(t, y), \nu_2(t, y), \ldots, \nu_n(t, y)]^T$ be the outer unit normal vector at $y \in \partial\Omega_{(t)}$. We define the conormal derivative $\partial/\partial\mathcal{N}$ and the parabolic operator $\mathcal{L}$ relative to $\mathcal{P}$ by

$$\frac{\partial v}{\partial \mathcal{N}}(t, y) := [A\nabla_y v(t, y)] \bullet \nu(t, y),$$

$$\mathcal{L}v(t, y) := \frac{\partial v}{\partial t}(t, y) - \mathcal{P}v(t, y).$$

Let f be a function on $\left\{(t, y) :\ 0 < t < T,\ y \in \cup_{0 \leq \tau \leq T}\Omega_{(\tau)} \cup \Omega\right\}$, ψ a function on $\{(t, y) :\ 0 < t < T,$ $y' \in B, y^n = 1\}$, and h a function on $\Omega_{(0)} \cup \Omega$. Now consider the following initial-boundary value problem on the deformed domain:

$$\begin{cases} \mathcal{L}v(t, y) = -f(t, y) & \text{on } \tilde{Q}_T \\ v(t, y) = 0 & \text{on } R_{0,T} \\ \dfrac{\partial v}{\partial \mathcal{N}}(t, y) = \begin{cases} \psi(t, y) & \text{on } R_{1,T} \\ 0 & \text{on } R_{2,T} \end{cases} \\ v(0, y) = h(y) & \text{on } \Omega_0. \end{cases} \tag{2.4}$$

We want to solve the following inverse problem: *determine the shape $S(t, y')$ from the data $v(t, y)$ $((t, y) \in R_{1,T})$.* Of course, the time interval $[0, T]$ can be replaced by another one; in what follows, we consider the case of the time interval $[0, T]$. In order to solve the inverse problem, we transform the initial-boundary value problem (2.4) to another one on the basic domain Ω as follows. For each $t > 0$, define a change of coordinates $\phi_{(t)} = \phi_{S(t,\,\cdot\,)}$ by

$$\begin{array}{rccl} \phi_{(t)}: & \Omega_{(t)} & \longrightarrow & \Omega \\ & y = (y', y^n) & \longmapsto & x = (x', x^n) = \phi_{(t)}(y) = \left(y', \dfrac{y^n - S(t,y')}{1 - S(t,y')}\right). \end{array} \tag{2.5}$$

Then the transformed operator is given as

$$\hat{\mathcal{P}}u(t,x) := \nabla_x \bullet (\hat{A}(t,y)\nabla_x u(t,x)) - \hat{\boldsymbol{b}}(t,x) \bullet \nabla_x u(t,x) - \hat{a}(t,x)u(t,x),$$

where

$$\begin{aligned} \hat{A}(t,x) \equiv \left[\hat{a}^{\alpha\beta}(t,x)\right] &:= (D_y\phi_{(t)})(\phi_{(t)}^{-1}(x))A(t,\phi_{(t)}^{-1}(x))(D_y\phi_{(t)})^T(\phi_{(t)}^{-1}(x)), \\ \hat{\boldsymbol{b}}(t,x) \equiv \left[\hat{b}^{\alpha}(t,x)\right] &:= (D_y\phi_{(t)})(\phi_{(t)}^{-1}(x))\boldsymbol{b}(t,\phi_{(t)}^{-1}(x)) \\ &\quad -(\det D_x\phi_{(t)}^{-1}(x))^{-1}\hat{A}(t,x)\nabla_x\left(\det D_x\phi_{(t)}^{-1}(x)\right) \\ &\quad -\left[0,\ldots,0,\frac{(1-x^n)S_t(t,x')}{1-S(t,x')}\right]^T, \\ \hat{a}(t,x) &:= a(t,\phi_{(t)}^{-1}(x)). \end{aligned}$$

If we set $u(t,x) = u(t,\phi_{(t)}(y)) = v(t,y)$, then $\mathcal{P}v(t,y) = \hat{\mathcal{P}}u(t,x)$. Denote by $\hat{\mathcal{L}}$ and $\partial/\partial\hat{\mathcal{N}}$ the parabolic operator and the conormal derivative relative to $\hat{\mathcal{P}}$, respectively; we have

$$\frac{\partial v}{\partial \mathcal{N}}(t,y) = (1 - S(t,x'))\sum_{\beta=1}^{n} \hat{a}^{n\beta}(t,x)\frac{\partial u}{\partial x^\beta}(t,x) = (1 - S(t,x'))\frac{\partial u}{\partial \hat{\mathcal{N}}}(t,x) \quad \text{on} \quad R_{1,T},$$

$$\frac{\partial v}{\partial \mathcal{N}}(t,y) = \frac{\partial u}{\partial \hat{\mathcal{N}}}(t,x) \quad \text{on} \quad R_{2,T}.$$

Let $Q_T := (0,T) \times \Omega$ and $S_{j,T} := [0,T] \times \Gamma_j$ $(j = 0,1,2)$ be space-time cylinders. Then the problem (2.4) is transformed into the following initial-boundary value problem on the basic domain:

$$\left\{\begin{array}{rll} \hat{\mathcal{L}}u(t,x) = -\hat{f}(t,x) & & \text{on } Q_T \\ u(t,x) = 0 & & \text{on } S_{0,T} \\ \dfrac{\partial u}{\partial \hat{\mathcal{N}}}(t,x) = \left\{\begin{array}{l} \hat{\psi}(t,x) \\ 0 \end{array}\right. & \begin{array}{l} \text{on} \\ \text{on} \end{array} & \begin{array}{l} S_{1,T} \\ S_{2,T} \end{array} \\ u(0,x) = \hat{h}(x) & & \text{on } \Omega, \end{array}\right. \tag{2.6}$$

where $\hat{f}(t,x) := f(t,\phi_{(t)}^{-1}(x))$, $\hat{\psi}(t,x) := \dfrac{1}{1 - S(t,x')}\psi(t,x)$ and $\hat{h}(x) := h(\phi_{(t)}^{-1}(x))$.

