Proceedings of the 30th Sapporo Symposium on
Partial Differential Equations

Edited by T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura

Y. Tonegawa, and K. Tsutaya

Sapporo, 2005

HOKKAIDO UNIVERSITY
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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 3 through August 5 in 2005 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broard spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 25 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura,
Y. Tonegawa, and K. Tsutaya

The 30th Sapporo Symposium on Partial Differential Equations
（第３０回偏微分方程式論札幌シンポジウム）

代表者:　　小澤 徹，　儀我 美一，　神保 秀一，　　中村 玄，　　利根川 吉廣，　津田谷 公利
Organizers: T. Ozawa, Y. Giga,　 S. Jimbo,　G. Nakamura, Y. Tonegawa, K. Tsutaya

1. Period (期間) August 3 , 2005 - August 5 , 2005

2. Venue (場所)

   Room 203, Faculty of Science Building #5, Hokkaido University

   北海道大学 理学部５号館大講義室 (203 号室)

3. Program （プログラム）

August 3, 2005 (Wednesday)

9:30-10:30	Vladimir GEORGIEV (Univ. of Pisa)
	Smoothing - Strichartz estimates for the Schrödinger equation with small magnetic potential
11:00-12:00	高橋 太（東北大） Futoshi TAKAHASHI (Tohoku Univ.)
	On the isoperimetric inequality for mappings with remainder term
14:00-14:30	Free discussion with speakers in the tea room
14:30-15:00	Piotr RYBKA (Warsaw Univ.)
	Driven crystalline curvature flow in the plane
15:15-15:45	滝本 和広（広大）Kazuhiro TAKIMOTO (Hiroshima Univ.)
	Boundary blowup solutions to curvature equations
16:00-16:30	寺澤 祐高（北大）Yutaka TERASAWA (Hokkaido Univ.)
	The Navier-Stokes equations with initial data in uniformly local L^p spaces
16:30-17:00	Free discussion with speakers in the tea room

August 4, 2005 (Thursday)

9:30-10:30	上見練太郎（はこだて未来大）Rentaro AGEMI (Future Univ.- Hakodate)
	The initial-boundary value problems for nonlinear elastic waves
11:00-12:00	久保英夫（阪大）Hideo KUBO (Osaka Univ.)
	Asymptotic behavior of solutions to semilinear systems of wave equations

CONTENTS

Program

14:00-14:30	Free discussion with speakers in the tea room

14:30-15:00　中村誠（東北大）Makoto NAKAMURA (Tohoku Univ.)
An application of Kubota-Yokoyama estimates to quasilinear wave equations with cubic terms in exterior domains

15:15-15:45　佐々木浩宣（北大）Hironobu SASAKI (Hokkaido Univ.)
Small data scattering for a Klein-Gordon equation with a cubic convolution

16:00-16:30　山内雄介（北大）Yusuke YAMAUCHI (Hokkaido Univ.)
Blow-up results for a reaction-diffusion system

16:30-17:00　Free discussion with speakers in the tea room

18:00-20:00　Reception at Sapporo Aspen Hotel (懇親会, 札幌アスペンホテル)

August 5, 2005 (Friday)

9:30-10:30　牛島健夫（東京理科大）Takeo USHIJIMA (Tokyo Univ. of Science)
Approximation of the Gauss curvature flow by a three-dimensional crystalline motion

11:00-12:00　Chao-Nien CHEN (National Changhua Univ. of Education)
Some variational methods for studying almost periodic differential equations

12:00-13:00　Free discussion with speakers in the tea room

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　8月4日18:00 からの懇親会は久保田幸次先生 (北海道大学名誉教授) の70歳の御祝いを兼ねています。多数のご参加をお待ち申し上げております。
The reception is also for celebrating Prof. Emeritus Kubota's seventieth birthday.

Venue : 札幌アスペンホテル (Sapporo Aspen Hotel)　Tel : (011)700-2111

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Symposium Homepage: http://coe.math.sci.hokudai.ac.jp/sympo/sapporo/program050803.html

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SMOOTHING - STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH SMALL MAGNETIC POTENTIAL

VLADIMIR GEORGIEV, ATANAS STEFANOV AND MIRKO TARULLI

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $A = (A_1(t,x), \cdots, A_n(t,x)), x \in \mathbb{R}^n, n \geq 3$ be a magnetic potential, such that $A_j(t,x), j = 1, \cdots, n$, are functions, and let the magnetic Laplacian operator be

$$\Delta_A = \sum_j (\partial_j + iA_j)^2 = \Delta + 2iA\nabla + idiv(A) - \left(\sum_j A_j^2\right)$$

Our goal is to study the dispersive properties of the corresponding Schrödinger equation

$$(1.1) \qquad \begin{cases} \partial_t u - i\Delta_A u = F(t,x), & t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(0,x) = f(x). \end{cases}$$

In this work, we will be concerned with the Strichartz and smoothing estimates for (1.1), when the vector potential A is small in certain sense. In fact, we aim at obtaining global scale invariant Strichartz and smoothing estimates, under appropriate scale invariant smallness assumptions on A.

In the "free" case $A = 0$, there exists vast literature on the subject. Introduce the mixed space-time norms

$$\|u\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q}.$$

We say that a pair of exponents (q,r) is Strichartz admissible, if $2 \leq q, r \leq \infty$, $2/q + n/r = n/2$ and $(q,r,n) \neq (2,\infty,2)$. Then, by result of Strichartz, Ginibre-Velo, and Keel-Tao,

$$(1.2) \qquad \|e^{it\Delta}f\|_{L^q L^r} \leq C\|f\|_{L^2}$$

$$(1.3) \qquad \|\int e^{-is\Delta}F(s,\cdot)ds\|_{L_x^2} \leq C\|F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

$$(1.4) \qquad \|\int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds\|_{L^q L^r} \leq C\|F\|_{L^{\tilde{q}'}L^{\tilde{r}'}},$$

where $(\tilde{q}, \tilde{r})$ is another Strichartz admissible pair and $q' = q/(q-1)$.

On the other hand, the smoothing estimates, were established by Kenig-Ponce-Vega in the seminal paper, [15], see also Ruiz-Vega'[94]. These were later extended to more general second order Schrödinger equations in [16]. We state them only for the solutions of the linear problem $u_t - i\Delta u = 0$ with initial data $u(0, x) = f(x)$

$$\sup_m 2^{-m/2} \left\| D_x^{1/2} u \right\|_{L_t^2 L^2(|x| \sim 2^m)} \leq C \|f\|_{L^2}$$

Motivated by the Strichartz estimates and these "local smoothing" norms, introduce the spaces X_k, defined by the norms[1]

$$\|\phi\|_{X_k} = \sup_{(q,r)-Str., q \neq 2} \|\phi_k\|_{L_t^q L_x^r} + 2^{k/2} \sup_m 2^{-m/2} \|\phi_k\|_{L_t^2 L^2(|x| \sim 2^m)},$$

where $\phi_k := P_k \phi$ is the kth Littlewood-Paley piece of ϕ (see Section 2.1 below).

Define also the Banach spaces X by the norm

$$\|\phi\|_X := \left(\sum_k \|\phi\|_{X_k}^2 \right)^{1/2}.$$

It is clear by the elementary properties of Besov spaces, that whenever $q, r \geq 2$, $\|\phi\|_{L^q L^r} \lesssim (\sum_k \|\phi_k\|_{L^q L^r}^2)^{1/2}$ and therefore $\sup_{(q,r)-Str.} \|\phi\|_{L^q L^r} \lesssim \|\phi\|_X$. It is also true (although not as obvious, see Lemma 2.1 below) that

$$\sup_m 2^{-m/2} \left\| D_x^{1/2} \phi \right\|_{L_t^2 L^2(|x| \sim 2^m)} \lesssim \|\phi\|_X.$$

Theorem 1.1. *If $n \geq 3$, then one can find a positive number $\varepsilon > 0$ so that for any (vector) potential $A = A(t, x)$ satisfying*

$$\|A\|_{L^\infty L^n} + \|\nabla A\|_{L^\infty L^{n/2}} + \left(\sum_m 2^m \sup_k \|A_{<k}\|_{L^\infty L^\infty(|x| \sim 2^m)} \right) \leq \varepsilon$$

there exists $C > 0$, such that for any $f \in S(\mathbb{R}^n)$ and any $F(t, x) \in S(\mathbb{R} \times \mathbb{R}^n)$ the solution $u(t, x)$ to (1.1) satisfies the estimate

$$\|u\|_X \leq C\|f\|_{L_x^2} + C\|F\|_{L_t^{q'} L_x^{r'}},$$

where (q, r) is Strichartz admissible with $q \neq 2$. In particular, the solutions to (1.1) satisfy

$$\sup_{(q,r)-Str., q \neq 2} \|u\|_{L^q L^r} \lesssim \|f\|_{L^2} + \|F\|_{L_t^{q'} L^{r'}}.$$

---

[1]The expressions $\phi \to \|\phi\|_{X_k}$ are not faithful norms, in the sense that may be zero, even for some $\phi \neq 0$. On the other hand, they satisfy all the other norm requirements and $\phi \to (\sum_k \|\phi_k\|_{X_k}^2)^{1/2}$ is a norm!

The main idea behind the proof is to show a global scale invariant estimates for solutions to the *linear* Schrödinger equation $u_t - i\Delta u = F$ with initial data f in the form

$$(1.5) \quad \begin{aligned} \|u\|_X &\leq C\|f\|_{L^2} + C\min_{H=H^1+H^2}(\|H^1\|_{L^{q'}L^{r'}} + \\ &+ (\sum_k 2^{-k}(\sum_m 2^{m/2}\|H_k^2\|_{L_t^2 L^2(|x|\sim 2^m)})^2)^{1/2}. \end{aligned}$$

The proof of Theorem 1.1 then proceeds in a standard way.

2. Preliminaries

2.1. **Fourier transform and Littlewood-Paley projections.** Define the Fourier transform and its inverse by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx$$

$$f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi$$

Introduce a positive, decreasing, smooth away from zero function $\chi : \mathbf{R}_+^1 \to \mathbf{R}_+^1$, supported in $\{\xi : 0 \leq \xi \leq 2\}$ and $\chi(\xi) = 1$ for all $|\xi| \leq 1$. Define $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$, which is positive and supported in the annulus $1/2 \leq |\xi| \leq 2$. We have that φ is smooth and $\sum_{k\in\mathcal{Z}} \varphi(2^{-k}\xi) = 1$ for all $\xi \neq 0$. In higher dimensions, we slightly abuse the notations and denote a function with similar properties by the same name, i.e. $\varphi(\xi) = \varphi(|\xi|)$, $\chi(x) = \chi(|x|)$ etc. Note that for $n > 1$, $\chi(x) : \mathbf{R}^n \to \mathbf{R}^1$ is a smooth function even at zero.

The kth Littlewood-Paley projection is defined as a multiplier type operator by $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$. Note that the kernel of P_k is integrable, smooth and real valued for every k. In particular, it is bounded on every $L^p : 1 \leq p \leq \infty$ and it commutes with differential operators. Another helpful observation is that for the differential operator D_x^s defined via the multiplier $|\xi|^s$, one has

$$D_x^s P_k u = 2^{ks}\tilde{P}_k u,$$

where $\tilde{P}_k$ is given by the multiplier $\tilde{\varphi}(2^{-k}\xi)$, where $\tilde{\varphi}(\xi) = \varphi(\xi)|\xi|^s$.
We also consider $P_{<k} := \sum_{l<k} P_l$, which essentially restricts the Fourier transform to frequencies $\lesssim 2^k$.

Define also the function $\psi(x) = \chi(x/4) - \chi(4x)$. Note that ψ has similar support properties as φ and $\psi(x)\varphi(x) = \varphi(x)$. Thus, we may also define the operators Z_k by $\widehat{Z_k f}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi)$. By the construction, $Z_k P_k = P_k$ and $Z_k = P_{k-2} + \ldots + P_{k+1}$.
Recall the Calderón commutator estimate (see for example the work of Rodnianski and Tao, [20])

$$\|[P_k, f]g\|_{L^r} \leq C2^{-k}\|\nabla f\|_{L^q}\|g\|_{L^p},$$

whenever $1 \leq r, p, q \leq \infty$ and $1/r = 1/q + 1/p$.

Also of interest will be the properties of products under the action of P_k. We have that for any two (Schwartz) functions f, g

$$
\begin{aligned}
P_k(fg) &= \sum_{l \geq k-2} P_k(f_l g_{l-2 \leq \cdot \leq l+2}) + \text{symmetric term} + \\
&+ P_k(f_{\leq k-4} g_{k-1 \leq \cdot \leq k+1}) + \text{symmetric term} = \\
&= f_{\leq k-4} g_k + [P_k, f_{\leq k-4}] g_{k-1 \leq \cdot k+1} + \text{symmetric terms} \\
&+ \sum_{l \geq k-2} P_k(f_l g_{l-2 \leq \cdot \leq l+2}) + \text{symmetric term}.
\end{aligned}
$$

In particular, we shall need an appropriate (product like!) expression for $P_k(A \nabla u)$. The main term is clearly when ∇u is in high frequency mode, while $\vec{A}$ is low frequency. More precisely, according to our considerations above,

$$
P_k(A \nabla u) = A_{\leq k-4} \nabla u_k + E^k,
$$

where

$$
\begin{aligned}
(2.1) \quad E^k &= [P_k, A_{\leq k-4}] \nabla u_{k-1 \leq \cdot k+1} + \sum_l P_k(A_l \cdot \nabla u_{l-2 \leq \cdot \leq l+2}) + \\
&+ \sum_l P_k(A_{l-2 \leq \cdot \leq l+2} \cdot \nabla u_l) + P_k(A_{k-1 \leq \cdot \leq k+1} \cdot \nabla u_{\leq k-4})
\end{aligned}
$$

Note that in terms of L^p behavior and Littlewood-Paley theory, one treats these error terms as if they were in the form $(\partial_x A)u$.

2.2. Besov spaces versions of the "local smoothing space".
We show that the space X introduced earlier is embedded into the "local smoothing space".

Lemma 2.1. *For every Schwartz function ϕ, there is a constant $C = C(n)$, so that*

$$
(2.2) \qquad \sup_m 2^{-m/2} \left\| D_x^{1/2} \phi(t, x) \right\|_{L_t^2 L^2(|x| \sim 2^m)} \leq C_n \|\phi\|_X.
$$

Proof. We prove only (2.2). Fix m. We first dispose with the easy case $\sup_m 2^{-m/2} \left\| D_x^{1/2} P_{<-m} \phi \right\|_{L_t^2 L^2(|x| \sim 2^m)}$.

We have by Hölder's inequality

$$
2^{-m/2} \left\| D_x^{1/2} P_{<-m} \phi \right\|_{L_t^2 L^2(|x| \sim 2^m)} \lesssim 2^{-m} \left\| \tilde{P}_{<-m} \phi \right\|_{L^2 L^2(|x| \sim 2^m)} \lesssim
$$

$$
\lesssim \left\| \tilde{P}_{<-m} \phi \right\|_{L^2 L^{2n/(n-2)}} \lesssim \|\phi\|_{L^2 L^{2n/(n-2)}} \lesssim \|\phi\|_X.
$$

For the remaining term, we have that

$$2^{-m/2}\left\|D_x^{1/2}P_{\geq -m}\phi\right\|_{L_t^2 L^2(|x|\sim 2^m)} = 2^{-m/2}\|D_x^{1/2}\sum_{k\geq -m}P_k\phi\|_{L_t^2 L^2(|x|\sim 2^m)} \lesssim$$

$$\lesssim 2^{-m/2}\|\sum_{k\geq -m}2^{k/2}\varphi(2^{-m}x)\tilde{P}_k\phi\|_{L_t^2 L_x^2}$$

Here, we have replaced $\|F\|_{L^2(|x|\sim 2^m)}$ by the comparable expression $\|\varphi(2^{-m}F\|_{L^2}$. This will be done frequently (and without much discussions) in the sequel in order to make use of the Plancherel's theorem, which is of course valid only in the global L^2 space.

To continue, we represent

$$\varphi(2^{-m}\cdot)\tilde{P}_k\phi = \tilde{P}_k(\varphi(2^{-m}\cdot)\phi(\cdot)) - [\tilde{P}_k, \varphi(2^{-m}\cdot)]\phi(\cdot).$$

We basically treat $\tilde{P}_k$ as a bounded operator (on every L^p space), while applying the Calderón commutator estimate for the second term. Recall that by definition by $\tilde{P}_k = \tilde{P}_k Z_k$. We have

$$2^{-m/2}\|\sum_{k\geq -m}2^{k/2}\varphi(2^{-m}x)\tilde{P}_k\phi\|_{L_t^2 L_x^2} \leq$$

$$\leq \sum_{k\geq -m}2^{-m/2}2^{k/2}\left\|[\tilde{P}_k, \varphi(2^{-m}x)]Z_k\phi\right\|_{L_t^2 L_x^2} +$$

$$+2^{-m/2}\left\|\sum_{k\geq -m}2^{k/2}\tilde{P}_k(\varphi(2^{-m}\cdot)Z_k\phi)\right\|_{L^2 L^2} \lesssim$$

$$\lesssim \sum_{k\geq -m}2^{-m/2}2^{-k/2}\left\|\nabla(\varphi(2^{-m}\cdot))\right\|_{L_x^n}\|Z_k\phi\|_{L_t^2 L_x^{2n/(n-2)}} +$$

$$+2^{-m/2}\left(\sum_k 2^k\left\|\tilde{P}_k(\varphi(2^{-m}x)Z_k\phi)\right\|_{L^2 L^2}^2\right)^{1/2} \lesssim$$

$$\lesssim \|\phi\|_{L^2 L_x^{2n/(n-2)}} + 2^{-m/2}(\sum_k 2^k\|\phi_k\|_{L_t^2 L^2(|x|\sim 2^m)}^2)^{1/2} \lesssim \|\phi\|_X.$$

$\square$

3. Estimates for the bilinear form $Q(F,G)$

Introduce the sesquilinear forms

$$Q(F,G) = \int_{s,t:t\geq s}\langle e^{i(t-s)\Delta}F(s), G(t)\rangle_{L^2(\mathbb{R}^n)}dsdt,$$

$$Q_T(F,G) = \int_{\mathbb{R}}\int_{\mathbb{R}}\langle e^{i(t-s)\Delta}F(s), G(t)\rangle_{L^2(\mathbb{R}^n)}\varphi\left(\frac{t-s}{T}\right)dsdt,$$

Note that the standard Strichartz estimates can be expressed in terms of Q

(3.1) $$|Q(F,G)| \leq C\|F\|_{L_t^{q_1'}L_x^{r_1'}}\|G\|_{L_t^{q_2'}L_x^{r_2'}},$$

for all Strichartz pairs (q_1, r_1), (q_2, r_2).

Our main goal is to obtain good bilinear estimates for $Q_T(F, G; \psi)$, which will be useful later on for the relevant estimates needed for $Q(F, G)$.

Our next lemma provides one possible estimate for $Q_T(G, G)$.

3.1. Estimates in the local smoothing space.

Lemma 3.1. *Let G be a Schwartz function and k be an integer. Then there exists a constant C depending only on the dimension, so that*

$$\sup_s \left(\int |\widehat{G}_{1,2}(s + 2\pi|\xi|^2, \xi)|^2 \varphi(2^{-k}\xi) d\xi \right)^{1/2} \le$$

$$\le C \sum_m 2^{-k/2} 2^{m/2} \left\| \varphi(2^{-m}\cdot) G_k \right\|_{L_t^2 L_x^2}.$$

Here $\widehat{G}_{1,2}(\tau, \xi)$ is meant to denote the space-time Fourier transform of G.

This lemma is essentially a dual version of the "smoothing estimate" of Kenig-Ponce-Vega and may have appeared in the literature, but we include its proof for completeness.

Proof. Start with the smoothing estimate, applied to a frequency localized data $Z_k f$. We have

$$\sup_m 2^{-m/2} \left\| e^{-it\Delta} D_x^{1/2} Z_k f \right\|_{L_t^2 L_{\{2^{m-1} \le |x| \le 2^{m+1}\}}^2} \le C \|f\|_{L^2}$$

Observe that $D^{1/2} Z_k = 2^{k/2} \tilde{Z}_k$, where $\widehat{\tilde{Z}_k f}(\xi) = \hat{f}(\xi) \tilde{\psi}(2^{-k}\xi)$ and $\tilde{\psi}(\xi) = |\xi|^{1/2} \psi(\xi)$.

By dualizing this inequality, we arrive at

$$\sum_m 2^{-m/2} 2^{k/2} \int [e^{-it\Delta} \tilde{Z}_k f] \overline{G_m(t, x)} dt dx \le$$

$$\le C \sum_m \|G_m\|_{L^2 L_{\{2^{m-1} \le |x| \le 2^{m+1}\}}^2} \|f\|_{L^2}.$$

Let for any $x : 2^{m-1} \le |x| \le 2^{m+1}$, $G(t, x) = \sum_m 2^{-m/2} 2^{k/2} G_m(t, x)$. We have

$$\int f(x) \left(\int e^{it\Delta} \tilde{Z}_k G(t, x) dt \right) dx \le$$

$$\le C \sum_m 2^{m/2} 2^{-k/2} \|G\|_{L^2 L_{\{2^{m-1} \le |x| \le 2^{m+1}\}}^2} \|f\|_{L^2},$$

It follows that

$$\left\| \int e^{it\Delta} \tilde{Z}_k G(t, x) dt \right\|_{L^2} \le C \sum_m 2^{m/2} 2^{-k/2} \left\| \varphi(2^{-m} x) G \right\|_{L^2 L^2}.$$

It is clear that for arbitrary real s, one has

$$(3.2) \qquad \left\| \int e^{it\Delta} e^{-2\pi i st} \tilde{Z}_k G(t,\cdot) dt \right\|_{L^2} \leq C \sum_m 2^{m/2} 2^{-k/2} \left\| \varphi(2^{-m}x)G \right\|_{L^2 L^2},$$

since the L_t^2 norm on the right is not affected by the unimodular factor $e^{-2\pi i st}$.

