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Approximation of the Gauss curvature flow by a three-dimensional crystalline motion *

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In this paper, we consider an approximation of the Gauss curvature flow in $\mathbb{R}^3$ by so-called crystalline algorithm.

1 The Gauss curvature flow

The Gauss curvature flow in $\mathbb{R}^d$ makes a *smooth strictly convex hypersurface* shrink with the outward normal velocity equals to the Gauss curvature with negative sign. Let us explain more precisely. Let $\{\Gamma(t)\}$ be a family of smooth strictly convex closed hypersurfaces, $\kappa_1 = \kappa_1(P,t), \kappa_2 = \kappa_2(P,t), \ldots, \kappa_{d-1} = \kappa_{d-1}(P,t)$ the principal curvatures of $\Gamma(t)$ at P on $\Gamma(t)$ where we use the sign convention that the all principal curvatures of the hypersurfaces are positive, and $\kappa = \kappa(P,t) = \kappa_1\kappa_2\cdots\kappa_{d-1}$ the Gauss curvature of $\Gamma(t)$ at P. We call $\Gamma(t)$ the solution of the Gauss curvature flow if and only if at every points P on $\Gamma(t)$, the relation

$$v(P,t) = -\kappa(P,t)$$

is satisfied, where $v = v(P,t)$ denotes the outward normal velocity of $\Gamma(t)$ at P.

To describe the Gauss curvature flow, we use the support function $h(\nu,t)$ of the convex hypersurface $\Gamma(t)$ which is defined by

$$h(\nu,t) = \sup\{\langle P,\nu\rangle \mid P \in \Gamma(t)\},$$

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where $\nu \in S^{d-1}$ is a unit vector and $\langle \ , \ \rangle$ denotes the usual inner product in $\mathbb{R}^d$. An intuitive meaning of the support function is the signed distance from the origin to the tangent hyperplane of $\Gamma(t)$ at the point where the unit outward normal vector is ν. Using the support function, the Gauss curvature flow can be described as

$$\frac{\partial h}{\partial t}(\nu, t) = -\kappa(P, t), \tag{1}$$

where $P \in \Gamma(t)$ is a point where $\langle P, \nu \rangle = h(\nu, t)$. We note that for all $T_* \in \mathbb{R}$

$$(d(T_* - t))^{1/d} S^{d-1}$$

is a self-similar shrinking solution of (1) for $t \in (-\infty, T_*)$.

We shall consider the evolution of $\Gamma(t)$ by (1) which starts from the initial hypersurface Γ_0. We set $\Omega(t)$ being the open set enclosed by $\Gamma(t)$. The existence of its solution until single point extinction was proved in [5] and [20]:

Theorem 1 *If the initial hypersurface Γ_0 is smooth and strictly convex, then there exists a unique solution $\Gamma(t)$ to the Gauss curvature flow, which stays smooth and strictly convex. Moreover, the solution converges to a point within a finite time, say T_0, and this extinction time T_0 is given by $T_0 = V(\Gamma_0 \cup \Omega_0)/(dV(B^d))$. Here, V denotes the Lebesgue measure on $\mathbb{R}^d$ and B^d the unit ball $\{x \in R^d \mid |x| \leq 1\}$.*

2 Crystalline Motion

The main object of this paper is so-called crystalline motion. This motion was introduced by Taylor [18] and Angenent & Gurtin [2] to analyze crystal growth mathematically. The most typical crystalline motion in $\mathbb{R}^2$ makes each edge of a polygon keep the same direction but move with *the normal speed inversely proportional to its length*. Several papers, e.g. [7], [8], [10], [11], [13], and [21], have shown the convergence of two-dimensional crystalline motions to curve shortening flows in the plane as the number of the edges goes to infinity. We particularly note that the results in [9] and [15] have given the convergence for general curves which are *not necessarily convex*. See [1] for the behaviour of convex polygons under crystalline motions in $\mathbb{R}^2$.

As for the higher dimensional case, while *a motion of a surface by the crystalline mean curvature* was proposed also in [18], Bellettini, Novaga, and Paolini [3, 4] pointed out that the comparison principle is not valid in general and so it might not be natural to assume that all points on each side move with the same normal velocity (some people called it *the facets stay facets ansatz*).