The correspondence between the shape S and a solution u of the above initial-boundary value problem (2.6) is nonlinear in general. As in Bryan and Caudill [1], for the first approximating approach to the original inverse problem, we examine a linearized problem such that the correspondence becomes a linear mapping. The linearized problem is provided by the idea of Gâteaux differentiation. Let ϵ be a real number and replace S by ϵS in the problem (2.6).

The corresponding operators and data are denoted with index ϵ. Then we write the resulting problem as

$$\left\{\begin{array}{lr} \hat{\mathcal{L}}_\epsilon u(t,x) = -\hat{f}_\epsilon(t,x) & \text{on } Q_T \\ u(t,x) = 0 & \text{on } S_{0,T} \\ \dfrac{\partial u}{\partial \hat{\mathcal{N}}_\epsilon}(t,x) = \left\{\begin{array}{l} \hat{\psi}_\epsilon(t,x) \\ 0 \end{array}\right. & \begin{array}{l}\text{on } S_{1,T} \\ \text{on } S_{2,T}\end{array} \\ u(0,x) = \hat{h}_\epsilon(x) & \text{on } \Omega. \end{array}\right. \tag{2.7}$$

Denote by u_ϵ a solution of (2.7). Since $\phi_{(0)}$ is the identity map, $\hat{\mathcal{L}}_0 = \mathcal{L}$ and $\partial/\partial\hat{\mathcal{N}}_0 = \partial/\partial\mathcal{N}$. Hence u_0 becomes a solution to the problem:

$$\left\{\begin{array}{lr} \mathcal{L}u_0(t,x) = -f(t,x) & \text{on } Q_T \\ u_0(t,x) = 0 & \text{on } S_{0,T} \\ \dfrac{\partial u_0}{\partial \mathcal{N}}(t,x) = \left\{\begin{array}{l} \psi(t,x) \\ 0 \end{array}\right. & \begin{array}{l}\text{on } S_{1,T} \\ \text{on } S_{2,T}\end{array} \\ u_0(0,x) = h(x) & \text{on } \Omega. \end{array}\right. \tag{2.8}$$

Differentiating (2.7) in ϵ at $\epsilon = 0$, we have the desired linearized problem for an unknown function w:

$$\left\{\begin{array}{lr} \mathcal{L}w(t,x) = \displaystyle\sum_{i=1}^n \frac{\partial (F^i)^\sharp}{\partial x^i}(t,x) - F^\sharp(t,x) & \text{on } Q_T \\ w(t,x) = 0 & \text{on } S_{0,T} \\ \dfrac{\partial w}{\partial \mathcal{N}}(t,x) = \left\{\begin{array}{l} \Psi^\sharp(t,x) - \displaystyle\sum_{i=1}^n (F^i)^\sharp \nu_i \\ -\displaystyle\sum_{i=1}^n (F^i)^\sharp \nu_i \end{array}\right. & \begin{array}{l}\text{on } S_{1,T} \\ \\ \text{on } S_{2,T}\end{array} \\ w(0,x) = h^\sharp(x) & \text{on } \Omega, \end{array}\right. \tag{2.9}$$

where

$$(F^i)^\sharp = \sum_{j=1}^n \left.\frac{d\hat{a}^{ij}_\epsilon}{d\epsilon}\right|_{\epsilon=0} \frac{\partial u_0}{\partial x^j}, \qquad F^\sharp = \sum_{i=1}^n \left.\frac{d\hat{b}^i_\epsilon}{d\epsilon}\right|_{\epsilon=0} \frac{\partial u_0}{\partial x^i} + \left.\frac{d\hat{a}_\epsilon}{d\epsilon}\right|_{\epsilon=0} u_0 + \left.\frac{d\hat{f}_\epsilon}{d\epsilon}\right|_{\epsilon=0},$$

$$\Psi^\sharp = \left.\frac{d\hat{\psi}_\epsilon}{d\epsilon}\right|_{\epsilon=0}, \qquad h^\sharp = \left.\frac{d\hat{h}_\epsilon}{d\epsilon}\right|_{\epsilon=0}.$$

Our main subject is to solve the inverse problem based on the linearized problem. Therefore we restate the inverse problem: *determine the shape* $S(t,x')$ *for* $(t,x') \in [0,T] \times B$ *from the data* $d(t,x') := w(t,x)$ *for* $(t,x) \in S_{1,T}$.

3 Weak form and an a priori estimate for generalized solutions

In this section, we treat the domain and operator in more general type. Let Ω be a Lipschitz domain in $\boldsymbol{R}^n$ ($n \geq 2$) and $\Gamma = \Pi \cup \Gamma' \cup \Gamma''$ a Lipschitz dissection of $\Gamma = \partial\Omega$ (see [4] for the definition). For $T > 0$, put $Q_T = (0,T) \times \Omega$, $S_T = [0,T] \times \Gamma$, $S'_T = [0,T] \times \Gamma'$ and $S''_T = [0,T] \times \Gamma''$.

Suppose that the following quantities defined on the closure of Q_T are provided:

(1) $A \equiv A(t,x) := [a^{ij}(t,x)]$ is an $n \times n$ real matrix-valued measurable function, which is symmetric and positive definite;

(2) $\boldsymbol{a} \equiv \boldsymbol{a}(t,x) = [a^1(t,x), \dots, a^n(t,x)]^T$ and $\boldsymbol{b} \equiv \boldsymbol{b}(t,x) = [b^1(t,x), \dots, b^n(t,x)]^T$ are n-dimensional real vector-valued measurable functions;

(3) $a \equiv a(t,x)$ is a real-valued measurable function.