We next compute the space Fourier transform of the function $H(x) = \int e^{it\Delta} e^{-ist} G(t,\cdot) dt$. We have

$$\hat{H}(\xi) = c \int e^{-4\pi^2 it|\xi|^2} e^{-2\pi i st} \widehat{G}_{1,2}(\tau,\xi) e^{2\pi it\tau} dt d\tau =$$

$$= c \int \widehat{G}_{1,2}(\tau,\xi) \delta(2\pi\tau - 2\pi s - 4\pi^2|\xi|^2) d\tau = c\widehat{G}_{1,2}(s + 2\pi|\xi|^2, \xi).$$

By the Plancherel's theorem, (3.2) and since $\tilde{\psi} \sim \psi$ on their supports, we conclude

$$\sup_s \left(\int |\widehat{G}_{1,2}(s + 2\pi|\xi|^2, \xi)| \psi(2^{-k}\xi) d\xi \right)^{1/2} \leq$$

$$\leq C \sum_m 2^{m/2} 2^{-k/2} \left\| \varphi(2^{-m}x)G \right\|_{L^2 L^2},$$

Clearly, if we apply the last inequality to $P_k G$, Lemma 3.1 follows. $\qquad \square$

Our next lemma gives an alternative representation of $Q_T(F, G)$.

Lemma 3.2. *We have the relation*

$$(3.3) \qquad Q_T(F,G) = \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{F}(\tau + 2\pi|\xi|^2, \xi) \overline{\widehat{G}(\tau + 2\pi|\xi|^2, \xi)} T\hat{\varphi}(T\tau) d\xi d\tau,$$

for some $c > 0$.

The proof of both Lemma 3.3 is straightforward.

By Lemma 3.2 and Hölder's inequality, we have

$$|Q(F_k, G_k)| \leq \left(\int |\hat{F}(\tau + 2\pi|\xi|^2, \xi)|^2 \varphi(2^{-k}\xi) d\xi T |\hat{\varphi}(T\tau)| d\tau \right)^{1/2} \times$$

$$\times \left(\int |\hat{G}(\tau + 2\pi|\xi|^2, \xi)|^2 \varphi(2^{-k}\xi) d\xi T |\hat{\varphi}(T\tau)| d\tau \right)^{1/2} \leq$$

$$\leq \sup_\tau \left(\int |\hat{F}(\tau + 2\pi|\xi|^2, \xi)|^2 \varphi(2^{-k}\xi) d\xi \right)^{1/2} \times$$

$$\times \sup_\tau \left(\int |\hat{G}(\tau + 2\pi|\xi|^2, \xi)|^2 \varphi(2^{-k}\xi) d\xi \right)^{1/2},$$

whence by Lemma 3.1 one gets

$$(3.4) \qquad \begin{aligned} |Q(F_k, G_k)| &\leq C_n \left(\sum_m 2^{-k/2} 2^{m/2} \|F_k\|_{L_t^2 L^2(|x|\sim 2^m)} \right) \times \\ &\times \left(\sum_m 2^{-k/2} 2^{m/2} \|G_k\|_{L_t^2 L^2(|x|\sim 2^m)} \right) \end{aligned}$$

The following lemma gives a different representation formula for $Q_T(F, G)$. This is crucial in proving bilinear estimates, when one factor is in the mixed Lebesgue spaces, and the other is in the local smoothing space.

Lemma 3.3.

$$\langle e^{iz\Delta} F, G \rangle = \frac{c_n}{z^{n/2}} \int \widehat{F^1}\left(\frac{x}{2\pi z}\right) \overline{G^1}(x) dx = \frac{c_n}{z^{n/2}} \int F^1(x) \overline{\widehat{G^1}}\left(\frac{x}{2\pi z}\right) dx,$$

where $F^1(x) = e^{i|x|^2/(2z)} F(x)$ *and similarly* $G^1(x) = e^{-i|x|^2/(2z)} G(x)$.

We state the main result that is the key point in our approach to proof Theorem 1.1.

3.2. Mixed estimates - one factor is in the local smoothing space, another one is in the mixed Lebesgue space.

Theorem 3.1. *There exists a constant* $C = C(n, \varphi)$ *so that for any integer* k, *any* $F(t,x) \in S(\mathbb{R} \times \mathbb{R}^n)$ *and* $G(t,x) \in S(\mathbb{R} \times \mathbb{R}^n)$

$$(3.5) \quad |Q(F_k, G_k)| \leq C \left(\sum_{m \in \mathbb{Z}} 2^{m/2} 2^{-k/2} \|\varphi(2^{-m} \cdot) F_k\|_{L_t^2 L_x^2} \right) \|G_k\|_{L_t^{q'} L_x^{r'}}.$$

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On the isoperimetric inequality for mappings with remainder term

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In this talk, we concern a sharp version of the classical isoperimetric inequality for mappings from $\mathbf{R}^2$ to $\mathbf{R}^3$.

First, let us introduce the following function spaces:

$$\mathcal{W} := \left\{ u \in L^1_{loc}(\mathbf{R}^2; \mathbf{R}^3) : \int_{\mathbf{R}^2} |\nabla u|^2 dx + \int_{\mathbf{R}^2} \frac{|u|^2}{(1+|x|^2)^2} dx < \infty \right\},$$

$$\overline{\mathcal{W}} := \left\{ u \in \mathcal{W} : \int_{\mathbf{R}^2} \frac{u}{(1+|x|^2)^2} dx = 0 \right\}.$$

Let $\Pi : \mathbf{S}^2 \to \mathbf{R}^2 \cup \{\infty\}$ denote the stereographic projection from the north pole and let

$$\Pi^{-1}(x_1, x_2) = \frac{2}{1+|x|^2} \begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbf{S}^2$$

be its inverse, then the space $\mathcal{W}$ can be written as

$$\mathcal{W} = \left\{ u \in L^1_{loc}(\mathbf{R}^2; \mathbf{R}^3) : u \circ \Pi \in H^1(\mathbf{S}^2; \mathbf{R}^3) \right\}.$$

Note that by Poincaré inequality, $(u, v)_{\overline{\mathcal{W}}} := \int_{\mathbf{R}^2} \nabla u \cdot \nabla v dx$ defines a scalar product on $\overline{\mathcal{W}}$. From now on, we set $\overline{u} := u - \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{u}{(1+|x|^2)^2} dx \in \overline{\mathcal{W}}$ for $u \in \mathcal{W}$.

Let Q denote the oriented volume functional

$$Q(u) := \int_{\mathbf{R}^2} u \cdot u_{x_1} \wedge u_{x_2} dx$$

where $u_{x_i} = \frac{\partial}{\partial x_i} u$ and $\wedge$ denotes the vector product in $\mathbf{R}^3$.

The following inequality is referred to as the classical *isoperimetric inequality for mappings*:

$$S|Q(u)|^{2/3} \leq \int_{\mathbf{R}^2} |\nabla u|^2 dx \quad \text{for } \forall u \in \overline{\mathcal{W}}. \tag{1}$$

Here

$$S = \inf_{u \in \overline{\mathcal{W}}, Q(u) \neq 0} \frac{\int_{\mathbf{R}^2} |\nabla u|^2 dx}{|Q(u)|^{2/3}} = (32\pi)^{1/3}$$

denotes the best (largest) possible value for which the classical isoperimetric inequality (1) holds true.

By simple calculation, we see the function $\overline{U_{\lambda,a}} \in \overline{\mathcal{W}}$ where

$$U_{\lambda,a}(x) = \frac{2\lambda}{\lambda^2 + |x-a|^2} \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ -\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

attains the infimum value S for any $\lambda > 0$ and $a = (a_1, a_2) \in \mathbf{R}^2$. Furthermore, if we set the 7-dimensional manifold

$$\mathcal{M} := \left\{ c\overline{RU_{\lambda,a}} : c \in \mathbf{R} \setminus \{0\}, R \in SO(3), \lambda > 0, a \in \mathbf{R}^2 \right\} \subset \overline{\mathcal{W}} \setminus \{0\}$$

where $SO(3) = \{R : 3 \times 3\text{matrix}, \; R^t = R^{-1}, \det(R) = 1\}$, then by a classification theorem of Brezis and Coron, we see that this manifold consists of all mappings that achieve the best isoperimetric constant in (1):

$$\mathcal{M} = \left\{ u \in \overline{\mathcal{W}} \setminus \{0\} : \int_{\mathbf{R}^2} |\nabla u|^2 dx = S|Q(u)|^{2/3} \right\}.$$

Now, main theorem in this talk is the following.

Theorem. *There exists a positive constant $C > 0$ such that*

$$\int_{\mathbf{R}^2} |\nabla u|^2 dx - S|Q(u)|^{2/3} \geq Cd(u, \mathcal{M})^2$$

holds for any $u \in \overline{\mathcal{W}}$. Here $d(u, \mathcal{M})$ denotes the distance of u from $\mathcal{M}$ in $\overline{\mathcal{W}}$;

$$d(u, \mathcal{M}) = \inf\{\|u - v\|_{\overline{\mathcal{W}}} : v \in \mathcal{M}\}.$$

In the proof, we follow the argument of Bianchi-Egnell [2] and Bartsch-Weth-Willem [1], in which the Sobolev inequality with remainder term was studied.

Key points are:

(1) Non-degeneracy of critical manifold (Isobe [3]).

(2) Relative compactness of the minimizing sequence for S up to translation and dilation.

On (1), we set a 6-dimensional submanifold in $\mathcal{M}$

$$\mathcal{Z} := \left\{ \pm\overline{RU_{\lambda,a}} : R \in SO(3), \lambda > 0, a \in \mathbf{R}^2 \right\}.$$

Next lemma is equivalent to the fact that $\mathcal{Z}$ is a non-degenerate critical manifold in $\overline{\mathcal{W}}$ of the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u|^2 dx + \frac{2}{3} Q(u), \quad u \in \overline{\mathcal{W}},$$

that is,

$$T_{\overline{RU_{\lambda,a}}} \mathcal{Z} = \ker D^2 E(\overline{RU_{\lambda,a}})$$

holds for any $\overline{RU_{\lambda,a}} \in \mathcal{Z}$.

Lemma 1.(Isobe) *There exists a constant $C_1 > 0$ such that*

$$\int_{\mathbf{R}^2} |\nabla \phi|^2 dx + 4 \int_{\mathbf{R}^2} \overline{RU_{\lambda,a}} \cdot \phi_{x_1} \wedge \phi_{x_2} dx \geq C_1 \int_{\mathbf{R}^2} |\nabla \phi|^2 dx$$

holds for any $\overline{RU_{\lambda,a}} \in \overline{\mathcal{W}}$ and any $\phi \in \overline{\mathcal{W}}$ with $\phi \perp span\{\overline{RU_{\lambda,a}}\} \oplus T_{\overline{RU_{\lambda,a}}}\mathcal{Z}$.

On (2), Concentration-Compactness argument of P.Lions applies to

$$I = \inf\{-|Q(v)| : v \in \overline{\mathcal{W}}, \int_{\mathbf{R}^2} |\nabla v|^2 dx = 1\} < 0,$$

thus we get the next lemma.

Lemma 2. *Let $(u^n) \subset \overline{\mathcal{W}}$ be any minimizing sequence for I. Then there exist $a_n \in \mathbf{R}^2$ and $\lambda_n \in \mathbf{R}_+$ such that the new minimizing sequence defined by*

$$\tilde{u}^n(\cdot) = \overline{u^n(\frac{\cdot - a_n}{\lambda_n})}$$

is relatively compact in $\overline{\mathcal{W}}$. In particular, there exists a minimizer for I in $\overline{\mathcal{W}}$.

From this lemma, we obtain the relative compactness of the minimizing sequence for S up to translation and dilation. In the proof of Lemma 2, the 2nd Concentration-Compactness Lemma (CCL II) for the best constant of the isoperimetric inequality ([4]) plays an important role.

Concentration-Compactness Lemma II for the isoperimetric inequality.

Let $(v^n) \subset \overline{\mathcal{W}}$ satisfy the followings:

- *$v^n \rightharpoonup v^0$ weakly in $\overline{\mathcal{W}}$ for some v^0,*

- *$|\nabla v^n|^2 dx \overset{*}{\rightharpoonup} \mu$ weakly in $\mathcal{M}(\mathbf{R}^2)$, where μ is a nonnegative finite Radon measure on $\mathbf{R}^2$,*

- *$T^n \to T$ in $\mathcal{D}'(\mathbf{R}^2)$ for some distribution T, where $T^n \in \mathcal{D}'(\mathbf{R}^2)$ is defined as*

$$T^n(\varphi) = \int_{\mathbf{R}^2} (\varphi v^n) \cdot v^n_{x_1} \wedge v^n_{x_2} dx, \quad \forall \varphi \in \mathcal{D}(\mathbf{R}^2).$$

Then we have T is a finite signed measure on $\mathbf{R}^2$ and there exist at most countable set (possibly empty) J, distinct points $\{x_j\}_{j \in J} \subset \mathbf{R}^2$, nonnegative numbers $\{\mu_j\}_{j \in J}$, real numbers $\{\nu_j\}_{j \in J}$ such that

(1) $\mu \geq |\nabla v^0|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j},$

(2) $T = T^0 + \sum_{j \in J} \nu_j \delta_{x_j}$ *in* $\mathcal{D}'(\mathbf{R}^2)$, *where* T^0 *is defined as* T *through* v^0,

(3) $S|\nu_j|^{2/3} \leq \mu_j$ $(j \in J)$,

(4) $S|T(\mathbf{R}^2)|^{2/3} \leq \mu(\mathbf{R}^2)$,

(5) *If* $v^0 \equiv 0$ *and* $\mu(\mathbf{R}^2) = S|T(\mathbf{R}^2)|^{2/3}$, *then* $card(J) = 1$ *and there exists some* x_0 *in* $\mathbf{R}^2$ *such that* $T = C\delta_{x_0}, \mu = SC^{2/3}\delta_{x_0}$ *for some* $C \in \mathbf{R} \setminus \{0\}$.

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Driven crystalline curvature flow in the plane

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We study the driven crystalline mean curvature flow

$$V = \kappa_\gamma + \sigma \quad \text{on } \Gamma(t), \qquad \Gamma(0) = \Gamma_0, \tag{1}$$

where V is the normal velocity of the curve, σ is special but non-constant, Γ is a rectangle (or a graph over an interval). For simplicity we assume that the sides of Γ are parallel to the coordinate axes and the origin is the center of symmetry of Γ.

We recall the definition of mean curvature κ_γ,

$$\kappa_\gamma = \text{div}\xi \tag{2}$$

where $\xi(x) = (\nabla\gamma)(\mathbf{n}(x))$. This definition is correct if γ is smooth, but we have good reasons to consider

$$\gamma(x_1, x_2) = |x_1|\gamma_T + |x_2|\gamma_\Lambda, \quad \gamma_\Lambda, \gamma_T > 0. \tag{3}$$

First we provide the motivation for considering such γ. Namely, in a system modeling the crystal growth in vapor (see [GR2] and references therein) Γ is the boundary of a circular cylinder Ω. Moreover, equation (1) is coupled to the quasi-steady approximation of the diffusion equation for vapor supersaturation σ,

$$0 = \Delta\sigma \qquad \text{in } \bigcup_{0 < t < T} \mathbb{R}^3 \setminus \Omega(t), \quad \lim_{|x| \to \infty} \sigma(x) = \sigma^\infty > 0 \tag{4}$$

$$\frac{\partial\sigma}{\partial\mathbf{n}} = V, \qquad \text{on } S(t) = \partial\Omega(t). \tag{5}$$

In this system $\mathbf{n}$ is the outer normal to Ω; $\beta = \beta(\mathbf{n})$.

Precisely, the situation we are interested in may be called the onset on singularity: the velocity V is no longer constant over facets of Γ (or $\partial\Omega$), i.e. roughly speaking, over its flat portions. Before we study the full problem, i.e. system (1), (4), (5) for being Γ the boundary of a cylinder we wish to consider a simplified setting: Γ is a rectangle (or a graph of a function) and σ is a fixed function independent of time. Our choice of γ is motivated by the physics of the problem (see e.g. [Ne]). Namely, the normals to Γ should belong to the set of energetically preferred orientations which is the set of normals to the Wulff shape of γ. Let us recall the definition of the Wulff shape,

$$W_\gamma = \{x \in \mathbb{R}^2 : \forall \mathbf{n} \in \mathbb{R}^2, |\mathbf{n}| = 1, x \cdot \mathbf{n} \le \gamma(\mathbf{n})\}.$$

For γ given by (3) we notice that W_γ is a rectangle, i.e.

$$W_\gamma = \{x \in \mathbb{R}^2 : |x_1| \leq \gamma(\mathbf{n}_R), |x_2| \leq \gamma(\mathbf{n}_\Lambda)\}.$$

Thus, if Γ a rectangle, then normals to Γ belong to the set of non-differentiability points of γ and we cannot any longer take $\xi(x) = \nabla\gamma(\mathbf{n}(x))$. This is the reason why we replace $\nabla\gamma(\mathbf{n})$ with the subdifferential $\partial\gamma(\mathbf{n})$ which is defined everywhere, because of convexity of γ. Thus, in addition to (1) we have to consider a section

$$\xi(x) \in \partial\gamma(\mathbf{n}(x)).$$

Making the right selection of ξ is one the major problems, but before solving it let us specify our assumptions on σ and Γ. In [GR1] we proved that if σ is a solution to (4)–(5), where Ω is a circular cylinder and V's are constant and positive, then σ is monotone over each facet (see (6) below). We expect that this property of solutions to (1), (4), (5) should hold past the singularity formation. This is why we adopt the assumption

$$x_i \frac{\partial\sigma}{\partial x_i}(t, x_1, x_2) > 0 \qquad \text{for } x_i > 0, \quad i = 1, 2. \tag{6}$$

For simplicity we limit our considerations to σ such that

$$\sigma(t, -x_1, x_2) = \sigma(t, x_1, x_2), \qquad \sigma(t, x_1, -x_2) = \sigma(t, x_1, x_2). \tag{7}$$

As we mentioned we will make our assumption of Γ more clear. Namely, we assume that $\Gamma(t)$ consists of the sum of four graphs and it is symmetric with respect to the coordinate axes, i.e. $\Gamma(t) = \Gamma(d^R(t, \cdot)) \cup \Gamma(-d^R(t, \cdot)) \cup \Gamma(d^\Lambda(t, \cdot)) \cup \Gamma(-d^\Lambda(t, \cdot))$, where d^R, d^Λ are Lipschitz continuous functions, to be specified below and $\Gamma(f)$ denotes the graph of f. Because of the symmetry assumptions, we further consider only $S_\Lambda = \Gamma(d^\Lambda(t, \cdot))$ and $S_R = \Gamma(d^R(t, \cdot))$. We will call them *facets*.

We notice that (1) (while taking into account (2)) is the Euler-Lagrange equation of the functionals

$$\mathcal{E}_i(\xi) = \frac{1}{2} \int_{S_i} |\text{div}_S\xi - \sigma|^2 d\mathcal{H}^1, \qquad i = \Lambda, R$$

on

$$\mathcal{D}_i = \{\xi \in L^\infty(S_i) : \text{div}_S\xi \in L^2(S_i), \xi(x) \in \partial\gamma(\mathbf{n}(x)), \xi|_{S_i \cap S_j} \in \partial\gamma(\mathbf{n}_\Lambda) \cap \partial\gamma(\mathbf{n}_R)\}, i = \Lambda, R,$$

where $\mathbf{n}_i$ is the normal to S_i.

The idea of considering a variational problem like (VP) appeared first in [BNP1], [GG] and [GR2].

We are now ready to define the notion of solution.

Definition. A solution to (1) is a couple $\Gamma(t), \xi(t))$, $\xi \in \mathcal{D}$, where $\Gamma(t)$ is a (special) Lipschitz curve, and ξ is a solution to

$$\mathcal{E}_i(\xi) = \min\{\mathcal{E}_i(\zeta) : \zeta \in \mathcal{D}_i\}, \qquad i = R, \Lambda. \tag{VP}$$

Our goal is to gain precise information about the onset of possible facet bending or breaking. For this reason we will consider S_i having at most three faceted regions, i.e. maximal intervals on which

$$\sigma - \text{div}_S\xi = const.$$

holds. Before we consider the rectangle we will look at **the case of being $\Gamma(t)$ a graph.** We restrict our attention to:

(a) $\Gamma(t) = \Gamma(d(t, \cdot))$;

(b) $d(t, -x) = d(t, x)$; $\frac{\partial d}{\partial x}(t, \pm L) = 0$;

(c) $\beta(\mathbf{n}) = \frac{1}{\max\{n_1, n_2\}}$, $\mathbf{n} = (n_1, n_2)$.

(d) $d(t, x) = R_0$ if $0 < x \le l_0$, $d(t, \cdot)$ is monotone on (l_0, l_1), $d(t, x) = R_1$ if $l_1 < x < L$.

We are interested in the situation when at the initial time $t = 0$ the curve $\Gamma(0)$ is as above, i.e. $\Gamma(0) = \Gamma(d(0, \cdot))$, $\xi(0)$ solves (VP) and $\xi(0, x)$ belongs to the boundary of the subdifferential $\partial\gamma(\mathbf{n}(x))$ for $x \in (-l_1, -l_0) \cup (l_0, l_1)$. This is the reason why the Euler-Lagrange equation for the single functional $\mathcal{E}$ **does not hold** on the intervals $(-l_1, -l_0)$, (l_0, l_1).

The surface divergence $\mathrm{div}_S \xi = \tau \cdot \frac{\partial \xi}{\partial \tau}$, ($\tau$ is a unit tangent to Γ) gets a simplified form, from which we may calculate σ. Namely on $(-L, L) \setminus ((-l_1, -l_0) \cup (l_0, l_1))$ we have $\mathrm{div}_S \xi = \frac{\partial \xi_1}{\partial x_1}$.

