Boundary blowup solutions to curvature equations

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. We consider the so-called *curvature equations* of the form

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = f(u)g(|Du|) \quad \text{in } \Omega, \tag{1}$$

with the following boundary condition

$$u(x) \to \infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0. \tag{2}$$

Here, for a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, and $S_k, k = 1, \ldots, n$, denotes the k-th elementary symmetric function, i.e.,

$$S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k}, \tag{3}$$

where the sum is taken over increasing k-tuples, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We study the existence and the asymptotic behavior near $\partial\Omega$ of a solution to (1)-(2).

The family of equations (1), $k = 1, \ldots, n$ contains some well-known and important equations.

The case $k = 1$ corresponds to the **mean curvature equation**;

The case $k = 2$ corresponds to the **scalar curvature equation**;

The case $k = n$ corresponds to **Gauss curvature equation**.

We remark that (1) is a quasilinear equation for $k = 1$ while it is a *fully non-linear* equation for $k \geq 2$. In the particular case that $k = n$, it is an equation of Monge-Ampère type. It is much harder to analyze fully non-linear equations, but the study of the classical Dirichlet problem for curvature equations in the case that $2 \leq k \leq n - 1$ has been developed in the last two decades, see for instance [4, 11, 24].

The condition (2) is called the "boundary blowup condition," and a solution which satisfies (2) is called a "boundary blowup solution," a "large solution," or an "explosive solution." The boundary blowup problems arise from physics, geometry and many branches of mathematics, see for instance

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[12, 19, 21]. The study of such problems for non-linear PDEs starts from the pioneering work of Bieberbach [3] and Rademacher [21] who considered $\Delta u = e^u$ in two and three dimensional domain respectively. For the case of semilinear equations, they have been extensively studied (see, for example, [13, 20] and [2, 6, 15, 16, 17, 19]). The case of quasilinear equations of divergence type to which the mean curvature equation ($k = 1$) belongs has been treated in [1, 7, 9]. The case of Monge-Ampère equations has been studied in [5, 10, 18]. However, to the best of our knowledge, there are no results concerning such problems for other fully non-linear PDEs, except for the work of Salani [22] who considered the case of Hessian equations.

Throughout the article, we assume the following conditions on f and g:

- Let $t_0 \in [-\infty, \infty)$. $f \in C^\infty(t_0, \infty)$ is a positive function and satisfies $f'(t) \geq 0$ for all $t \in (t_0, \infty)$.
- If $t_0 > -\infty$, then $f(t) \to 0$ as $t \to t_0 + 0$; otherwise (i.e., if $t_0 = -\infty$),

$$\int_{-\infty}^{t} f(s)\, ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

- $g \in C^\infty[0, \infty)$ is a positive function.

The first condition assures us that the comparison principle for solutions to (1) holds. The typical examples of f are $f(t) = t^p$ ($p > 0$), $t_0 = 0$ and $f(t) = e^t$, $t_0 = -\infty$.

In the subsequent two sections, we state our main results.

2 Existence results

We recall the notion of k-convexity. Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega \in C^2$. For $k = 1, \ldots, n-1$, we say that Ω is *k-convex* (resp. *uniformly k-convex*) if the vector of the principal curvatures of $\partial\Omega$, $\kappa' = (\kappa'_1, \ldots, \kappa'_{n-1})$, satisfies $S_j(\kappa') \geq 0$ (resp. > 0) for $j = 1, \ldots, k$ and for every $x \in \partial\Omega$. We note that a C^2 domain is $(n-1)$-convex (resp. uniformly $(n-1)$-convex) if and only if it is convex (resp. strictly convex).

First, we shall establish the existence of a boundary blowup solution to the curvature equation (1). We focus on the case $k \geq 2$, because for $k = 1$ the existence has been already studied in [9].

Theorem 1. *Let $2 \leq k \leq n-1$. We suppose that Ω, f and g satisfy the following conditions.*

(A1) *Ω is a bounded and uniformly k-convex C^∞ domain.*

(A2) *There exists a constant $T > 0$ such that g is non-increasing in $[T, \infty)$, and $\lim_{t\to\infty} g(t) = 0$.*

(A3) *Set* $\tilde{g}(t) = g(t)/t$ *and* $F(t) = \int_{t_0}^{t} f(s)\, ds$. *Then*

$$\int^{\infty} \frac{dt}{\tilde{g}^{-1}\left(\frac{1}{F(t)}\right)} < \infty. \tag{5}$$

(A4) *Set*

$$H(t) = \int_0^t \frac{s^k}{g(s)\,(1+s^2)^{(k+2)/2}}\, ds. \tag{6}$$

Then $\lim_{t\to\infty} H(t) = \infty$.

