Boundary blowup solutions to curvature equations

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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. We consider the so-called curvature equations of the form

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = f(u)g(|Du|) \quad \text{in } \Omega,$$

with the following boundary condition

$$u(x) \to \infty \quad \text{as } \text{dist}(x, \partial \Omega) \to 0.$$

Here, for a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of the graph of the function $u$, and $S_k, k = 1, \ldots, n$, denotes the $k$-th elementary symmetric function, i.e.,

$$S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k},$$

where the sum is taken over increasing $k$-tuples, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We study the existence and the asymptotic behavior near $\partial \Omega$ of a solution to (1)-(2).

The family of equations (1), $k = 1, \ldots, n$ contains some well-known and important equations.

The case $k = 1$ corresponds to the mean curvature equation;

The case $k = 2$ corresponds to the scalar curvature equation;

The case $k = n$ corresponds to Gauss curvature equation.

We remark that (1) is a quasilinear equation for $k = 1$ while it is a fully non-linear equation for $k \geq 2$. In the particular case that $k = n$, it is an equation of Monge-Ampère type. It is much harder to analyze fully non-linear equations, but the study of the classical Dirichlet problem for curvature equations in the case that $2 \leq k \leq n - 1$ has been developed in the last two decades, see for instance [4, 11, 24].

The condition (2) is called the “boundary blowup condition,” and a solution which satisfies (2) is called a “boundary blowup solution,” a “large solution,” or an “explosive solution.” The boundary blowup problems arise from physics, geometry and many branches of mathematics, see for instance

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The study of such problems for non-linear PDEs starts from the pioneering work of Bieberbach [3] and Rademacher [21] who considered \( \Delta u = e^u \) in two and three dimensional domain respectively. For the case of semilinear equations, they have been extensively studied (see, for example, [13, 20] and [2, 6, 15, 16, 17, 19]). The case of quasilinear equations of divergence type to which the mean curvature equation \((k = 1)\) belongs has been treated in [1, 7, 9]. The case of Monge-Ampère equations has been studied in [5, 10, 18]. However, to the best of our knowledge, there are no results concerning such problems for other fully non-linear PDEs, except for the work of Salani [22] who considered the case of Hessian equations.

Throughout the article, we assume the following conditions on \( f \) and \( g \):

- Let \( t_0 \in [-\infty, \infty) \). \( f \in C^\infty(t_0, \infty) \) is a positive function and satisfies \( f'(t) \geq 0 \) for all \( t \in (t_0, \infty) \).
- If \( t_0 > -\infty \), then \( f(t) \to 0 \) as \( t \to t_0 + 0 \); otherwise (i.e., if \( t_0 = -\infty \)),
  \[
  \int_{-\infty}^{t} f(s) \, ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{4}
  \]
- \( g \in C^\infty[0, \infty) \) is a positive function.

The first condition assures us that the comparison principle for solutions to (1) holds. The typical examples of \( f \) are \( f(t) = t^p \) \((p > 0)\), \( t_0 = 0 \) and \( f(t) = e^t \), \( t_0 = -\infty \).

In the subsequent two sections, we state our main results.

### 2 Existence results

We recall the notion of \( k \)-convexity. Let \( \Omega \subset \mathbb{R}^n \) be a domain with boundary \( \partial \Omega \in C^2 \). For \( k = 1, \ldots, n-1 \), we say that \( \Omega \) is \( k \)-convex (resp. uniformly \( k \)-convex) if the vector of the principal curvatures of \( \partial \Omega \), \( \kappa' = (\kappa'_1, \ldots, \kappa'_{n-1}) \), satisfies \( S_j(\kappa') \geq 0 \) (resp. \( > 0 \)) for \( j = 1, \ldots, k \) and for every \( x \in \partial \Omega \). We note that a \( C^2 \) domain is \((n - 1)\)-convex (resp. uniformly \((n - 1)\)-convex) if and only if it is convex (resp. strictly convex).