In this paper, we introduce a three-dimensional crystalline motion for *convex polyhedra* and show its convergence to Gauss curvature flow in $\mathbb{R}^3$. As for the different way of approximation to the Gauss curvature flow, we mention [12, 14]. Our crystalline motion in $\mathbb{R}^3$ makes each side of a polyhedron move with *the normal speed inversely proportional to its area.* This motion is a three-dimensional version of the most typical two-dimensional one, which was introduced in [18]. Our motion should be said as *a motion of a surface by the crystalline Gauss curvature* and we find out that the comparison principle is available for this motion (see Lemma 3).

The precise definition of our crystalline motion is as follows:

Let $\tilde{W}$, which represents the anisotropy of the problem and is called the Wulff shape, be an N-sided convex polyhedron in $\mathbb{R}^3$ including the origin as its interior point. We also call $\tilde{W}$ the Wulff polyhedron to emphasize that the Wulff shape is a polyhedron. Since $\tilde{W}$ is an N-sided convex polyhedron in $\mathbb{R}^3$, there exist N unit vectors $\nu_1, \nu_2, \dots, \nu_N \in S^2$ such that

$$\tilde{W} = \bigcap_{i=1}^{N} \{P \in \mathbb{R}^3 \mid \langle P, \nu_i \rangle \le \tilde{h}_i\}, \quad \tilde{h}_i = \sup\{\langle P, \nu_i \rangle \mid P \in \partial\tilde{W}\}.$$

We call the set $\tilde{\Gamma}_i = \tilde{W} \cap \{P \in \mathbb{R}^3 \mid \langle P, \nu_i \rangle = \tilde{h}_i\}$ i-th side of $\tilde{W}$ and $\tilde{h}_i$ the height from the origin of $\tilde{\Gamma}_i$. We set $\tilde{h} = \tilde{h} = (\tilde{h}_i)_{i=1,2,\dots,N} \in \mathbb{R}^N$. We note that the unit outward normal vector on $\tilde{\Gamma}_i$ is ν_i and the support function of $\partial\tilde{W}$ coincide with $\tilde{h}_i$ at $\nu = \nu_i$. Let $\tilde{A}_i = A(\tilde{\Gamma}_i)$ be the area of $\tilde{\Gamma}_i$.

We call that an N-sided convex polyhedron and its boundary Γ are *a $\tilde{W}$-admissible polyhedron* and *a $\tilde{W}$-admissible surface*, respectively, if and only if the outward normal vector of the i-th side, say Γ_i, of Γ is ν_i for all i. For a $\tilde{W}$-admissible surface Γ, the height from the origin $h = (h_i)_{i=1,2,\dots,\mathbb{N}} \in \mathbb{R}^N$ is defined by $h_i = \sup\{\langle P, \nu_i \rangle \mid P \in \Gamma\}$ and $A_i = A(\Gamma_i)$ denotes the area of Γ_i. Clearly, $\tilde{W}$ is a $\tilde{W}$-admissible polyhedron and so $\partial\tilde{W}$ is a $\tilde{W}$-admissible surface.

Then, a crystalline motion of a $\tilde{W}$-admissible surface $\Gamma(t)$ is defined by the system of ordinary differential equations

$$\frac{dh_i(t)}{dt} = -\tilde{h}_i \frac{\tilde{A}_i}{A_i(t)}, \quad 1 \le i \le N. \tag{2}$$

We call this flow *the $\tilde{W}$-crystalline flow* and a family $\{\Gamma(t)\}$ of $\tilde{W}$-admissible surfaces which satisfies (2) a solution to the $\tilde{W}$-crystalline flow. The quantity $\frac{\tilde{A}_i}{A_i(t)}$ might be regarded as the crystalline Gauss curvature of the i-th side of $\Gamma(t)$. We note that for all $T_* \in \mathbb{R}$

$$(3(T_* - t))^{1/3} \partial \tilde{W}$$

is a self-similar shrinking solution of (2) for $t \in (-\infty, T_*)$. This self-similar solution will be used in the comparison argument of the proof of our main result below.