The minimization problem (VP) is in fact of obstacle type, hence by the general theory (see [KS])

$$0 = \frac{\partial \xi_1}{\partial x}(\pm l_i) \quad i = 0, 1. \tag{8}$$

This fact allows us to deduce

Proposition 1. Suppose σ satisfies (6) and (7). If (d, ξ) is a solution to (1), $|d_x(t, x)| \le 1$ and it satisfies (a-d) above, then:

(a) Equation (1) take the form

$$\dot{R}_0 = \int_0^{l_0} \sigma(s)\, ds + \frac{\gamma(\mathbf{n}_R)}{l_0} \quad \text{on} \quad [0, l_0]$$

$$d_t = \sigma \quad \text{on} \quad (l_0, l_1) \tag{9}$$

$$\dot{R}_1 = \int_{l_1}^{L} \sigma(s)\, ds + \frac{2\gamma(\mathbf{n}_R)}{L - l_1} \quad \text{on} \quad [l_1, L].$$

(b) The velocities satisfy the inequality $\dot{R}_1 > \dot{R}_0 > 0$ and the following condition holds

$$\sigma(l_0) = \int_0^{l_0} \sigma(s)\, ds + \frac{\gamma(\mathbf{n}_R)}{l_0}, \qquad \sigma(l_1) = \int_{l_1}^{L} \sigma(s)\, ds - \frac{2\gamma(\mathbf{n}_R)}{L - l_1}. \tag{TC}$$

The details will be presented elsewhere, see [GR3]. The very simple form of $(9)_2$ is the result of (c) and $|d_x(t, x)| \le 1$. Let us notice that (TC) is a consequence of (8). Moreover in order to solve the resulting system (9) we need to know the time evolution of $l_i(t)$, $i = 0, 1$. Fortunately, we may determine it in advance.

Corollary 1. Let us suppose that the assumptions of Proposition hold. In particular, σ is independent from time, then $l_0(t) = l_{00}$, $l_1(t) = l_{10}$.

We have now identified all the necessary ingredients to solve (9).

Theorem 1. Let us suppose that $\Gamma(0) = \Gamma(d(0, \cdot))$ and (a)-(d) hold. Then, there exists a local in time solution of (9).

While local in time existence to a system of ODE's and its continuous dependence upon data is obvious, we note that the constraint $|d_x(t, x)| \le 1$ influences the existence time. Propagation of (8) (i.e. (TC)) is not clear, this is important because we first construct d, then ξ which is supposed to be a solution to an obstacle problem. We also stress that $d(t, \cdot)$ is a continuous function due to continuity of $d(0, \cdot)$ and (TC). For the full proof of this result we refer the reader to [GR3].

We note that this Theorem does not guarantee uniqueness of solutions. This may be proved by applying the methods of the semigroup theory similar to that used in [FG]. Namely, one can show that the solution (d, ξ) we constructed is a solution to the differential inclusion,

$$\dot{u} + \partial\Phi(u) \ni 0,$$

where Φ is appropriately chosen. The general theory (see e.g.[Br]) yields uniqueness (and existence too).

Once we understood the graph evolution we may turn our attention to **the case of a rectangle**. More precisely, we assume that $\Gamma(t)$ consists of the sum of four graphs and it is symmetric with respect to the coordinate axes, i.e. $\Gamma(t) = \Gamma(d^R(t, \cdot)) \cup \Gamma(-d^R(t, \cdot)) \cup \Gamma(d^\Lambda(t, \cdot)) \cup \Gamma(-d^\Lambda(t, \cdot))$. Thus, the evolution system consists of two sets like (9) for $d^R(t, \cdot)$ (defined over $[-L(t), L(t)]$) and $d^\Lambda(t, \cdot)$ (defined over $[-R(t), R(t)]$). Each of the functions d^R, d^Λ is constant over $[-L(t), L(t)] \setminus ((-l_1^R, -l_0^R) \cup (l_0^R, l_1^R))$ (respectively, over $[-R(t), R(t)] \setminus ((-l_1^\Lambda, -l_0^\Lambda) \cup (l_0^\Lambda, l_1^\Lambda)$. Thus, those systems are coupled by $L(t)$, $R(t)$, $l_i^j(t)$, $i = 0, 1$, $j = \Lambda, R$. The details get more complicated.

We are mostly interested in the behavior of solutions on the intervals (l_0^j, l_1^j), $j = \Lambda, R$. We can rely upon the following result.

Proposition 2. If $x_i \frac{\partial \sigma}{\partial x_i}(t, x_1, x_2) > 0$ for $x_i > 0$, $i = 1, 2$, then function $t \mapsto l_0^j(t)$ is decreasing while the function $t \mapsto l_1^j(t)$ is increasing, $j = R, \Lambda$.

Thus, in particular we can consider the following data $l_0^i(0) = l_{00}^i = l_{10}^i = l_1^i(0)$. Thus we are able to state our main result.

Theorem 2. Let us suppose that σ satisfies (6), (7) and $\Gamma(0)$ is a rectangle with $l_0^i(0) = l_1^i(0)$, $i = R, \Lambda$. Moreover, we assume that (TC) is satisfied at $t = 0$. Then, there exists a solution to (1) and (TC) is satisfied for all $t > 0$.

This result is proved in [GR3]. Let us stress that (TC) at time t means that at that time instant the Cahn-Hoffman vector is a solution to obstacle problem (TC). Moreover, by Theorem 2. this property propagates in time. Unfortunately, the methods of this theorem do not permit us to conclude uniqueness of solutions. This requires different methods.

We conclude by saying that in our setting facets, which loose their stability bend.

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Boundary blowup solutions to curvature equations

Kazuhiro Takimoto[1]

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. We consider the so-called *curvature equations* of the form

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = f(u)g(|Du|) \quad \text{in } \Omega, \tag{1}$$

with the following boundary condition

$$u(x) \to \infty \quad \text{as dist}(x, \partial\Omega) \to 0. \tag{2}$$

Here, for a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, and $S_k, k = 1, \ldots, n$, denotes the k-th elementary symmetric function, i.e.,

$$S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k}, \tag{3}$$

where the sum is taken over increasing k-tuples, $1 \le i_1 < i_2 < \cdots < i_k \le n$. We study the existence and the asymptotic behavior near $\partial\Omega$ of a solution to (1)-(2).

The family of equations (1), $k = 1, \ldots, n$ contains some well-known and important equations.

The case $k = 1$ corresponds to the **mean curvature equation**;

The case $k = 2$ corresponds to the **scalar curvature equation**;

The case $k = n$ corresponds to **Gauss curvature equation**.

We remark that (1) is a quasilinear equation for $k = 1$ while it is a *fully non-linear* equation for $k \ge 2$. In the particular case that $k = n$, it is an equation of Monge-Ampère type. It is much harder to analyze fully non-linear equations, but the study of the classical Dirichlet problem for curvature equations in the case that $2 \le k \le n - 1$ has been developed in the last two decades, see for instance [4, 11, 24].

The condition (2) is called the "boundary blowup condition," and a solution which satisfies (2) is called a "boundary blowup solution," a "large solution," or an "explosive solution." The boundary blowup problems arise from physics, geometry and many branches of mathematics, see for instance

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[12, 19, 21]. The study of such problems for non-linear PDEs starts from the pioneering work of Bieberbach [3] and Rademacher [21] who considered $\Delta u = e^u$ in two and three dimensional domain respectively. For the case of semilinear equations, they have been extensively studied (see, for example, [13, 20] and [2, 6, 15, 16, 17, 19]). The case of quasilinear equations of divergence type to which the mean curvature equation ($k = 1$) belongs has been treated in [1, 7, 9]. The case of Monge-Ampère equations has been studied in [5, 10, 18]. However, to the best of our knowledge, there are no results concerning such problems for other fully non-linear PDEs, except for the work of Salani [22] who considered the case of Hessian equations.

Throughout the article, we assume the following conditions on f and g:

- Let $t_0 \in [-\infty, \infty)$. $f \in C^\infty(t_0, \infty)$ is a positive function and satisfies $f'(t) \geq 0$ for all $t \in (t_0, \infty)$.

- If $t_0 > -\infty$, then $f(t) \to 0$ as $t \to t_0 + 0$; otherwise (i.e., if $t_0 = -\infty$),

$$\int_{-\infty}^{t} f(s)\, ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

- $g \in C^\infty[0, \infty)$ is a positive function.

The first condition assures us that the comparison principle for solutions to (1) holds. The typical examples of f are $f(t) = t^p$ ($p > 0$), $t_0 = 0$ and $f(t) = e^t$, $t_0 = -\infty$.

In the subsequent two sections, we state our main results.

2 Existence results

We recall the notion of k-convexity. Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega \in C^2$. For $k = 1, \ldots, n-1$, we say that Ω is k-convex (resp. uniformly k-convex) if the vector of the principal curvatures of $\partial\Omega$, $\kappa' = (\kappa'_1, \ldots, \kappa'_{n-1})$, satisfies $S_j(\kappa') \geq 0$ (resp. > 0) for $j = 1, \ldots, k$ and for every $x \in \partial\Omega$. We note that a C^2 domain is $(n-1)$-convex (resp. uniformly $(n-1)$-convex) if and only if it is convex (resp. strictly convex).

First, we shall establish the existence of a boundary blowup solution to the curvature equation (1). We focus on the case $k \geq 2$, because for $k = 1$ the existence has been already studied in [9].

Theorem 1. *Let $2 \leq k \leq n-1$. We suppose that Ω, f and g satisfy the following conditions.*

(A1) *Ω is a bounded and uniformly k-convex C^∞ domain.*

(A2) *There exists a constant $T > 0$ such that g is non-increasing in $[T, \infty)$, and $\lim_{t \to \infty} g(t) = 0$.*

(A3) *Set $\tilde{g}(t) = g(t)/t$ and $F(t) = \int_{t_0}^{t} f(s)\,ds$. Then*

$$\int^{\infty} \frac{dt}{\tilde{g}^{-1}\left(\frac{1}{F(t)}\right)} < \infty. \tag{5}$$

(A4) *Set*

$$H(t) = \int_{0}^{t} \frac{s^k}{g(s)\,(1 + s^2)^{(k+2)/2}}\,ds. \tag{6}$$

Then $\lim_{t \to \infty} H(t) = \infty$.

(A5) *Set $\varphi(t) = g(t)(1 + t^2)^{k/2}$. Then $\varphi(t)$ is a convex function in $[0, \infty)$.*

(A6) $\limsup_{t \to \infty} g'(t)t^2 < \infty$.

Then there exists a viscosity solution to (1)-(2).

The strategy of the proof of this theorem is as follows (we refer the readers to [23] for details). We note that **comparison principles** for viscosity solutions play important roles.

$\boxed{\textbf{Step 1.}}$ We show that there exists a classical solution to the Dirichlet problem

$$\begin{cases} H_k[u_n] = f(u_n)g(|Du_n|) & \text{in } \Omega, \\ u_n \equiv n & \text{on } \partial\Omega, \end{cases} \tag{7}$$

for every $n \in \mathbb{N}$ with $n > t_0$. It is enough to derive the C^2-a priori estimate for (7) (see [8, 14]).

$\boxed{\textbf{Step 2.}}$ We prove that $\lim_{n \to \infty} u_n (=: u)$ exists and is a viscosity solution to (1)-(2).

Next we obtain the following non-existence result.

Theorem 2. *Let $2 \leq k \leq n - 1$. We define two functions $\bar{g}, \bar{h}$ by*

$$\bar{g}(t) = \max_{s \geq t} g(s), \quad \bar{h}(t) = \frac{t}{\sqrt{1 + t^2}} \left(\frac{\binom{n-1}{k}}{\bar{g}(t)} \right)^{1/k}. \tag{8}$$

We assume that $\lim_{t \to \infty} g(t) = 0$. If there exists $R \geq \inf_{x \in \Omega} \sup_{y \in \Omega} |x - y|$ such that

$$\int^{\infty} \frac{dt}{\bar{h}^{-1}\left(f(t)^{1/k}R\right)} < \infty, \tag{9}$$

then (1)-(2) has no solutions.

Example 1. Let $2 \leq k \leq n - 1$ and p, q be positive constants. Suppose Ω is a bounded and uniformly k-convex C^∞ domain. We consider these three equations:

$$H_k[u] = \frac{u^p}{(1 + |Du|^2)^{q/2}} \quad \text{in } \Omega, \tag{10}$$

$$H_k[u] = \frac{e^{pu}}{(1 + |Du|^2)^{q/2}} \quad \text{in } \Omega, \tag{11}$$

$$H_k[u] = \frac{e^{pu}}{e^{q|Du|}} \quad \text{in } \Omega. \tag{12}$$

It follows from Theorem 1 and Theorem 2 that

- The equation (10) has a boundary blowup solution provided $p > q$ and $1 \leq q \leq k - 1$.

- The equation (11) has a boundary blowup solution provided $1 \leq q \leq k - 1$.

- The equation (12) does not have any boundary blowup solutions.

Remark 1. Theorem 2 indicates that as far as (10) is concerned, p is necessarily greater than q in order for a boundary blowup solution to exist. In this case, our condition (A3) reduces to $p > q$ as well. We conjecture that (10) has a boundary blowup solution provided we assume only $1 \leq q < p$.

The case $k = n$, which corresponds to Gauss curvature equation, is excluded from Theorem 1. We state the existence result for the case $k = n$.

Theorem 3. *Let Ω be a bounded and strictly convex C^∞ domain, and $k = n$. We assume that the condition (A3) is satisfied and that $\limsup_{t \to \infty} g(t)t < \infty$. Then there exists a viscosity solution to (1)-(2).*

3 Asymptotic behavior near the boundary

In this section we establish the asymptotic behavior of a boundary blowup solution near the boundary when the domain is strictly convex. We shall prove the following.

Theorem 4. *Let $1 \leq k \leq n - 1$. We assume that (A2) and (A3) in Theorem 1 and the conditions given below are satisfied.*

(B1) *Ω is a bounded and strictly convex C^∞ domain.*

(B2) $t_0 = -\infty$, or $t_0 > -\infty$ and $f^{1/k}$ is Lipschitz continuous at t_0.

(B3) There exists a constant $T' > 0$ such that f is a convex function in $[T', \infty)$.

(B4) Set $h(t) = \dfrac{t}{g(t)^{1/k}\sqrt{1+t^2}}$. Then there exists a constant $\alpha > 0$ such that $h(t)/t^\alpha$ is non-decreasing in $(0, \infty)$.

(B5) $\displaystyle\lim_{t\to\infty} \frac{g(t)}{(1+t^2)g'(t)} = 0$.

Then there exist positive constants C_1, C_2 such that every solution u to (1)-(2) satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega) \le \psi(u(x)) \le C_2 \operatorname{dist}(x, \partial\Omega), \qquad (13)$$

where ψ is defined by

$$\psi(t) = \int_t^\infty \frac{ds}{h^{-1}\left(f(s)^{1/k}\right)}. \qquad (14)$$

We state the idea of the proof. Since Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$, there exist positive numbers R_1, R_2 with $R_1 < R_2$ satisfying the following condition: for every $z \in \partial\Omega$, there are two balls $B_{1,z}, B_{2,z}$ whose radii are R_1 and R_2 respectively such that $B_{1,z} \subset \Omega \subset B_{2,z}$ and $\partial B_{1,z} \cap \partial B_{2,z} = \{z\}$.

Let v_1 (resp. v_2) be a radially symmetric solution to (1) with $v_1(x) \to \infty$ as $\operatorname{dist}(x, \partial B_{1,z}) \to 0$ (resp. $v_2(x) \to \infty$ as $\operatorname{dist}(x, \partial B_{2,z}) \to 0$). The condition (B4) guarantees the existence of v_1 and v_2. By the comparison principle, we see that

$$v_2 \le u \le v_1 \quad \text{in } B_{1,z}. \qquad (15)$$

In view of (15), it suffices to study the asymptotic behavior of the radially symmetric solution near the boundary. The assertion follows from the claim that if $u = u(|x|)$ is a radially symmetric solution to (1)-(2) in $B_R(0)$ with $R > 0$, then there exist constants $C_1, C_2 > 0$ which are independent of r such that

$$C_1(R - r) \le \psi(u(r)) \le C_2(R - r) \qquad (16)$$

when r is near R.

Example 2. Let $1 \le k \le n - 1$ and $p, q > 0$. Suppose Ω is a bounded and strictly convex C^∞ domain. Then Theorem 4 implies that

- A boundary blowup solution u to (10) (if it exists) satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \leq u(x) \leq C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \quad \text{near } \partial\Omega \qquad (17)$$

  for some constants $C_1, C_2 > 0$, provided $p \geq k$ and $p > q$.

- A boundary blowup solution u to (11) (if it exists) satisfies

$$u(x) = -\frac{q}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \qquad (18)$$

  provided $q > 0$.

We state our result concerning the asymptotic behavior of a solution to (1)-(2) near $\partial\Omega$ for the case $k = n$. We mention that

$$H(t) = \int_0^t \frac{s^n}{g(s)\,(1+s^2)^{(k+2)/2}}\,ds \qquad (19)$$

in this case, and introduce the following condition:

(B6) There exists a constant $\alpha > 0$ such that $H(t)/t^\alpha$ is non-decreasing.

Theorem 5. *Let $k = n$. We assume the conditions (A3), (B1), (B2) and (B6). Then there exist positive constants C_1, C_2 such that every solution u to (1)-(2) satisfies*

$$C_1 \operatorname{dist}(x, \partial\Omega) \leq \Psi(u(x)) \leq C_2 \operatorname{dist}(x, \partial\Omega), \qquad (20)$$

where Ψ is defined by

$$\Psi(t) = \int_t^\infty \frac{ds}{H^{-1}(F(s))}. \qquad (21)$$

Example 3. Let $k = n$ and $p, q > 0$. Suppose Ω is a bounded and strictly convex C^∞ domain. Then Theorems 3 and 5 implies that

- If $p > q \geq 1$, then there exists a boundary blowup solution to (10). Moreover, the solution u satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \leq u(x) \leq C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \quad \text{near } \partial\Omega \qquad (22)$$

  for some constants $C_1, C_2 > 0$, provided $p \geq n$ and $p > q > 1$.

- A boundary blowup solution u to (11) exists and satisfies

$$u(x) = -\frac{q-1}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \qquad (23)$$

  provided $q > 1$.

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THE NAVIER-STOKES EQUATIONS WITH INITIAL DATA IN UNIFORMLY LOCAL L^p SPACES

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In this talk, we consider the Cauchy problem for the incompressible homogeneous Navier-Stokes equations with viscosity 1 in $\mathbb{R}^d$ where $d \geq 2$. The equation is of the form

$$\text{(NS)} \quad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0, & t > 0, \quad x \in \mathbb{R}^d, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Here $u = (u^1, u^2, \cdots, u^d)$ is the unknown velocity vector field and p is the unknown pressure scalar field.

Our main purpose here is to solve (NS) for initial data which may not decay at space infinity but not necessarily locally bounded. There are many works which construct mild solutions of the Navier-Stokes equations on various function spaces (e.g. [22], [4], [10], [21], [16], [17], [24], [7], [6], [8], [13], [23], [30], [5]). E. B. Fabes, B. F. Jones and N. M. Rivière [10], T. Kato [21], Y. Giga and T. Miyakawa [16] constructed mild solutions of (NS) with initial data in L^p space where p is largerthan the space dimension d. Moreover in [21] and [16], the case $p = d$ is discussed. However, all functions in L^p spaces decay at space infinity when p is finite. When one considers nondecaying flows at space infinity as we would like to do, the function space for initial data should be a space of functions which may not decay at space infinity. The L^∞ space, considered by J. R. Cannon and G. H. Knightly [4], M. Cannone [6], Y. Giga, K. Inui and S. Matsui [13] is of course such a kind of function spaces, and Besov spaces with negative regularity considered by M. Cannone and Y. Meyer [7], M. Cannone [6], H. Kozono and M. Yamazaki [24], etc. are such kinds of function spaces too. However there is no work constructing mild solutions with initial data in uniformly local type spaces, which naturally contain functions which may not decay at space infinity. (For the definition of uniformly local Triebel-Lizorkin spaces and Besov Spaces, e.g. [33]). In this paper, we shall construct the mild solutions of (NS) with initial data in uniformly local L^p spaces where p is grater than or equal to the space dimension d. The method is quite simlar to that of E. B. Fabes, B. F. Jones and N. M. Rivière [10], T. Kato [21], Y. Giga

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and T. Miyakawa [16] except that we use the convolution type estimate
we newly obain instead of Young's inequality for for convolutions. Uni-
formly local L^p spaces consist of functions which are locally in L^p and
its L^p norm in any Eucledian ball with radius 1 are uniformly bounded.
When p is finite, they obviously contain functions which does not decay
at space infinity but not necessarily locally bounded.

Uniformly local L^p spaces were used by J. Ginibre and G. Velo [18] for
complex Ginzburg-Landau equations and used by P G Lemarié-Rieusset
[25], [26] and Y. Taniuchi [31] for the equation of the fluid mechanics.
In his work [25] and [26], Lemarié-Rieusset constructed in the three di-
mensional Euclidean space a suitable weak solution which is *local in time*
with arbitrary initial data in uniformly local L^2 space, and furthermore
he constructed a suitable weak solution which is *global in time* with arbi-
trary initial data in the closure of compactly supported smooth functions
in uniformly local L^2 space. Y. Taniuchi [31] obtained estimates of $2 - D$
vorticity equations. However he only considers L^p_{uloc}-L^p_{uloc} type estimates
of convolution type operators, while we also treat $L^p_{uloc} - L^q_{uloc}$ type esti-
mates of convolution type operator in which the indices p and q may be
different. Let us be more precise. We consider the equations (NS) with
initial data in $L^p_{uloc,\rho}$ space for positive number ρ and $p \in [1, \infty]$. When
p is finite, the space $L^p_{uloc,\rho}(\mathbb{R}^d)$ is defined as follows.

