Recovery of Boundaries and Types for Multiple Obstacles from the Far-field Pattern

Jin Cheng† Jijun Liu‡ Gen Nakamura§

† Department of Mathematics, Fudan University
Shanghai, 200433, P.R.China, **email:** jcheng@ms.fudan.edu.cn
‡ Department of Mathematics, Southeast University
Nanjing, 210096, P.R.China, **email:** jjliu@seu.edu.cn
§ Department of Mathematics, Faculty of Science
Hokkaido University, Sapporo, 060-0810, Japan
email: gnaka@math.sci.hokudai.ac.jp

Abstract

We consider an inverse scattering problem for multiple obstacles $D = \cup_{j=1}^{N} D_j \subset R^3$ with different types of boundary of D_j. By constructing an indicator function from the far-field pattern of scattered wave, we can firstly determine the boundary location for all obstacles, then identify the boundary type for each obstacle, as well as the boundary impedance in case of Robin-type obstacles. The reconstruction procedures for these identifications are also given. Comparing with the existing probing method which is applied to identify one obstacle in generally, we should analyze the behavior of both the imaginary part and the real part of the indicator function so that we can identify the type of multiple obstacles.

Keywords: Inverse scattering, probe method, uniqueness, indicator
AMS Classifications: 35R30,35J05,76Q05

1 Introduction

Let D be a bounded domain in R^3 such that $D = \cup_{j=1}^{N} D_j$, $\overline{D_i} \cap \overline{D_j} = \emptyset$ $(i \neq j)$. Each D_j is a simply connected domain with C^2 boundary ∂D_j. The scattering of time-harmonic acoustic plane waves by the obstacle D with some boundary is modelled as an exterior boundary value problem for the Helmholtz equation. That is, for a given incident plane wave $u^i(x) = e^{ikx\cdot d}$, $d \in S^2 = \{\xi \in R^3 : |\xi| = 1\}$, the total wave field $u = u^i + u^s \in H^1_{loc}(R^3 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } R^3 \setminus \overline{D} \\ Bu(x,t) = 0, & \text{on } \partial D \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{r}\right), & r = |x| \longrightarrow \infty \end{cases} \tag{1.1}$$

where B is a boundary operator corresponding to different types of the obstacle D, that is,

$$Bu = \begin{cases} u & \text{if } \partial D_j \text{ is sound-soft,} \\ \frac{\partial u}{\partial \nu} & \text{if } \partial D_j \text{ is sound-hard,} \\ \frac{\partial u}{\partial \nu} + i\sigma(x)u & \text{if } \partial D_j \text{ is Robin-type,} \end{cases} \tag{1.2}$$

where ν is the unit normal on ∂D directed into the exterior of D, $\sigma(x) > 0$ is the boundary impedance coefficient. By the results in [5], we know that there exists a unique solution for the forward scattering problem (1.1).

For the incident field $u^i(x) = e^{ikx\cdot d}$, the far-field pattern $u^\infty(\theta, d)$ can be defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u^\infty(\theta, d) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \longrightarrow \infty,$$

where $\theta, d \in S^2$.

Generally, the inverse scattering problem corresponding to (1.1) is to identify the boundary ∂D and also $\sigma(x)$ in case of Robin-type boundary, from a knowledge of far-field pattern. If D is just one obstacle, then identifying ∂D for each kind of boundary conditions has been discussed thoroughly. For example, if D is sound-soft (Dirichlet boundary condition on ∂D) or sound-hard (Neumann boundary condition on ∂D), the problems have been studied by many researchers, see [3], [6], [8], [9], [11], [14], [18]. In the case of obstacle with Robin-type boundary, the problem of reconstructing $\sigma(x)$, when ∂D is given, has also been studied , see [4], [6], [16], [17]. For the inverse scattering problem of determining both ∂D and boundary impedance, an approximate determination (or reconstruction) of the shape of D and boundary impedance was discussed in [20] by using the asymptotic behavior of the low frequency scattered waves associated with three different incident waves (or frequencies). In [13], one numerical method is proposed to determine both ∂D and impedance $\sigma(x)$. In [1] and [2], the authors proved the uniqueness result of recovering ∂D for a Robin-type obstacle with unknown boundary impedance from the far-field pattern, by applying the probe method introduced by M. Ikehata (see [8], [9], [10], [11] and [12] for example). Moreover, it has also been noticed that the probe method, as well as the point-source method proposed in [19], can be applied to determine the boundaries of multiple obstacles, if their boundary types are the same (sound-soft or sound-hard). Now, we propose a new problem: if there are many obstacles with different types of boundary such as sound-soft, sound-hard, as well as Robin-type, can we still identify their locations as well as the type of boundary for each obstacle?

This is the main topic of this paper. Our answer to this problem is "yes". More precisely, our result can be stated as follows.

Theorem 1.1 *Let D be a bounded domain consisting of finite obstacles D_j $(j = 1, 2, \cdots, N)$, namely, $D = \cup_{j=1}^N D_j$. We assume that each obstacle D_j is simply connected bounded domain with C^2 boundary ∂D_j and $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$. For given incident plane waves $u^i(x, d) = e^{ikx\cdot d}$, consider the following scattering problem for total wave field $u(x, d) = u^i(x, d) + u^s(x, d)$:*

$$\begin{cases} \Delta u + k^2 u = 0, & in\ R^3 \setminus \overline{D} \\ B_j u(x, t) = 0, & on\ \partial D_j, j = 1, 2, \cdots, N \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{r}\right), & r = |x| \longrightarrow \infty, \end{cases} \tag{1.3}$$

where B_j is one of the boundary operator in (1.2) for $j = 1, 2, \cdots, N$. Assume that $\overline{D} \subset \Omega$ for some known sphere Ω and $0 < \sigma_j(x) \in C(\partial D_j)$ for Robin-type obstacles D_j. If there exists at least one Robin-type obstacle, then from the far-field pattern $u^\infty(\theta, d)$ for all $\theta, d \in S^2$, we can

(1) determine the number of obstacles N,

(2) locate ∂D_j for $j = 1, 2, \cdots, N$,

(3) identify the type of each obstacle D_j,

(4) determine $\sigma_j(x)$ for the Robin-typed obstacles D_j.

Our main tool to deal this problem is still the probe method. This method locates the shape of an obstacle by constructing the indicator function and analyzing its behavior. However, there are some new ingredients in this paper. In the case of multiple obstacles, we not only have to determine the location of each obstacle, but also we have to determine the number of obstacles and identify the type of each obstacle. This is the major and important difference between the multiple-obstacle inversion and single-obstacle one. Especially, we should catch some characteristics of the indicator function such that we can distinguish the sound-hard obstacle and obstacle with Robin-type boundary, since in most cases, we can consider the Neumann boundary as the special case of Robin boundary with $\sigma(x) = 0$. Then the most important ingredient of this paper is that we succeeded in providing a method distinguishing sound-hard boundary and Robin-type boundary. More precisely, we can determine the positions of obstacles and identify sound-soft boundary from the real part of the indicator function, while distinguishing the sound-hard boundary from Robin-type boundary is done by considering the imaginary part of the indicator function. In order to carry out this, the most important and difficult thing is to rewrite the indicator function in an appropriate form and analyze its behavior. The number of obstacles can be obtained immediately when we get the whole image of all obstacles.

