ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SEMILINEAR SYSTEMS OF WAVE EQUATIONS

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1. Introduction

In this note we consider asymptotic behavior of solutions to the Cauchy problem for semilinear systems of wave equations:

$$\partial_t^2 u_i - c_i^2 \Delta u_i = F_i(\partial u) \qquad \text{in} \quad \mathbf{R}^3 \times (0,\infty), \tag{1.1}$$

where $i = 1, \cdots, N$, $c_i > 0$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\partial = (\partial_0, \partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_0 = \partial_t = \partial/\partial t$ and $u(x,t) = (u_1(x,t), \cdots, u_N(x,t))$ is a real-valued unknown function. Besides, $F_i \in C^1(\mathbb{R}^{4N})$ is a given function satisfying

$$F_i(0) = \nabla F_i(0) = 0.$$

Our purpose here is to show that there are examples of nonlinearities F such that the corresponding equation (1.1) cannot be regarded as a perturbation from the system of homogeneous wave equations, even if we restrict our attention to small amplitude solutions. The results presented in the section 2 was obtained by a joint work with Professors Kôji Kubota and Hideaki Sunagawa, and the results in the section 3 was done by a joint work with Professor Soichiro Katayama.

We wish to explain the precise meaning of our purpose. Suppose that the Cauchy problem for (1.1) admits a unique global solution u. We say the equation (1.1) can be regarded as a perturbation from the system of homogeneous wave equations:

$$\partial_t^2 v_i - c_i^2 \Delta v_i = 0 \qquad \text{in} \quad \mathbf{R}^3 \times (0,\infty), \tag{1.2}$$

if the global solution u tend to the solution $v = (v_1, \cdots, v_N)$ of (1.2) as $t \to \infty$. This kind of asymptotic behavior is well studied in connection with the so-called nonlinear scattering theory in the energy space. (see, e.g. [19, Chapter 6] and the references cited therein). Nevertheless, there is another possibility that the effect of the nonlinearity remains so strong in sufficiently large time that the global solution u cannot approach to any free solutions. To our knowledge, there are only few results which suggest that such a phenomenon occurs for nonlinear wave equations (see e.g. Alinhac [3, 4], Lindblad–Rodnainski [16, 17]). Therefore our main goal of this note is to show that there exist small amplitude solutions to the Cauchy problem for (1.1) with a certain F whose large time behavior might be different from that of any free solutions.

We conclude this section by recalling a sufficient condition to ensure the *small data global existence* for (1.1) when nonlinearity F_i is sufficiently smooth. For the case $c_1 = \cdots = c_N$, such a condition was introduced by Klainerman [11]. We say $F(\partial u)$ satisfies the *null condition*, if and only if the quadratic part of it can be written as a linear combination of the following null forms

$$Q_0(u_j, u_k; c_i) =(\partial_t u_j)(\partial_t u_k) - c_i^2(\nabla u_j) \cdot (\nabla u_k), \tag{1.3}$$

$$Q_{ab}(u_j, u_k) =(\partial_a u_j)(\partial_b u_k) - (\partial_b u_j)(\partial_a u_k) \quad (0 \le a < b \le 3). \tag{1.4}$$

We remark that Christodoulou [5] also established the same result, independently. Moreover, the global solution u to the Cauchy problem for (1.1) satisfying the *null condition* approaches to some free solution (see Kubo–Ohta [14, Section 6]). On the contrary, the *null condition* is necessary to ensure *small data global existence* if we consider the scalar case, i.e., $N = 1$. In fact, the *blow-up* result was obtained by Alinhac [2].

The *null condition* is extended to the multiple speeds case (i.e., the speeds $c_1, \dots, c_N$ do not necessarily coincide with each other) so that the *small data global existence* for (1.1) holds (see Kovalyov [12], Agemi–Yokoyama [1], Yokoyama [21], Sideris – Tu [18], Kubota – Yokoyama [15], Katayama [7], [8], [9], Katayama – Yokoyama [10] and so on). For example, in addition to null forms, terms like $(\partial_a u_j)(\partial_b u_k)$ with $c_j \ne c_k$ are allowed to be included for the multiple speeds case. The precise conditions for the multiple speeds case are somewhat complicated, and we do not go into details here. Instead of this, we shall discuss an extension of the *null condition* for the case of the common propagation speeds with $N \ge 2$.