(1)

$$L^p_{uloc,\rho}(\mathbb{R}^d) := \{f \in L^1_{loc}(\mathbb{R}^d); \|f\|_{L^p_{uloc,\rho}} := \sup_{x \in \mathbb{R}^d} \Big(\int_{|x-y|<\rho} |f(y)|^p dy \Big)^{\frac{1}{p}} < \infty \}.$$

For simplicity of notations we set $L^\infty_{uloc,\rho}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$. When p is
finite, the space $L^p_{uloc,\rho}$ naturally contains both the space L^p and the
space L^∞. The space $L^p_{uloc,\rho}$ also contains all L^p-periodic functions, i.e.,
periodic functions which are locally p-th integrable in $\mathbb{R}^d$. We include the
parameter ρ here, since the exisitence time estimate of the mild solutions
can be different if ρ is different. Moreover varing ρ, we can reproduce T.
Kato's global esistence result for small initial data ; More precisely, one
can construct a unique mild solution globally in time if L^d norm of the
initial data are sufficiently small.

To solve (NS) we convert the equations to the integral equation of the
form

(2) $$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u) ds.$$

Here $e^{t\Delta}$ is the heat semigroup, $\mathbf{P}$ is the Helmholtz projection, and $f \otimes$
$g := (f_i g_j)_{1 \leq i,j \leq d}$ is a tensor product of $f = (f_1, f_2, \cdots, f_d)$ and $g = (g_1, g_2, \cdots, g_d)$. The solution of the integral equation (2) is called the
mild solution of (NS) with initial data u_0. The precise meaning and well-
definedness of each term follows from the $L^p_{uloc} - L^q_{uloc}$ type estimates of

convolution type operators which we shall obtain. Now we would like to state our main results.

Theorem 0.1. *(Existence and uniqueness)*

(i) Let $p \in (d, \infty]$ and $\rho > 0$. Then, for all $u_0 \in (L^p_{uloc,\rho}(\mathbb{R}^d))^d$ so that $\nabla \cdot u_0 = 0$, there exist a positive T^* and a unique mild solution $u \in L^\infty((0, T^*); (L^p_{uloc})^d) \cap C((0, T^*); (L^p_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0, T^*) \times \mathbb{R}^d$. The existence time T^* is bounded from below as

$$T^{*\frac{1}{2}+\frac{d}{2p}} \rho^{-\frac{2d}{p}} + T^{*\frac{1}{2}-\frac{d}{2p}} \geq \frac{\gamma}{||u_0||_{L^p_{uloc,\rho}}},$$

where γ is a positive constant depending only on d and p.

(ii) Let $\rho > 0$. For all $u_0 \in (\overline{\cup_{p>d} L^p_{uloc,\rho}(\mathbb{R}^d)}^{||\cdot||_{L^d_{uloc,\rho}}})^d$ so that $\nabla \cdot u_0 = 0$, there exist a positive T^* and a mild solution $u \in L^\infty((0, T^*); (L^d_{uloc,\rho})^d) \cap C((0, T^*); (L^d_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0, T^*) \times \mathbb{R}^d$. This solution may be chosen so that for all $T \in (0, T^*)$ we have

$$\sup_{0<t<T} t^{\frac{1}{2}} ||u(t, \cdot)||_{L^\infty} < \infty, \text{ and } \lim_{t \to 0} t^{\frac{1}{2}} ||u(t, \cdot)||_{L^\infty} = 0.$$

With this extra condition on the L^∞ norm, such a solution is unique. The existence time T^* is estimated from below as

$$T^* \geq \min\{\rho^2, \ \alpha\},$$

where α is a positive number satisfying

$$\sup_{0<t<\alpha} t^{\frac{1}{4}} ||e^{t\Delta} u_0||_{L^{2d}_{uloc,\rho}} \leq \gamma$$

where γ is a positive constant depending only on d.

(iii) Let $\rho > 0$. There exists $\epsilon > 0$ such that for all $u_0 \in (L^d_{uloc,\rho})^d$ with $||u_0||_{L^d_{uloc,\rho}} \leq \epsilon$, there exist a positive T^* and a unique mild solution $u \in L^\infty([0, T^*); (L^d_{uloc,\rho})^d) \cap C((0, T^*); (L^d_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0, T^*) \times \mathbb{R}^d$ so that $u(0, \cdot) = u_0$. The existence time T^* is estimated as

$$T^{*\frac{1}{4}} \rho^{-\frac{1}{2}} + 1 \geq \frac{\gamma}{||u_0||_{L^d_{uloc,\rho}}},$$

where γ is a positive constant depending only on d.

H. Koch and D. Tataru [23] showed that for any given $T > 0$, one can construct a local mild solution of (NS) which exists at least until time T if the bmo^{-1} norm of the initial data is sufficiently small. Especially, they constructed local mild solutions for any initial data in vmo^{-1}. They also showed that one can construct global mild solutions for small initial data in BMO^{-1}. The definitions of bmo^{-1}, vmo^{-1} and BMO^{-1} are the following. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we set

$$||f||_{BMO_T^{-1}} := \sup_{x \in \mathbb{R}^n, 0 < R^2 < T} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} \int_0^{R^2} |e^{t\Delta} f(y)|^2 dt dy \right)^{\frac{1}{2}}.$$

Then

$$BMO^{-1} := \{ f \in \mathcal{S}'(\mathbb{R}^n); ||f||_{BMO^{-1}} := ||f||_{BMO_\infty^{-1}} < \infty \},$$

$$bmo^{-1} := \{ f \in \mathcal{S}'(\mathbb{R}^n); ||f||_{bmo^{-1}} := ||f||_{BMO_1^{-1}} < \infty \},$$

$$vmo^{-1} := \{ f \in bmo^{-1}; \lim_{T \to 0} ||f||_{BMO_T^{-1}} = 0 \}.$$

The inclusion relations of $L_{uloc,\rho}^p$, bmo^{-1}, vmo^{-1} and BMO^{-1} are as follows.

$$L_{uloc,\rho}^p \subset vmo^{-1} \text{ if } p > d,$$

$$L_{uloc,\rho}^p \subset bmo^{-1} \text{ if } p \geq d,$$

$$L^d \subset BMO^{-1}.$$

Related to mild solutions constructed by H. Koch and D. Tataru [23], H. Miura [29] showed some uniqueness theorems of mild solutions of (NS). Although our function spaces $L_{uloc,\rho}^p$ when $d \leq p \leq +\infty$ are contained in bmo^{-1}, our results are useful since the definition of $L_{uloc,\rho}^p$ is very simple and it obviously contains some functions which may have singularities and may not decay at space infinity. Moreover, the convergences of mild solutions to initial data when time goes to zero are relatively simple in our case. For describing the convergences of mild solutions to initial data, we define the subspace $\mathcal{L}_{uloc,\rho}^p$ as the colsure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^d)$ in the space $L_{uloc,\rho}^p$, i.e.,

$$(3) \qquad \mathcal{L}_{uloc,\rho}^p := \overline{BUC(\mathbb{R}^d)}^{||\cdot||_{L_{uloc,\rho}^p}}.$$

Remark that the subspace $\mathcal{L}_{uloc,\rho}^\infty(\mathbb{R}^d)$ is the space $BUC(\mathbb{R}^d)$. The space $\mathcal{L}_{uloc,\rho}^p$ is useful since we can show that the solutions converge to the initial data in $L_{uloc,\rho}^p$ norm if the initial data belong to $\mathcal{L}_{uloc,\rho}^p$. In fact, we have the following theorem.

Theorem 0.2. *(Convergence to initial data)*
(i) Let $p \in (d, \infty]$ and $\rho > 0$. Let $u \in L^\infty((0,T); (L_{uloc,\rho}^p)^d)$ be a unique mild solution with initial data $u_0 \in L_{uloc,\rho}^p$. Then, for any compact set $K \subset \mathbb{R}^d$, we have

$$(4) \qquad \lim_{t \to 0} ||u(t) - u_0||_{L^p(K)} = 0$$

holds. Moreover,

$$(5) \qquad \lim_{t \to 0} ||u(t) - u_0||_{L^p_{uloc,\rho}} = 0$$

holds if and only if $u_0 \in \mathcal{L}^p_{uloc,\rho}$.

(ii) Let $\rho > 0$. Let u be a unique mild solution in $L^\infty((0,T); (L^d_{uloc,\rho})^d)$ which satisfies $t^{\frac{1}{2}} u(t) \in L^\infty((0,T); (L^\infty)^d)$, $\lim_{t \to 0} t^{\frac{1}{2}} ||u(t)||_{L^\infty} = 0$. Then,

$$(6) \qquad \lim_{t \to 0} ||u(t) - u_0||_{L^d(K)} = 0$$

holds. Moreover, under the above condition,

$$(7) \qquad \lim_{t \to 0} ||u(t) - u_0||_{L^d_{uloc,\rho}} = 0$$

holds if and only if $u_0 \in \mathcal{L}^d_{uloc,\rho}$.

One of the keys to our results is $L^p_{uloc,\rho}$-$L^q_{uloc,\rho}$ estimate we newly obtain, which we will show in the following theorem. We say that a function is radial decreasing if it is radial symmetric and nonincreasing.

Theorem 0.3.
Let $1 \le q \le p \le \infty$. Let $F(x)$, $H(x)$ be two real-valued functions in $\mathbb{R}^d$ and let $|F(x)| \le H(x)$ hold. Furthermore, assume that H is a bounded, integrable and radial decreasing function in $\mathbb{R}^d$.

We set $F_{t,m}(x) = t^{-\frac{d}{2}+m} F(x/t^{\frac{1}{2}})$ for $t > 0$, $m \ge 0$. Then, for any function $g \in L^q_{uloc,\rho}(\mathbb{R}^d)$, we can define pointwise

$$F_{t,m} * g(x) = \int_{\mathbb{R}^d} F_{t,m}(x-y)g(y)dy.$$

Furthermore we have the estimate

$$(8) \qquad ||F_{t,m} * g||_{L^p_{uloc,\rho}} \le \Big(\frac{C_1 ||H||_1}{t^m \rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2 ||H||_r}{t^{m+\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \Big) ||g||_{L^q_{uloc,\rho}},$$

where r is the number satisfying $\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1$, and C_1, C_2 are positive constants depending only on d.

Let us state the outline of the proof of this theorem. By rescaling, we may assume that $\rho = 1$. To obtain this estimate, we decompose $\mathbb{R}^d$ into countable cubes whose centers are lattice points, i.e.,

$$\mathbb{R}^d = \cup_{k \in \mathbb{Z}^d} S(k, \frac{1}{2}),$$

where $S(x, \theta) := \{y \; ; \; \max_{1 \le i \le d} |y_i - x_i| \le \theta\}$. We can decompose any measurable function f in $\mathbb{R}^d$ into

$$f(x) = \sum_{k \in \mathbb{Z}^d} \chi_{S(k,\frac{1}{2})}(x) f(x), \quad a.e. \ x \in \mathbb{R}^d,$$

where $\chi_A(x)$ is the characteristic function of a subset A in $\mathbb{R}^d$. We decompose both a convolution kernel H and a convoluted function g in this

way. Here we assume that H and g are both nonnegative. Furthemore, we assume that H is bounded, integrable, radial symmetric and nonincreasing function and g is L^q_{uloc} function (We say that a function is radial decreasing if it is radial symmetric and nonincreasing). Using Young's inequality for convolutions and the relation supp $f_1 * f_2 \subset \mathrm{supp} f_1 + \mathrm{supp} f_2$ when $f_1 * f_2$ is well-defined, we can estimate L^p_{uloc} norm of the function $H * g$ which is proven to be defined pointwise.

This talk is based on a joint work with Mr. Yasunori Maekawa, Hokkaido University.

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The initial-boundary value probrems for nonlinear elastic waves

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Let $\varphi(t, x)$ be a smooth deformation of a material that involves with time. The unknown of the problem is a displacement from the reference configuration, $u(t, x) = \varphi(t, x) - x$. The displacement gradient is then the matrix $G = \nabla u$ with components $G_{il} = \partial_l u^i$, where the spatial gradient will be denoted by ∇ or grad. Since we assume that the materials are isotropic, homogeneous and hyperelastic, the potential energy density is characterized by a stored energy function $\sigma = \sigma(\kappa_1, \kappa_2, \kappa_3)$, where $\kappa_1, \kappa_2, \kappa_3$ are principal invariants of the strain matrix $C = G + {}^tG + G\,{}^tG$. Thus the motion for the displacement is governed by a nonlinear system

$$(1) \qquad \partial_t^2 u - \mathrm{div}\, \frac{\partial \sigma}{\partial G} = 0 \qquad \text{in} \quad D,$$

where D is a domain in $\mathbf{R}^3$ with smooth boundary ∂D. It is natural from (1) that boundary values of displacement u satisfy the condition

$$(2) \qquad \frac{\partial \sigma}{\partial G} \cdot n = 0 \qquad \text{on} \quad \partial D,$$

where n stands for outer unit normal to ∂D. We assume initial data for u of the form

$$(3) \qquad u(0, x) = \varepsilon f(x), \quad \partial_t u(0, x) = \varepsilon g(x),$$

where $\varepsilon > 0$ is small and f and g have compact support.

It is known that the linear part Lu of (1) becomes

$$(4) \qquad \partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\mathrm{grad}\, \mathrm{div}\, u,$$

where the material constants c_1 and $c_2(c_1 > c_2)$ correspond to the propagation speeds of longtudinal and transverse waves,respectively,and are defined by the Lamé constants λ, μ;

$$c_1 = (\lambda + 2\mu)^{1/2}, \qquad c_2 = \mu^{1/2}.$$

It is known from physical requirement that $\mu > 0, 3\lambda + 2\mu > 0$. The nonlinear part $F(\nabla u, \nabla^2 u)$of (1) is linear in $\nabla^2 u$ and the first order terms in ∇u can be written in the form (see [1]),

(5) $\quad d_1 \text{grad}(\text{div } u)^2 + d_2(\text{grad}|\text{rot } u|^2 - 2\text{rot}(\text{div } u \text{ rot } u)) + Q(u, \nabla u).$

Here $Q(u, \nabla u)$ is a linear combination of null forms $Q_{lm}(\partial_n u^j, u^k)$. Thus the nonlinear system (1) is equivalent to

(6) $$Lu = F(\nabla u, \nabla^2 u)$$

In 1984,F.John[4] treated radial solutions of the form

(7) $$u^i = x_i \psi(t, r), \qquad r = |x|.$$

The nonlinear system (6) reduces to a single second-order equation for ψ of the form

(8) $$\partial_t^2 \psi = c^2(\partial_r^2 \psi + 4r^{-1}\partial_r \psi) + r^{-2}A(\psi, r\partial_r \psi)$$
$$c^2 = c_1^2 + 2d_1 r\partial_r \psi + d_3 \psi,$$

where $A(\psi, r\partial_r \psi) = d_{11}(r\partial_r \psi)^2 + d_{12}\psi(r\partial_r \psi) + d_{22}\psi^2 +$ third order terms and higher. We assume initial data for ψ of the form

(9) $$\psi(0, r) = \varepsilon\rho(r), \qquad \partial_t \psi(0, r) = \varepsilon\eta(r),$$

where ρ and η have compact support.F.John proved that if the genuine non-linearity condition $d_1 \neq 0$,then the second derivatives of the solution ψ for the initial value problem (8) (9) blow up at a finite time for sufficiently small ε.

We return to the original initial problem (1) (3). In 2000, T.Sideris[5] and the author[1],[2] proved independently the existence of a unique global in time small smooth solution to the initial value problem, provided the genuine nonlinearity condition does not hold, i.e., $d_1 = 0$. The author derived also in [1] the "null condition" for nonlinear term of general type $F(\partial u, \partial^2 u)$ $(\partial u = (\partial_t, \nabla u))$ from John-Shatah observation and proved then the condition $d_1 = 0$ means the null condition for $F(\nabla u, \nabla^2 u)$.There are some open problems. The first is to show global existence of smooth solutions for

$F(\partial u, \partial^2 u)$ satisfying the null condition.When an isotropic homogeneous material satisfies not necessary hyperelastic, we obtain a more general system than (1) from Rivlin-Ericksen theorem ([3]). The second is to show global existence of smooth solutions for this equation, which has almost same nonlinearity as (5) but has not a symmetric property getting energy inequarity of "higher order". We can not expect that initial data are always sufficiently smooth. The third is then to show existence of weak solutions in energy level.

We study now initial-boundary value problem in the exterior or interior domain of the unit sphere $|x| = 1$ for the displacement u of the form (7). The boundary condition (2) reduces to a single first order equation for ψ of the form

(10) $\qquad c_1^2 \partial_r \psi + (3\lambda + 2\mu)\psi = d_1(\partial_r \psi)^2 + B(\psi, \partial_r \psi) \qquad$ on $\quad |x| = 1,$

where $B(\psi, \partial_r \psi) = b_{12}\psi\partial_r\psi + b_{22}\psi^2 +$ third order terms and higher.

We first consider the extrior probrem in $D : |x| \geq 1$. Since the displacement u of the form (7) is unbounded, we use a standard method in study of initial value problem (1) (3). Set

$$\begin{aligned} Q &= |\partial_t u|^2 + c_2^2|\nabla u|^2 + (c_1^2 - c_2^2)(\text{div } u)^2 \\ Q_j &= -2c_2^2 \,{}^t(\partial_t u)(\partial_j u) - 2(c_1^2 - c_2^2)\partial_t u^j \text{div } u. \end{aligned}$$

We then obtain an identity:

(11) $$\partial_t Q + \sum_{j=1}^{3} \partial_j Q_j = 2 \,{}^t(\partial_t u)Lu$$

We find from(7) and definitions of c_1, c_2 that Q is equivalent to

$$e(t, r) = (r\partial_t \psi)^2 + (r\partial_r \psi)^2 + \psi^2$$

and, on the boundary $\partial D : |x| = 1$,

$$\sum_{j=!}^{3} x_j Q_j = \partial_t \psi(c_1^2 \partial_r \psi + (3c_1^2 - c_2^2)\psi) = 2\mu\psi\partial_t\psi.$$

Here we use the linear boundary condition $c_1^2 \partial_r \psi + (3\lambda + 2\mu)\psi = 0$.By integrating the identity (11) over $[0, t] \times D$,we find from facts stated above that

(12) $$\begin{aligned} &\int_1^\infty e(t, r)dr + c_2^2\psi(t, 1)^2 \\ &\leq C \left(\int_1^\infty e(0, r)dr + c_2^2\psi(0, 1)^2 \right) + 2\int_0^t dt \int_D {}^t(\partial_t u)Ludx. \end{aligned}$$

Thus, in the case of exterior problem, the energy inequality holds for any material.

Finally we consider the interior problem in $D : |x| \leq 1$. The equation $\partial_t^2 \psi - c_1^2(\partial_r^2 \psi + 4r^{-1}\partial_r \psi) = 0$ in (8) is the wave equation for radial solution in "five" dimensions. However this equation has "three" dimensional character in L^2 sense. By changing the unknown from ψ to $r^2\psi$, we find that

$$(13) \qquad r^2(\partial_t^2 \psi - c_1^2(\partial_r^2 \psi + 4r^{-1}\partial_r \psi)) = \partial_t^2(r^2\psi) - c_1^2 \partial_r^2(r^2\psi) + 2c_1^2\psi.$$

Then we obtain an identity similar to (11):

$$(14) \qquad\qquad \partial_t \bar{Q} + \partial_r \bar{Q}_1 = 2\partial_t(r^2\psi)\bar{L}(r^2\psi),$$

where $\bar{L} = \partial_t^2 - c_1^2\partial_r^2 + c_1^2 r^{-2}$ and

$$\begin{aligned} \bar{Q} &= (\partial_t(r^2\psi))^2 + c_1^2(\partial_r(r^2\psi))^2 + 2c_1^2 r^2\psi^2 \\ \bar{Q}_1 &= -2c_1^2\partial_t(r^2\psi)\partial_r(r^2\psi) \end{aligned}$$

We find by using the linear boundary condition that, on the boudary $\partial D : r = 1$,

$$\bar{Q}_1 = -2c_1^2\partial_t\psi(\partial_r\psi + 2\psi) = 2(\lambda - 2\mu)\psi\partial_t\psi.$$

We assume that

$$(15) \qquad\qquad\qquad\qquad \lambda > 2\mu$$

We then find by integrating the identity (14) over $[0, t] \times [0, 1]$ that energy equality holds:

$$(16) \quad \begin{aligned} &\int_0^1 \bar{Q}(t, r)dr + (\lambda - 2\mu)\psi(t, 1)^2 \\ &= \int_0^1 \bar{Q}(0, r)dr + (\lambda - 2\mu)\psi(0, 1)^2 + 2\int_0^t dt \int_0^1 \partial_t(r^2\psi)\bar{L}(r^2\psi)dr. \end{aligned}$$

We remark that the materials Rubber, Lead, Aluminium, Copper and etc. satisfy the condition (15) but Glass, Iron, Steel and etc. do not satisfy the condition (15).

In conclusion, the author's conjecture is that the initial-boundary value problem has a unique global radial solution if $d_1 = 0$ in the exterior problem and if $\lambda > 2\mu$ in the interior problem ?.

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SEMILINEAR SYSTEMS OF WAVE EQUATIONS

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1. INTRODUCTION

In this note we consider asymptotic behavior of solutions to the Cauchy problem for semilinear systems of wave equations:

$$\partial_t^2 u_i - c_i^2 \Delta u_i = F_i(\partial u) \qquad \text{in} \quad \mathbf{R}^3 \times (0, \infty), \tag{1.1}$$

where $i = 1, \cdots, N$, $c_i > 0$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\partial = (\partial_0, \partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_0 = \partial_t = \partial/\partial t$ and $u(x,t) = (u_1(x,t), \cdots, u_N(x,t))$ is a real-valued unknown function. Besides, $F_i \in C^1(\mathbb{R}^{4N})$ is a given function satisfying

$$F_i(0) = \nabla F_i(0) = 0.$$

Our purpose here is to show that there are examples of nonlinearities F such that the corresponding equation (1.1) cannot be regarded as a perturbation from the system of homogeneous wave equations, even if we restrict our attention to small amplitude solutions. The results presented in the section 2 was obtained by a joint work with Professors Kôji Kubota and Hideaki Sunagawa, and the results in the section 3 was done by a joint work with Professor Soichiro Katayama.