We will give a mathematically rigorous reconstruction procedure for recovering ∂D_j for $j = 1, 2, \cdots, N$. Then the uniqueness of identifying ∂D_j and the determination of number of obstacles from $u^\infty(d, \theta)$ for all $d, \theta \in S^2$ becomes obvious from the reconstruction. Since our reconstruction procedure is point wise, it is enough to consider the case that D consists of 3 obstacles with sound-soft, sound-hard and Robin-type boundary respectively, and to illustrate the reconstruction procedure for identifying the location and type for each obstacle. This does not loose any generality. More precisely, we assume that D_1, D_2, D_3 are sound soft, sound hard and Robin-type, respectively. Once we have identified the location and type of each obstacle, we determine $\sigma_3(x)$ on ∂D_3 by the moment method. So, henceforth we assume $N = 3$.

Remark 1.1 *For our problem, if $\sigma(x) \in C(\partial D_3)$, the well-posedness for this direct problem can be established from the standard scattering theory. That is, we can apply the radiation condition to get the uniqueness (Theorem 3.12, [5]) and use the combined single-layer and double-layer theory to get the existence of the solution.*

Remark 1.2 *We can also identify D_1 and D_2 in case of $D_3 = \emptyset$ if we assume the unique solvability of the boundary value problem (2.2) given later.*

Remark 1.3 *Our reconstruction method considers the inverse scattering problem with multiple obstacles with different boundary types by firstly transform the far-field pattern of scattered wave to a Dirichlet-to-Neumann map defined on the surface of a ball, and then apply this D-to-N map as inversion data to recover the obstacles by probe method. From the numerical point of view, we should consider the influence of noisy in far-field pattern and error in computation on our final inversion results. The main difficulty is that we should solve an integral equation of second kind with hyper-singularity (see (2.4) in the sequel) from noisy far-field to construct the D-to-N map. This problem can be solved by the technique proposed in [15]. The numerical test of this problem as well as its influence on our inversion algorithm is being consideration.*

Our paper is organized as follows:

- Section 2: Preliminary results
- Section 3: Probe method
- Section 4: Moment method for determining $\sigma(x)$
- Section 5: Some estimates
- Section 6: Singularity Analysis

2 Preliminary results

In this section, we give some known results for the probe method, which are necessary for our paper.

Without loss of generality, we assume that $\overline{D} \subset B(0, \frac{R}{2})$ for some constant $R > 0$. We also assume that 0 is not a Dirichlet eigenvalue of $\Delta + k^2$ in $\Omega := B(0, R)$ for given $k > 0$.

Proposition 2.1 *The scattered solution $u^s(x, d)$ for $|x| > \frac{R}{2}$ can be determined uniquely from $u^\infty(d, \theta)$.*

The physical background for this proposition is obvious, that is, the far-field pattern of scattered wave determines the near-field outside the obstacle completely. This procedure has nothing to do with the boundary conditions of scatterers. For the proof, see Theorem 3.6 and Corollary 3.8 in [5], or [1], [2].

Let $G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ be the fundamental solution of the Helmholtz equation. For each $y \in R^3 \setminus \overline{D}$, we define $E(\cdot, y) \in H^1_{loc}(R^3 \setminus \overline{D})$ as the solution to

$$\begin{cases} \Delta E + k^2 E = 0, & \text{in } R^3 \setminus \overline{D} \\ B_i E(x, y) = -B_i G(x, y) & \text{on } \partial D_i, \quad i = 1, 2, 3 \\ \frac{\partial E}{\partial r} - ikE = O\left(\frac{1}{r}\right), & r = |x| \longrightarrow \infty. \end{cases} \tag{2.1}$$

Proposition 2.2 *For $x, y \in \partial\Omega$, $E(x, y)$, $\frac{\partial}{\partial\nu(x)}E(x, y)$ and $\frac{\partial}{\partial\nu(y)}E(x, y)$ can be determined from $u^\infty(d, \theta)$ for all $d, \theta \in S^2$.*

The proof for $D = D_1 \cup D_2 \cup D_3$ given here is an analogy to that given in [1] for $D = D_3$.

Proof: In fact, since we can chose R, so we assume that 0 is not the Dirichlet eigenvalue of $\Delta + k^2$. Therefore $\{e^{ikx.d} | d \in S^2\}$ is dense in $L^2(\partial\Omega)$([6], Theorem 5.5). For any fixed $y \in \partial B(0, R_1)(R_1 > R)$, there exists a sequence $\{\alpha_j^n(y), d_j^n(y)\}$ such that

$$\sum_{1 \le j \le m_n(y)} \alpha_j^n(y) e^{ikx.d_j^n(y)} \to G(x - y) \quad in \quad L^2(\partial\Omega).$$

as $n \to \infty$. On the other hand, since both $\sum_{1 \le j \le m_n(y)} \alpha_j^n(y) e^{ikx.d_j^n(y)}$ and $G(x - y)$ satisfy Helmholtz equation in Ω, by the result in [6](Theorem 5.4), we know that $\sum_{1 \le j \le m_n(y)} \alpha_j^n(y) e^{ikx.d_j^n(y)} \to G(x - y)$ uniformly on any compact subset of Ω (together with all their derivatives). Therefore it follows that

$$B_i \sum_{1 \le j \le m_n(y)} \alpha_j^n(y) e^{ikx \cdot d_j^n(y)} \to B_i G(x - y)$$

in $L^2(\partial D_i), i = 1, 2, 3$ as $n \to \infty$. Since $u^s(x, d) \in H^1(R^3 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u^s + k^2 u^s = 0, & in \quad R^3 \setminus \overline{D} \\ B_i u^s = -B_i e^{ikx \cdot d}, & on \quad \partial D_i, \quad i = 1, 2, 3 \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{r}\right), & r = |x| \to \infty, \end{cases}$$

by the continuous dependance of direct scattering problem, we have

$$\sum_{1 \le j \le m_n(y)} \alpha_j^n(y) u^s(x, d_j^n(y)) \to E(x, y) \text{ uniformly on } R/2 < |x| < 2R$$

for $y \in \partial B(0, R_1)$. Now we get from Proposition 1 that, for $y \in \partial B(0, R_1)$ and $R/2 < |x| < 2R$, $E(x, y)$, $\frac{\partial E(x,y)}{\partial \nu(x)}$ $\frac{\partial E(x,y)}{\partial \nu(y)}$ can be determined by $\{u^\infty(d, \theta) : \theta, d \in S^2\}$. Since R_1 is arbitrary, we complete the proof by letting $R_1 \to R$. $\square$

Consider a solution $u(x) \in H^1(\Omega \setminus \overline{D})$ to the following boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j u(x, t) = 0, & \text{on } \partial D_j, j = 1, 2, 3 \\ u(x) = f, & \text{on } \partial \Omega \end{cases} \tag{2.2}$$

for given $f \in H^{1/2}(\partial \Omega)$.

Since we have used $\overline{D}$ to indicate the closure of domain D, we will use $\tilde{z}$ to indicate the complex conjugate of complex number z in the sequel.