We can prove the well-posedness of this flow by the classical theorem of the existence and the uniqueness of the solution of ordinary differential equations.

Theorem 2 *Let $\tilde{W}$ be a convex polyhedron in $\mathbb{R}^3$ including the origin as its interior point, and Γ_0 a $\tilde{W}$-admissible surface. Then, there exists a unique solution $\Gamma(t)$ to $\tilde{W}$-crystalline flow with $\Gamma(0) = \Gamma_0$. Moreover, the enclosed volume vanishes at the maximal existence time $T = V(\Gamma_0 \cup \Omega_0)/(3V(\tilde{W})) \in (0, +\infty)$. Here, Ω_0 is the open set enclosed by Γ_0 and V denotes the three-dimensional volume.*

We also note that the comparison lemma holds for the $\tilde{W}$-crystalline flow.

Lemma 3 *Let $\tilde{W}$ be a convex polyhedron in $\mathbb{R}^3$ including the origin as its interior point, and $\Gamma'(t)$ and $\Gamma(t)$ solutions to $\tilde{W}$-crystalline flow for $t \in [0, T)$. Then, $\Gamma'(0) \subset \Gamma(0) \cup \Omega(0)$ implies $\Gamma'(t) \subset \Gamma(t) \cup \Omega(t)$ for all $t \in [0, T)$. Here, $\Omega(t)$ is the open set enclosed by $\Gamma(t)$.*

3 Mainr result

Now let us consider a sequence of convex polyhedra $\tilde{W}^k$ and that of the $\tilde{W}^k$-crystalline flows. Here and hereafter, the parameter $k \in \mathbb{N}$ indicates the accuracy of the approximation and the larger integer k corresponds to the better approximation. We note that, for example, N^k in (A1) below does not mean k-th power of N. Our main purpose is to show that this sequence of crystalline flows converges to the Gauss curvature flow under the assumptions below. First we assume that

$$\text{(A1)} \qquad \begin{array}{l}\text{the Wulff polyhedron } \tilde{W}^k \text{ has } N^k\text{-sides} \\ \text{and is symmetric with respect to the origin}\end{array}$$

and the sequence of the Wulff shapes $\{\tilde{W}^k\}$ converges to the unit ball $B^3 = \{P \in \mathbb{R}^3 \mid |P| \leq 1\}$ in the Hausdorff distance, namely,

$$\text{(A2)} \qquad \lim_{k\to\infty} d_H(\tilde{W}^k, B^3) = 0.$$

Here $d_H(A_1, A_2)$ is the Hausdorff distance between sets A_1 and A_2. We use the convention of $d_H(\emptyset, \emptyset) = 0$ and $d_H(\emptyset, A) = d_H(A, \emptyset) = +\infty$ provided $A \neq \emptyset$. Second we assume that

$$\text{(A3)} \qquad \text{the initial surface } \Gamma_0^k \text{ is a } \tilde{W}^k\text{-admissible surface}$$

and it converges to a smooth and strictly convex surface Γ_0:

$$\text{(A4)} \qquad \lim_{k\to\infty} d_H(\Gamma_0^k, \Gamma_0) = 0.$$

Let $\Gamma(t)$ and T_0 be the solution of (1) which starts from the smooth strictly convex surface Γ_0 and its extinction time, respectively. We set $\Gamma(T_0) = \lim_{t\uparrow T_0} \Gamma(t)$ and $\Gamma(t) = \emptyset$ for $t > T_0$. Let $\Gamma^k(t)$ and T^k be the solution of (2) with $\tilde{W} = \tilde{W}^k$ (namely, solution to the $\tilde{W}^k$-crystalline flow) which starts from Γ_0^k and its extinction time, respectively. We set $\Gamma^k(T^k) = \lim_{t\uparrow T^k} \Gamma^k(t)$ and $\Gamma^k(t) = \emptyset$ for $t > T^k$. We also set $\Omega(t) = \emptyset$ for $t \geq T_0$ and $\Omega^k(t) = \emptyset$ for $t \geq T^k$.