First, we shall establish the existence of a boundary blowup solution to the curvature equation (1). We focus on the case \( k \geq 2 \), because for \( k = 1 \) the existence has been already studied in [9].

**Theorem 1.** Let \( 2 \leq k \leq n-1 \). We suppose that \( \Omega, f \) and \( g \) satisfy the following conditions.

(A1) \( \Omega \) is a bounded and uniformly \( k \)-convex \( C^\infty \) domain.

(A2) There exists a constant \( T > 0 \) such that \( g \) is non-increasing in \([T, \infty)\), and \( \lim_{t \to \infty} g(t) = 0 \).

(A3) If \( f \) is a positive function in \( (t_0, \infty) \), then \( f'(t) \geq 0 \) for all \( t \in (t_0, \infty) \).

(A4) If \( f \) is a positive function in \( (-\infty, t_1) \), then \( f(t) \to 0 \) as \( t \to t_1 + 0 \), and \( f(t) \to \infty \) as \( t \to -\infty \).

(A5) If \( g \) is a positive function in \([0, \infty)\), then \( g(0) = 0 \) and \( g(t) \to \infty \) as \( t \to \infty \).

(A6) If \( f \) is a positive function in \([t_0, \infty)\), then \( f(t) \to \infty \) as \( t \to \infty \) and \( f(t) \to 0 \) as \( t \to t_0 + 0 \).

(A7) If \( g \) is a positive function in \([0, \infty)\), then \( g(0) = 0 \) and \( g(t) \to \infty \) as \( t \to \infty \).

(A8) If \( f \) is a positive function in \((t_0, \infty)\), then \( f(t) \to \infty \) as \( t \to \infty \) and \( f(t) \to 0 \) as \( t \to t_0 + 0 \).

(A9) If \( g \) is a positive function in \([0, \infty)\), then \( g(0) = 0 \) and \( g(t) \to \infty \) as \( t \to \infty \).

The existence of a boundary blowup solution to (1) can be established.
(A3) Set $\tilde{g}(t) = g(t)/t$ and $F(t) = \int_{t_0}^t f(s) \, ds$. Then
\[
\int_{\mathbb{R}} \frac{dt}{\tilde{g}^{-1} \left( \frac{1}{1+F(t)} \right)} < \infty. \tag{5}
\]

(A4) Set
\[
H(t) = \int_0^t \frac{s^k}{g(s) \left(1 + s^2\right)^{(k+2)/2}} \, ds. \tag{6}
\]
Then $\lim_{t \to \infty} H(t) = \infty$.

(A5) Set $\varphi(t) = g(t)(1 + t^2)^{1/2}$. Then $\varphi(t)$ is a convex function in $[0, \infty)$.

(A6) $\limsup_{t \to \infty} g'(t)^2 < \infty$.

Then there exists a viscosity solution to (1)-(2).

The strategy of the proof of this theorem is as follows (we refer the readers to [23] for details). We note that comparison principles for viscosity solutions play important roles.

**Step 1.** We show that there exists a classical solution to the Dirichlet problem
\[
\begin{cases}
H_k[u_n] = f(u_n)|Du_n| & \text{in } \Omega, \\
u_n \equiv n & \text{on } \partial \Omega,
\end{cases} \tag{7}
\]
for every $n \in \mathbb{N}$ with $n > t_0$. It is enough to derive the $C^2$-a priori estimate for (7) (see [8, 14]).

**Step 2.** We prove that $\lim_{n \to \infty} u_n (=: u)$ exists and is a viscosity solution to (1)-(2).

Next we obtain the following non-existence result.

**Theorem 2.** Let $2 \leq k \leq n - 1$. We define two functions $\tilde{g}, \tilde{h}$ by
\[
\begin{align*}
\tilde{g}(t) &= \max_{s \geq t} g(s), \\
\tilde{h}(t) &= \frac{t}{\sqrt{1 + t^2}} \left( \frac{\left\lfloor \frac{n-1}{k} \right\rfloor}{\tilde{g}(t)} \right)^{1/k}.
\end{align*} \tag{8}
\]
We assume that $\lim_{t \to \infty} g(t) = 0$. If there exists $R \geq \inf_{x \in \Omega} \sup_{y \in \Omega} |x - y|$ such that
\[
\int_{\mathbb{R}} \frac{dt}{\tilde{h}^{-1} \left( f(t)^{1/k} R \right)} < \infty, \tag{9}
\]
then (1)-(2) has no solutions.