Lemma 2.1 *If $D_3 \neq \emptyset$, then there exists a unique solution to (2.2) for any $f \in H^{1/2}(\partial \Omega)$.*

Proof Firstly, we prove the uniqueness. It is enough to prove that $f = 0$ implies $u = 0$ in $\Omega \setminus \overline{D}$. For $f = 0$, it is easy to see from (2.2) that

$$0 = \int_{\Omega \setminus \overline{D}} (\Delta u + k^2 u) \tilde{u} dx = \int_{\partial D} \tilde{u} \frac{\partial u}{\partial \nu} ds - \int_{\Omega \setminus \overline{D}} (\nabla u \cdot \nabla \tilde{u} - k^2 u \tilde{u}) dx,$$

$$0 = \int_{\Omega \setminus \overline{D}} (\Delta \tilde{u} + k^2 \tilde{u}) u dx = \int_{\partial D} u \frac{\partial \tilde{u}}{\partial \nu} ds - \int_{\Omega \setminus \overline{D}} (\nabla \tilde{u} \cdot \nabla u - k^2 \tilde{u} u) dx$$

due to $u = 0$ on $\partial \Omega$. Subtracting these two equalities and noticing the boundary conditions of u on ∂D_1 and ∂D_2 lead to

$$\int_{\partial D_3} \left(\tilde{u} \frac{\partial u}{\partial \nu} - u \frac{\partial \tilde{u}}{\partial \nu} \right) ds = 0.$$

Now the boundary condition in ∂D_3 leads to

$$\int_{\partial D_3} \sigma(x) |u(x)|^2 ds = 0$$

from which we get $u = 0$ on ∂D_3 and $\frac{\partial u}{\partial \nu} = 0$ on ∂D_3 from the boundary condition. Now the uniqueness of the Cauchy problem for the Helmholtz equations outside D_3 implies $u = 0$ in $\Omega \setminus \overline{D}$. On the other hand, by the integral equation method for the scattering problem ([5], [6]), we know that the direct problem (2.2) can be transformed into a Fredholm integral equation of the second kind, therefore the uniqueness implies the existence due to the Fredholm alternative theorem. □

Remark 2.1 *The existence of D_3 is important to the proof of our uniqueness. If we do not have an obstacle with Robin-type boundary, i.e., $D = D_1 \cup D_2$, then we have to assume the uniqueness of*

$$\begin{cases} \Delta u + k^2 u = 0 & in\ \Omega \setminus (D_1 \cup D_2) \\ B_i u = 0 & on\ \partial D_i, i = 1, 2 \\ u(x) = f & on\ \partial\Omega. \end{cases} \tag{2.3}$$

Define the Dirichlet-to-Neumann map $\Lambda_{\partial D,\sigma}$ formally by

$$\Lambda_{\partial D,\sigma} : f \longrightarrow \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

where $u \in H^1(\Omega \setminus \overline{D})$ is the solution of (2.2) for $f \in H^{1/2}(\partial\Omega)$. In the next Lemma, we show the relations between the far-field patterns and the Dirichlet-to-Neumann map.

Lemma 2.2 *Let u be the solution to (2.2) for $f \in H^{1/2}(\partial\Omega)$. Then, $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ can be obtained from $f(x)$ and $u^\infty(d, \theta)$ for $\theta, d \in S^2$.*

Proof Let $x_0 \in \partial B(0, R_0)$ for $R/2 < R_0 < R$. By the Green's formula, we have that, for $G_D = G_D(x, x_0) = G(x, x_0) + E(x, x_0)$,

$$\begin{aligned} u(x_0) &= \int_{\partial\Omega} \left(G_D \frac{\partial u}{\partial \nu_1} - u \frac{\partial G_D}{\partial \nu_1} \right) ds + \sum_{j=1}^{3} \int_{\partial D_j} \left(G_D \frac{\partial u}{\partial \nu_1} - u \frac{\partial G_D}{\partial \nu_1} \right) ds \\ &= \int_{\partial\Omega} \left(G_D \frac{\partial u}{\partial \nu_1} - f \frac{\partial G_D}{\partial \nu_1} \right) ds, \end{aligned}$$

because $B_j G_D = B_j u = 0$ on ∂D_j, where ν_1 is the outward normal to the boundary of domain $\Omega \setminus \overline{D}$.

Taking the normal derivatives of u on $\partial B(0, R_0)$ in above expression and letting $R_0 \to R$, by the properties of the single layer potential and double layer potential ([5]), we have

$$\frac{1}{2} \frac{\partial u(x_0)}{\partial \nu_1(x_0)} = \int_{\partial\Omega} \frac{\partial G_D(x, x_0)}{\partial \nu_1(x_0)} \frac{\partial u(x)}{\partial \nu_1(x)} ds - \frac{\partial}{\partial \nu_1(x_0)} \int_{\partial\Omega} f(x) \frac{\partial G_D(x, x_0)}{\partial \nu_1(x)} ds \tag{2.4}$$

for $x_0 \in \partial B(0,R) = \partial\Omega$. The equation (2.4) is a Fredholm integral equation of the second kind with respect to $\frac{\partial u(x)}{\partial \nu(x)}|_{\partial\Omega}$. There exists a unique solution due to the unique solvability of (2.2).

By Proposition 2.2, we know that, for $x, y \in \partial\Omega$, $\frac{\partial E(x,y)}{\partial \nu(x)}$ and $\frac{\partial E(x,y)}{\partial \nu(y)}$ can be obtained from $u^\infty(d,\theta)$, $\theta, d \in S^2$. Therefore $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ can be obtained from $u^\infty(d,\theta)$, $\theta, d \in S^2$ and $f(x)$.

The proof is complete. □

From this lemma, we see that the original inverse problem can be restated as the problem of reconstructing the shapes of the 3 obstacles and the boundary impedance of D_3 from the Dirichlet-to-Neumann map $\Lambda_{\partial D,\sigma}$.

Remark 2.2 *The Dirichlet-to-Neumann map $\Lambda_{\partial D,\sigma}$ can be defined by the following weak form.*

For any $g \in H^{1/2}(\partial\Omega)$, take any $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Then it follows

$$\begin{aligned}\langle \Lambda_{\partial D,\sigma} f, g\rangle &= \int_{\Omega\setminus\overline{D}} (\nabla u \cdot \nabla v - k^2 uv)dx + \int_{\partial D} v\frac{\partial u}{\partial \nu} ds(x) \\ &= \int_{\Omega\setminus\overline{D}} (\nabla u \cdot \nabla v - k^2 uv)dx + \int_{\partial D_1} v\frac{\partial u}{\partial \nu} ds(x) - \\ &\quad \int_{\partial D_3} i\sigma(x) uv ds(x) \end{aligned} \tag{2.5}$$

for any $f \in H^{1/2}(\partial\Omega)$, where u is the solution to (2.2) for $f \in H^{1/2}(\partial\Omega)$.

Corresponding to the case $D = \emptyset$, we can formally define the Dirichlet-to-Neumann map $\Lambda_{0,0} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ by

$$\Lambda_{0,0} : f \longrightarrow \frac{\partial u_1}{\partial \nu}|_{\partial\Omega}$$

where $u_1(x) \in H^1(\Omega)$ is the solution to

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0, & \text{in } \Omega \\ u_1(x) = f \in H^{1/2}(\partial\Omega), & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Here note that by the assumption 0 is not the Dirichlet eigenvalue of the operator $\Delta + k^2$ in Ω, (2.6) is uniquely solvable.