(A5) *Set* $\varphi(t) = g(t)(1+t^2)^{k/2}$. *Then* $\varphi(t)$ *is a convex function in* $[0, \infty)$.

(A6) $\limsup_{t\to\infty} g'(t)t^2 < \infty$.

Then there exists a viscosity solution to (1)-(2).

The strategy of the proof of this theorem is as follows (we refer the readers to [23] for details). We note that **comparison principles** for viscosity solutions play important roles.

Step 1. We show that there exists a classical solution to the Dirichlet problem

$$\begin{cases} H_k[u_n] = f(u_n) g(|Du_n|) & \text{in } \Omega, \\ u_n \equiv n & \text{on } \partial\Omega, \end{cases} \tag{7}$$

for every $n \in \mathbb{N}$ with $n > t_0$. It is enough to derive the C^2-a priori estimate for (7) (see [8, 14]).

Step 2. We prove that $\lim_{n\to\infty} u_n (=: u)$ exists and is a viscosity solution to (1)-(2).

Next we obtain the following non-existence result.

Theorem 2. *Let* $2 \le k \le n-1$. *We define two functions* $\bar{g}, \bar{h}$ *by*

$$\bar{g}(t) = \max_{s \ge t} g(s), \quad \bar{h}(t) = \frac{t}{\sqrt{1+t^2}} \left(\frac{\binom{n-1}{k}}{\bar{g}(t)} \right)^{1/k}. \tag{8}$$

We assume that $\lim_{t\to\infty} g(t) = 0$. *If there exists* $R \ge \inf_{x\in\Omega} \sup_{y\in\Omega} |x-y|$ *such that*

$$\int^{\infty} \frac{dt}{\bar{h}^{-1}\left(f(t)^{1/k} R\right)} < \infty, \tag{9}$$

then (1)-(2) has no solutions.

Example 1. Let $2 \le k \le n-1$ and p, q be positive constants. Suppose Ω is a bounded and uniformly k-convex C^∞ domain. We consider these three equations:

$$H_k[u] = \frac{u^p}{(1+|Du|^2)^{q/2}} \quad \text{in } \Omega, \tag{10}$$

$$H_k[u] = \frac{e^{pu}}{(1+|Du|^2)^{q/2}} \quad \text{in } \Omega, \tag{11}$$

$$H_k[u] = \frac{e^{pu}}{e^{q|Du|}} \quad \text{in } \Omega. \tag{12}$$

It follows from Theorem 1 and Theorem 2 that

- The equation (10) has a boundary blowup solution provided $p > q$ and $1 \le q \le k-1$.
- The equation (11) has a boundary blowup solution provided $1 \le q \le k-1$.
- The equation (12) does not have any boundary blowup solutions.

Remark 1. Theorem 2 indicates that as far as (10) is concerned, p is necessarily greater than q in order for a boundary blowup solution to exist. In this case, our condition (A3) reduces to $p > q$ as well. We conjecture that (10) has a boundary blowup solution provided we assume only $1 \le q < p$.

The case $k = n$, which corresponds to Gauss curvature equation, is excluded from Theorem 1. We state the existence result for the case $k = n$.

Theorem 3. *Let Ω be a bounded and strictly convex C^∞ domain, and $k = n$. We assume that the condition (A3) is satisfied and that* $\limsup_{t\to\infty} g(t)t < \infty$. *Then there exists a viscosity solution to (1)-(2).*

3 Asymptotic behavior near the boundary

In this section we establish the asymptotic behavior of a boundary blowup solution near the boundary when the domain is strictly convex. We shall prove the following.

Theorem 4. *Let $1 \le k \le n-1$. We assume that (A2) and (A3) in Theorem 1 and the conditions given below are satisfied.*

(B1) *Ω is a bounded and strictly convex C^∞ domain.*

(B2) $t_0 = -\infty$, *or* $t_0 > -\infty$ *and* $f^{1/k}$ *is Lipschitz continuous at* t_0.

(B3) *There exists a constant* $T' > 0$ *such that* f *is a convex function in* $[T', \infty)$.