**Example 1.** Let $2 \leq k \leq n - 1$ and $p, q$ be positive constants. Suppose $\Omega$ is a bounded and uniformly $k$-convex $C^\infty$ domain. We consider these three equations:

\begin{align*}
H_k[u] &= \frac{u^p}{(1 + |Du|^2)^{q/2}} \quad \text{in } \Omega, \quad (10) \\
H_k[u] &= \frac{e^{pu}}{(1 + |Du|^2)^{q/2}} \quad \text{in } \Omega, \quad (11) \\
H_k[u] &= \frac{e^{pu}}{e^{q|Du|}} \quad \text{in } \Omega. \quad (12)
\end{align*}

It follows from Theorem 1 and Theorem 2 that

- The equation (10) has a boundary blowup solution provided $p > q$ and $1 \leq q \leq k - 1$.
- The equation (11) has a boundary blowup solution provided $1 \leq q \leq k - 1$.
- The equation (12) does not have any boundary blowup solutions.

**Remark 1.** Theorem 2 indicates that as far as (10) is concerned, $p$ is necessarily greater than $q$ in order for a boundary blowup solution to exist. In this case, our condition (A3) reduces to $p > q$ as well. We conjecture that (10) has a boundary blowup solution provided we assume only $1 \leq q < p$.

The case $k = n$, which corresponds to Gauss curvature equation, is excluded from Theorem 1. We state the existence result for the case $k = n$.

**Theorem 3.** Let $\Omega$ be a bounded and strictly convex $C^\infty$ domain, and $k = n$. We assume that the condition (A3) is satisfied and that $\limsup_{t \to \infty} g(t)t < \infty$. Then there exists a viscosity solution to (1)-(2).

3 Asymptotic behavior near the boundary

In this section we establish the asymptotic behavior of a boundary blowup solution near the boundary when the domain is strictly convex. We shall prove the following.

**Theorem 4.** Let $1 \leq k \leq n - 1$. We assume that (A2) and (A3) in Theorem 1 and the conditions given below are satisfied.

1. $\Omega$ is a bounded and strictly convex $C^\infty$ domain.
(B2) \( t_0 = -\infty \), or \( t_0 > -\infty \) and \( f^{1/k} \) is Lipschitz continuous at \( t_0 \).

(B3) There exists a constant \( T' > 0 \) such that \( f \) is a convex function in \([T', \infty)\).

(B4) Set \( h(t) = \frac{t}{g(t)^{1/k} \sqrt{1 + t^2}} \). Then there exists a constant \( \alpha > 0 \) such that \( h(t)/t^\alpha \) is non-decreasing in \((0, \infty)\).

(B5) \[ \lim_{t \to \infty} \frac{g(t)}{(1 + t^2)g'(t)} = 0. \]

Then there exist positive constants \( C_1, C_2 \) such that every solution \( u \) to (1)-(2) satisfies

\[ C_1 \text{dist}(x, \partial \Omega) \leq \psi(u(x)) \leq C_2 \text{dist}(x, \partial \Omega), \]

where \( \psi \) is defined by

\[ \psi(t) = \int_t^\infty \frac{ds}{h^{-1}(f(s)^{1/k})}. \]

We state the idea of the proof. Since \( \Omega \) is a bounded and strictly convex domain with boundary \( \partial \Omega \in C^\infty \), there exist positive numbers \( R_1, R_2 \) with \( R_1 < R_2 \) satisfying the following condition: for every \( z \in \partial \Omega \), there are two balls \( B_{1,z}, B_{2,z} \) whose radii are \( R_1 \) and \( R_2 \) respectively such that \( B_{1,z} \subset \Omega \subset B_{2,z} \) and \( \partial B_{1,z} \cap \partial B_{2,z} = \{z\} \).