The weak formula of $\Lambda_{0,0}$ is given by

$$\langle \Lambda_{0,0} f, g\rangle = \int_\Omega (\nabla u_1 \cdot \nabla v - k^2 u_1 v)dx, \tag{2.7}$$

where u_1 is the solution of (2.6) for $f \in H^{1/2}(\partial\Omega)$ and $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = g$ for $g \in H^1(\partial\Omega)$. For the solution u of (2.2) and the solution u_1 of (2.6), we have

Lemma 2.3 *Let $u \in H^1(\Omega \setminus \overline{D})$ and $u_1 \in H^1(\Omega)$ be the solutions to (2.2) and (2.6), respectively. There exists a constant $C = C(k, R, \sigma_0)$ such that, for all $f \in H^{1/2}(\partial\Omega)$,*

$$\|u - u_1\|_{H^1(\Omega\setminus\overline{D})} \leq C \|u_1\|_{H^1(D)}$$

where $\sigma_0 > 0$ is a constant satisfying $0 < \sigma(x) \leq \sigma_0$, $x \in \partial D_3$.

The proof of Lemma 2.3 is almost the same as that given in [10]. But for the readers convenience we give the proof in Section 5.

3 Probe Method

Definition 1 *For any continuous curve $c = \{c(t)\,|\,0 \leq t \leq 1\}$, if it satisfies $c(0), c(1) \in \partial\Omega$ and $c(t) \in \Omega \quad (0 < t < 1)$, then we call c a needle in Ω.*

Definition 2 *For any needle c in Ω, we call*

$$t(c, D) = \sup\{0 < t < 1 | c(s) \in \Omega \setminus \overline{D} \text{ for all } 0 < s < t\}$$

geometric impact parameter(GIP). It is obvious that $t(c, D) = 1$ if c does not touch any point on ∂D.

From this definition, we know if a needle c touches $\overline{D}$, then $t(c, D) < 1$ and $t(c, D)$ is the first hitting time, i.e., $c(t(c, D)) \in \partial D$ and $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < t(c, D)$.

Since $\Omega \setminus \overline{D}$ is connected, we have a reconstruction algorithm for ∂D in terms of the geometric impact parameter and the needle, i.e.,

$$\partial D = \{c(t)\,|\,t = t(c, D),\ c \text{ is a needle and } t(c, D) < 1\}. \tag{3.1}$$

In order to reconstruct ∂D, it suffices to consider the problem of calculating the GIP for each needle from the Dirichlet-to-Neumann map.

Lemma 3.1 *Suppose that Γ is an arbitrary open set of $\partial\Omega$. For each $0 < t < 1$, there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $H^1(\Omega)$, which satisfies*

$$\Delta v_n + k^2 v_n = 0$$

such that $supp(v_n|_{\partial\Omega}) \subset \Gamma$ and

$$v_n \longrightarrow G(\cdot - c(t)) \quad in \quad H^1_{loc}(\Omega \setminus \{c(t')|0 < t' \leq t\}).$$

This result comes from the Runge approximation theorem, see [8], [9].

Remark 3.1 *Usually the Runge approximation is not constructive, because its proof is done by using the unique continuation and Hahn-Banach theorem. However, for the Helmholtz equation, it is possible to make the Runge approximation constructive by using the translation theory (see [7]).*

It is obvious that $v_n|_{\partial\Omega}$ depends on $c(t)$. We denote it by $v_n|_{\partial\Omega} = f_n(\cdot, c(t))$, where $f_n(\cdot, c(t)) \in H^{1/2}(\partial\Omega)$ and $supp(f_n(\cdot, c(t)) \subset \Gamma$.

Definition 3 *For a given needle c in Ω and $0 < t < 1$, we define the indicator function*

$$I(t, c) = \lim_{n \longrightarrow \infty} \langle \widetilde{\overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))}}, f_n(\cdot, c(t)) \rangle, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, $\widetilde{\overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))}}$ is the complex conjugate of $(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))$.

Next we show that $\Re I(t, c)$ and $\Im I(t, c)$ ($\Re, \Im$ denote the real part and imaginary part respectively) can be used to calculate GIP from which the locations of 3 obstacles can be determined, and we can also identify the type of each obstacle.

Theorem 3.1 *For a given needle $c(t)$ in Ω, it follows that*
(A) $c(t(c, D)) \in \partial D$ if and only if
(1) $I(t, c)$ exists for all $0 \le t < t(c, D)$ and

$$|\Re(I(t, c))| < +\infty, \qquad for \quad 0 \le t < t(c, D),$$

(2) $\lim_{t \to t(c,D)-} |\Re I(t, c)| = +\infty$.
(B) when $c(t(c, D)) \in \partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$, we can identify ∂D_i for $i = 1, 2, 3$ by

$$\lim_{t \to t(c,D)-} \Re I(t, c) = +\infty \Longleftrightarrow c(t(c, D)) \in \partial D_1,$$

$$\lim_{t \to t(c,D)-} \Re I(t, c) = -\infty \text{ and } \lim_{t \to t(c,D)-} \Im I(t, c) < +\infty \Longleftrightarrow c(t(c, D)) \in \partial D_2,$$

$$\lim_{t \to t(c,D)-} \Re I(t, c) = -\infty \text{ and } \lim_{t \to t(c,D)-} \Im I(t, c) = +\infty \Longleftrightarrow c(t(c, D)) \in \partial D_3.$$

Remark 3.2 *The result (A) gives a criterion for the geometric impact parameter $t(c, D)$ for a fixed needle $c(t)$. Furthermore, since $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$, we can identify ∂D_i according to (B).*

Proof For a given needle $c(t)$, by Lemma 3.1, we know that there exists a sequences $\{v_n(x)\} \in H^1(\Omega)$ which satisfies

$$\begin{cases} \Delta v_n + k^2 v_n = 0, & \text{in } \Omega \\ v_n = f_n(\cdot, c(t)), & \text{on } \partial\Omega \ ; \qquad supp(f_n(\cdot, c(t))) \subset \Gamma, \end{cases}$$

and

$$v_n \longrightarrow G(\cdot - c(t)) \qquad in \quad H^1_{loc}(\Omega \setminus \overline{\{c(t')|0 < t' \le t\}}) \quad (n \longrightarrow \infty).$$

Let $u_n(x) \in H^1(\Omega \setminus \overline{D})$ satisfy

$$\begin{cases} \Delta u_n + k^2 u_n = 0, & \text{in } \Omega \setminus \overline{D} \\ B_i u_n = 0, & \text{on } \partial D_i, \quad i = 1, 2, 3 \\ u_n = f_n, & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

then $w_n = u_n - v_n|_{\Omega \setminus \overline{D}} \in H^1(\Omega \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta w_n + k^2 w_n = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j w_n = -B_j v_n, & \text{on } \partial D_j, \quad j = 1, 2, 3 \\ w_n = 0, & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.3 and Lemma 3.1, we know that, for $c(t) \notin \overline{D}$, it holds that

$$w_n \longrightarrow w \quad \text{in} \quad H^1(\Omega \setminus \overline{D}), \qquad n \longrightarrow \infty, \tag{3.4}$$

where $w = w(x, c(t))$ satisfies

$$\begin{cases} \Delta w + k^2 w = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j w = -B_j G(\cdot - c(t)), & \text{on } \partial D_j \quad j = 1, 2, 3 \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