We wish to explain the precise meaning of our purpose. Suppose that the Cauchy problem for (1.1) admits a unique global solution u. We say the equation (1.1) can be regarded as a perturbation from the system of homogeneous wave equations:

$$\partial_t^2 v_i - c_i^2 \Delta v_i = 0 \qquad \text{in} \quad \mathbf{R}^3 \times (0, \infty), \tag{1.2}$$

if the global solution u tend to the solution $v = (v_1, \cdots, v_N)$ of (1.2) as $t \to \infty$. This kind of asymptotic behavior is well studied in connection with the so-called nonlinear scattering theory in the energy space. (see, e.g. [19, Chapter 6] and the references cited therein). Nevertheless, there is another possibility that the effect of the nonlinearity remains so strong in sufficiently large time that the global solution u cannot approach to any free solutions. To our knowledge, there are only few results which suggest that such a phenomenon occurs for nonlinear wave equations (see e.g. Alinhac [3, 4], Lindblad–Rodnainski [16, 17]). Therefore our main goal of this note is to show that there exist small amplitude solutions to the Cauchy

problem for (1.1) with a certain F whose large time behavior might be different from that of any free solutions.

We conclude this section by recalling a sufficient condition to ensure the *small data global existence* for (1.1) when nonlinearity F_i is sufficiently smooth. For the case $c_1 = \cdots = c_N$, such a condition was introduced by Klainerman [11]. We say $F(\partial u)$ satisfies the *null condition*, if and only if the quadratic part of it can be written as a linear combination of the following null forms

$$Q_0(u_j, u_k; c_i) = (\partial_t u_j)(\partial_t u_k) - c_i^2 (\nabla u_j) \cdot (\nabla u_k), \tag{1.3}$$

$$Q_{ab}(u_j, u_k) = (\partial_a u_j)(\partial_b u_k) - (\partial_b u_j)(\partial_a u_k) \quad (0 \le a < b \le 3). \tag{1.4}$$

We remark that Christodoulou [5] also established the same result, independently. Moreover, the global solution u to the Cauchy problem for (1.1) satisfying the *null condition* approaches to some free solution (see Kubo–Ohta [14, Section 6]). On the contrary, the *null condition* is necessary to ensure *small data global existence* if we consider the scalar case, i.e., $N = 1$. In fact, the *blow-up* result was obtained by Alinhac [2].

The *null condition* is extended to the multiple speeds case (i.e., the speeds $c_1, \ldots, c_N$ do not necessarily coincide with each other) so that the *small data global existence* for (1.1) holds (see Kovalyov [12], Agemi–Yokoyama [1], Yokoyama [21], Sideris – Tu [18], Kubota – Yokoyama [15], Katayama [7], [8], [9], Katayama – Yokoyama [10] and so on). For example, in addition to null forms, terms like $(\partial_a u_j)(\partial_b u_k)$ with $c_j \ne c_k$ are allowed to be included for the multiple speeds case. The precise conditions for the multiple speeds case are somewhat complicated, and we do not go into details here. Instead of this, we shall discuss an extension of the *null condition* for the case of the common propagation speeds with $N \ge 2$.

2. Example, I

This section is concerned with the Cauchy problem for semilinear systems of wave equations:

$$\begin{cases} \partial_t^2 u_1 - c_1^2 \Delta u_1 = |\partial_t u_2|^p & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\ \partial_t^2 u_2 - c_2^2 \Delta u_2 = |\partial_t u_1|^q & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \end{cases} \tag{2.1}$$

where c_1, $c_2 > 0$, $1 < p \le q$. First we recall known results concerning the *small data global existence* and *blowup* for the Cauchy problem for (2.1). Yokoyama [21] proved that when $c_1 \ne c_2$, the problem admits a unique global smooth solution when $p = q = 2$ and the initial data are in $C_0^\infty(\mathbf{R}^3)$ and sufficiently small. On the other hand, Deng showed in Theorem 3.3 of [6] that if $c_1 = c_2$ and $q(p-1) \le 2$, then, in general, a classical solution to the problem blows up in finite time however small the initial data are. It is remarkable that the above condition is valid for $p = q = 2$. Recently, Xu [20] proved the blowup result when $c_1 \ne c_2$ and $6(pq-1)/(p+q+2) \le 1$.

Thus we see from these results that the feature of the problem (2.1) depend not only on the exponents p, q but also on the propagation speeds c_1, c_2.

In order to extend the existence result due to [21] for general p, $q > 1$, we consider only radially symmetric solution to the Cauchy problem for (2.1). To be more specific, we seek solutions to the problem in $\mathcal{X} \times \mathcal{X}$, where $\mathcal{X}$ is defined by

$$\mathcal{X} = \{w(x,t) \in C(\mathbf{R}^3 \times [0,\infty)) \; ; \; \text{there is } u(r,t) \in X^2 \text{ such that} \qquad (2.2)$$
$$w(x,t) = u(|x|,t) \quad \text{for } (x,t) \in \mathbf{R}^3 \times [0,\infty) \quad \text{and} \quad \lim_{|x|\to\infty} w(x,0) = 0\}$$

with

$$X^2 = \{u(r,t) \in C^1(\mathbf{R} \times [0,\infty)) \; ; \; ru(r,t) \in C^2(\mathbf{R} \times [0,\infty)), \qquad (2.3)$$
$$u(-r,t) = u(r,t) \text{ for } (r,t) \in \mathbf{R} \times [0,\infty)\}.$$

Note that $\mathcal{X} \subset C^1(\mathbb{R}^3 \times [0,\infty)) \cap C^2((\mathbb{R}^3 \setminus \{0\}) \times [0,\infty))$, because $\partial_r u(r,t) = 0$ for $r = 0$ if $u \in X^2$. Therefore the solution which we shall obtain is an "almost" classical solution.

While, we consider the following type of initial condition:

$$u_j(x,0) = f_j(|x|), \quad (\partial_t u_j)(x,0) = g_j(|x|) \quad \text{for} \quad x \in \mathbf{R}^3 \quad (j = 1,2), \qquad (2.4)$$

and introduce a class of the initial data Y as follows:

$$Y = \{(f,g) \in C^1(\mathbf{R}) \times C(\mathbf{R}) \; ; \; rf(r) \in C^2(\mathbf{R}), \; rg(r) \in C^1(\mathbf{R}), \qquad (2.5)$$
$$f(-r) = f(r), \; g(-r) = g(r) \text{ for } r \in \mathbf{R}\}.$$

This space is consistent with X^2 in the sense that the solution v to the Cauchy problem for the homogeneous wave equation

$$\partial_t^2 v - c^2 \Delta v = 0 \quad \text{in} \quad \mathbf{R}^3 \times (0,\infty) \qquad (2.6)$$

belongs to X^2, if the initial data $(f,g) \in Y$ satisfy such a decay condition as

$$M_\kappa(f,g) := \sup_{r>0} (1+r)^\kappa \|(f(r),g(r))\| < \infty,$$

where $\kappa > 0$ and

$$\|(f(r),g(r))\| = |f(r)| + (1+r)(|f'(r)| + |g(r)|) + r(|f''(r)| + |g'(r)|).$$

Moreover we have the following estimate:

$$[v(r,t)](1 + |r - ct|)^\kappa \le CM_\kappa(f,g)$$

for $(r,t) \in \mathbf{R} \times [0,\infty)$, where we put

$$[v(r,t)] = |v(r,t)| + (1+r) \sum_{|\alpha|=1} |\partial_{r,t}^\alpha v(r,t)| + r \sum_{|\alpha|=2} |\partial_{r,t}^\alpha v(r,t)|.$$

In the application we choose κ as κ_1 or κ_2 which are defined as follows:

$$\kappa_1 = p - 1, \tag{2.7}$$

$$\kappa_2 = \min(q - 1, q(p - 1)) \quad \text{if} \quad c_1 \neq c_2 \text{ or } p > 2, \tag{2.8}$$

$$\kappa_2 = q(p - 1) - 1 \quad \text{if} \quad c_1 = c_2 \text{ and } 1 < p < 2. \tag{2.9}$$

In addition, if $c_1 = c_2$, $p = 2$ and $q(p - 1) > 2$ holds, then we take such κ_2 as $1 < \kappa_2 < q - 1$. Then we have the following existence result.

Theorem 2.1. *Let* $1 < p \leq q$ *and suppose that*

$$q(p - 1) > 2, \tag{2.10}$$

if $c_1 = c_2$, *and that*

$$q(p - 1) > 1, \tag{2.11}$$

if $c_1 \neq c_2$. *Assume* $(f_j, g_j) \in Y$ *and* $M_{\kappa_j}(f_j, g_j) \leq \varepsilon$ *for* $j = 1, 2$ *and* $\varepsilon > 0$.

Then there are positive constants ε_0 *and* C_0 (*depending only on* c_1, c_2, p *and* q) *such that for any* ε *with* $0 < \varepsilon \leq \varepsilon_0$, *there exists uniquely a solution* $(u_1, u_2) \in \mathcal{X} \times \mathcal{X}$ *of the Cauchy problem (2.1) and (2.4) satisfying*

$$[\widetilde{u_1}(r, t)](1 + |r - c_1 t|)^{\kappa_1} + [\widetilde{u_2}(r, t)](1 + |r - c_2 t|)^{\kappa_2} \leq 2C_0 \varepsilon, \tag{2.12}$$

if $c_1 \neq c_2$ *or* $p > 2$, *and*

$$[\widetilde{u_1}(r, t)]\{(1 + r + t)^{[1 - \kappa_1]+}\}^{-1}(1 + |r - c_1 t|) \tag{2.13}$$
$$+ [\widetilde{u_2}(r, t)](1 + |r - c_2 t|)^{\kappa_2} \leq 2C_0 \varepsilon,$$

if $c_1 = c_2$ *and* $1 < p \leq 2$. *Here we denoted* $u_1(x, t) = \widetilde{u_1}(|x|, t)$, $u_2(x, t) = \widetilde{u_2}(|x|, t)$, *and* $A^{[a]+} = A^a$ *if* $a > 0$; $A^{[0]+} = 1 + \log A$ *for* $A \geq 1$.

This result shows that the condition given by [6] is sharp if $c_1 = c_2$ and that it can be relaxed if $c_1 \neq c_2$. But it is still an open question what will happen when $c_1 \neq c_2$, $q(p - 1) \leq 1$ and the condition given by [20] does not fulfilled.

From now on we denote by (u_1, u_2) the global solution of the Cauchy problem (2.1) and (2.4) obtained in Theorem 2.1 and assume that $0 < \varepsilon \leq \varepsilon_0$. Our next step is to examine the large time behavior of (u_1, u_2). We define θ_1, θ_2 by

$$\theta_j = \kappa_j - 1 \quad \text{if } c_1 = c_2 \; ; \quad \theta_j = \kappa_j - (1/2) \quad \text{if } c_1 \neq c_2, \tag{2.14}$$

where κ_1 and κ_2 are defined by (2.7), (2.8) and (2.9). Since $\kappa_2 > 1/2$ by the definition and (2.11), we find that there exists uniquely a solution $v_2 \in \mathcal{X}$ of (2.6) with $c = c_2$ satisfying

$$\|u_2(t) - v_2(t)\|_{E(c_2)} \leq C \varepsilon^q (1 + t)^{-\theta_2} \quad \text{for} \quad t \geq 0, \tag{2.15}$$

and $\|v_2(0)\|_{E(c_2)} < \infty$, where $C = C(c_1, c_2, p, q)$ is a positive constant and

$$\|u(t)\|_{E(c)}^2 = \frac{1}{2} \int_{\mathbf{R}^3} (|\partial_t u(x, t)|^2 + c^2 |\nabla u(x, t)|^2) dx.$$

Now we are in a position to state our main result. Suppose that $\theta_1 \geq 0$, i.e., if $c_1 = c_2$, then $p \geq 2$; otherwise $p \geq 3/2$ (for the remaining case, we refer to [13]). As an unperturbed system, we choose

$$\begin{cases} \partial_t^2 v_1 - c_1^2 \Delta v_1 = |\partial_t v_2|^p & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\ \partial_t^2 v_2 - c_2^2 \Delta v_2 = 0 & \text{in} \quad \mathbf{R}^3 \times (0, \infty). \end{cases} \tag{2.16}$$

In other words, our proposal is to regard (2.1) as a perturbation from the *"modified free system"* (2.16), but in general not from the free system

$$\begin{cases} \partial_t^2 w_1 - c_1^2 \Delta w_1 = 0 & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\ \partial_t^2 w_2 - c_2^2 \Delta w_2 = 0 & \text{in} \quad \mathbf{R}^3 \times (0, \infty). \end{cases} \tag{2.17}$$

Theorem 2.2. *Assume that p, q and (f_j, g_j) $(j = 1, 2)$ fulfill the hypotheses of Theorem 2.1. Suppose that $\theta_1 \geq 0$. Then there exists uniquely a solution $(v_1, v_2) \in \mathcal{X} \times \mathcal{X}$ of (2.16) satisfying (2.15) and*

$$\|u_1(t) - v_1(t)\|_{E(c_1)} \leq C\varepsilon^{p+q-1}(1+t)^{-\theta} \quad \text{for} \quad t \geq 0. \tag{2.18}$$

Here θ is a positive number such that if $c_1 \neq c_2$, $\theta_1 > 0$ and $q < 2$, then $\theta = \theta_1 + \max\{\theta_2 + (p-1)(q-2), 0\}$; otherwise $\theta = \theta_1 + \theta_2$, where $\theta_2 (> 0)$ is defined by (2.14) with $j = 2$. Besides, C is a constant depending only on c_1, c_2, p and q.

If we suppose in addition that $\theta_1 > 0$, then there exists uniquely a solution $(w_1, w_2) \in \mathcal{X} \times \mathcal{X}$ of (2.17) satisfying

$$\|v_1(t) - w_1(t)\|_{E(c_1)} \leq C\varepsilon^p (1+t)^{-\theta_1} \quad \text{for} \quad t \geq 0. \tag{2.19}$$

Therefore, combining (2.18) with (2.19), we see that u_1 tends to w_1 in the energy norm as $t \to \infty$, hence (2.1) can be regarded simply as a perturbation from the free system (2.17) in this case.

Therefore, the case $\theta_1 = 0$ is of our special interest. To simplify the situation, we assume that the initial data are linear in ε. Namely,

$$f_j(r) = \varepsilon\varphi_j(r), \quad g_j(r) = \varepsilon\psi_j(r) \quad \text{for} \quad r \in \mathbf{R} \tag{2.20}$$

with $(\varphi_j, \psi_j) \in Y$ and $M_{\kappa_j}(\varphi_j, \psi_j) \leq 1$. Then we have the following.

Theorem 2.3. *Let $c_1 = c_2$, $p = 2$ and (2.10) hold. Suppose that (f_j, g_j) are as in the above and that*

$$r\psi_2(r) - (r\varphi_2(r))' \neq 0 \quad at \quad r = r_0 \tag{2.21}$$

for a positive number r_0. Then there are positive numbers C, ε_1 and t_0 such that for $0 < \varepsilon \leq \varepsilon_1$ and $t \geq t_0$ we have

$$C^{-1}\varepsilon^p (\log t) - \|u_1(0)\|_{E(c_1)} \leq \|u_1(t)\|_{E(c_1)} \leq \|u_1(0)\|_{E(c_1)} + C\varepsilon^p (\log t). \tag{2.22}$$

Under the assumptions in Theorem 2.3, it is impossible that u_1 has a free profile w_1 with $\|w_1(0)\|_{E(c_1)} < \infty$. Indeed, if not, then $\lim_{t \to \infty} \|u_1(t)\|_{E(c_1)} = \|w_1(0)\|_{E(c_1)}$. Clearly, this contradicts (2.22).

Remark. We can extend the theorems presented in this section to the case where the nonlinearity of the first equation in (2.1) is replaced by $|\partial_t u_2|^{p-1} \partial_t u_2$ or $|\nabla u_2|^p$. In addition, we can admit the linear combination of these terms as the nonlinearity in the theorems except Theorem 2.3, as well.

3. Example, II

The aim of this section is to show that the following semilinear system :

$$
\begin{cases}
\partial_t^2 u_1 - \Delta u_1 = (\partial_1 u_1)(\partial_1 u_2 - \partial_2 u_1) & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\
\partial_t^2 u_2 - \Delta u_2 = (\partial_2 u_1)(\partial_1 u_2 - \partial_2 u_1) & \text{in} \quad \mathbf{R}^3 \times (0, \infty),
\end{cases}
\tag{3.1}
$$

cannot be regarded as a perturbation from the free system (2.17). Observe that the quadratic nonlinearity is of critical order concerning the *small data global existence* and *blowup* due to [6] and that the nonlinearities in (3.1) does not satisfy the *null condition*. Therefore it seems hopeless to have a global solution for the problem. Nevertheless, Alinhac [4] introduced some algebraic condition for (1.1) including the *null condition*, and proved the global existence result for (1.1) satisfying his condition with small initial data

$$
u_j(0, x) = \varepsilon f_j(x), \quad (\partial_t u_j)(0, x) = \varepsilon g_j(x) \quad \text{for} \quad x \in \mathbf{R}^3.
\tag{3.2}
$$

The system (3.1) is nothing else an example satisfying the condition, hence the Cauchy problem (3.1) and (3.2) admits a unique global smooth solution (u_1, u_2). We underline that he suggests, without any rigorous proof, that his global solutions does not tends to any solution of the free system in general.

The key of the proof given in [4] is to introduce an auxiliary function $w = \partial_1 u_2 - \partial_2 u_1$. Then we have

$$
\partial_t^2 w - \Delta w = Q_{12}(w, u_1),
\tag{3.3}
$$

where $Q_{12}(w, u_1) = (\partial_1 w)(\partial_2 u_1) - (\partial_2 w)(\partial_1 u_1)$, which is one of the null forms. Now, using (3.3), we can rewrite the system (3.1) as

$$
\begin{cases}
\partial_t^2 u_1 - \Delta u_1 = w(\partial_1 u_1) & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\
\partial_t^2 u_2 - \Delta u_2 = w(\partial_2 u_1) & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\
\partial_t^2 w - \Delta w = Q_{12}(w, u_1) & \text{in} \quad \mathbf{R}^3 \times (0, \infty)
\end{cases}
\tag{3.4}
$$

with initial data (3.2) for $j = 1, 2$ and

$$
w(x, 0) = \varepsilon f_3(x), \quad (\partial_t w)(x, 0) = \varepsilon g_3(x) \quad \text{for} \quad x \in \mathbf{R}^3.
\tag{3.5}
$$

where

$$
f_3 = \partial_1 f_2 - \partial_2 f_1, \quad g_3 = \partial_1 g_2 - \partial_2 g_1.
\tag{3.6}
$$

Note that the system (3.4) still does not satisfy the *null condition*, because the first and second equations in (3.4) are not written in terns of the null forms. While, the third equation in (3.4) is written in terns of the null form, hence there exists uniquely a solution v_3 of (2.6) with $c = 1$ satisfying

$$\lim_{t \to \infty} \|w(t) - v_3(t)\|_{E(1)} = 0 \tag{3.7}$$

and $\|w_2(0)\|_{E(1)} < \infty$. Having this in mind, we suppose that (3.4) can be regarded as a perturbation from

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 = v_3 \left(\partial_1 v_1\right) & \text{in } \mathbf{R}^3 \times (0, \infty), \\ \partial_t^2 v_2 - \Delta v_2 = v_3 \left(\partial_2 v_1\right) & \text{in } \mathbf{R}^3 \times (0, \infty), \\ \partial_t^2 v_3 - \Delta v_3 = 0 & \text{in } \mathbf{R}^3 \times (0, \infty). \end{cases} \tag{3.8}$$

Actually we have the following result.

Theorem 3.1. *For any initial data f_1, f_2, g_1 and $g_2 \in C_0^\infty(\mathbb{R}^3)$, there exists uniquely a solution (v_1, v_2, v_3) of (3.8) satisfying (3.7) and*

$$\lim_{t \to \infty} \|u_j(t) - v_j(t)\|_{E(1)} = 0 \quad (j = 1, 2), \tag{3.9}$$

where (u_1, u_2) is the solution to the Cauchy problem (3.1) and (3.2).

Finally we state a result which shows that the asymptotic profile of (u_1, u_2) is actually different from any solutions of the free system.

Theorem 3.2. *There exist initial data f_1, f_2, g_1 and $g_2 \in C_0^\infty(\mathbb{R}^3)$ such that*

$$\lim_{t \to \infty} \|u_j(t)\|_{E(1)} = \infty \quad (j = 1, 2) \tag{3.10}$$

holds for the solution (u_1, u_2) to the Cauchy problem (3.1) and (3.2).

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AN APPLICATION OF KUBOTA-YOKOYAMA ESTIMATES TO QUASILINEAR WAVE EQUATIONS WITH CUBIC TERMS IN EXTERIOR DOMAINS

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Abstract. Small global solutions for quasilinear wave equations are considered in three space dimensions in exterior domains. The obstacles are compact with smooth boundary and the local energy near the obstacles is assumed to decay exponentially with a possible loss of regularity. The null condition is needed to show global solutions for quadratic nonlinearities.