2. Example, I

This section is concerned with the Cauchy problem for semilinear systems of wave equations:

$$\begin{cases} \partial_t^2 u_1 - c_1^2 \Delta u_1 = |\partial_t u_2|^p & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \\ \partial_t^2 u_2 - c_2^2 \Delta u_2 = |\partial_t u_1|^q & \text{in} \quad \mathbf{R}^3 \times (0, \infty), \end{cases} \tag{2.1}$$

where $c_1,\ c_2 > 0$, $1 < p \le q$. First we recall known results concerning the *small data global existence* and *blowup* for the Cauchy problem for (2.1). Yokoyama [21] proved that when $c_1 \ne c_2$, the problem admits a unique global smooth solution when $p = q = 2$ and the initial data are in $C_0^\infty(\mathbf{R}^3)$ and sufficiently small. On the other hand, Deng showed in Theorem 3.3 of [6] that if $c_1 = c_2$ and $q(p-1) \le 2$, then, in general, a classical solution to the problem blows up in finite time however small the initial data are. It is remarkable that the above condition is valid for $p = q = 2$. Recently, Xu [20] proved the blowup result when $c_1 \ne c_2$ and $6(pq-1)/(p+q+2) \le 1$.

Thus we see from these results that the feature of the problem (2.1) depend not only on the exponents p, q but also on the propagation speeds c_1, c_2.

In order to extend the existence result due to [21] for general $p,\ q > 1$, we consider only radially symmetric solution to the Cauchy problem for (2.1). To be more specific, we seek solutions to the problem in $\mathcal{X} \times \mathcal{X}$, where $\mathcal{X}$ is defined by

$$\begin{aligned} \mathcal{X} = \{ & w(x,t) \in C(\mathbf{R}^3 \times [0,\infty))\ ;\ \text{there is}\ \ u(r,t) \in X^2\ \ \text{such that} \\ & w(x,t) = u(|x|,t) \quad \text{for}\ (x,t) \in \mathbf{R}^3 \times [0,\infty) \quad \text{and} \quad \lim_{|x|\to\infty} w(x,0) = 0\} \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} X^2 = \{u(r,t) \in C^1(\mathbf{R} \times [0,\infty))\ ;\ ru(r,t) \in C^2(\mathbf{R} \times [0,\infty)), \\ u(-r,t) = u(r,t)\ \ \text{for}\ (r,t) \in \mathbf{R} \times [0,\infty)\}. \end{aligned} \tag{2.3}$$

Note that $\mathcal{X} \subset C^1(\mathbb{R}^3 \times [0,\infty)) \cap C^2((\mathbb{R}^3 \setminus \{0\}) \times [0,\infty))$, because $\partial_r u(r,t) = 0$ for $r = 0$ if $u \in X^2$. Therefore the solution which we shall obtain is an "almost" classical solution.

While, we consider the following type of initial condition:

$$u_j(x,0) = f_j(|x|), \quad (\partial_t u_j)(x,0) = g_j(|x|) \quad \text{for} \quad x \in \mathbf{R}^3 \quad (j = 1,2), \tag{2.4}$$

and introduce a class of the initial data Y as follows:

$$\begin{aligned} Y = \{(f,g) \in C^1(\mathbf{R}) \times C(\mathbf{R})\ ;\ rf(r) \in C^2(\mathbf{R}),\ rg(r) \in C^1(\mathbf{R}), \\ f(-r) = f(r),\ g(-r) = g(r)\ \text{for}\ r \in \mathbf{R}\}. \end{aligned} \tag{2.5}$$

This space is consistent with X^2 in the sense that the solution v to the Cauchy problem for the homogeneous wave equation

$$\partial_t^2 v - c^2 \Delta v = 0 \quad \text{in} \quad \mathbf{R}^3 \times (0,\infty) \tag{2.6}$$

belongs to X^2, if the initial data $(f,g) \in Y$ satisfy such a decay condition as

$$M_\kappa(f,g) := \sup_{r>0}\,(1+r)^\kappa \,|||(f(r),g(r))||| < \infty,$$

where $\kappa > 0$ and

$$|||(f(r),g(r))||| = |f(r)| + (1+r)(|f'(r)| + |g(r)|) + r(|f''(r)| + |g'(r)|).$$