For such quantities, consider the linear differential operators $\mathcal{A}^i$ and $\mathcal{P}$ defined by

$$\begin{aligned}
\mathcal{A}^i u(t,x) &:= \sum_{j=1}^{n} a^{ij}(t,x)\frac{\partial u}{\partial x^j}(t,x) + a^i(t,x)u(t,x), \\
\mathcal{P}u(t,x) &:= \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left[\mathcal{A}^i u(t,x)\right] - \sum_{i=1}^{n} b^i(t,x)\frac{\partial u}{\partial x^i}(t,x) - a(t,x)u(t,x) \\
&\equiv \nabla_x \bullet (A(t,x)\nabla_x u(t,x) + \boldsymbol{a}(t,x)u(t,x)) - \boldsymbol{b}(t,x)\bullet\nabla_x u(t,x) - a(t,x)u(t,x).
\end{aligned}$$

Then we also denote by $\mathcal{L}$ and $\dfrac{\partial}{\partial \mathcal{N}}$ the parabolic operator and the conormal derivative relative to $\mathcal{P}$, that is,

$$\mathcal{L}u(t,x) := \frac{\partial u}{\partial t}(t,x) - \mathcal{P}u(t,x), \qquad \frac{\partial u}{\partial \mathcal{N}}(t,x) := \sum_{i=1}^{n} \mathcal{A}^i u(t,x)\nu_i(x),$$

where $\nu = \nu(x) = [\nu_1(x), \nu_2(x), \dots, \nu_n(x)]^T$ is the unit outer normal vector at $x \in \partial\Omega$. Consider the following initial-boundary value problem

$$\begin{cases}
\mathcal{L}u(t,x) = \displaystyle\sum_{i=1}^{n} \frac{\partial f^i}{\partial x^i}(t,x) - f(t,x) & \text{on } Q_T \\
u(t,x) = 0 & \text{on } S'_T \\
\dfrac{\partial u}{\partial \mathcal{N}}(t,x) + \sigma(t,x)u(t,x) = \psi(t,x) & \text{on } S''_T \\
u(0,x) = h(x) & \text{on } \Omega.
\end{cases} \tag{3.1}$$

Before introducing the concept of generalized solutions of (3.1), we provide some notations for norms and spaces of integrable functions on Q_T and S_T. For $1 \le q, r < \infty$, Denote by $L^{q,r}(Q_T)$ the space of functions $u(t,x)$ on Q_T such that

$$\|u\|_{q,r;Q_T} \equiv \|u\|_{L^{q,r}(Q_T)} := \left\{\int_0^T \left(\int_\Omega |u(t,x)|^q\,dx\right)^{\frac{r}{q}} dt\right\}^{\frac{1}{r}} < \infty.$$

Similarly we define the norm $\|u\|_{q,r,S_T}$ and the space $L^{q,r}(S_T)$ by using the surface measure $d\tilde{x}$ on Γ instead of the Lebesgue measure on Ω. In particular, when $q = r$, $L^{q,q}(Q_T) = L^q(Q_T)$; $\|u\|_{q,q;Q_T} = \|u\|_{q,Q_T} \equiv \|u\|_{L^q(Q_T)}$ and $\|u\|_{q,q;S_T} = \|u\|_{q,S_T} \equiv \|u\|_{L^q(S_T)}$. In the case of $r = \infty$, we also define the corresponding norms and spaces in the usual manner. Then we introduce the spaces of test functions and generalized solutions as in [3]:

$$\begin{aligned}
H^{1,1}(Q_T) &:= H^1((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega)), \\
V^{0,1}(Q_T) &:= C([0,T]; L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))
\end{aligned}$$

with norm $\|\ \|_{Q_T}$, where

$$\|u\|_{Q_T} := \max_{0 \le t \le T} \|u(t,\cdot)\|_{2,\Omega} + \|\nabla_x u\|_{2,Q_T}.$$

In the followings, γ denotes the trace operator to the boundary Γ.

Definition 3.1 *We call $u \in V^{0,1}(Q_T)$ a generalized solution of (3.1) if u satisfies*

(i) $u|_{S'_T} = 0$, *that is,* $\gamma\eta(t,\cdot) = 0$ *on* Γ' *for almost every* $t \in [0,T]$;

(ii) for every $\eta \in H^{1,1}(Q_T)$ *with* $\eta|_{S'_T} = 0$,

$$\begin{aligned}
&\int_\Omega u(T,x)\eta(T,x)dx - \int_{Q_T} u\eta_t\, dxdt \\
&+ \int_{Q_T} \{(A\nabla_x u + \boldsymbol{a}u) \cdot \nabla_x \eta + (\boldsymbol{b} \cdot \nabla_x u + au)\,\eta\}\, dxdt \\
&+ \int_{Q_T} (\boldsymbol{f} \cdot \nabla_x \eta + f\eta)\, dxdt + \int_{S_T} \sigma\,\gamma u\,\gamma\eta\, d\tilde{x}dt - \int_{S_T} (\psi + \gamma\boldsymbol{f} \cdot \nu)\,\gamma\eta\, d\tilde{x}dt \\
&= \int_\Omega h(x)\eta(0,x)\, dx, \qquad (3.2)
\end{aligned}$$

where $\boldsymbol{f} := (f^1, \ldots, f^n)^T$ *and* $\gamma\boldsymbol{f} := (\gamma f^1, \ldots, \gamma f^n)^T$.

For later use, it is convenient to generalize the equation (3.2) with replacing $\gamma\boldsymbol{f}$ by an arbitrarily given function $\boldsymbol{\psi} := (\psi^1, \ldots, \psi^n)^T$ defined on S_T:

$$\begin{aligned}
&\int_\Omega u(T,x)\eta(T,x)dx - \int_{Q_T} u\eta_t\, dxdt \\
&+ \int_{Q_T} \{(A\nabla_x u + \boldsymbol{a}u) \cdot \nabla_x \eta + (\boldsymbol{b} \cdot \nabla_x u + au)\,\eta\}\, dxdt \\
&+ \int_{Q_T} (\boldsymbol{f} \cdot \nabla_x \eta + f\eta)\, dxdt + \int_{S_T} \sigma\,\gamma u\,\gamma\eta\, d\tilde{x}dt - \int_{S_T} (\psi + \boldsymbol{\psi} \cdot \nu)\,\gamma\eta\, d\tilde{x}dt \\
&= \int_\Omega h(x)\eta(0,x)\, dx. \qquad (3.3)
\end{aligned}$$

We notice that all the integrals in (3.3) are well-defined under following assumption.