1. **Introduction.** This is a note on the joint work with Jason Metcalfe and Christopher D. Sogge [31]. We consider the global existence of solutions for quadratic and cubic quasilinear wave equations with Dirichlet conditions exterior to compact obstacles. The obstacles are assumed to have smooth boundary and the local energy near the obstacles is needed to decay exponentially as the time tends to infinity. Strictly mentioned, the exponential decay rate is not necessary in our argument, but several polynomial decay rate are required. In this setting, Keel-Smith-Sogge have shown in [18], [19], [20], the global and almost global solutions for semilinear and quasilinear wave equations for star-shaped obstacles. Metcalfe-Sogge [34] have shown the global solutions for general obstacles including star-shaped, nontrapping, or some of trapping obstacles in terms that there is the local energy decay. They have used the null conditions to show the global solutions which put restrictions on the interaction of waves of same speeds. In [32], we have considered the global solutions for the same obstacles in [34], but the null conditions in [34] are generalized to put restrictions only on the interactions of the waves of same speeds in the wave equations of the same speeds. We have used the low energy method which appeared in Sideris-Tu [42] for boundaryless cases to treat such null conditions.

For higher dimensions, Shibata-Tsutsumi have shown in [39, 38] global solutions for general quadratic nonlinearities in du when the dimensions $n \geq 6$. Hayashi has shown in [8] global solutions in the exterior of a sphere for $n \geq 4$ for some restricted class of quadratic nonlinearities. Metcalfe-Sogge have shown in [33] global solutions for quadratic quasilinear wave equations exterior of nontrapping obstacles for $n \geq 4$.

Let us introduce our obstacles $\mathcal{K} \subset \mathbb{R}^3$ precisely. We assume that $\mathcal{K}$ is compact with smooth boundary, but not necessarily connected. By scaling, without loss of generality, we may assume

$$\mathcal{K} \subset \{ x \in \mathbb{R}^3 : |x| < 1 \}, \ 0 \in \mathcal{K} \backslash \partial\mathcal{K}.$$

The additional assumption is the exponential local energy decay with a possible loss of regularity which is described as follows. If u is a solution to

$$(1.1) \quad \begin{cases} \Box u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K} \\ u(t,\cdot)|_{\partial\mathcal{K}} = 0 \\ u(0,\cdot) = f, \ \partial_t u(0,\cdot) = g, & \text{supp } f \cup \text{supp } g \subset \{\mathbb{R}^3 \backslash \mathcal{K}, |x| \leq 4\}, \end{cases}$$

then there must be constants $c, C > 0$ so that

$$(1.2) \qquad \|u'(t, \cdot)\|_{L^2(x \in \mathbb{R}^3 \backslash \mathcal{K}, |x| \leq 4)} \leq C e^{-ct} \sum_{|\alpha| \leq 1} \|\partial_x^\alpha u'(0, \cdot)\|_2,$$

where $\square$ denotes the D'Alembertian, i.e. $\square = \partial_t^2 - \Delta$, and Δ denotes the Laplacian, i.e. $\Delta = \sum_{j=1}^n \partial_{x_j}^2$. Throughout this paper, we assume this local energy decay estimate for $\mathcal{K}$.

Remark. (1) Lax, Morawetz and Phillips have shown (1.2) without a loss of regularity, namely $|\alpha| = 0$ in the RHS, when $\mathcal{K}$ is star-shaped in [28] (see also [29, V. Theorem 3.2]).

(2) Morawetz, Ralston and Strauss have shown (1.2) without a loss of regularity ($|\alpha| = 0$) when $\mathcal{K}$ is bounded, connected and nontrapping in [36, (3.1)]. Here if the lengths of all rays in $B_1(0) \backslash \mathcal{K}$ are bounded, then waves are not trapped and (1.2) holds without a loss of regularity. They also treat the multi-dimensional cases. See Melrose [30] for further results. Ralston [37] has shown that (1.2) could not hold without a loss of regularity when there are trapped rays.

(3) Ikawa has shown (1.2) with an additional loss of regularity, namely $|\alpha| \leq \ell$ with $\ell \geq 1$ in the RHS, when $\mathcal{K}$ is some kinds of trapping obstacles. He has shown (1.2) with $\ell = 6$ when $\mathcal{K}$ consists of two disjoint strictly convex bodies in [14], and (1.2) with $\ell = 2$ when $\mathcal{K}$ consists of sufficiently separated several disjoint strictly convex bodies in [15]. Since we have the standard energy preservation

$$\|u'(t, \cdot)\|_{L^2(\mathbb{R}^3 \backslash \mathcal{K})} = \|u'(0, \cdot)\|_{L^2(\mathbb{R}^3 \backslash \mathcal{K})},$$

we can reduce the estimate (1.2) with an additional regularity, $|\alpha| \leq \ell(\geq 1)$, to the estimate for $|\alpha| \leq 1$ with different constants c and C by the interpolation. Therefore we can treat the above obstacles by the condition (1.2).

(4) We remark that we do not require exponential decay. The order e^{-ct} could be replaced with $(1 + t)^{-1-\delta-m}$ for $\delta > 0$ and $m \geq 0$, where we need $1 + \delta$ for the integral ability and m is the number of L we need in our argument.

We consider quadratic and cubic quasilinear systems of the form

$$(1.3) \qquad \begin{cases} \vec{\square}\, u = F(u, \partial u, \partial^2 u), & (t, x) \in (0, \infty) \times (\mathbb{R}^3 \backslash \mathcal{K}) \\ u(t, \cdot)|_{\partial \mathcal{K}} = 0 \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g, \end{cases}$$

where

$$\vec{\square} = (\square_{c_1}, \square_{c_2}, \dots, \square_{c_D}), \quad 0 < c_1 < \cdots < c_D \quad \square_{c_I} = \partial_t^2 - c_I^2 \Delta.$$

The nonlinear term $F = (F^I)_{1 \leq I \leq D}$ is the sum of quadratic and cubic terms which has the form

$$F^I(u, \partial u, \partial^2 u) = B^I(\partial u) + Q^I(\partial u, \partial^2 u) + R^I(u, \partial u, \partial^2 u) + P^I(u, \partial u)$$

where

$$B^I(\partial u) = \sum_{\substack{1 \leq J, K \leq D \\ 0 \leq j, k \leq 3}} A_{jk}^{IJK} \partial_j u^J \partial_k u^K, \qquad Q^I(\partial u, \partial^2 u) = \sum_{\substack{1 \leq J, K \leq D \\ 0 \leq j, k, \ell \leq 3}} B_{jk\ell}^{IJK} \partial_j u^J \partial_k \partial_\ell u^K,$$

$$R^I(u, du, d^2 u) = \sum_{\substack{1 \leq J \leq D \\ 0 \leq j, k \leq 3}} C_{jk}^{IJ}(u, \partial u) \partial_j \partial_k u^J,$$

where $C_{jk}^{IJ}(u, \partial u) = O(|u|^2 + |\partial u|^2)$, and $P(u, \partial u) = O(|u|^3 + |\partial u|^3)$.

For energy estimates, we assume symmetry conditions for quasilinear terms

$$B_{jk\ell}^{IJK} = B_{jk\ell}^{JIK} = B_{j\ell k}^{IJK}, \qquad C_{jk}^{IJ} = C_{jk}^{JI} = C_{kj}^{IJ}.$$

To obtain global existence, we also require that the equations satisfy the following null conditions which only involves the self-interactions of each wave family :

$$(1.4) \qquad \sum_{0 \le j,k \le 3} A_{jk}^{III} \xi_j \xi_k = 0 \quad \text{whenever} \quad \xi_0^2 = c_I^2(\xi_1^2 + \xi_2^2 + \xi_3^2), \quad I = 1, \ldots, D,$$

$$(1.5) \qquad \sum_{0 \le j,k,l \le 3} B_{jkl}^{III} \xi_j \xi_k \xi_l = 0 \quad \text{whenever} \quad \xi_0^2 = c_I^2(\xi_1^2 + \xi_2^2 + \xi_3^2), \quad I = 1, \ldots, D.$$

A typical example which satisfies (1.4) and (1.5) is given by

$$(1.6) \quad C((\partial_t u^I)^2 - c_I^2(\nabla u^I)^2) + \sum_{j=0}^{3} C_j \partial_j ((\partial_t u^I)^2 - c_I^2(\nabla u^I)^2)$$

$$+ \sum_{j=0}^{3} C_j' \partial_j u^I (\partial_t^2 u^I - c_I^2 \Delta u^I), \quad C, \ C_j, \ C_j' \in \mathbb{R}.$$

(See [18, (1.7), (1.10)].)

We briefly remark on the null condition for the boundaryless case in three space dimensions. John has shown in [16] that the nontrivial solution of single wave equation

$$\Box u = (\partial_t u)^2,$$

which data have the compact support, blows up in finite time. On the other hand, Christodoulou in [5] and Klainerman in [22] have shown independently the global solutions for small data when the nonlinear term satisfies the null condition. A typical example of such equation is given by

$$\Box u = a\{(\partial_t u)^2 - (\nabla u)^2\}, \quad a \in \mathbb{R}.$$

Alinhac has shown in [2] that the null condition is necessary to show the global solutions when the nonlinear term is quadratic quasilinear excluding u itself. Kovalyov pointed out in [24] that when we consider the systems of wave equations with different speeds, the situation become different and the systems tend to have global solutions for small data. A typical example which has global solutions for small data is given by

$$\begin{cases} (\partial_t^2 - c_1^2 \Delta)u_1 = a(\partial_t u_2)^2 \\ (\partial_t^2 - c_2^2 \Delta)u_2 = b(\partial_t u_1)^2 \end{cases} \quad a, b \in \mathbb{R}, \quad c_1 \ne c_2 > 0.$$

For further historical sketch, we refer to the section 6 in [25] or [26].

We refer to compatibility conditions. For the solution u of (1.3), the functions $\{\partial_t^j u(0, x)\}_{j \ge 0}$ are called compatible functions. The compatible functions are functions of spatial variables and $\partial_t^j u(0, x)$ are expressed by $\{\partial_x^\alpha f\}_{|\alpha| \le j}$ and $\{\partial_x^\alpha g\}_{|\alpha| \le j-1}$. We say that the compatibility conditions of order s are satisfied if $\partial_t^j u(0, x)|_{\partial \mathcal{K}} = 0$ for all $0 \le j \le s$ (See [18, Definition 9.2]). Additionally, we say that $(f, g) \in C^\infty$ satisfies the compatibility conditions to infinite order if the compatibility conditions are satisfied to any order $s \ge 0$.

We can now state our main result:

Theorem 1.1. *Let $\mathcal{K}$ be a fixed compact obstacle with smooth boundary that satisfies (1.2). Assume that $F(u, \partial u, \partial^2 u)$ and $\overset{\rightarrow}{\Box}$ are as above and that $(f, g) \in C^\infty(\mathbb{R}^3 \backslash \mathcal{K})$ satisfy the compatibility conditions to infinite order. Then there is a constant $\varepsilon > 0$, and an integer $N > 0$ so that if*

$$(1.7) \qquad \sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_x^\alpha f\|_2 + \sum_{|\alpha| \leq N-1} \|\langle x \rangle^{1+|\alpha|} \partial_x^\alpha g\|_2 \leq \varepsilon$$

then (1.3) has a unique solution $u \in C^\infty([0, \infty) \times \mathbb{R}^3 \backslash \mathcal{K})$.

2. Pointwise Estimates.

We consider pointwise estimates in this section.

Lemma 2.1. *Let F, f and g be any functions.*

Let u be a solution to

$$\begin{cases} (\partial_t^2 - \Delta)u(t, x) = F(t, x), & (t, x) \in [0, \infty) \times \mathbb{R}^3 \backslash \mathcal{K} \\ u(t, x)|_{x \in \partial \mathcal{K}} = 0 \\ u(0, x) = f(x), & \partial_t u(0, x) = g(x). \end{cases}$$

Then for any $M \geq 0$ and $\mu_0 \geq 0$

$$(2.1) \quad (1 + t + |x|) \sum_{\substack{|\alpha| + \mu \leq M \\ \mu \leq \mu_0}} |L^\mu Z^\alpha u(t, x)| \leq C \sum_{\substack{j + \mu + |\alpha| \leq M+8 \\ \mu \leq \mu_0 + 2, \ j \leq 1}} \|(\langle x \rangle^j \partial_{t,x}^j L^\mu Z^\alpha u)(0, x)\|_{L_x^2}$$

$$+ C \int_0^t \int_{\mathbb{R}^3 \backslash \mathcal{K}} \sum_{\substack{|\alpha| + \mu \leq M+7 \\ \mu \leq \mu_0 + 1}} |L^\mu Z^\alpha F(s, y)| \frac{dy \, ds}{|y|}$$

$$+ C \int_0^t \sum_{\substack{|\alpha| + \mu \leq M+4 \\ \mu \leq \mu_0 + 1}} \|L^\mu \partial^\alpha F(s, y)\|_{L^2(|y| < 4)} \, ds.$$

The following estimates are the special version to treat the inhomogeneity F near the light cones, which follows from the Huygens principle.

Lemma 2.2. *Let F be any function.*

Let u be a solution to

$$\begin{cases} (\partial_t^2 - c_I^2 \Delta)u(t, x) = F(t, x), & (t, x) \in [0, \infty) \times \mathbb{R}^3 \backslash \mathcal{K} \\ u(t, x)|_{x \in \partial \mathcal{K}} = 0 \\ u(t, \cdot) = 0 \ \ for \ \ t \leq 0. \end{cases}$$

Assume

$$\mathrm{supp} F \subset \{(t, x); t \geq 1 \vee \frac{6}{c_1}, \ \frac{c_1 t}{10} \leq |x| \leq 10 c_D t\}.$$

Then for any $M \geq 0$ and $\mu_0 \geq 0$

$$(2.2) \quad \sup_{|x| \leq c_1 t/2} (1 + t) \sum_{\substack{\mu + |\alpha| \leq M \\ \mu \leq \mu_0}} |L^\mu Z^\alpha u(t, x)| \leq C \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3 \backslash \mathcal{K}} \sum_{\substack{|\alpha| + \mu \leq M+7 \\ \mu \leq \mu_0 + 1}} |L^\mu Z^\alpha F(s, y)| dy$$

$$+ C \sup_{0 \leq s \leq t} (1 + s) \sum_{\substack{|\alpha| + \mu \leq M+3 \\ \mu \leq \mu_0}} \|L^\mu \partial^\alpha F(s, y)\|_{L^2(|y| < 4)}.$$

We also need the following $L^\infty - L^\infty$ estimates to treat the inhomogeneity away from the light cones, which are special (more elementary) version of Kubota-Yokoyama estimates (see Kubota-Yokoyama [27, Theorem 3.4] for the boundaryless case).

Lemma 2.3. *Let F, f and g be any functions.*

Let u be a solution to

$$
\begin{cases}
(\partial_t^2 - c_I^2 \Delta) u(t, x) = F(t, x), & (t, x) \in [0, \infty) \times \mathbb{R}^3 \backslash \mathcal{K} \\
u(t, x)|_{x \in \partial \mathcal{K}} = 0 \\
u(0, x) = f(x), & \partial_t u(0, x) = g(x).
\end{cases}
$$

Assume

$$
(2.3) \qquad \operatorname{supp} F \subset \{(t, x); 0 \le t \le 2, \ |x| \le 2\} \cup \{(t, x); |x| \le \frac{c_I t}{5} \ or \ |x| \ge 5 c_I t \ \}.
$$

Then for any $\theta > 0$, $M \ge 0$ and $\mu_0 \ge 0$, there exists a constant $C = C(\theta, M, \mu_0, \mathcal{K})$ such that

(2.4)

$$
\sup_{|x| \le c_I t/2} (1 + t) \sum_{\substack{\mu + |\alpha| \le M \\ \mu \le \mu_0}} |L^\mu Z^\alpha u(t, x)| \le C \sum_{\substack{\mu + |\alpha| \le M + 7 \\ \mu \le \mu_0 + 1, \ j \le 1}} \| (\langle x \rangle^j \partial_{t,x}^j (r \partial_r)^\mu Z^\alpha u)(0, x) \|_{L_x^2}
$$

$$
+ C \sup_{\substack{s \ge 0 \\ y \in \mathbb{R}^3 \backslash \mathcal{K}}} \langle y \rangle^{2 - \theta} (1 + s + |y|)^{1 + \theta} \sum_{\substack{|\alpha| + \mu \le M \\ \mu \le \mu_0}} |L^\mu Z^\alpha F(s, y)|
$$

$$
+ C \sup_{\substack{s \ge 0 \\ y \in \mathbb{R}^3 \backslash \mathcal{K}}} \langle y \rangle^{2 - \theta} (1 + s + |y|)^{1 + \theta} \sum_{\substack{|\alpha| + \mu \le M + 4 \\ \mu \le \mu_0}} |L^\mu \partial^\alpha F(s, y)|.
$$

Remark 2.4. *The original Kubota-Yokoyama estimates are expressed as follows (see [27, Theorem 3.4]). Let u be the solution of the problem*

$$
\begin{cases}
(\partial_t^2 - c_I^2 \Delta) u^I = F^I & in \ (0, \infty) \times \mathbb{R}^3, \ I = 1, \cdots, D, \\
u^I(0, x) = \partial_t u^I(0, x) = 0.
\end{cases}
$$

Then for any $\mu > 0$, $\nu \ge 0$ and $I = 1, \cdots, D$, the following estimate hold :

$(2.5) \quad (1 + t + |x|)(1 + ||x| - c_I t|)^\nu |u^I(t, x)|$

$$
\le C L_\nu \sup_{(s, y) \in D^I(t, x)} |y| (1 + s + |y|)^{1 + \mu + \nu} z_\mu(s, y) |F^I(s, y)|,
$$

where

$$
L_\nu \equiv 1 \ for \ \nu > 0, \qquad L_0 \equiv \left(1 + \log \frac{1 + |x| + c_I t}{1 + ||x| - c_I t|} \right)
$$

$$
D^I(t, x) \equiv \left\{ (s, y) \in \mathbb{R} \times \mathbb{R}^3 \mid 0 \le s \le t, \ ||x| - c_I(t - s)| \le |y| \le |x| + c_I(t - s) \right\}
$$

$$
\Lambda_J \equiv \left\{ (s, y) \in \mathbb{R} \times \mathbb{R}^3 \mid s \ge 1, |y| \ge 1, \ ||y| - c_J s| \le \frac{\min_{1 \le K, L \le D} |c_K - c_L|}{3} \right\}
$$

$$
z_\mu(s, y) \equiv \begin{cases}
(1 + ||y| - c_J s|)^{1 - \mu} & if \ (s, y) \in \Lambda_J, \ \exists J = 1, \cdots, D \\
(1 + |y|)^{1 - \mu} & if \ (s, y) \in ((0, \infty) \times \mathbb{R}^3) \backslash (\cup_{1 \le J \le D} \Lambda_J).
\end{cases}
$$

3. Sobolev-type Estimates.

We need the following Sobolev inequalities. Boundaryless cases are due to Klainerman-Sideris [23], Sideris [40], and Hidano-Yokoyama [10]. The following is the exterior domain analog of the boundaryless cases.

Lemma 3.1. *Let $c > 0$, $0 \le \theta \le 1/2$ be any constants.*

For any function $u \in C_0^\infty((0, \infty) \times \mathbb{R}^3 \backslash \mathcal{K})$ with the Dirichlet condition $u|_{\partial\mathcal{K}} = 0$, and any $M \ge 0$, $\mu_0 \ge 0$

$$(3.1) \quad \langle x \rangle^{1/2+\theta} \langle ct - |x| \rangle^{1-\theta} \sum_{\substack{\mu+|\alpha| \le M \\ \mu \le \mu_0}} |L^\mu Z^\alpha u'(t,x)| \le C \sum_{\substack{\mu+|\alpha| \le M+2 \\ \mu \le \mu_0+1}} \|L^\mu Z^\alpha u'(t,x)\|_{L_x^2}$$

$$+ C \sum_{\substack{\mu+|\alpha| \le M+1 \\ \mu \le \mu_0}} \|\langle t + |x| \rangle L^\mu Z^\alpha \Box_c u(t,x)\|_{L_x^2}$$

$$+ C(1+t) \sum_{\mu \le \mu_0} \|L^\mu u'(t,x)\|_{L^2(|x|<2)}.$$

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Small data scattering for a Klein-Gordon equation with a cubic convolution

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Abstract

We consider the scattering problem for the Klein-Gordon equation with cubic convolution nonlinearity. We give some estimates for the nonlinearity, and prove the existence of the scattering operator, which improves the known results in some sense. Our proof is based on the Strichartz estimates for the inhomogeneous Klein-Gordon equation.

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = F_\gamma(u) \tag{1}$$

in space-time $\mathbb{R} \times \mathbb{R}^n$, where u is a real-valued or a complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}^n$. The nonlinearity $F_\gamma(u)$ is a cubic convolution term $F_\gamma(u) = (V_\gamma * |u|^2)u$ with

$$|V_\gamma(x)| \le C|x|^{-\gamma}. \tag{2}$$

Here, $0 < \gamma < n$ and $*$ denotes the convolution in the space variables. The term $F_\gamma(u)$ is an approximative expression of the nonlocal interaction of specific elementary particles. Menzala and Strauss started to study this equation in [1].

In order to treat the scattering problem, we define the scattering operator for (1). First, we list some notation to give the definition. Let H^s be

the usual Sobolev space $(1 - \Delta)^{-s/2} L_2(\mathbb{R}^n)$ and let $H^{s,\sigma}$ be the weighted Sobolev space $(1-\Delta)^{-s/2} \langle x \rangle^{-\sigma} L_2(\mathbb{R}^n)$. A Hilbert space $X^{s,\sigma}$ is denoted by $H^{s,\sigma} \oplus H^{s-1,\sigma}$. For a positive number δ and a Banach space A, we denote the set $\{ a \in A; \|a|A\| \leq \delta \}$ by $B(\delta; A)$. Then the scattering operator is defined as the mapping $S : B(\delta; X^{s,\sigma}) \ni (f_-, g_-) \mapsto (f_+, g_+) \in X^{s,0}$ if the following condition holds for some $\delta > 0$:

For any $(f_-, g_-) \in B(\delta; X^{s,\sigma})$, there uniquely exist a time-global solution $u \in C(\mathbb{R}; H^s)$ of (1), and data $(f_+, g_+) \in X^{s,0}$ such that $u(t)$ approaches $u_\pm(t)$ in H^s as t tends to $\pm\infty$, where $u_\pm(t)$ are solutions of linear Klein-Gordon equations whose initial data are $(f_\pm, g_\pm)$, respectively.