Let \( v_1 \) (resp. \( v_2 \)) be a radially symmetric solution to (1) with \( v_1(x) \to \infty \) as \( \text{dist}(x, \partial B_{1,z}) \to 0 \) (resp. \( v_2(x) \to \infty \) as \( \text{dist}(x, \partial B_{2,z}) \to 0 \)). The condition (B4) guarantees the existence of \( v_1 \) and \( v_2 \). By the comparison principle, we see that

\[ v_2 \leq u \leq v_1 \quad \text{in} \quad B_{1,z}. \]  

In view of (15), it suffices to study the asymptotic behavior of the radially symmetric solution near the boundary. The assertion follows from the claim that if \( u = u(|x|) \) is a radially symmetric solution to (1)-(2) in \( B_R(0) \) with \( R > 0 \), then there exist constants \( C_1, C_2 > 0 \) which are independent of \( r \) such that

\[ C_1(R - r) \leq \psi(u(r)) \leq C_2(R - r) \]  

when \( r \) is near \( R \).

**Example 2.** Let \( 1 \leq k \leq n - 1 \) and \( p, q > 0 \). Suppose \( \Omega \) is a bounded and strictly convex \( C^\infty \) domain. Then Theorem 4 implies that
A boundary blowup solution $u$ to (10) (if it exists) satisfies
\[ C_1 \text{dist}(x, \partial \Omega)^{-\frac{p}{q-1}} \leq u(x) \leq C_2 \text{dist}(x, \partial \Omega)^{-\frac{p}{q-1}} \quad \text{near } \partial \Omega \] (17)
for some constants $C_1, C_2 > 0$, provided $p \geq k$ and $p > q$.

A boundary blowup solution $u$ to (11) (if it exists) satisfies
\[ u(x) = -\frac{q}{p} \log \text{dist}(x, \partial \Omega) + O(1) \quad \text{near } \partial \Omega, \] (18)
provided $q > 0$.

We state our result concerning the asymptotic behavior of a solution to (1)-(2) near $\partial \Omega$ for the case $k = n$. We mention that
\[ H(t) = \int_0^t \frac{s^n}{g(s)(1 + s^2)^{(k+2)/2}} \, ds \] (19)
in this case, and introduce the following condition:

(B6) There exists a constant $\alpha > 0$ such that $H(t)/t^\alpha$ is non-decreasing.

**Theorem 5.** Let $k = n$. We assume the conditions (A3), (B1), (B2) and (B6). Then there exist positive constants $C_1, C_2$ such that every solution $u$ to (1)-(2) satisfies
\[ C_1 \text{dist}(x, \partial \Omega) \leq \Psi(u(x)) \leq C_2 \text{dist}(x, \partial \Omega), \] (20)
where $\Psi$ is defined by
\[ \Psi(t) = \int_t^\infty \frac{ds}{H^{-1}(F(s))}. \] (21)

**Example 3.** Let $k = n$ and $p, q > 0$. Suppose $\Omega$ is a bounded and strictly convex $C^\infty$ domain. Then Theorems 3 and 5 implies that

- If $p > q \geq 1$, then there exists a boundary blowup solution to (10). Moreover, the solution $u$ satisfies
  \[ C_1 \text{dist}(x, \partial \Omega)^{-\frac{q-1}{p-q+2}} \leq u(x) \leq C_2 \text{dist}(x, \partial \Omega)^{-\frac{q-1}{p-q+2}} \quad \text{near } \partial \Omega \] (22)
  for some constants $C_1, C_2 > 0$, provided $p \geq n$ and $p > q > 1$.

- A boundary blowup solution $u$ to (11) exists and satisfies
  \[ u(x) = -\frac{q-1}{p} \log \text{dist}(x, \partial \Omega) + O(1) \quad \text{near } \partial \Omega, \] (23)
  provided $q > 1$. 
References