Assumption 3.1 *(1) There exists a constant $\nu > 0$ such that for almost every $(t,x) \in Q_T$ and every $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \boldsymbol{R}^n$,*

$$\sum_{i,j=1}^{n} a^{ij}(t,x)\xi_i\xi_j \geq \nu\,|\xi|^2 .$$

(2) (a)

$$\sum_{i=1}^{n} (a^i)^2,\ \sum_{i=1}^{n} (b^i)^2,\ a \in L^{q_0,r_0}(Q_T),$$

with exponent (q_0, r_0) such that

$$q_0 > \frac{n}{2},\quad r_0 \geq \frac{2q_0}{2q_0 - n}.$$

(b)

$$\sigma \in L^{q_2,r_2}(S_T),$$

with exponent (q_2, r_2) *such that*

$$q_2 > n-1, \quad r_2 \geq \frac{2q_2}{q_2 - n + 1} \quad (n \geq 3);$$
$$q_2 > 1, \quad r_2 > \frac{2q_2}{q_2 - 1} \quad (n = 2).$$

(3) (a)

$$h \in L^2(\Omega), \quad f^i \in L^2(Q_T), \quad \psi^i \in L^2(S_T) \qquad (i = 1, 2, \ldots, n).$$

(b)

$$f \in L^{q_1, r_1}(Q_T),$$

with exponent (q_1, r_1) *such that*

$$\frac{2n}{n+2} \leq q_1 < 2, \quad r_1 \geq \frac{4q_1}{(n+4)q_1 - 2n} \quad (n \geq 3);$$
$$1 < q_1 < 2, \quad r_1 \geq \frac{2q_1}{3q_1 - 2} \quad (n = 2)$$

or

$$q_1 \geq 2, \quad r_1 > 1.$$

(4)

$$\psi \in L^{q_3, r_3}(S_T),$$

with exponent (q_3, r_3) *such that*

$$\frac{2(n-1)}{n} \leq q_3 \leq 2, \quad r_3 \geq \frac{4q_3}{(n+2)q_3 - 2(n-1)} \quad (n \geq 3);$$
$$1 < q_3 \leq 2, \quad r_3 \geq \frac{2q_3}{2q_3 - 1} \quad (n = 2)$$

or

$$q_3 \geq 2, \quad r_3 \geq \frac{4}{3} \quad (n \geq 3); \quad q_3 \geq 2, \quad r_3 > \frac{4}{3} \quad (n = 2).$$

Under Assumption 3.1, there exist positive constants μ_1 and μ_2 such that

$$\left\| \sum_{i=1}^{n} (a^i)^2 \right\|_{L^{q_0, r_0}(Q_T)} \leq \mu_1, \quad \left\| \sum_{i=1}^{n} (b^i)^2 \right\|_{L^{q_0, r_0}(Q_T)} \leq \mu_1,$$

and

$$\|\sigma\|_{L^{q_2, r_2}(S_T)} \leq \mu_2.$$

In the followings, we fix such constants.

We extend the energy inequality in [3] (see Chap.III, §2) to the case of mixed boundary conditions on a general bounded Lipschitz domain and, using it, we obtain the following a priori estimate for solutions of (3.3).

Theorem 3.1 *Suppose that Assumption 3.1 is fulfilled. Then there exists a positive constant C depending only on ν, μ_1, μ_2, n such that*

$$\|u\|_{Q_T} \le C\left\{\|h\|_{2,\Omega} + \|\boldsymbol{f}\|_{2,Q_T} + \|f\|_{q_1,r_1;Q_T} + \|\psi\|_{q_3,r_3;S_T} + \|\boldsymbol{\psi}\|_{2,S_T}\right\} \tag{3.4}$$

for every solution u of (3.3) with $u|_{S'_T} = 0$. In particular, if $f \in L^2(Q_T)$ and $\psi \in L^2(S_T)$, then $\|f\|_{q_1,r_1;Q_T}$ and $\|\psi\|_{q_3,r_3;S_T}$ in the estimate (3.4) can be replaced by $\|f\|_{2,Q_T}$ and $\|\psi\|_{2,S_T}$, respectively.

4 Unique identification and reconstruction of the shape

In this section, we return the same situation on the domain and operator as in §2. Before stating the results, let me introduce a condition for the coefficients and free terms of the initial-boundary value problem (2.4).

Assumption 4.1 *The coefficients and free terms of the initial-boundary value problem (2.4) satisfies the following conditions.*

(1) The coefficients and free terms are bounded.

(2) There exists a constant $\nu > 0$ such that for almost every $(t, y) \in Q_T$ and every $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \boldsymbol{R}^n$,

$$\sum_{i,j=1}^{n} a^{ij}(t, y)\xi_i\xi_j \ge \nu\,|\xi|^2 .$$

(3) The coefficients a^{ij} $(i, j = 1, \ldots, n)$ are of class C^1 and the other coefficients are Lipschitz continuous in y and their Lipschitz constants are bounded in t.

(4) $S = S(t, y')$ is Lipschitz continuous in (t, y').

(5) h is Lipschitz continuous and its trace γh to the boundary $\partial\Omega_0$ vanishes on the lower surface: $y^n = S(0, y')$.

(6) a^{ij}, b^i $(i, j = 1, \ldots, n)$ and ψ have derivatives a^{ij}_t, b^i_t, ψ_t in t and satisfy

$$\int_0^T \operatorname*{ess\,sup}_{y\in\Omega} \left|\frac{\partial a^{ij}}{\partial t}(t, y)\right| dt \;<\; \infty, \tag{4.1}$$

$$\int_0^T \operatorname*{ess\,sup}_{y\in\Omega} \left|\frac{\partial b^i}{\partial t}(t, y)\right|^2 dt \;<\; \infty, \tag{4.2}$$

$$\int_0^T \operatorname*{ess\,sup}_{y\in\Gamma} \left|\frac{\partial \psi}{\partial t}(t, y)\right| dt \;<\; \infty. \tag{4.3}$$

Then the following unique identification theorem is obtained.