We call that "$(S, X^{s,\sigma})$ is well-defined" if we can define the scattering operator $S : B(\delta; X^{s,\sigma}) \to X^{s,0}$ for some $\delta > 0$.

By Mochizuki [2], it is shown that if $n \geq 3$, $s \geq 1$, $\gamma < n$ and $2 \leq \gamma \leq 2s + 2$, then $(S, X^{s,0})$ is well-defined. By using the methods of Mochizuki and Motai [3] and Strauss [5], we see that if $n \geq 2$, $s \geq 1$, $4/3 < \gamma < 2$ and $\sigma > 1/3$, then $(S, X^{s,\sigma})$ is well-defined. In view of the condition of σ, there is a jump at $\gamma = 2$. Our aim of this paper is to fill the gap. By using the Strichartz estimate for pre-admissible pair (see [4]) and the complex interpolation method for the weighted Sobolev space (see, e.g., [6]), we show that $(S, X^{s,\sigma})$ is well-defined if $4/3 < \gamma < 2$ and $\sigma > (2 - \gamma)/2$, which improves the condition above.

In order to state our results, we give notation which will be used in this paper.

For $s \in \mathbb{R}$ and $(1/p, 1/q) \in [0,1] \times [0,1]$, let H_p^s be the Sobolev space $(1 - \Delta)^{-s/2} L_p(\mathbb{R}^n)$. For $s \in \mathbb{R}$, we set $E^s[u](t) = \|(u(t), \partial_t u(t)) | X^{s,0}\|$. For $s_0 \in \mathbb{R}$ and $Q = (1/q, 1/r) \in [0,1] \times [0,1]$, $L(s_0, Q)$ is denoted by $L_q(\mathbb{R}; H_r^{s_0}(\mathbb{R}^n))$. Put $\omega = \sqrt{1 - \Delta}$ and $U(t) = \exp(\pm it\omega)$. For a Banach space A, $\mathcal{B}^0(\mathbb{R}; A)$ is the set of all A-valued, continuous and bounded functions on $\mathbb{R}$. Moreover, if f in $\mathcal{B}^0(\mathbb{R}; A)$ has its derivative, and if $\partial_t f \in \mathcal{B}^0(\mathbb{R}; A)$, then we write $f \in \mathcal{B}^1(\mathbb{R}; A)$. For $s \in \mathbb{R}$, $\mathcal{H}^s$ denotes $\mathcal{B}^0(\mathbb{R}; H^s) \cap \mathcal{B}^1(\mathbb{R}; H^{s-1})$ with the norm $\|u|\mathcal{H}^s\| = \|u|L(s, (0, 1/2))\| + \|\partial_t u|L(s - 1, (0, 1/2))\|$. Furthermore, we set

$$\underline{\mathcal{H}}^s = \{ u \in \mathcal{H}^s; \text{ there exist } f, g \in \mathcal{S}(\mathbb{R}^n) \text{ such that}$$
$$u(t) = \cos t\omega f + \omega^{-1} \sin t\omega g, \omega^{-1} \partial_t u(t) \in \mathcal{H}^s \}.$$

We call $u = u(t, x)$ a free solution if $u \in \underline{\mathcal{H}}^s$ for some $s \in \mathbb{R}$. For a free

solution u_0, $u \in \acute{\mathcal{S}}(\mathbb{R}^n)$ is said to be a u_0-solution if

$$u(t) = u_0(t) + \int_0^t \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau.$$

For $s, s_0 \in \mathbb{R}$ and $Q = (1/q, 1/r) \in [0,1] \times [0,1]$, we denote $L(s_0, Q) \cap \mathcal{H}^s$ and $L(s_0, Q) \cap \underline{\mathcal{H}^s}$ by $Z(s_0, s, Q)$ and $\underline{Z}(s_0, s, Q)$, respectively. Define $1/q_\varepsilon = 1/3 - \varepsilon$ and $1/r_\theta = 1/2 - (1+\theta)/3n$. Assume that $4/3 < \gamma < 2$. Then we can easily show that there exist sufficiently small $\varepsilon(\gamma) > 0$ and $\theta(\gamma) \in (0,1)$ such that

$$\frac{1}{6} < \frac{n}{2}\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big) < \frac{1}{q_{\varepsilon(\gamma)}} < n\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big),$$

$$\gamma = 2 - 2\Big\{\frac{2}{q_{\varepsilon(\gamma)}} - n\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big)\Big\}.$$

For $Q_\gamma = (1/q_{\varepsilon(\gamma)}, 1/r_{\theta(\gamma)})$, we set

$$s(Q_\gamma) = \max\Big\{\frac{n+2}{n}\Big(1 - \frac{3}{q_{\varepsilon(\gamma)}}\Big), \frac{2-\gamma}{4}\Big\}.$$

We are now ready to state our main result.

Theorem 1. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $\sigma > (2-\gamma)/2$, $s \geq 1$ and put $s_\gamma = s(Q_\gamma)$, $Z = Z(s_\gamma + s - 1, s, Q_\gamma)$, $u_\star(t) = \cos t\omega f_\star + \omega^{-1} \sin t\omega f_\star$, where $\star$ denotes either $0, +$ or $-$. Then there exist some positive numbers η_0 and η_- satisfying the following properties:*

(i) If $(f_0, g_0) \in B(\eta_0; X^{s,\sigma})$, then there uniquely exist $u \in Z$ and $(f_+, g_+), (f_-, g_-) \in X^{s,0}$ such that u is a $u_0 - $ solution and we have

$$\lim_{t \to \pm\infty} E^s[u - u_\pm](t) = 0. \tag{3}$$

Moreover, the operators $V_\pm : B(\eta_0; X^{s,\sigma}) \ni (f_0, g_0) \mapsto (f_\pm, g_\pm) \in X^{s,0}$ are well defined, injective and continuous.

(ii) If $(f_-, g_-) \in B(\eta_-; X^{s,\sigma})$, there uniquely exist $u \in Z$ and $(f_+, g_+) \in X^{s,0}$ such that u satisfies

$$u(t) = u_-(t) + \int_t^\infty \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau \tag{4}$$

and (3) holds.
Moreover, the scattering operator $S : B(\eta_-; X^{s,\sigma}) \ni (f_-, g_-) \mapsto (f_+, g_+) \in X^{s,0}$ is well defined, injective and continuous.

Sketch of the proof.

In order to prove main theorem, we first show that there exists a unique solution in $L = L(s_\gamma + s - 1, Q_\gamma)$ if a data u_- in L is sufficiently small.

Lemma 2. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $s \geq 1$. Then there exists some $\delta > 0$ satisfying as follows: If $u_0 \in B(\delta; L)$, then there uniquely exists $u \in L$ such that we have*

$$u(t) = u_0(t) + \int_{-\infty}^t \frac{\sin(t - \tau)\omega}{\omega} F(u(\tau))d\tau, \tag{5}$$

$$\|u|L\| \leq \frac{4}{3}\|u_0|L\|, \tag{6}$$

$$\left\| \int_J U(t - \tau)F(u(\tau))d\tau | L(s - 1, (0, 1/2)) \right\| \leq \frac{1}{3}\|u_0|L\|. \tag{7}$$

From Lemma 2, we see that a mapping

$$B(\delta; L) \ni u_- \mapsto u_+ \in L$$

is well-defined. Therefore, it remains to show that

$$\|U(t)\phi|L\| \lesssim \|\phi|X^{s,\sigma}\|. \tag{8}$$

The estimate (8) is given by using the following lemma.

Lemma 3. *Assume that $n \geq 2$, $\max(0, 1/2 - 1/n) < 1/r < 1/2$ and $(n/2 - n/r)/2 < 1/q < (n/2 - n/r)$. Then we have*

$$\|U(\cdot)f|L_q L_r\| \lesssim \|f|H^{s,\sigma}\| \tag{9}$$

if

$$s > \frac{n+2}{n}\left\{\frac{2}{q} - n\left(\frac{1}{2} - \frac{1}{r}\right)\right\}$$

and

$$\sigma > \frac{2}{q} - n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Lemma 3 can be shown from the complex interpolation method for the weighted Sobolev space and the Strichartz estimate for the free Klein-Gordon equation.

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BLOW-UP RESULTS
FOR A REACTION-DIFFUSION SYSTEM

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We consider the Cauchy problem for the reaction-diffusion system:

$$(1) \quad \begin{cases} u_t - \Delta u = |x|^{\sigma_1} u^{p_1} v^{q_1}, & x \in \mathbf{R}^N, \ t > 0, \\ v_t - \Delta v = |x|^{\sigma_2} u^{p_2} v^{q_2}, & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0, & x \in \mathbf{R}^N, \\ v(x,0) = v_0(x) \geq 0, \not\equiv 0, & x \in \mathbf{R}^N, \end{cases}$$

where p_j, $q_j \geq 0$, $\sigma_j > \max\{-2, -N\}$ $(j = 1, 2)$, $p_1, q_2 \neq 1$.

Our aim is to show conditions for the nonexistence of global solutions of the system (1) in three cases $p_1, q_2 < 1$, $q_2 < 1 < p_1$, or $p_1, q_2 > 1$. The conditions are about the relation between the exponents p_j, q_j, σ_j, and the initial data.

There are some papers on the Cauchy problem for semilinear reaction-diffusion systems. In [1], Escobedo and Herrero proved the existence and nonexistence of global solutions, so-called the Fujita-type result, for $\sigma_1 = \sigma_2 = p_1 = q_2 = 0$, p_2, $q_1 \geq 1$, $p_2 q_1 > 1$. As an extension of [1], Mochizuki and Huang [3] showed the Fujita-type result for $p_1 = q_2 = 0$, $0 \leq \sigma_1 < N(p_2 - 1)$, $0 \leq \sigma_2 < N(q_1 - 1)$, p_2, $q_1 \geq 1$, $p_2 q_1 > 1$. Both of the results show that the interaction between the unknown functions in the nonlinear terms determines the behavior of solutions of the system.

In [2], Escobedo and Levine showed an interesting result for $\sigma_1 = \sigma_2 = 0$, p_1, p_2, q_1, $q_2 \geq 0$. Under the assumption that $p_2 + q_2 \geq p_1 + q_1 > 0$, they showed that if $p_1 > 1$, the solutions of the system behave like a solution of the single equation $u_t - \Delta u = u^{p_1 + q_1}$.

In fact, the same result as [2] holds in our problem, that is, if $p_1 > 1$, the solutions of the system behave like a solution of the single equation

$u_t - \Delta u = |x|^{\sigma_1} u^{p_1 + q_1}$ under the assumption that $(p_2 + q_2 - 1)/(\sigma_2 + 2) \geq (p_1 + q_1 - 1)/(\sigma_1 + 2)$.

The iteration method of [2] is often used to show blow up for reaction-diffusion systems. However, the method does not seem applicable for our problem because the nonlinear terms have the variable coefficients $|x|^{\sigma_j}$. We improve the argument in [3] and apply it to our problem. The argument in [3] is to transform the system of PDEs into the ordinary differential inequalities. In our problem, multiplying the equation by negative power of unknown function makes the transformation possible.

For simplicity, let

$$\alpha = \frac{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2)}{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}}, \quad \beta = \frac{p_2(\sigma_1 + 2) + (1 - p_1)(\sigma_2 + 2)}{2\{p_2 q_1 - (1 - p_1)(1 - q_2)\}}.$$

For $a > 0$, we define the function spaces:

$$I_a = \{w \in BC(\boldsymbol{R}^N); w(x) \geq 0, \liminf_{|x| \to \infty} |x|^a w(x) > 0\}.$$

Now, we state our main results.

Theorem 1. *Let $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$.*
(i) *If $\max(\alpha, \beta) \geq N/2$, then no nontrivial global solutions exist.*
(ii) *If $u_0 \in I_a$ $(a < 2\alpha)$ or $v_0 \in I_b$ $(b < 2\beta)$, then no global solutions exist.*
(iii) *For any $\nu > 0$, there exists large $C > 0$ such that no global solutions with $u_0(x) \geq C \exp(-\nu |x|^2)$ exist.*

Theorem 2. *Let $p_1 > 1$, $q_2 < 1$.*
(i) *If $\alpha \geq N/2$ or $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$, then no nontrivial global solutions exist.*
(ii) *If $u_0 \in I_a$ $(a < \max((\sigma_1 + 2 - Nq_1)/(p_1 - 1),$*
$-\{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2) - p_2 q_1 N\}/\{(1 - p_1)(1 - q_2)\})$, then no global solutions exist.
(iii) *For any $\nu > 0$, there exists large $C > 0$ such that no global solutions with $u_0(x) \geq C \exp(-\nu |x|^2)$ exist.*

Theorem 3. *Let $p_1 > 1$, $q_2 > 1$.*
(i) *If $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$ or $p_2 + q_2 \leq 1 + (2 + \sigma_2)/N$, then no*

nontrivial global solutions exist.

(ii) *If $u_0 \in I_a$ ($a < (\sigma_1 + 2 - Nq_1)/(p_1 - 1)$) or $v_0 \in I_b$ ($b < (\sigma_2 + 2 - Np_2)/(q_2 - 1)$), then no global solution exist.*

(iii) *For any $\nu > 0$, there exists large $C > 0$ such that no global solutions with $u_0(x) \geq C \exp(-\nu|x|^2)$ exist.*

We can also rewrite the theorems into the way in Escobedo-Levine [2].

Corollary 4. *Assume that*

$$(2) \qquad \frac{p_1 + q_1 - 1}{\sigma_1 + 2} \leq \frac{p_2 + q_2 - 1}{\sigma_2 + 2},$$

and let $p_1 < 1$, $q_2 \neq 1$.
(i) *If $\max(\alpha, \beta) \geq N/2$, then no nontrivial global solutions exist.*
(ii) *If $0 < \max(\alpha, \beta) < N/2$, then no global solutions exist for large data.*

Corollary 5. *Assume (2), and let $p_1 > 1$, $q_2 \neq 1$.*
(i) *If $p_1 + q_1 \leq 1 + (2 + \sigma_1)/N$, then no nontrivial global solutions exist.*
(ii) *If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then no global solutions exist for large data.*

Outline of the proof. We give the outline of the proof of Theorem 1 (i). Let $\alpha \geq N/2$, and assume that (u, v) are global solutions for (1). Since $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$, we can take a positive constant $k > 0$ such that $(1 - q_2)/p_2 < k < q_1/(1 - p_1)$. For this constant k, we fix positive constants r_1, $r_2 > 0$ satisfying

$$r_2 = kr_1,$$

$$r_1 < \min\{1 - p_1, \ p_2\},$$

$$r_2 < \min\{1 - q_2, \ q_1\},$$

$$r_1 \sigma_1 < \frac{N(q_1 - k(1 - p_1))}{k},$$

$$r_2 \sigma_2 < \frac{N(kp_2 - (1 - q_2))}{k}.$$

For $\varepsilon > 0$, define the cut off function

$$\rho_\varepsilon(x) = \begin{cases} \varepsilon^{\frac{N}{2}} \exp\left(-\dfrac{1}{1 - \varepsilon|x|^2}\right) & (|x| < \varepsilon^{-\frac{1}{2}}), \\ 0 & (|x| \geq \varepsilon^{-\frac{1}{2}}), \end{cases}$$

and set

$$F_\varepsilon(t) = \int_{\mathbf{R}^N} u(x,t)^{r_1} \rho_\varepsilon(x) dx,$$

$$G_\varepsilon(t) = \int_{\mathbf{R}^N} v(x,t)^{r_2} \rho_\varepsilon(x) dx.$$

Then we have

$$F_\varepsilon'(t) \geq -C_1 \varepsilon F_\varepsilon(t) + C_2 \varepsilon^{-\frac{\sigma_1}{2}} F_\varepsilon(t)^{-\frac{(1-p_1)-r_1}{r_1}} G_\varepsilon(t)^{\frac{q_1}{r_2}},$$

$$G_\varepsilon'(t) \geq -C_3 \varepsilon G_\varepsilon(t) + C_4 \varepsilon^{-\frac{\sigma_2}{2}} F_\varepsilon(t)^{\frac{p_2}{r_1}} G_\varepsilon(t)^{-\frac{(1-q_2)-r_2}{r_2}},$$

where $C_j > 0$ ($j = 1, 2, 3, 4$). Changing variables and using the phase field argument in [3], we obtain the upper bounds of $F_\varepsilon(t)$ and $G_\varepsilon(t)$:

(3)
$$F_\varepsilon(t) \leq A\varepsilon^{\alpha r_1},$$

$$G_\varepsilon(t) \leq B\varepsilon^{\beta r_2},$$

where $A, B > 0$.

Next, we show lower bound of $F_\varepsilon(t)$. We introduce the system of integral equations associated to (1):

$$u(t) = S(t)u_0 + \int_0^t S(t-s)|\cdot|^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds,$$

$$v(t) = S(t)v_0 + \int_0^t S(t-s)|\cdot|^{\sigma_2} u(s)^{p_2} v(s)^{q_2} ds,$$

where

$$S(t)f(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy.$$

Then we get

$$u(x,t) \geq \begin{cases} C_5(1+t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2t}\right), & (\alpha > \frac{N}{2}), \\ C_6(1+t)^{-\frac{N}{2}} \log(1+t) \exp\left(-\frac{C_7|x|^2}{t}\right), & (\alpha = \frac{N}{2}), \end{cases}$$

where $C_j > 0$ $(j = 5, 6, 7)$ and $t_1 > 0$. From the definition of $F_\varepsilon(t)$, we have the lower bound

$$(4) \qquad F_\varepsilon(\varepsilon^{-1}) \geq \begin{cases} C_8 \varepsilon^{\frac{Nr}{2}}, & (\alpha > \frac{N}{2}), \\ C_9 \varepsilon^{\frac{Nr}{2}} \log(1 + \varepsilon^{-1}), & (\alpha = \frac{N}{2}), \end{cases}$$

where $C_j > 0$ $(j = 8, 9)$. This contradicts the upper bound of F_ε for small $\varepsilon > 0$.

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Approximation of the Gauss curvature flow by a three-dimensional crystalline motion *

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In this paper, we consider an approximation of the Gauss curvature flow in $\mathbb{R}^3$ by so-called crystalline algorithm.

1   The Gauss curvature flow

The Gauss curvature flow in $\mathbb{R}^d$ makes a *smooth strictly convex hypersurface* shrink with the outward normal velocity equals to the Gauss curvature with negative sign. Let us explain more precisely. Let $\{\Gamma(t)\}$ be a family of smooth strictly convex closed hypersurfaces, $\kappa_1 = \kappa_1(P,t), \kappa_2 = \kappa_2(P,t), \ldots, \kappa_{d-1} = \kappa_{d-1}(P,t)$ the principal curvatures of $\Gamma(t)$ at P on $\Gamma(t)$ where we use the sign convention that the all principal curvatures of the hypersurfaces are positive, and $\kappa = \kappa(P,t) = \kappa_1\kappa_2\cdots\kappa_{d-1}$ the Gauss curvature of $\Gamma(t)$ at P. We call $\Gamma(t)$ the solution of the Gauss curvature flow if and only if at every points P on $\Gamma(t)$, the relation

$$v(P,t) = -\kappa(P,t)$$

is satisfied, where $v = v(P,t)$ denotes the outward normal velocity of $\Gamma(t)$ at P.

To describe the Gauss curvature flow, we use the support function $h(\nu, t)$ of the convex hypersurface $\Gamma(t)$ which is defined by

$$h(\nu, t) = \sup\{\langle P, \nu \rangle \mid P \in \Gamma(t)\},$$

where $\nu \in S^{d-1}$ is a unit vector and $\langle \ , \ \rangle$ denotes the usual inner product in $\mathbb{R}^d$. An intuitive meaning of the support function is the signed distance from the origin to the tangent hyperplane of $\Gamma(t)$ at the point where the unit outward normal vector is ν. Using the support function, the Gauss curvature flow can be described as

$$\frac{\partial h}{\partial t}(\nu, t) = -\kappa(P, t),$$

(1)

where $P \in \Gamma(t)$ is a point where $\langle P, \nu \rangle = h(\nu, t)$. We note that for all $T_* \in \mathbb{R}$

$$(d(T_* - t))^{1/d} S^{d-1}$$

is a self-similar shrinking solution of (1) for $t \in (-\infty, T_*)$.

We shall consider the evolution of $\Gamma(t)$ by (1) which starts from the initial hypersurface Γ_0. We set $\Omega(t)$ being the open set enclosed by $\Gamma(t)$. The existence of its solution until single point extinction was proved in [5] and [20]:

Theorem 1 *If the initial hypersurface Γ_0 is smooth and strictly convex, then there exists a unique solution $\Gamma(t)$ to the Gauss curvature flow, which stays smooth and strictly convex. Moreover, the solution converges to a point within a finite time, say T_0, and this extinction time T_0 is given by $T_0 = V(\Gamma_0 \cup \Omega_0)/(dV(B^d))$. Here, V denotes the Lebesgue measure on $\mathbb{R}^d$ and B^d the unit ball $\{x \in R^d \mid |x| \leq 1\}$.*

2   Crystalline Motion

The main object of this paper is so-called crystalline motion. This motion was introduced by Taylor [18] and Angenent & Gurtin [2] to analyze crystal growth mathematically. The most typical crystalline motion in $\mathbb{R}^2$ makes each edge of a polygon keep the same direction but move with *the normal speed inversely proportional to its length*. Several papers, e.g. [7], [8], [10], [11], [13], and [21], have shown the convergence of two-dimensional crystalline motions to curve shortening flows in the plane as the number of the edges goes to infinity. We particularly note that the results in [9] and [15] have given the convergence for general curves which are *not necessarily convex*. See [1] for the behaviour of convex polygons under crystalline motions in $\mathbb{R}^2$.