Moreover we have the following estimate:

$$[v(r,t)](1 + |r - ct|)^\kappa \le CM_\kappa(f,g)$$

for $(r,t) \in \mathbf{R} \times [0,\infty)$, where we put

$$[v(r,t)] = |v(r,t)| + (1+r) \sum_{|\alpha|=1} |\partial_{r,t}^\alpha v(r,t)| + r \sum_{|\alpha|=2} |\partial_{r,t}^\alpha v(r,t)|.$$

In the application we choose κ as κ_1 or κ_2 which are defined as follows:

$$\kappa_1 = p-1, \tag{2.7}$$

$$\kappa_2 = \min(q-1, q(p-1)) \quad \text{if} \quad c_1 \neq c_2 \text{ or } p > 2, \tag{2.8}$$

$$\kappa_2 = q(p-1)-1 \qquad \text{if} \quad c_1 = c_2 \text{ and } 1 < p < 2. \tag{2.9}$$

In addition, if $c_1 = c_2$, $p = 2$ and $q(p-1) > 2$ holds, then we take such κ_2 as $1 < \kappa_2 < q-1$. Then we have the following existence result.

Theorem 2.1. *Let $1 < p \le q$ and suppose that*

$$q(p-1) > 2, \tag{2.10}$$

if $c_1 = c_2$, and that

$$q(p-1) > 1, \tag{2.11}$$

if $c_1 \neq c_2$. Assume $(f_j, g_j) \in Y$ and $M_{\kappa_j}(f_j, g_j) \le \varepsilon$ for $j = 1, 2$ and $\varepsilon > 0$.

Then there are positive constants ε_0 and C_0 (depending only on c_1, c_2, p and q) such that for any ε with $0 < \varepsilon \le \varepsilon_0$, there exists uniquely a solution $(u_1, u_2) \in \mathcal{X} \times \mathcal{X}$ of the Cauchy problem (2.1) and (2.4) satisfying

$$[\widetilde{u_1}(r,t)](1 + |r - c_1 t|)^{\kappa_1} + [\widetilde{u_2}(r,t)](1 + |r - c_2 t|)^{\kappa_2} \le 2C_0 \varepsilon, \tag{2.12}$$

if $c_1 \neq c_2$ or $p > 2$, and

$$\begin{aligned} &[\widetilde{u_1}(r,t)]\{(1 + r + t)^{[1-\kappa_1]_+}\}^{-1}(1 + |r - c_1 t|) \\ &\qquad + [\widetilde{u_2}(r,t)](1 + |r - c_2 t|)^{\kappa_2} \le 2C_0 \varepsilon, \end{aligned} \tag{2.13}$$

if $c_1 = c_2$ and $1 < p \le 2$. Here we denoted $u_1(x,t) = \widetilde{u_1}(|x|,t)$, $u_2(x,t) = \widetilde{u_2}(|x|,t)$, and $A^{[a]_+} = A^a$ if $a > 0$; $A^{[0]_+} = 1 + \log A$ for $A \ge 1$.

This result shows that the condition given by [6] is sharp if $c_1 = c_2$ and that it can be relaxed if $c_1 \neq c_2$. But it is still an open question what will happen when $c_1 \neq c_2$, $q(p-1) \le 1$ and the condition given by [20] does not fulfilled.

From now on we denote by (u_1, u_2) the global solution of the Cauchy problem (2.1) and (2.4) obtained in Theorem 2.1 and assume that $0 < \varepsilon \le \varepsilon_0$. Our next step is to examine the large time behavior of (u_1, u_2). We define θ_1, θ_2 by

$$\theta_j = \kappa_j - 1 \quad \text{ if } c_1 = c_2 \ ; \quad \theta_j = \kappa_j - (1/2) \quad \text{if } c_1 \neq c_2, \tag{2.14}$$

where κ_1 and κ_2 are defined by (2.7), (2.8) and (2.9). Since $\kappa_2 > 1/2$ by the definition and (2.11), we find that there exists uniquely a solution $v_2 \in \mathcal{X}$ of (2.6) with $c = c_2$ satisfying

$$\|u_2(t) - v_2(t)\|_{E(c_2)} \le C \varepsilon^q (1 + t)^{-\theta_2} \quad \text{ for } \quad t \ge 0, \tag{2.15}$$

and $\|v_2(0)\|_{E(c_2)} < \infty$, where $C = C(c_1, c_2, p, q)$ is a positive constant and

$$\|u(t)\|_{E(c)}^2 = \frac{1}{2} \int_{\mathbf{R}^3} (|\partial_t u(x,t)|^2 + c^2 |\nabla u(x,t)|^2) dx.$$