Theorem 4.1 *In addition to Assumption 4.1, assume that $f(t, y) = 0$. Moreover suppose that one of the followings is fulfilled:*

(i) The shape S is independent of the time variable t, and $h \not\equiv 0$ on Ω or $\psi \not\equiv 0$ on $S_{1,T}$.

(ii) For every open interval $I \subset (0, T)$, $\psi \not\equiv 0$ on $I \times \Gamma_1$.

Then, the data $d(t,x') \equiv w(t,x)$ ($(t,x) \in S_{1,T}$) determine $S(t,x')$ ($(t,x') \in [0,T] \times B$) uniquely.

A key result to obtain the unique identification theorem is the following integral equality on the boundary.

Lemma 4.1 *Suppose that Assumption 4.1 is satisfied. Let φ be a generalized solution of the problem:*

$$\begin{cases} \mathcal{L}^*\varphi(t,x) = 0 & \text{on } Q_T \\ \varphi(t,x) = 0 & \text{on } S_{0,T} \\ \dfrac{\partial \varphi}{\partial \mathcal{N}^*}(t,x) = 0 & \text{on } S_{2,T} \\ \varphi(T,x) = 0 & \text{on } \Omega, \end{cases} \tag{4.4}$$

where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$ and $\partial/\partial\mathcal{N}^*$ denotes the asocitated conormal derivative. Then

$$\int_{S_{1,T}} d\, \frac{\partial \varphi}{\partial \mathcal{N}^*}\, d\tilde{x}dt = -\int_{S_{0,T}} S\, (a^{dd})^{-1} \frac{\partial u_0}{\partial \mathcal{N}} \frac{\partial \varphi}{\partial \mathcal{N}^*}\, d\tilde{x}dt. \tag{4.5}$$

Combining the lemma with the denseness of the Dirichlet to Neumann map from $S_{1,T}$ to $S_{0,T}$ with respect to the problem (4.4), we can verify the theorem.

Next we proceed to discuss a reconstruction procedure of the shape. To describe the result, we need the following function space on $B_T := [0,T] \times B$:

$$H^{1,1}_{,0}(B_T) := H^1(0,T;H^0(B)) \cap H^0(0,T;H^1_0(B)),$$

where for an open set O, $H^1(O)$ denotes the usual Sobolev space on O based on the L^2–space $L^2(O) \equiv H^0(O)$ and also $H^1_0(O)$ stands for the space of Sobolev functions with null trace on the boundary.

We reconstruct the shape $S(t,x')$ approximately from given data $d(t,x')$ as follows:

1. Take a basis $\{S_i\}_{i=1}^{\infty}$ of $H^{1,1}_{,0}(B_T)$ such that for each $t \in (0,T)$, supp $S_i(t,\,\cdot\,) \subset B$ ($i = 1,2,\ldots$). Denote by $w_i(t,x)$ the solution of (2.9) in the case where $S(t,x') = S_i(t,x')$. Then evaluate the response $d_i(t,x') := w_i(t,x)$ for $(t,x) \in S_{1,T}$.
2. For arbitrarily fixed $m \in \boldsymbol{N}$, choose a sequence $(\overline{a}_1,\ldots,\overline{a}_m)$ by the method of least squares:
$$\underset{(a_1,\ldots,a_m)}{\arg\min} \left\| d - \sum_{i=1}^{m} a_i d_i \right\|^2_{L^2(S_{1,T})} = (\overline{a}_1,\ldots,\overline{a}_m). \tag{4.6}$$
3. For each $m \in \boldsymbol{N}$, the m–th approximation $S_{(m)}$ of S is given by
$$S_{(m)}(t,x') := \sum_{i=1}^{m} \overline{a_i} S_i(t,x'). \tag{4.7}$$

Then we have the following result on the convergence and stability of the reconstruction procedure.

Theorem 4.2 *(1) There is a dense linear subspace $\mathcal{S}$ of $H^{1,1}_{,0}(B_T)$ such that for every $S \in \mathcal{S}$*

$$\lim_{m\to\infty} S_{(m)} = S \quad in \quad H^{1,1}_{,0}(B_T). \tag{4.8}$$

(2) *For given S and $S' \in H^{1,1}_{,0}(B_T)$, denote by d and d' the corresponding responses on $S_{1,T}$ respectively. And for each $m \in \boldsymbol{N}$, let $(\overline{a}_1, \ldots, \overline{a}_m)$ and $(\overline{a}'_1, \ldots, \overline{a}'_m)$ be the sequences determined by the method of least squares from d and d' respectively. Then it holds*

$$\max_{1 \leq i \leq m} |\overline{a}_i - \overline{a}'_i| \leq K_m \|d - d'\|_{L^2(S_{1,T})} \tag{4.9}$$

with a positive constant K_m determined by $d_1, \ldots, d_m$.

References

[1] Kurt Bryan and Lester F. Caudill Jr., *Stability and reconstruction for an inverse problem for the heat equation*, Inverse Problems **14** (1998) 1429–1453

[2] H. Kawakami, Y. Moriyama and M. Tsuchiya, *An estimation problem for the boundary shape of a domain via parabolc equations*, in preparation

[3] O. A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva, *Lincar and Quasilinear Equations of Parabolic Type*, (English transl.) AMS, 1968

[4] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000

[5] Jean-Claude Saut and Bruno Sheurer, *Unique continuation for some evolution equation*, J. Differential Equations, **66**, 118-139 (1987)

Common property resource and private capital accumulation with random jump

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Abstract

We consider an economic model of exploitation of a common property resource, when agents can also invest in private and productive capital. In this model, we assume that the common pool is under uncertainty in the sense that it could have a sudden increase or decrease in the process of extraction. We want to see whether there is an optimal solution to this model and moreover, we shall also calculate the exhaustion (ruin) probability of the economy.