On the other hand, by the calculation in Section 6, we have two kinds of expressions for $\langle \overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n}^{\sim}, f_n \rangle$, i.e.,

$$\begin{aligned} &\langle \overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))}^{\sim}, f_n(\cdot, c(t)) \rangle \\ &= -\int_{\Omega \setminus \overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx + \\ &\int_{\partial D_3} \left(2i\sigma v_n \tilde{w}_n + i\sigma |v_n|^2 + i\sigma |w_n|^2 \right) ds + \\ &\int_{\partial D_1} \left[\left(v_n \frac{\partial \tilde{w}_n}{\partial \nu} - \tilde{w}_n \frac{\partial v_n}{\partial \nu} \right) + \left(v_n \frac{\partial \tilde{v}_n}{\partial \nu} - \tilde{w}_n \frac{\partial w_n}{\partial \nu} \right) \right] ds. \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\langle \overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n(\cdot, c(t))}^{\sim}, f_n(\cdot, c(t)) \rangle \\ &= \int_{\Omega \setminus \overline{D}} \left(|\nabla w_n|^2 - k^2 |w_n|^2 \right) dx + \int_D \left(|\nabla v_n|^2 - k^2 |v_n|^2 \right) dx - \\ &\int_{\partial D_2} \left[\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} \right] ds - \int_{\partial D_3} \left[\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} - i\sigma |u_n|^2 \right] ds \end{aligned} \tag{3.7}$$

Let n tend to infinity in (3.6). Then, by (3.4), we have

$$\begin{aligned}
-I(t,c) &= \int_D \{|\nabla G(\cdot - c(t))|^2 - k^2|G(\cdot - c(t))|^2\}dx \\
&+ \int_{\Omega\setminus\overline{D}} \{|\nabla w|^2 - k^2|w|^2\}dx \\
&-i \int_{\partial D_3} \sigma(x)\{|G(\cdot - c(t))|^2 + |w|^2\}ds - 2i\int_{\partial D_3} \sigma(x)\tilde{w}Gds - \\
&\int_{\partial D_1}\left[\left(G\frac{\partial \tilde{w}}{\partial \nu} - \tilde{w}\frac{\partial G}{\partial \nu}\right) + \left(G\frac{\partial \tilde{G}}{\partial \nu} - \tilde{w}\frac{\partial w}{\partial \nu}\right)\right]ds. \qquad (3.8)
\end{aligned}$$

If $t(c,D) = 1$, then, by the definition of $t(c,D)$, we know that $c(t)$ does not touch ∂D, i.e. $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < 1$. Since $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < 1$ and $c(1) \in \partial\Omega$, it is easy to verify that

$$\lim_{t\longrightarrow 1} \Re I(t,c) \neq -\infty, \qquad |\Re I(t,c)| < +\infty, \qquad \text{for} \quad 0 \leq t < 1.$$

If $t(c,D) < 1$, then we know that $c(t) \in \Omega \setminus \overline{D}$ for $0 < t < t(c,D)$ and $x_0 = c(t(c,D)) \in \partial D$.

Likewise before, since $c(t) \in \Omega \setminus \overline{D}\,(0 \leq t < t(c,D))$, we have $|\Re(I(t,c))| < +\infty$ for $0 \leq t < t(c,D)$. On the other hand,

$$\begin{aligned}
- \quad & \Re I(t,c) = \int_D [|\nabla G(\cdot - c(t))|^2 - k^2|G(\cdot - c(t))|^2]dx \\
+ \quad & \int_{\Omega\setminus\overline{D}} (|\nabla w|^2 - k^2|w|^2)dx + 2\int_{\partial D_3} \sigma(x)\Im(\tilde{w}G(\cdot - c(t)))ds - \\
& \Re\left[\int_{\partial D_1}\left(G\frac{\partial \tilde{w}}{\partial \nu} - \tilde{w}\frac{\partial G}{\partial \nu} + G\frac{\partial \tilde{G}}{\partial \nu} - \tilde{w}\frac{\partial w}{\partial \nu}\right)ds\right] \\
\geq \quad & \int_D |\nabla G(\cdot - c(t))|^2 dx - k^2\int_D |G(\cdot - c(t))|^2 dx - k^2\int_{\Omega\setminus\overline{D}} |w|^2 dx \\
+ \quad & 2\int_{\partial D_3} \sigma(x)\Im(\tilde{w}G(\cdot - c(t)))ds - \\
& \Re\left[\int_{\partial D_1}\left(G\frac{\partial \tilde{w}}{\partial \nu} - \tilde{w}\frac{\partial G}{\partial \nu} + G\frac{\partial \tilde{G}}{\partial \nu} - \tilde{w}\frac{\partial w}{\partial \nu}\right)ds\right] \qquad (3.9)
\end{aligned}$$

We can identify ∂D from the real part of indicator function $I(t,c)$. In fact, according to the result of singularity analysis about $w(x,x_0)$ and $G(x - x_0)$ for $x_0 \in \partial D$ given in Theorem 5.1 and Theorem 5.2 below, we have from (3.9),

$$\lim_{t\longrightarrow t(c,D_j)-} \Re(I(t,c)) = -\infty.$$

if $j=2,3$. On the other hand, consider the real part of the limit of the real part of (3.7) as $n\to\infty$. It is easy to find that the real part will tend to $+\infty$ when $c(t)\to\partial D_1$, since $\int_D|\nabla G(\cdot-c(t))|^2dx$ will blow up, while the integrals on the boundary are bounded. These facts imply that we can distinguish the sound-soft boundary D_1 from the other two kinds of boundaries (sound-hard and Robin-type). Now we want to distinguish ∂D_2 and ∂D_3 furthermore. For this purpose, we need to consider the imaginary part of (3.7). In fact, it yields from (3.6) that

$$\Im(\overbrace{\langle(\Lambda_{\partial D,\sigma}-\Lambda_{0,0})f_n}^{\sim},f_n\rangle)=\int_{\partial D_3}\sigma(x)|u_n|^2ds. \tag{3.10}$$

Now we estimate the behavior of the imaginary part of indicator function. Remind our previous notations, we get

$$\begin{aligned}
&\lim_{n\to\infty}\Im(\overbrace{\langle(\Lambda_{\partial D,\sigma}-\Lambda_{0,0})f_n}^{\sim},f_n\rangle)=\lim_{n\to\infty}\int_{\partial D_3}\sigma(x)|u_n|^2ds\\
&=\lim_{n\to\infty}\int_{\partial D_3}\sigma(x)|v_n+w_n|^2ds\\
&=\int_{\partial D_3}\sigma(x)|(G(x-c(t)))+w(x,c(t))|^2ds,
\end{aligned}\tag{3.11}$$

where w is the function defined by (3.5). According to the singularity analysis in section 5 and section 6, we know that $|G(x-c(t))+w(x,c(t))|$ is estimated by $|G(x-c(t))|$. Hence, from (3.11) and the estimate for G in section 5, we have

$$\lim_{t\to t(c,D_3)-}\Im I(t,c)=\lim_{t\to t(c,D_3)-}\int_{\partial D_3}\sigma(x)|(G(x-c(t)))+w(x,c(t))|^2ds=+\infty,$$

$$\lim_{t\to t(c,D_2)-}\Im I(t,c)=\lim_{t\to t(c,D_2)-}\int_{\partial D_3}\sigma(x)|G(x-c(t))+w(x,c(t))|^2ds<\infty.$$

Since D_2 and D_3 are separated, these behavior of $\Im I(t,c)$ enable us to distinguish ∂D_3 and ∂D_2.

The proof is complete. □

Now we give the reconstruction procedure for the shape and type of each obstacle. It can be realized by the following steps:

- Calculate the Dirichlet-to-Neumann map $\Lambda_{\sigma,D}$ from the far field patterns $u^\infty(d,\theta),\ d,\theta\in S^2$.
- For any given needle $c(t)$, calculate the sequences v_n and $f_n(\cdot,c)$.
- Calculate $\overbrace{\langle(\Lambda_{\partial D,\sigma}-\Lambda_{0,0})f_n(\cdot,c(t))}^{\sim},f_n(\cdot,c(t))\rangle$.