Now our main result is the next theorem:

Theorem 4 *Assume (A1), (A2), (A3), and (A4). Then the solution* $\Gamma^k(t)$ *to the* $\tilde{W}^k$*-crystalline flow with the initial surface* $\Gamma^k(0) = \Gamma_0^k$ *converges to the solution* $\Gamma(t)$ *to (1) with the initial surface* $\Gamma(0) = \Gamma_0$ *locally uniformly in* $t \in [0, T_0)$*:*

$$\lim_{k\to\infty} \sup_{0\leq s\leq t} d_H(\Gamma^k(s), \Gamma(s)) = 0.$$

Here T_0 *is the extinction time of* $\Gamma(t)$*.*

4 Outline of the proof

In this section we explain the outline of the proof of Theorem 4.

We recall the result of K. Ishii and H. M. Soner [15]. They were concerned with the two-dimensional crystalline motion whose Wulff shape is a regular polygon centered at the origin, and showed its convergence to the curve shortening flow as the Wulff polygon tends to the unit disc. Their method, which is a kind of perturbed test function method, works to prove our theorem. In their case, they used a disc as a test function for the solution to the curve shortening flow, and then chose a suitable dilation of the Wulff polygon approximating the disc as one for the solution to the crystalline motion. In our case, however, a surface has two principal curvatures at each point. Therefore, we need to use an ellipsoid as a test function to the Gauss curvature flow, and then choose a $\tilde{W}^k$-admissible polyhedron approximating the ellipsoid in some nice sense. Seeking such a nice polyhedron would be just a Minkowski problem (see Lemma 9), since this problem concerns the existence, uniqueness, and stability of convex surfaces with preassigned Gauss curvature as a function of the outer normal (e.g. [16]). As for the perturbed test function method, we refer [6]. To our knowledge the first successful applications of this method to viscosity solutions appeared in this paper.

Throughout this section, we assume (A1), (A2), (A3), and (A4).

For $k \in \mathbb{N}$, let $\{\Gamma^k(t)\}_{t\geq 0}$ be the solution of the $\tilde{W}^k$-crystalline flow and let $\Omega^k(t)$ be the open set enclosed by $\Gamma^k(t)$. For $t \geq 0$, we define semicontinuous envelopes

$$\hat{\Omega}(t) = \bigcap_{\varepsilon>0, N\in\mathbb{N}} \mathrm{cl}\left(\bigcup_{|s-t|\leq\varepsilon, s\geq 0,\, k\geq N} \left(\Gamma^k(s) \cup \Omega^k(s)\right) \right),$$

$$\underline{\Omega}(t) = \bigcup_{\varepsilon>0, N\in\mathbb{N}} \mathrm{int}\left(\bigcap_{|s-t|\leq\varepsilon, s\geq 0,\, k\geq N} \Omega^k(s) \right).$$

Here, for a set A, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ mean the closure of A and the interior of A, respectively. In [17], the properties of the sets like $\hat{\Omega}(t)$ and $\underline{\Omega}(t)$ are noted. Let $\Gamma(t)$ be the solution to the equation (1) and $\Omega(t)$ the open set which is enclosed by $\Gamma(t)$.

We note that because of $\dot{h}_i < 0$

$$\Omega(t_2) \subset \Omega(t_1),\ \underline{\Omega}(t_2) \subset \underline{\Omega}(t_1) \quad \text{and} \quad \hat{\Omega}(t_2) \subset \hat{\Omega}(t_1) \quad \text{for} \quad t_2 \geq t_1 \geq 0$$

hold. We set

$$\hat{T} = \sup\{t \,|\, \hat{\Omega}(t) \neq \emptyset\},\ \underline{T} = \sup\{t \,|\, \underline{\Omega}(t) \neq \emptyset\}, \quad \text{and} \quad T_0 = \sup\{t \,|\, \Omega(t) \neq \emptyset\}.$$

By the definition of $\hat{\Omega}(t)$ and $\underline{\Omega}(t)$, we have

$$\hat{T} \geq \underline{T} \text{ and } \mathrm{cl}(\underline{\Omega}(t)) \subset \hat{\Omega}(t). \tag{3}$$

If we prove that

$$\hat{\Omega}(t) \subset \mathrm{cl}(\Omega(t)) \text{ and } \Omega(t) \subset \underline{\Omega}(t) \text{ for all } t \in [0, T_0), \tag{4}$$

which is the result of Lemma 8 below, then we obtain the convergence result.