(B4) *Set* $h(t) = \dfrac{t}{g(t)^{1/k}\sqrt{1+t^2}}$. *Then there exists a constant* $\alpha > 0$ *such that* $h(t)/t^{\alpha}$ *is non-decreasing in* $(0, \infty)$.

(B5) $\displaystyle\lim_{t\to\infty} \frac{g(t)}{(1+t^2)g'(t)} = 0.$

Then there exist positive constants C_1, C_2 *such that every solution* u *to (1)-(2) satisfies*

$$C_1 \operatorname{dist}(x, \partial\Omega) \le \psi(u(x)) \le C_2 \operatorname{dist}(x, \partial\Omega), \tag{13}$$

where ψ *is defined by*

$$\psi(t) = \int_t^{\infty} \frac{ds}{h^{-1}\left(f(s)^{1/k}\right)}. \tag{14}$$

We state the idea of the proof. Since Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^{\infty}$, there exist positive numbers R_1, R_2 with $R_1 < R_2$ satisfying the following condition: for every $z \in \partial\Omega$, there are two balls $B_{1,z}, B_{2,z}$ whose radii are R_1 and R_2 respectively such that $B_{1,z} \subset \Omega \subset B_{2,z}$ and $\partial B_{1,z} \cap \partial B_{2,z} = \{z\}$.

Let v_1 (resp. v_2) be a radially symmetric solution to (1) with $v_1(x) \to \infty$ as $\operatorname{dist}(x, \partial B_{1,z}) \to 0$ (resp. $v_2(x) \to \infty$ as $\operatorname{dist}(x, \partial B_{2,z}) \to 0$). The condition (B4) guarantees the existence of v_1 and v_2. By the comparison principle, we see that

$$v_2 \le u \le v_1 \quad \text{in } B_{1,z}. \tag{15}$$

In view of (15), it suffices to study the asymptotic behavior of the radially symmetric solution near the boundary. The assertion follows from the claim that if $u = u(|x|)$ is a radially symmetric solution to (1)-(2) in $B_R(0)$ with $R > 0$, then there exist constants $C_1, C_2 > 0$ which are independent of r such that

$$C_1(R - r) \le \psi(u(r)) \le C_2(R - r) \tag{16}$$

when r is near R.

Example 2. Let $1 \le k \le n-1$ and $p, q > 0$. Suppose Ω is a bounded and strictly convex C^{∞} domain. Then Theorem 4 implies that

- A boundary blowup solution u to (10) (if it exists) satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \le u(x) \le C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \quad \text{near } \partial\Omega \tag{17}$$

for some constants $C_1, C_2 > 0$, provided $p \ge k$ and $p > q$.

- A boundary blowup solution u to (11) (if it exists) satisfies

$$u(x) = -\frac{q}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \tag{18}$$

provided $q > 0$.

We state our result concerning the asymptotic behavior of a solution to (1)-(2) near $\partial\Omega$ for the case $k = n$. We mention that

$$H(t) = \int_0^t \frac{s^n}{g(s)\,(1+s^2)^{(k+2)/2}}\, ds \tag{19}$$

in this case, and introduce the following condition:

(B6) There exists a constant $\alpha > 0$ such that $H(t)/t^\alpha$ is non-decreasing.

Theorem 5. *Let $k = n$. We assume the conditions (A3), (B1), (B2) and (B6). Then there exist positive constants C_1, C_2 such that every solution u to (1)-(2) satisfies*

$$C_1 \operatorname{dist}(x, \partial\Omega) \le \Psi(u(x)) \le C_2 \operatorname{dist}(x, \partial\Omega), \tag{20}$$

where Ψ is defined by

$$\Psi(t) = \int_t^\infty \frac{ds}{H^{-1}(F(s))}. \tag{21}$$

Example 3. Let $k = n$ and $p, q > 0$. Suppose Ω is a bounded and strictly convex C^∞ domain. Then Theorems 3 and 5 implies that

- If $p > q \ge 1$, then there exists a boundary blowup solution to (10). Moreover, the solution u satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \le u(x) \le C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \quad \text{near } \partial\Omega \tag{22}$$

for some constants $C_1, C_2 > 0$, provided $p \ge n$ and $p > q > 1$.

- A boundary blowup solution u to (11) exists and satisfies

$$u(x) = -\frac{q-1}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \tag{23}$$

provided $q > 1$.

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