- Calculate $I(c,t)$ and

$$\partial D_1 = \{c(t_0):\ |\Re I(c,t)| < +\infty \quad \text{for} \quad 0 \le t < t_0; \lim_{t\to t_0-} \Re I(c,t) = +\infty\};$$

$$\begin{aligned}\partial D_2 = \{\ & c(t_0):\ |\Re I(c,t)|,\, |\Im I(c,t)| < +\infty \quad \text{for} \quad 0 \le t < t_0; \\ & \lim_{t\to t_0-} \Re I(t,c) = -\infty \text{ and } \lim_{t\to t_0-} \Im I(t,c) < +\infty\};\end{aligned}$$

$$\begin{aligned}\partial D_3 = \{\ & c(t_0):\ |\Re I(c,t)|,\, |\Im I(c,t)| < +\infty \quad \text{for} \quad 0 \le t < t_0; \\ & \lim_{t\to t_0-} \Re I(t,c) = -\infty \text{ and } \lim_{t\to t_0-} \Im I(t,c) = +\infty\}.\end{aligned}$$

The rest of the part of the proof of Theorem 1.1 is to reconstruct boundary impedance on D_3. This will be given in the next section.

4 Moment method for determining $\sigma(x)$

In this section, we reconstruct the boundary impedance $\sigma(x)$. Since in the previous section, we have reconstructed ∂D from the far field patterns $u^\infty(d,\theta)$, $d,\theta \in S^2$, therefore in this section we assume that $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$ is known.

Consider the boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j u(x) = 0, & \text{on } \partial D_j, \quad j = 1,2 \\ \frac{\partial u}{\partial \nu} + i\sigma(x) u = 0, & \text{on } \partial D \\ u(x) = f, & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

for a given $f(x) \in H^{1/2}(\partial\Omega)$.

Lemma 4.1 *Suppose that $u_j(x) \in H^1(\Omega\setminus\overline{D})$, $j = 1,2,\cdots$ satisfy (4.1) with $f = f_j$. Put $\phi_j(x) = u_j(x)|_{\partial D_3}$. If*

$$\overline{span\{f_j(x)\}} = H^{1/2}(\partial\Omega), \tag{4.2}$$

then we have

$$\overline{span\{\phi_j(x)\}} = H^{1/2}(\partial D_3).$$

Proof Assume that $f(x) \in H^{-1/2}(\partial D_3)$ which satisfies

$$\int_{\partial D_3} \phi_j \tilde{f} ds = 0, \qquad j = 1,2,\cdots, \tag{4.3}$$

we want to prove that $f(x) = 0$. Here $\int_{\partial D_3} \phi_j \tilde{f} ds$ denotes the pairing $\langle \tilde{f}, \phi_j \rangle$ between $H^{-1/2}(\partial D_3)$ and $H^{1/2}(\partial D_3)$.

Consider the following boundary value problem

$$\begin{cases} \Delta v + k^2 v = 0, & \text{in } \Omega \setminus \overline{D} \\ B_j v(x) = 0, & \text{on } \partial D_j, \quad j = 1, 2 \\ \frac{\partial v}{\partial \nu} + i\sigma(x) v = \tilde{f}, & \text{on } \partial D_3 \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Since $\sigma(x) > 0$, likewise the proof of Lemma 2.1, we know there exists a unique solution $v \in H^1(\Omega \setminus \overline{D})$ for (4.4).

By the Green's formula, we know that

$$\begin{aligned} 0 &= \int_{\Omega\setminus\overline{D}} (v\Delta u_j - u_j \Delta v) dx \\ &= \int_{\partial\Omega} \left(\frac{\partial u_j}{\partial \nu_1} v - \frac{\partial v}{\partial \nu_1} u_j \right) + \int_{\partial D} \left(\frac{\partial u_j}{\partial \nu_1} v - \frac{\partial v}{\partial \nu_1} u_j \right), \end{aligned} \tag{4.5}$$

where ν_1 is the outward normal of domain $\Omega \setminus \overline{D}$.

Noticing $\nu_1 = -\nu$ on ∂D and $v|_{\partial\Omega} = 0$, we have

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} u_j ds = \int_{\partial D_3} \left(i\sigma v u_j + \frac{\partial v}{\partial \nu} u_j \right) ds$$

due to the boundary conditions on ∂D_1 and ∂D_2. Therefore, it holds that

$$\int_{\partial\Omega} f_j \frac{\partial v}{\partial \nu} ds = \int_{\partial D_3} \phi_j \tilde{f} ds = 0, \qquad j = 1, 2, \cdots$$

Since $\overline{span\{f_j(x)\}} = H^{1/2}(\partial\Omega)$, we obtain

$$\frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0.$$

By the uniqueness of the Cauchy problem for the Helmholtz equations in domain $\Omega \setminus \overline{D}$, we have $v(x) = 0$ in $\Omega \setminus \overline{D}$. Then by (4.4), we know $f(x) = 0$. The proof is complete. □

On the other hand, we can obtain $u_j|_{\partial D}$ and $\frac{\partial u_j}{\partial \nu}|_{\partial D}$ by solving the the following Cauchy problem

$$\begin{cases} \Delta u_j + k^2 u_j = 0, & \text{in } \Omega \setminus \overline{D} \\ u_j = f_j, \frac{\partial u_j}{\partial \nu} = \Lambda_{\partial D, \sigma} f_j, & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

for a given $f_j(x)$, hence both $u_j|_{\partial D_3}$ and $\frac{\partial u_j}{\partial \nu}|_{\partial D_3}$ are obtained.

Now, by integrating the Robin-type boundary condition over ∂D_3, we have that the impedance $\sigma(x)$ satisfies

$$\int_{\partial D_3} i\sigma(x) u_j ds = -\int_{\partial D_3} \frac{\partial u_j}{\partial \nu} ds, \qquad j = 1, 2, \cdots \tag{4.7}$$

Here note that $span\{u_j|_{\partial D_3}\}$ is dense in $H^{1/2}(\partial D_3)$ by Lemma 4.1, hence $\sigma(x)$ can be determined uniquely from this moment problem.

Now the recovery of the impedance $\sigma(x)$ can be realized by the following steps:

- Choose f_j, $j = 1, 2, \cdots$ such that $\overline{span\{f_j\}_{j=1}^{\infty}} = H^{1/2}(\partial\Omega)$.
- For every f_j, solve the Cauchy problem (4.6) and obtain $u_j|_{\partial D_3}$ and $\frac{\partial u_j}{\partial \nu}|_{\partial D_3}$.
- Solve the moment problem (4.7) to get $\sigma(x)$.

5 Some Estimates

In this section we give the the proof of Lemma 2.3 and estimate of $\|w\|_{L^2(\Omega\setminus\overline{D})}$.