1 The Model

There are n identical agents having common access to a stock of natural resource, denoted by $S(t)$. Each agent i also owns a private capital stock $K_i(t)$. Agent i extracts the amount $R_i(t)$ of the common resource stock (i=1,...,n). Extraction is costless. Assume that each individual extracts equal amount, so that the resource stock is governed by

$$dS(t) = dJ(t) - nR_i dt. \tag{1.1}$$

where $J(t)$ is a pure jump process given by

$$J(t) = \int_0^t \int_{\mathbf{R}\setminus\{0\}} S(s-) \cdot zN(ds, dz). \tag{1.2}$$

$N(\cdot)$ is a Poisson Random measure with intensity

$$E[N(A)] = \lambda \int_A \sigma(z) dt dz,$$

where $A \in B(\mathbf{R}_+ \times \mathbf{R})$, a Borel set and $\lambda > 0$. $\lambda\sigma(z)dz$ is called Lévy measure and we assume that

$$\int_{-\infty}^{\infty} |z^2| \wedge 1 \; \sigma(z)dz < \infty.$$

Assume that agent i consumes $C_i(t)$, and remaining quantity is invested to accumulate his physical capital. The rate of accumulation of the privately owned capital stock is thus

$$dK_i(t) = (R_i^{1-\beta} K_i^{\alpha} - C_i)dt, \tag{1.3}$$

where $0 < \alpha, \beta < 1$. Each agent wishes to maximize the integral of the stream of discounted utility

$$\text{(1.4)} \qquad \max \int_0^\infty (1-\alpha)^{-1} C_i^{1-\alpha} e^{-\rho t} dt$$

subject to (1.1) and (1.3), where $\rho(>0)$ is a discount factor.

2 The cooperative outcome

If the agents cooperate each other, they will collectively seek to maximize the same level of their welfare. They will choose the rate of extraction per agent R_i and consumption per agent C_i to maximize (1.4) subject to (1.1) and (1.3) with the initial and boundary conditions $S(0) = S > a, K_i(0) = K_i > 0, P\left(\lim_{t\to\infty} S(t) < \infty\right) = 1, \lim_{t\to\infty} K_i(t) \geq 0$, where a is a positive constant.

Define the value function $V(S, K_i)$ for this maximization problem by

$$\text{(2.1)} \qquad V(S, K_h) = \max_{C_i, R_i \geq 0} E\left[\int_0^{\tau_a} (1-\alpha)^{-1} C_i^{1-\alpha} e^{-\rho t} dt + g(S(\tau_a), K_i(\tau_a))\right],$$

where g is a given function which represents the evaluation of existing stocks, τ_a is the first exit time of the system $(S(t), K_i(t))$ from D, where $D = \{(S, K); S > a, K > 0\}$.

It is known that this optimization problem is equivalent to the following Hamilton-Jacobi-Bellman equation;

(2.2)

$$\rho V(S, K_i) = \max_{C_i, R_i > 0} \left\{ \begin{array}{l} (1-\alpha)^{-1} C_i^{1-\alpha} + V_{K_i}(R_i^{1-\beta} K_i^{\alpha} - C_i) \\ +\lambda \int_{\mathbf{R}} \{V(S + Sz, K_i) - V(S, K_i)\} \sigma(z) dz + V_S(-nR_i) \end{array} \right\}$$

for $(S, K_i) \in D$, and

$$V(S, K_i) = g(S, K_i) \quad \text{for} \quad (S, K_i) \in \partial D,$$

Instead of solving it directly, assume that the partial differential equation has a simple solution

$$V(S, K_i) = AK_i^{1-\alpha} + BS^{1-\beta},$$

where A and B are positive constants to be determined. We shall also discuss the ruin probability $P(\tau_a \leq t)$.

References

[1] Fujisaki,M., Ibuki,K. and Katayama,S. *Renewable common property resource and private capital accumulation*, Shōdai-Ronshū, vol. 55, pp.1-6, Kobe Univ. Comm., 2004.

[2] Gaier,J., Grandis,P and Schachermayer,W. *Asymptotic ruin probabilities and optimal investment*, The Annales of Applied Probability, vol.13, no.3, pp.1054-1076, 2003.

Semimartingales from the Fokker-Planck equation

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1 Monge-Kantorovich problem

Problem: General references are [1, 14, 15],

What is the best way to move a sand pile from one place to another ?

Let $L : \mathbf{R}^d \mapsto [0, \infty)$. They study the minimizer of

$$T(P_0, P_1) := \inf\Big\{E\Big[\int_0^1 L\Big(\frac{d\varphi(t)}{dt}\Big)dt\Big]\Big|P\varphi(t)^{-1} = P_t(t = 0, 1)\Big\}. \qquad (1.1)$$

The so-called Duality Theorem for $T(P_0, P_1)$ plays a crucial role.

Theorem 1.1 (Duality Theorem)

$$\begin{aligned} T_K(P_0, P_1) \;&=\; \sup\Big\{\int_{\mathbf{R}^d} \varphi(1, y)P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x)P_0(dx)\Big| \qquad (1.2) \\ &\qquad \varphi(1, y) - \varphi(0, x) \le L(y - x), \varphi(t, \cdot) \in C_b(\mathbf{R}^d)(t = 0, 1)\Big\}. \end{aligned}$$

From Thm 1.1, we have

$$T_K(P_0, P_1) = \sup\Big\{\int_{\mathbf{R}^d} u(1, y; \varphi)P_1(dy) - \int_{\mathbf{R}^d} \varphi(x)P_0(dx)\Big|\varphi \in C_b(\mathbf{R}^d)\Big\},$$

where for $L^*(z) := \sup_{u \in \mathbf{R}^d}(< z, u > -L(u))$,

$$\begin{aligned} \frac{\partial u(t, x; \varphi)}{\partial t} + L^*(D_x u(t, y; \varphi)) \;&=\; 0 \quad ((t, x) \in (0, 1) \times \mathbf{R}^d), \\ u(0, x; \varphi) \;&=\; \varphi(x). \end{aligned}$$

2 Stochastic control version

In this section we discuss a Markov control version of the Monge-Kantorovich problem (see [11, 12]).