As for the higher dimensional case, while *a motion of a surface by the crystalline mean curvature* was proposed also in [18], Bellettini, Novaga, and Paolini [3, 4] pointed out that the comparison principle is not valid in general and so it might not be natural to assume that all points on each side move with the same normal velocity (some people called it *the facets stay facets ansatz*).

In this paper, we introduce a three-dimensional crystalline motion for *convex polyhedra* and show its convergence to Gauss curvature flow in $\mathbb{R}^3$. As for the different way of approximation to the Gauss curvature flow, we mention [12, 14]. Our crystalline motion in $\mathbb{R}^3$ makes each side of a polyhedron move with *the normal speed inversely proportional to its area*. This motion is a three-dimensional version of the most typical two-dimensional one, which was introduced in [18]. Our motion should be said as *a motion of a surface by the crystalline Gauss curvature* and we find out that the comparison principle is available for this motion (see Lemma 3).

The precise definition of our crystalline motion is as follows:

Let $\tilde{W}$, which represents the anisotropy of the problem and is called the Wulff shape, be an N-sided convex polyhedron in $\mathbb{R}^3$ including the origin as its interior point. We also call $\tilde{W}$ the Wulff polyhedron to emphasize that the Wulff shape is a polyhedron. Since $\tilde{W}$ is an N-sided convex polyhedron in $\mathbb{R}^3$, there exist N unit vectors $\nu_1, \nu_2, \ldots, \nu_N \in S^2$ such that

$$\tilde{W} = \bigcap_{i=1}^{N} \{P \in \mathbb{R}^3 \mid \langle P, \nu_i \rangle \le \tilde{h}_i\}, \quad \tilde{h}_i = \sup\{\langle P, \nu_i \rangle \mid P \in \partial \tilde{W}\}.$$

We call the set $\tilde{\Gamma}_i = \tilde{W} \cap \{P \in \mathbb{R}^3 \mid \langle P, \nu_i \rangle = \tilde{h}_i\}$ i-th side of $\tilde{W}$ and $\tilde{h}_i$ the height from the origin of $\tilde{\Gamma}_i$. We set $\tilde{h} = \tilde{h} = (\tilde{h}_i)_{i=1,2,\ldots,N} \in \mathbb{R}^N$. We note that the unit outward normal vector on $\tilde{\Gamma}_i$ is ν_i and the support function of $\partial \tilde{W}$ coincide with $\tilde{h}_i$ at $\nu = \nu_i$. Let $\tilde{A}_i = A(\tilde{\Gamma}_i)$ be the area of $\tilde{\Gamma}_i$.

We call that an N-sided convex polyhedron and its boundary Γ are a $\tilde{W}$-*admissible polyhedron* and a $\tilde{W}$-*admissible surface*, respectively, if and only if the outward normal vector of the i-th side, say Γ_i, of Γ is ν_i for all i. For a $\tilde{W}$-admissible surface Γ, the height from the origin $h = (h_i)_{i=1,2,\ldots,N} \in \mathbb{R}^N$ is defined by $h_i = \sup\{\langle P, \nu_i \rangle \mid P \in \Gamma\}$ and $A_i = A(\Gamma_i)$ denotes the area of Γ_i. Clearly, $\tilde{W}$ is a $\tilde{W}$-admissible polyhedron and so $\partial \tilde{W}$ is a $\tilde{W}$-admissible surface.

Then, a crystalline motion of a $\tilde{W}$-admissible surface $\Gamma(t)$ is defined by

the system of ordinary differential equations

$$(2) \qquad \frac{dh_i(t)}{dt} = -\tilde{h}_i \frac{\tilde{A}_i}{A_i(t)}, \quad 1 \leq i \leq N.$$

We call this flow *the $\tilde{W}$-crystalline flow* and a family $\{\Gamma(t)\}$ of $\tilde{W}$-admissible surfaces which satisfies (2) a solution to the $\tilde{W}$-crystalline flow. The quantity $\frac{\tilde{A}_i}{A_i(t)}$ might be regarded as the crystalline Gauss curvature of the i-th side of $\Gamma(t)$. We note that for all $T_* \in \mathbb{R}$

$$(3(T_* - t))^{1/3}\partial\tilde{W}$$

is a self-similar shrinking solution of (2) for $t \in (-\infty, T_*)$. This self-similar solution will be used in the comparison argument of the proof of our main result below.

We can prove the well-posedness of this flow by the classical theorem of the existence and the uniqueness of the solution of ordinary differential equations.

Theorem 2 *Let $\tilde{W}$ be a convex polyhedron in $\mathbb{R}^3$ including the origin as its interior point, and Γ_0 a $\tilde{W}$-admissible surface. Then, there exists a unique solution $\Gamma(t)$ to $\tilde{W}$-crystalline flow with $\Gamma(0) = \Gamma_0$. Moreover, the enclosed volume vanishes at the maximal existence time $T = V(\Gamma_0 \cup \Omega_0)/(3V(\tilde{W})) \in (0, +\infty)$. Here, Ω_0 is the open set enclosed by Γ_0 and V denotes the three-dimensional volume.*

We also note that the comparison lemma holds for the $\tilde{W}$-crystalline flow.

Lemma 3 *Let $\tilde{W}$ be a convex polyhedron in $\mathbb{R}^3$ including the origin as its interior point, and $\Gamma'(t)$ and $\Gamma(t)$ solutions to $\tilde{W}$-crystalline flow for $t \in [0, T)$. Then, $\Gamma'(0) \subset \Gamma(0) \cup \Omega(0)$ implies $\Gamma'(t) \subset \Gamma(t) \cup \Omega(t)$ for all $t \in [0, T)$. Here, $\Omega(t)$ is the open set enclosed by $\Gamma(t)$.*

3 Mainr result

Now let us consider a sequence of convex polyhedra $\tilde{W}^k$ and that of the $\tilde{W}^k$-crystalline flows. Here and hereafter, the parameter $k \in \mathbb{N}$ indicates the accuracy of the approximation and the larger integer k corresponds to the

better approximation. We note that, for example, N^k in (A1) below does not mean k-th power of N. Our main purpose is to show that this sequence of crystalline flows converges to the Gauss curvature flow under the assumptions below. First we assume that

(A1)   the Wulff polyhedron $\tilde{W}^k$ has N^k-sides
        and is symmetric with respect to the origin

and the sequence of the Wulff shapes $\{\tilde{W}^k\}$ converges to the unit ball $B^3 = \{P \in \mathbb{R}^3 \mid |P| \le 1\}$ in the Hausdorff distance, namely,

(A2) $$\lim_{k \to \infty} d_H(\tilde{W}^k, B^3) = 0.$$

Here $d_H(A_1, A_2)$ is the Hausdorff distance between sets A_1 and A_2. We use the convention of $d_H(\emptyset, \emptyset) = 0$ and $d_H(\emptyset, A) = d_H(A, \emptyset) = +\infty$ provided $A \ne \emptyset$. Second we assume that

(A3)   the initial surface Γ_0^k is a $\tilde{W}^k$-admissible surface

and it converges to a smooth and strictly convex surface Γ_0:

(A4) $$\lim_{k \to \infty} d_H(\Gamma_0^k, \Gamma_0) = 0.$$

Let $\Gamma(t)$ and T_0 be the solution of (1) which starts from the smooth strictly convex surface Γ_0 and its extinction time, respectively. We set $\Gamma(T_0) = \lim_{t \uparrow T_0} \Gamma(t)$ and $\Gamma(t) = \emptyset$ for $t > T_0$. Let $\Gamma^k(t)$ and T^k be the solution of (2) with $\tilde{W} = \tilde{W}^k$ (namely, solution to the $\tilde{W}^k$-crystalline flow) which starts from Γ_0^k and its extinction time, respectively. We set $\Gamma^k(T^k) = \lim_{t \uparrow T^k} \Gamma^k(t)$ and $\Gamma^k(t) = \emptyset$ for $t > T^k$. We also set $\Omega(t) = \emptyset$ for $t \ge T_0$ and $\Omega^k(t) = \emptyset$ for $t \ge T^k$.

Now our main result is the next theorem:

Theorem 4 *Assume (A1), (A2), (A3), and (A4). Then the solution $\Gamma^k(t)$ to the $\tilde{W}^k$-crystalline flow with the initial surface $\Gamma^k(0) = \Gamma_0^k$ converges to the solution $\Gamma(t)$ to (1) with the initial surface $\Gamma(0) = \Gamma_0$ locally uniformly in $t \in [0, T_0)$:*
$$\lim_{k \to \infty} \sup_{0 \le s \le t} d_H(\Gamma^k(s), \Gamma(s)) = 0.$$

Here T_0 is the extinction time of $\Gamma(t)$.

4 Outline of the proof

In this section we explain the outline of the proof of Theorem 4.

We recall the result of K. Ishii and H. M. Soner [15]. They were concerned with the two-dimensional crystalline motion whose Wulff shape is a regular polygon centered at the origin, and showed its convergence to the curve shortening flow as the Wulff polygon tends to the unit disc. Their method, which is a kind of perturbed test function method, works to prove our theorem. In their case, they used a disc as a test function for the solution to the curve shortening flow, and then chose a suitable dilation of the Wulff polygon approximating the disc as one for the solution to the crystalline motion. In our case, however, a surface has two principal curvatures at each point. Therefore, we need to use an ellipsoid as a test function to the Gauss curvature flow, and then choose a $\tilde{W}^k$-admissible polyhedron approximating the ellipsoid in some nice sense. Seeking such a nice polyhedron would be just a Minkowski problem (see Lemma 9), since this problem concerns the existence, uniqueness, and stability of convex surfaces with preassigned Gauss curvature as a function of the outer normal (e.g. [16]). As for the perturbed test function method, we refer [6]. To our knowledge the first successful applications of this method to viscosity solutions appeared in this paper.

Throughout this section, we assume (A1), (A2), (A3), and (A4).

For $k \in \mathbb{N}$, let $\{\Gamma^k(t)\}_{t \geq 0}$ be the solution of the $\tilde{W}^k$-crystalline flow and let $\Omega^k(t)$ be the open set enclosed by $\Gamma^k(t)$. For $t \geq 0$, we define semicontinuous envelopes

$$\hat{\Omega}(t) = \bigcap_{\varepsilon > 0, N \in \mathbb{N}} \mathrm{cl}\left(\bigcup_{|s-t| \leq \varepsilon, s \geq 0, \, k \geq N} \left(\Gamma^k(s) \cup \Omega^k(s)\right)\right),$$

$$\underline{\Omega}(t) = \bigcup_{\varepsilon > 0, N \in \mathbb{N}} \mathrm{int}\left(\bigcap_{|s-t| \leq \varepsilon, s \geq 0, \, k \geq N} \Omega^k(s)\right).$$

Here, for a set A, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ mean the closure of A and the interior of A, respectively. In [17], the properties of the sets like $\hat{\Omega}(t)$ and $\underline{\Omega}(t)$ are noted. Let $\Gamma(t)$ be the solution to the equation (1) and $\Omega(t)$ the open set which is enclosed by $\Gamma(t)$.

We note that because of $\dot{h}_i < 0$

$$\Omega(t_2) \subset \Omega(t_1), \quad \underline{\Omega}(t_2) \subset \underline{\Omega}(t_1) \quad \text{and} \quad \hat{\Omega}(t_2) \subset \hat{\Omega}(t_1) \quad \text{for} \quad t_2 \geq t_1 \geq 0$$

hold. We set

$$\hat{T} = \sup\{t \,|\, \hat{\Omega}(t) \neq \emptyset\}, \quad \underline{T} = \sup\{t \,|\, \underline{\Omega}(t) \neq \emptyset\}, \quad \text{and} \quad T_0 = \sup\{t \,|\, \Omega(t) \neq \emptyset\}.$$

By the definition of $\hat{\Omega}(t)$ and $\underline{\Omega}(t)$, we have

$$(3) \qquad\qquad\qquad \hat{T} \geq \underline{T} \text{ and } \mathrm{cl}(\underline{\Omega}(t)) \subset \hat{\Omega}(t).$$

If we prove that

$$(4) \qquad\quad \hat{\Omega}(t) \subset \mathrm{cl}(\Omega(t)) \text{ and } \Omega(t) \subset \underline{\Omega}(t) \text{ for all } t \in [0, T_0),$$

which is the result of Lemma 8 below, then we obtain the convergence result.

We show these inclusions (4) by the following steps. First we show that $\hat{\Omega}(t)$ and $\underline{\Omega}(t)$ are are sub and super-solutions, respectively, of (1) in visocosity sense (Lemma 5 and Lemma 6). Second comparing initial states, we show the inclusions among $\hat{\Omega}(0)$, $\underline{\Omega}(0)$, and Ω_0 (Lemma 7). Finally, from the Lemmas 5,6, and 7, we obtain the desired inclusions (Lemma 8). The second part is not difficult. The final part is a rather standard argument. In the fisrt part, we need a help from the theory of Minkowski problem (Lemma 9).

Lemma 5 *Let $(P_0, t_0) \in \mathbb{R}^3 \times (0, +\infty)$ and $\{O(t)\}_{t \in (0, t_0]}$ be a family of closed sets with smoothly evolving and strictly convex boundaries. If $P_0 \in \partial\hat{\Omega}(t_0) \cap \partial O(t_0)$ and $\hat{\Omega}(t) \subset O(t)$ for all $t \in (0, t_0]$, then the inequality*

$$(5) \qquad\qquad\qquad V_O(P_0, t_0) \leq -\kappa_O(P_0, t_0)$$

holds. Here, $V_O(P_0, t_0)$ and $\kappa_O(P_0, t_0)$ is the normal velocity and the Gauss curvature of $\partial O(t_0)$ at P_0, respectively.

Lemma 6 *Let $(P_0, t_0) \in \mathbb{R}^3 \times (0, +\infty)$ and $\{O(t)\}_{t \in (0, t_0]}$ be a family of closed sets with smoothly evolving and strictly convex boundaries. If $P_0 \in \partial\underline{\Omega}(t_0) \cap \partial O(t_0)$ and $\mathrm{int}(O(t)) \subset \underline{\Omega}(t)$ for all $t \in (0, t_0]$, then the inequality*

$$(6) \qquad\qquad\qquad V_O(P_0, t_0) \geq -\kappa_O(P_0, t_0)$$

holds. Here, $V_O(P_0, t_0)$ and $\kappa_O(P_0, t_0)$ is the normal velocity and the Gauss curvature of $\partial O(t_0)$ at P_0, respectively.

Lemma 7 $\hat{\Omega}(0) \subset \mathrm{cl}(\Omega_0)$ *and* $\Omega_0 \subset \underline{\Omega}(0)$ *hold.*

Lemma 8 $\underline{T} = \hat{T} = T_0$. $\hat{\Omega}(t) \subset \mathrm{cl}(\Omega(t))$ *and* $\Omega(t) \subset \underline{\Omega}(t)$ *hold for all* $t \in [0, T_0)$.

For positive numbers a and b, we set

$$E = E(a, b) = \{(x, y, z) \mid ax^2 + by^2 + z^2 \leq 1\}.$$

For this ellipsoid E we have the following lemma.

Lemma 9 *Let E be the ellipsoid defined as above. For any $k \in \mathbb{N}$, there uniquely exists a $\tilde{W}^k$-admissible polyhedron E^k symmetric with respect to the origin such that*

$$(7) \qquad\qquad \kappa^E(\nu_i^k) = \frac{\tilde{A}_i^k}{A_i^{E^k}}$$

holds for all $1 \leq i \leq N^k$. *Moreover,*

$$(8) \qquad\qquad \lim_{k \to \infty} d_H(E^k, E) = 0$$

holds. Here, ν_i^k denotes the outward normal vector of the i-th side of $\tilde{W}^k$, $\kappa^E(\nu)$ Gauss curvature of E at the point where the outward normal vector is ν, $\tilde{A}_i^k$ the area of the i-th side of $\tilde{W}^k$, $A_i^{E^k}$ the area of the i-th side of E^k, respectively.

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Some Variational Methods for Studying Almost Periodic Differential Equations

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Variational method is a useful tool to study nonlinear differential equations of the form

$$- \triangle u + F(x, u) = 0, x \in \Omega. \qquad (1)$$

As heterogeneities occur in every natural environment, Eq.(1) has been extensively studied in case F is periodic in each component of x. The interested reader may consult [CR] for studying standing pulses of nonlinear Schrodinger equations and [AJM1-2, RS1-2] for studying connecting orbits of Allen-Cahn type equation.

A natural extension of periodic function is almost periodic function. Such a class of functions can be defined as follows:

Definition. (i) A set P is called relatively dense in $\mathbb{R}$ if there exists a positive number ρ such that every interval of length ρ contains at least one element of P. (ii) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Given $\varepsilon > 0$, a positive number τ is called an ε-period of g if

$$\sup_{\theta \in \mathbb{R}} |g(\theta + \tau) - g(\theta)| \leq \varepsilon.$$

(iii) g is almost periodic if for every $\varepsilon > 0$ there exists a relatively dense set of ε-periods of g.

Clearly, periodic functions are trivial examples of almost periodic function.

A simple example of almost periodic function is $\cos\theta + \cos\sqrt{2}\theta$. Such a function is often called quasi-periodic function, which belongs to a subclass of almost periodic functions. We refer to [F] for more detailed properties of almost periodic function.

To study the spatially heterogeneous effect, we take a cylindric-shaped domain defined by $\Omega = \{(x_1, \hat{x})| |\hat{x}| < g(x_1), x_1 \in \mathbb{R}\}$, where g is a positive almost periodic function and $\hat{x} \in \mathbb{R}^{N-1}$. In order to avoid complicated notation involved in illustration, our attention will be mainly restricted to

$$J(u) = \int_\Omega [\frac{1}{2}(|\nabla u|^2 + u^2) - \int_0^u f(\xi)d\xi]dx. \tag{2}$$

Each critical point of J on $H^1(\Omega)$ is a solution of

(P) $$\triangle u + u - f(u) = 0$$

under the homogeneous Neumann boundary conditions. Such a solution can be viewed as a pulse in a channel.

A simple but interesting example of $f(\xi)$ is $|\xi|^{p-1}\xi$. For $p \in (1, \frac{N+2}{N-2})$, we know from Sobolev inequality that

$$\|v\|_{L^{p+1}} \le C\|v\|_{H^1}.$$

For an unbounded domain Ω, whether the minimization problem

$$\inf_{v \in H^1(\Omega)} \frac{\|v\|_{L^{p+1}(\Omega)}}{\|v\|_{H^1(\Omega)}} \tag{3}$$

can actually be attained by an element of $H^1(\Omega)$ in general is not known. The answer of this question is closely related to the critical points of J. Observed that $u \equiv 0$ is a local minimum of J. If the Palais-Smale condition holds, the Mountain Pass Lemma [AR, R1] would imply a critical value β defined by

$$\beta = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) | \gamma(0) = 0$ and $J(\gamma(1)) < 0\}$. It is known [CT1,CR] that in case g is periodic the Palais-Smale condition does not hold. Nevertheless, there exists a Palais-Smale sequence which converges to a solution u_0 of (P) at critical level β. Furthermore, it is easy to see that a rescaling of this function u_0 assumes the minimum of (3). We note that, for any domain Ω, if u_0 is a critical point associated with the mountain pass mini-max value, the same way of rescaling transfers a solution u_0 to a minimizer of (3).

If Ω is enclosed by a almost periodic function g, the situation become more delicate. By Bochner's criterion [B, F], g is almost periodic if and only if the set of its translates $\{g(\cdot + s) | s \in \mathbb{R}\}$ is precompact in the space of continuous bounded functions on the real line, endowed with L^∞-norm. Set $H(g) = \{h |$ there exists a sequence of translates of g which converges to h uniformly in $\mathbb{R}\}$. For general almost periodic functions, there are functions in $H(g)$ which are not translates of g. As an example, if $g(\theta) = \cos\theta + \cos\sqrt{2}\theta$ there is a sequence $\{\theta_m\}$ such that $\lim_{m\to\infty} g(\theta_m) = -2$, however $g(\theta) > -2$ for all $\theta \in \mathbb{R}$. This phenome non reflects the theory of almost periodic differential equations interesting and at the same time difficult.

It is assumed that f satisfies the following conditions:

(f1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $\lim_{\xi \to 0} \frac{f(\xi)}{\xi} = 0$.

(f2) There is a constant c_1 such that $|f'(\xi)| \le c_1(1 + |\xi|^{p-1})$ where $1 < p < \frac{N+2}{N-2}$ if $N > 2$ and $1 < p < \infty$ if $N = 2$.

(f3) There is a $\mu > 0$ such that $0 < (\mu + 2) \int_0^\xi f(y)dy \le \xi f(\xi)$ if $\xi \ne 0$.

Theorem 1. Suppose (f1)-(f3) are satisfied, there exist infinite number of solutions to (P). Furthermore, if $\frac{f(\xi)}{\xi}$ is an increasing function of ξ for $\xi \in (0, \infty)$, then for any $\varepsilon > 0$ there exists a solution with critical level inside the interval $[\beta, \beta + \varepsilon]$.

If g is periodic, Theorem 1 is nothing new, since translates of a solution by multiple periods are also solutions of (P). In case g is almost periodic but not periodic, our method provides a way to stress out infinitely many solutions distingulished by the locations of the pulses. Theorem 1 also indicates that for some almost periodic (but not periodic) function g, the minimization problem (3) possesses a minimizer.

The proof of Theorem 1 is based on the investigation of Palais-Smale sequences generated by negative pseudo-gradient flow. Detailed analysis can be found in [CT2]. The same methods can be used to deal with a class of elliptic equations treated in [S1-2].

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