Proof of Lemma 2.3

Let $p(x) = u(x) - u_1(x)|_{\Omega\setminus\overline{D}}$. It is easy to verify that $p(x)$ satisfies

$$\begin{cases} \Delta p + k^2 p = 0, & \text{in } \Omega\setminus\overline{D} \\ B_j p = -B_j u_1, & \text{on } \partial D_j, j = 1,2,3 \\ p(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

By the well-posedness of the boundary value problem (5.1), we know that the solution $p(x) \in H^1(\Omega\setminus\overline{D})$ depends continuously on the boundary data on $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$. Therefore, there exists a constant $C > 0$ such that

$$\|p\|_{H^1(\Omega\setminus\overline{D})} \le C\{\|B_1 u_1\|_{H^{1/2}(\partial D_1)} + \|B_2 u_1\|_{H^{-1/2}(\partial D_2)} + \|B_3 u_1\|_{H^{-1/2}(\partial D_3)}\} \tag{5.2}$$

On the other hand, the trace theorem for $u_1(x)$ on the domain D yields

$$\|B_1 u_1\|_{H^{1/2}(\partial D_1)} + \|B_2 u_1\|_{H^{-1/2}(\partial D_2)} + \|B_3 u_1\|_{H^{-1/2}(\partial D_3)} \le C\,\|u_1\|_{H^1(D)}$$

due to $0 < \sigma(x) < \sigma_0$, and the proof is complete. □

Theorem 5.1 *There exists a constant C independent of D such that*

$$\|w\|_{L^2(\Omega\setminus\overline{D})} \le C.$$

Proof We adapt the proof of [10] to our case. First we define a function $v(x)$ by

$$\begin{cases} \Delta v + k^2 v = w, & \text{in } \Omega\setminus\overline{D} \\ B_j v = 0, & \text{on } \partial D_j, \quad j = 1,2 \\ \frac{\partial v}{\partial \nu} + i\widetilde{\sigma} v = 0, & \text{on } \partial D_3 \\ v(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Then, from the well-posedness of this boundary problem, we have

$$\|v\|_{H^2(\Omega\backslash\overline{D})} \le C\,\|w\|_{L^2(\Omega\backslash\overline{D})}\,. \tag{5.4}$$

Since $\Omega\backslash\overline{D}$ is a domain in R^3 with C^2 boundary, by the Sobolev embedding theorems, we know that $H^2(\Omega\setminus\overline{D})$ can be embedded into $B^{1/2}(\Omega\setminus\overline{D})$ (Hölder space with exponent 1/2) and $\|v\|_{B^{1/2}} \le C\,\|v\|_{H^2}$. So we have

$$\|v\|_{B^{1/2}} \le C\,\|w\|_{L^2}\,.$$

From this inequality, we know that

$$\begin{cases} |v(x)-v(y)| \le C|x-y|^{1/2}\,\|w\|_{L^2(\Omega\backslash\overline{D})}\,, & x,y\in\Omega\setminus\overline{D} \\ \|v\|_{L^\infty(\Omega\backslash\overline{D})} \le C\,\|w\|_{L^2(\Omega\backslash\overline{D})}\,. \end{cases} \tag{5.5}$$

Remind the definition of weak solutions w and v to (3.5) and (5.3) respectively, by the Green formula and the boundary conditions for v, w, we have

$$\begin{aligned} 0 &= \int_{\Omega\backslash\overline{D}}(\Delta+k^2)\tilde{w}(x)v(x)dx \\ &= -\int_{\partial D} v\frac{\partial\tilde{w}}{\partial\nu}ds - \int_{\Omega\backslash\overline{D}}\left(\nabla\tilde{w}(x)\nabla v(x) - k^2\tilde{w}(x)v(x)\right)dx \\ &= \int_{\partial D_2} v\frac{\partial\tilde{G}(x-c(t))}{\partial\nu}ds + \int_{\partial D_3} v\left[\tilde{w}\widetilde{i\sigma} + \left(\frac{\partial}{\partial\nu}+\widetilde{i\sigma}\right)\tilde{G}\right]ds - \\ &\quad \int_{\Omega\backslash\overline{D}}\left(\nabla\tilde{w}(x)\nabla v(x) - k^2\tilde{w}(x)v(x)\right)dx, \end{aligned}$$

which yields

$$\begin{aligned} &\int_{\Omega\backslash\overline{D}}|w(x)|^2dx = \int_{\Omega\backslash\overline{D}}(\Delta+k^2)v\tilde{w}dx \\ =\ & \int_{\partial D_1}\tilde{G}\frac{\partial v}{\partial\nu}ds + \int_{\partial D_3}\tilde{w}\widetilde{i\sigma}vds - \int_{\Omega\backslash\overline{D}}\left(\nabla\tilde{w}(x)\cdot\nabla v(x) - k^2\tilde{w}(x)v(x)\right)dx \\ =\ & \int_{\partial D_1}\tilde{G}\frac{\partial v}{\partial\nu}ds - \int_{\partial D_2} v\frac{\partial\tilde{G}}{\partial\nu}ds - \int_{\partial D_3} v\left(\frac{\partial}{\partial\nu}+\widetilde{i\sigma}\right)\tilde{G}ds \\ =\ & \int_{\partial D_1}\tilde{G}\frac{\partial v}{\partial\nu}ds - \int_{\partial D_3} v\widetilde{i\sigma}\tilde{G}ds - v(c(t))\left[\int_{\partial D_2}+\int_{\partial D_3}\right]\frac{\partial\tilde{G}}{\partial\nu}ds - \\ & \left[\int_{\partial D_2}+\int_{\partial D_3}\right](v(x)-v(c(t)))\frac{\partial\tilde{G}(x-c(t))}{\partial\nu}ds. \end{aligned} \tag{5.6}$$

On the other hand, if $y\notin\overline{D}_j$, we have

$$\int_{\partial D_j}\frac{\partial}{\partial\nu}\tilde{G}(x-y)ds + k^2\int_{D_j}\tilde{G}(x-y)dx = \int_{D_j}(\Delta+k^2)\tilde{G}(x-y)dx = 0.$$

Therefore (5.6) leads to

$$
\begin{aligned}
& \|w\|^2_{L^2(\Omega\setminus\overline{D})} \\
= \; & \int_{\partial D_1} \tilde{G}\frac{\partial v}{\partial \nu} ds - \int_{\partial D_3} vi\widetilde{\sigma G} ds + k^2 v(c(t)) \left(\int_{D_2} + \int_{D_3} \right) \tilde{G}(x-c(t))dx - \\
& \left(\int_{\partial D_2} + \int_{\partial D_3} \right) (v(x) - v(c(t))) \frac{\partial \tilde{G}(x-c(t))}{\partial \nu} ds
\end{aligned} \tag{5.7}
$$

On one hand, (5.5) tells us

$$
\frac{|v(x)-v(y)|}{|x-y|} \le C \frac{1}{|x-y|^{1/2}} \|w\|_{L^2(\Omega\setminus\overline{D})},
$$

which implies

$$
\left| \int_{\partial D_1} \tilde{G}\frac{\partial v}{\partial \nu} ds \right| \le C \|w\|_{L^2(\Omega\setminus\overline{D})} \int_{\partial D_1} |\tilde{G}(x-c(t))| \frac{1}{|x-c(t)|^{1/2}} ds \le C \|w\|_{L^2(\Omega\setminus\overline{D})}
$$

as $c(t) \longrightarrow \partial D_1$. On the other hand, the integrals $\int_{\partial D} |\tilde{G}(x-c(t))|ds$, $\int_{\partial D} |x - c(t)|^{1/2} |\frac{\partial}{\partial \nu}\tilde{G}(x-c(t))|ds$ and $\int_D |\tilde{G}(x-c(t))|dx$ are bounded as $c(t) \longrightarrow \partial D$. Therefore by (5.5) and (5.7), we have

$$
\|w\|^2_{L^2(\Omega\setminus\overline{D})} \le C \|w\|_{L^2(\Omega\setminus\overline{D})}.
$$

The proof is complete. □

Theorem 5.2 *Assume $x_0 \in \partial D$ and $c(t) \in (\Omega \setminus \overline{D}) \cap \partial B(x_0, \delta)$ for some $\delta > 0$, where $B(x_0,\delta)$ is an open ball centered at x_0 with radius δ, then there exists some constant $C > 0$ such that for δ small enough the following estimates hold:*

$$
\int_D |\nabla G(x-c(t))|^2 dx \ge \frac{C}{\delta}, \qquad \int_D |G(x-c(t))|^2 dx \le C,
$$

$$
\int_{\partial D} |G(x-c(t))|^2 ds \le C|\ln \delta|, \quad \int_{\partial D} |w(x,c(t))|^2 ds \le C \int_{\partial D} |G(x-c(t))|^2 ds,
$$

where D should be D_j for $j = 1,2,3$ in the former three estimates and for $j = 2,3$ in the forth estimate, the constants $C > 0$ may be different.