Let $L(t,x;u) \in C([0,1] \times \mathbf{R}^d \times \mathbf{R}^d : [0,\infty))$ and be convex in u. Our stochastic control problem is

$$\begin{aligned} V(P_0,P_1) \quad := \quad & \inf\Big\{E\Big[\int_0^1 L(t,X^u(t);u(t))dt\Big]\Big| \\ & \qquad dX^u(t) = u(t)dt + dW(t), PX^u(t)^{-1} = P_t(t=0,1)\Big\}. \end{aligned}$$

Here $u(t)$ is progressively measurable and $W(t)$ is a Wiener process on a complete filtered probability space.

The duality theorem for $V(P_0,P_1)$ is

$$V(P_0,P_1) = \sup\Big\{\int_{\mathbf{R}^d}\varphi(1,y)P_1(dy) - \int_{\mathbf{R}^d}\varphi(0,x)P_0(dx)\Big\}, \tag{2.1}$$

where the supremum is taken over all classical solutions φ, to the following HJB equation:

$$\begin{aligned} \frac{\partial\varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + H(t,x;D_x\varphi(t,x)) \quad &= \quad 0((t,x) \in (0,1) \times \mathbf{R}^d), \qquad (2.2) \\ \varphi(1,\cdot) \quad &\in \quad C_b^\infty(\mathbf{R}^d), \end{aligned}$$

$$H(t,x;z) := \sup_{u \in \mathbf{R}^d}\{< z,u > -L(t,x;u)\}. \tag{2.3}$$

(A.0) (i) P_0 and P_1 are Borel probability measures on $\mathbf{R}^d$,
(ii) $L(t,x;u) \in C([0,1] \times \mathbf{R}^d \times \mathbf{R}^d : [0,\infty))$ and is convex in u,
(iii) $V(P_0,P_1)$ is finite,
(A.1). There exists $\delta > 1$ such that

$$\liminf_{|u|\to\infty} \frac{\inf\{L(t,x;u) : (t,x) \in [0,1] \times \mathbf{R}^d\}}{|u|^\delta} > 0.$$

(A.2).

$$\Delta L(\varepsilon_1, \varepsilon_2) := \sup \frac{L(t,x;u) - L(s,y;u)}{1 + L(s,y;u)} \to 0 \quad \text{as } \varepsilon_1,\ \varepsilon_2 \to 0,$$

where the supremum is taken over all (t,x) and (s,y), $\in [0,1] \times \mathbf{R}^d$, for which $|t-s| \le \varepsilon_1$, $|x-y| < \varepsilon_2$ and all $u \in \mathbf{R}^d$.
(A.3). (i)$L(t,x;u) \in C^3([0,1] \times \mathbf{R}^d \times \mathbf{R}^d)$,
(ii) $D_u^2 L(t,x;u)$ is positive definite for all $(t,x,u) \in [0,1] \times \mathbf{R}^d \times \mathbf{R}^d$,
(iii) $\sup\{L(t,x;o) : (t,x) \in [0,1] \times \mathbf{R}^d\}$ is finite,
(iv) $|D_x L(t,x;u)|/(1 + L(t,x;u))$ is bounded,
(v) $\sup\{|D_u L(t,x;u)| : (t,x) \in [0,1] \times \mathbf{R}^d, |u| \le R\}$ is finite for all $R > 0$.
(A.4). (i) $\Delta L(0,\infty)$ is finite, or (ii) $\delta = 2$ in (A.1).
Remark 2.1. Typical examples are $(1 + |u|^2)^{\delta/2}$ for $\delta > 1$ and $|u|^\delta$ for $\delta \ge 2$.

Proposition 2.1 *Suppose that (A.0)-(A.2) hold. Then* $V(P_0, P_1)$ *has a minimizer.*

Theorem 2.1 (Duality Theorem) *Suppose that (A.0,i), (A.1)-(A.4) hold. Then the duality theorem (2.1) holds.*

Corollary 2.1 *Suppose that (A.0)-(A.4) hold. Then for any minimizer* $\{X(t)\}_{0 \le t \le 1}$ *of* $V(P_0, P_1)$*, there exists a measurable* $b_X(t,x)$ *such that*

$$dX(t) = b_X(t, X(t))dt + dW(t). \tag{2.4}$$

Next we only study the case where (A.4, ii) holds. (A typical example is $|u|^\delta$ for $\delta \ge 2$.)

Proposition 2.2 *(i) Suppose that (A.0)-(A.2) and (A.4, ii) hold. Then* $V(P_0, P_1)$ *has a Markovian minimizer.*
(ii) Suppose in addition that for any $(t,x) \in [0,1] \times \mathbf{R}^d$, $L(t,x;u)$ *is strictly convex in* u*. Then the minimizer is unique.*

3 Semimartingales from the Fokker-Planck equations.

As an application of the duality theorem for the stochastic optimal control problem, we consider the following: for a weak solution of the Fokker-Planck equation with a p-th integrable drift vector ($p > 1$), we show the existence of a semimartingale of which one-dimensional marginal distributions are given by this solution.

More precisely, let $b : [0, 1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ be measurable and $\{P_t(dx)\}_{0 \le t \le 1}$, $\subset \mathcal{M}_1(\mathbf{R}^d)$, satisfy the following Fokker-Planck equation: for $f \in C_b^{1,2}([0, 1] \times \mathbf{R}^d)$ and $t \in [0, 1]$,

$$\begin{aligned} & \int_{\mathbf{R}^d} f(t, x) P_t(dx) - \int_{\mathbf{R}^d} f(0, x) P_0(dx) \qquad (3.1) \\ = & \int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\partial f(s, x)}{\partial s} + \frac{1}{2} \triangle f(s, x) + < b(t, x), D_x f(s, x) > \Big) P_s(dx). \end{aligned}$$

Inspired by Born's probabilistic interpretation of the solution to Schrödinger's equation, Nelson proposed the problem of the construction of a Markov process $\{X(t)\}_{0 \le t \le 1}$ for which the following holds (see [13]):

$$X(t) = X(0) + \int_0^t b(s, X(s)) ds + W(t) \quad (t \in [0, 1]), \qquad (3.2)$$

$$P(X(t) \in dx) = P_t(dx) \quad (t \in [0, 1]), \qquad (3.3)$$

where $\{W(t)\}_{0 \le t \le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$-Wiener process.