Now we are in a position to state our main result. Suppose that $\theta_1 \geq 0$, i.e., if $c_1 = c_2$, then $p \geq 2$; otherwise $p \geq 3/2$ (for the remaining case, we refer to [13]). As an unperturbed system, we choose

$$\begin{cases} \partial_t^2 v_1 - c_1^2 \Delta v_1 = |\partial_t v_2|^p & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 v_2 - c_2^2 \Delta v_2 = 0 & \text{in} \quad \mathbf{R}^3 \times (0,\infty). \end{cases} \tag{2.16}$$

In other words, our proposal is to regard (2.1) as a perturbation from the "*modified free system*" (2.16), but in general not from the free system

$$\begin{cases} \partial_t^2 w_1 - c_1^2 \Delta w_1 = 0 & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 w_2 - c_2^2 \Delta w_2 = 0 & \text{in} \quad \mathbf{R}^3 \times (0,\infty). \end{cases} \tag{2.17}$$

Theorem 2.2. *Assume that p, q and (f_j, g_j) $(j = 1, 2)$ fulfill the hypotheses of Theorem 2.1. Suppose that $\theta_1 \geq 0$. Then there exists uniquely a solution $(v_1, v_2) \in \mathcal{X} \times \mathcal{X}$ of (2.16) satisfying (2.15) and*

$$\|u_1(t) - v_1(t)\|_{E(c_1)} \leq C\varepsilon^{p+q-1}(1+t)^{-\theta} \quad \textit{for} \quad t \geq 0. \tag{2.18}$$

Here θ is a positive number such that if $c_1 \neq c_2$, $\theta_1 > 0$ and $q < 2$, then $\theta = \theta_1 + \max\{\theta_2 + (p-1)(q-2), 0\}$; otherwise $\theta = \theta_1 + \theta_2$, where $\theta_2\,(> 0)$ is defined by (2.14) with $j = 2$. Besides, C is a constant depending only on c_1, c_2, p and q.

If we suppose in addition that $\theta_1 > 0$, then there exists uniquely a solution $(w_1, w_2) \in \mathcal{X} \times \mathcal{X}$ of (2.17) satisfying

$$\|v_1(t) - w_1(t)\|_{E(c_1)} \leq C\varepsilon^p(1+t)^{-\theta_1} \quad \text{for} \quad t \geq 0. \tag{2.19}$$

Therefore, combining (2.18) with (2.19), we see that u_1 tends to w_1 in the energy norm as $t \to \infty$, hence (2.1) can be regarded simply as a perturbation from the free system (2.17) in this case.

Therefore, the case $\theta_1 = 0$ is of our special interest. To simplify the situation, we assume that the initial data are linear in ε. Namely,

$$f_j(r) = \varepsilon\varphi_j(r), \quad g_j(r) = \varepsilon\psi_j(r) \quad \text{for} \quad r \in \mathbf{R} \tag{2.20}$$

with $(\varphi_j, \psi_j) \in Y$ and $M_{\kappa_j}(\varphi_j, \psi_j) \leq 1$. Then we have the following.

Theorem 2.3. *Let $c_1 = c_2$, $p = 2$ and (2.10) hold. Suppose that (f_j, g_j) are as in the above and that*

$$r\psi_2(r) - (r\varphi_2(r))' \neq 0 \quad \textit{at} \quad r = r_0 \tag{2.21}$$

for a positive number r_0. Then there are positive numbers C, ε_1 and t_0 such that for $0 < \varepsilon \leq \varepsilon_1$ and $t \geq t_0$ we have

$$C^{-1}\varepsilon^p(\log t) - \|u_1(0)\|_{E(c_1)} \leq \|u_1(t)\|_{E(c_1)} \leq \|u_1(0)\|_{E(c_1)} + C\varepsilon^p(\log t). \tag{2.22}$$

Under the assumptions in Theorem 2.3, it is impossible that u_1 has a free profile w_1 with $\|w_1(0)\|_{E(c_1)} < \infty$. Indeed, if not, then $\lim_{t\to\infty} \|u_1(t)\|_{E(c_1)} = \|w_1(0)\|_{E(c_1)}$. Clearly, this contradicts (2.22).

Remark. We can extend the theorems presented in this section to the case where the nonlinearity of the first equation in (2.1) is replaced by $|\partial_t u_2|^{p-1}\partial_t u_2$ or $|\nabla u_2|^p$. In addition, we can admit the linear combination of these terms as the nonlinearity in the theorems except Theorem 2.3, as well.