We show these inclusions (4) by the following steps. First we show that $\hat{\Omega}(t)$ and $\underline{\Omega}(t)$ are are sub and super-solutions, respectively, of (1) in visocosity sense (Lemma 5 and Lemma 6). Second comparing initial states, we show the inclusions among $\hat{\Omega}(0)$, $\underline{\Omega}(0)$, and Ω_0 (Lemma 7). Finally, from the Lemmas 5,6, and 7, we obtain the desired inclusions (Lemma 8). The second part is not difficult. The final part is a rather standard argument. In the fisrt part, we need a help from the theory of Minkowski problem (Lemma 9).

Lemma 5 *Let $(P_0, t_0) \in \mathbb{R}^3 \times (0, +\infty)$ and $\{O(t)\}_{t\in(0,t_0]}$ be a family of closed sets with smoothly evolving and strictly convex boundaries. If $P_0 \in \partial\hat{\Omega}(t_0) \cap \partial O(t_0)$ and $\hat{\Omega}(t) \subset O(t)$ for all $t \in (0, t_0]$, then the inequality*

$$V_O(P_0, t_0) \leq -\kappa_O(P_0, t_0) \tag{5}$$

holds. Here, $V_O(P_0, t_0)$ and $\kappa_O(P_0, t_0)$ is the normal velocity and the Gauss curvature of $\partial O(t_0)$ at P_0, respectively.

Lemma 6 *Let $(P_0, t_0) \in \mathbb{R}^3 \times (0, +\infty)$ and $\{O(t)\}_{t\in(0,t_0]}$ be a family of closed sets with smoothly evolving and strictly convex boundaries. If $P_0 \in \partial\underline{\Omega}(t_0) \cap \partial O(t_0)$ and $\mathrm{int}(O(t)) \subset \underline{\Omega}(t)$ for all $t \in (0, t_0]$, then the inequality*

$$V_O(P_0, t_0) \geq -\kappa_O(P_0, t_0) \tag{6}$$

holds. Here, $V_O(P_0, t_0)$ and $\kappa_O(P_0, t_0)$ is the normal velocity and the Gauss curvature of $\partial O(t_0)$ at P_0, respectively.

Lemma 7 *$\hat{\Omega}(0) \subset \mathrm{cl}(\Omega_0)$ and $\Omega_0 \subset \underline{\Omega}(0)$ hold.*

Lemma 8 *$\underline{T} = \hat{T} = T_0$. $\hat{\Omega}(t) \subset \mathrm{cl}(\Omega(t))$ and $\Omega(t) \subset \underline{\Omega}(t)$ hold for all $t \in [0, T_0)$.*

For positive numbers a and b, we set

$$E = E(a, b) = \{(x, y, z) \mid ax^2 + by^2 + z^2 \le 1\}.$$

For this ellipsoid E we have the following lemma.

Lemma 9 *Let E be the ellipsoid defined as above. For any $k \in \mathbb{N}$, there uniquely exists a $\tilde{W}^k$-admissible polyhedron E^k symmetric with respect to the origin such that*

$$\kappa^E(\nu_i^k) = \frac{\tilde{A}_i^k}{A_i^{E^k}} \tag{7}$$

holds for all $1 \le i \le N^k$. Moreover,

$$\lim_{k \to \infty} d_H(E^k, E) = 0 \tag{8}$$

holds. Here, ν_i^k denotes the outward normal vector of the i-th side of $\tilde{W}^k$, $\kappa^E(\nu)$ Gauss curvature of E at the point where the outward normal vector is ν, $\tilde{A}_i^k$ the area of the i-th side of $\tilde{W}^k$, $A_i^{E^k}$ the area of the i-th side of E^k, respectively.

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