The first result was given by Carlen [3] (see also [16]). It was generalized, by Mikami [7], to the case where the second order differential operator has variable coefficients. The further generalization and almost complete resolution was made by Cattiaux and Léonard [4-6] (see also [2, 8, 9] for the related topics).

In these papers, they assumed that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t, x)|^2 P_t(dx) < \infty \qquad (3.4)$$

for some b for which (3.1) holds. This is called the **finite energy condition**.

In this section we consider Nelson's problem under a weaker assumption than (3.4). More precisely, we assume the following: there exists $p > 1$ for which

$$\inf\left\{\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^p P_t(dx) \middle| b \text{ satisfies } (9.1)\right\} < \infty. \tag{3.5}$$

We call (3.5) the **generalized finite energy condition**.

In [7] we made use of the duality theorem (Theorem 2.1) for $L =< a(t,x)^{-1}u, u >$ for which the minimizer is the h-path process. Put

$$\begin{aligned} v(P_0, P_1) \quad := \quad & \inf\left\{\int_0^1 dt \int_{\mathbf{R}^d} L(t,x; b(t,x)) P(t,dx) \right| \\ & \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d), P(t,dx) = P_t(dx)(t = 0,1), \\ & (b(t,x), P(t,dx)) \text{ satisfies } (3.1)\right\}. \end{aligned} \tag{3.6}$$

By Theorem 2.1, we obtain, under (3.5), the following:

Theorem 3.1 **([12])** *Suppose that (A.1)-(A.4) hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0, P_1) = v(P_0, P_1). \tag{3.7}$$

For $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$, put

$$\mathbf{V}(\mathbf{P}) := \inf\left\{E\left[\int_0^1 L(t, X^u(t); u(t)) dt\right] \middle| PX^u(t)^{-1} = P_t (0 \le t \le 1)\right\}, \tag{3.8}$$

$$\mathbf{v}(\mathbf{P}) := \inf\left\{\int_0^1 dt \int_{\mathbf{R}^d} L(t,x; b(t,x)) P_t(dx) | b(t,x) \text{ satisfies } (3.1)\right\}. \tag{3.9}$$

Using a similar result to (3.7) on small time intervals $\subset [0,1]$, we obtain

Theorem 3.2 **([12])** *Suppose that (A.1)-(A.4) hold. Then*
(i) for any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}) (\in [0, \infty]). \tag{3.10}$$

(ii) For any $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$*, for which* $\mathbf{v}(\mathbf{P})$ *is finite, there exist a unique minimizer* $b_o(t,x)$ *of* $\mathbf{v}(\mathbf{P})$ *and a minimizer* X *of* $\mathbf{V}(\mathbf{P})$*. In particular, for any minimizer* X *of* $\mathbf{V}(\mathbf{P})$*,*

$$b_X(t, X(t)) = b_o(t, X(t)) \tag{3.11}$$

and (3.2)-(3.3) with $b = b_o$ *hold.*

References

[1] Ambrosio L, Lecture notes on optimal transport problems, in *Mathematical Aspects of Evolving Interfaces, Funchal, Portugal, 2000 (Colli P, Rodrigues JF, eds.), Lecture Notes in Math.* ***1812*** (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 2003) 1–52.

[2] Blumenthal RM and Corson HH, On continuous collections of measures, in *6th Berkeley Sympos. Math. Statist. Probab., Berkeley 1970/1971 (Le Cam L, Neyman J, Scott EL, eds.),* ***2*** (Univ. California press, Berkeley, 1972) pp. 33-40.

[3] Carlen EA, Conservative diffusions, Commun. Math. Phys. **94** (1984) 293-315.

[4] Cattiaux P and Léonard C, Minimization of the Kullback information of diffusion processes, Ann. Inst. H. Poincar Probab. Statist. **30** (1994) 83-132; Correction : Ann. Inst. H. Poincar Probab. Statist. **31** (1995) 705–707.

[5] Cattiaux P and Léonard C, Large deviations and Nelson processes, Forum Math. **7** (1995) 95–115.

[6] Cattiaux P and Léonard C, Minimization of the Kullback information for some Markov processes, in *Sminaire de Probabilits, XXX, (Azema J, Emery M, Yor, M, eds.), Lecture Notes in Math.* ***1626*** (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1996) 288–311.

[7] Mikami T, Variational processes from the weak forward equation, Commun. Math. Phys. **135** (1990) 19–40.

[8] Mikami T, Markov marginal problems and their applications to Markov optimal control, in *Stochastic Analysis, Control, Optimization and Applications, (McEneaney WM, Yin GG, Zhang Q., eds.) A Volume in Honor of W. H. Fleming,* (Boston: Birkhäuser 1999) 457-476.

[9] Mikami T, Optimal control for absolutely continuous stochastic processes and the mass transportation problem, Elect. Comm. in Probab. **7** (2002) 199–213.

[10] Mikami T, Monge's problem with a quadratic cost by the zero-noise limit of h-path processes, Probab. Theory Related Fields **129** (2004) 245–260.

[11] Mikami T and Thieullen M, Duality Theorem for Stochastic Optimal Control Problem, Hokkaido University preprint series, #652 (2004).

[12] Mikami T, Semimartingales from the Fokker-Planck equation, preprint. Hokkaido University preprint series, #724 (2005).

[13] Nelson E, Dynamical theories of Brownian motion (Princeton University Press, Princeton, 1967).

[14] Rachev ST and Rüschendorf L, *Mass transportation problems, Vol. I: Theory, Vol. II: Application* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1998).

[15] Villani C, *Topics in Optimal Transportation.* Graduate Studies in Mathematics Vol. 58 (Amer. Math. Soc., Providence, RI, 2003).

[16] Zheng WA, Tightness results for laws of diffusion processes application to stochastic mechanics. Ann. Inst. Henri Poincaré **21** (1985) 103-124.