3. Example, II

The aim of this section is to show that the following semilinear system :

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = (\partial_1 u_1)(\partial_1 u_2 - \partial_2 u_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 u_2 - \Delta u_2 = (\partial_2 u_1)(\partial_1 u_2 - \partial_2 u_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \end{cases} \tag{3.1}$$

cannot be regarded as a perturbation from the free system (2.17). Observe that the quadratic nonlinearity is of critical order concerning the *small data global existence* and *blowup* due to [6] and that the nonlinearities in (3.1) does not satisfy the *null condition*. Therefore it seems hopeless to have a global solution for the problem. Nevertheless, Alinhac [4] introduced some algebraic condition for (1.1) including the *null condition*, and proved the global existence result for (1.1) satisfying his condition with small initial data

$$u_j(0,x) = \varepsilon f_j(x), \quad (\partial_t u_j)(0,x) = \varepsilon g_j(x) \quad \text{for} \quad x \in \mathbf{R}^3. \tag{3.2}$$

The system (3.1) is nothing else an example satisfying the condition, hence the Cauchy problem (3.1) and (3.2) admits a unique global smooth solution (u_1, u_2). We underline that he suggests, without any rigorous proof, that his global solutions does not tends to any solution of the free system in general.

The key of the proof given in [4] is to introduce an auxiliary function $w = \partial_1 u_2 - \partial_2 u_1$. Then we have

$$\partial_t^2 w - \Delta w = Q_{12}(w, u_1), \tag{3.3}$$

where $Q_{12}(w,u_1) = (\partial_1 w)(\partial_2 u_1) - (\partial_2 w)(\partial_1 u_1)$, which is one of the null forms. Now, using (3.3), we can rewrite the system (3.1) as

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 = w(\partial_1 u_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 u_2 - \Delta u_2 = w(\partial_2 u_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 w - \Delta w = Q_{12}(w, u_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty) \end{cases} \tag{3.4}$$

with initial data (3.2) for $j = 1, 2$ and

$$w(x,0) = \varepsilon f_3(x), \quad (\partial_t w)(x,0) = \varepsilon g_3(x) \quad \text{for} \quad x \in \mathbf{R}^3. \tag{3.5}$$

where

$$f_3 = \partial_1 f_2 - \partial_2 f_1, \ g_3 = \partial_1 g_2 - \partial_2 g_1. \tag{3.6}$$

Note that the system (3.4) still does not satisfy the *null condition*, because the first and second equations in (3.4) are not written in terns of the null forms. While, the third equation in (3.4) is written in terns of the null form, hence there exists uniquely a solution v_3 of (2.6) with $c = 1$ satisfying

$$\lim_{t\to\infty} \|w(t) - v_3(t)\|_{E(1)} = 0 \tag{3.7}$$

and $\|w_2(0)\|_{E(1)} < \infty$. Having this in mind, we suppose that (3.4) can be regarded as a perturbation from

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 = v_3\,(\partial_1 v_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 v_2 - \Delta v_2 = v_3\,(\partial_2 v_1) & \text{in} \quad \mathbf{R}^3 \times (0,\infty), \\ \partial_t^2 v_3 - \Delta v_3 = 0 & \text{in} \quad \mathbf{R}^3 \times (0,\infty). \end{cases} \tag{3.8}$$

Actually we have the following result.

Theorem 3.1. *For any initial data f_1, f_2, g_1 and $g_2 \in C_0^\infty(\mathbb{R}^3)$, there exists uniquely a solution (v_1, v_2, v_3) of (3.8) satisfying (3.7) and*

$$\lim_{t\to\infty} \|u_j(t) - v_j(t)\|_{E(1)} = 0 \quad (j = 1, 2), \tag{3.9}$$

where (u_1, u_2) is the solution to the Cauchy problem (3.1) and (3.2).

Finally we state a result which shows that the asymptotic profile of (u_1, u_2) is actually different from any solutions of the free system.

Theorem 3.2. *There exist initial data f_1, f_2, g_1 and $g_2 \in C_0^\infty(\mathbb{R}^3)$ such that*

$$\lim_{t\to\infty} \|u_j(t)\|_{E(1)} = \infty \qquad (j = 1, 2) \tag{3.10}$$

holds for the solution (u_1, u_2) to the Cauchy problem (3.1) and (3.2).

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