Proof: Except the fourth estimate, the proofs of the estimates are given in [2]. For the readers convenience we repeat them. Denote the tangent plane of ∂D at point x_0 by $T(x_0, \partial D)$. From the expressions of Green's function, we have

$$
|\nabla G(x-c(t))|^2 = O\left(\frac{1}{|x-c(t)|^4} \right), \quad |G(x-c(t))|^2 = O\left(\frac{1}{|x-c(t)|^2} \right). \tag{5.8}
$$

We have that, for δ small enough,

$$\begin{aligned}\int_D \frac{1}{|x-c(t)|^4}|dx &\geq \int_{D\cap B(x_0,\delta)} \frac{1}{|x-c(t)|^4}|dx \geq \int_{D\cap B(x_0,\delta)} \frac{1}{(2\delta)^4}dx \\ &= \frac{1}{(2\delta)^4}\int_{D\cap B(x_0,\delta)} dx \geq \frac{1}{(2\delta)^4}\frac{1}{4}\int_{B(x_0,\delta)} dx = \frac{C}{\delta}. \qquad (5.9)\end{aligned}$$

Hence we have obtained the first estimate. The second estimate is obvious.

On the other hand, let $c'(t) \in \Omega \setminus \overline{D}$ satisfy

$$c'(t) \in \partial B(x_0,\delta), \quad c'(t) - x_0 \text{ is perpendicular to } T(x_0, \partial D).$$

Then,

$$\int_{\partial D} |G(x-c(t))|^2 dx \leq C\left(\int_{\partial D1} + \int_{\partial D2}\right)\frac{1}{|x-c(t)|^2}dx,$$

where

$$\partial D1 := \partial D \cap \{x \in R^3 \mid |x-c'(t)|^2 \geq \frac{1}{|\ln\delta|}\},$$

$$\partial D2 := \partial D \cap \{x \in R^3 \mid |x-c'(t)|^2 \leq \frac{1}{|\ln\delta|}\}.$$

Since $\frac{1}{2}\frac{1}{|\log\delta|} - \sqrt{2} \geq 0$ for small enough δ and

$$|c(t)-c'(t)| \leq \sqrt{2}\delta, \qquad |x-c(t)| \geq \frac{1}{2}\frac{1}{|\log\delta|},$$

we have

$$\int_{\partial D1} \frac{1}{|x-c'(t)|^2}ds \leq |\ln\delta| \int_{\partial D1} ds \leq C|\ln\delta|. \qquad (5.10)$$

For the second integral, since

$$\partial D2' = \{x \in R^3 \mid x \in T(x_0,\partial D), |x-x_0|^2 \leq \frac{1}{|\ln\delta|^2} - \delta^2\}$$

approximates $\partial D2$ for small $\delta > 0$, we know that

$$\begin{aligned}\int_{\partial D2} \frac{1}{|x-c(t)|^2}ds &\leq 2\int_{\partial D2'} \frac{1}{|x-c(t)|^2}ds = \int_{\partial D2'} \frac{1}{|x-x_0|^2+\delta^2}ds \\ &= 2\int_0^{2\pi}\int_0^{\sqrt{|\ln\delta|^{-2}-\delta^2}} \frac{rdrd\theta}{r^2+\delta^2} \\ &= 4\pi(|\ln\delta| - \ln(|\ln\delta)|) \leq C|\ln\delta| \qquad (5.11)\end{aligned}$$

for $\delta > 0$ small enough. Then the third estimate follows from (5.10)-(5.11).

The fourth estimate will be given in the next section. The proof is complete. □

6 Singularity Analysis

6.1 Expression of $\Lambda_{\partial D,\sigma} - \Lambda_{0,0}$

Here we prove (3.6). Let $v(x) \in H^1(\Omega \setminus \overline{D})$. From the definition of the weak solution of u_n to (3.3), we have

$$0 = \int_{\partial\Omega} v \frac{\partial u_n}{\partial \nu_1} ds + \int_{\partial D} v \frac{\partial u_n}{\partial \nu_1} ds - \int_{\Omega\setminus\overline{D}} (\nabla u_n \nabla v - k^2 v u_n) dx.$$

Hence, reminding the boundary condition of u_n, we get

$$\int_{\partial\Omega} v \frac{\partial u_n}{\partial \nu_1} ds = \int_{\Omega\setminus\overline{D}} (\nabla u_n \nabla v - k^2 v u_n) dx + \int_{\partial D_1} \frac{\partial u_n}{\partial \nu} v ds - \int_{\partial D_3} i\sigma u_n v ds. \tag{6.1}$$

Firstly, we take $v = \tilde{w}_n$ in this expression, we get that

$$0 = \int_{\Omega\setminus\overline{D}} (\nabla u_n \nabla \tilde{w}_n - k^2 \tilde{w}_n u_n) dx + \int_{\partial D_1} \frac{\partial u_n}{\partial \nu} \tilde{w}_n ds - \int_{\partial D_3} i\sigma u_n \tilde{w}_n ds. \tag{6.2}$$

due to $w_n|_\Omega = 0$. On the other hand, by taking $v = \tilde{v}_n$ also in the above expression, we have

$$\langle \Lambda_{\partial D,\sigma} f_n, \tilde{f}_n \rangle = \int_{\Omega\setminus\overline{D}} (\nabla u_n \nabla \tilde{v}_n - k^2 \tilde{v}_n u_n) dx + \int_{\partial D_1} \frac{\partial u_n}{\partial \nu} \tilde{v}_n ds - \int_{\partial D_3} i\sigma u_n \tilde{v}_n ds. \tag{6.3}$$

Analogously, we have

$$\langle \Lambda_{0,0} f_n, \tilde{f}_n \rangle = \int_{\partial\Omega} \frac{\partial v_n}{\partial \nu_1} \tilde{v}_n ds = \int_{\Omega} (\nabla v_n \nabla \tilde{v}_n - k^2 v_n \tilde{v}_n) dx. \tag{6.4}$$

From the above expression, we get that

$$\begin{aligned} \overbrace{\langle (\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n, \tilde{f}_n \rangle}^{\widetilde{\quad}} = &\int_{\Omega\setminus\overline{D}} (\nabla \tilde{w}_n \nabla v_n - k^2 \tilde{w}_n v_n) dx + \int_{\partial D_1} \frac{\partial \tilde{u}_n}{\partial \nu} v_n ds - \\ &\int_{D} (|\nabla v_n|^2 - k^2 |v_n|^2) dx - \int_{\partial D_3} \widetilde{i\sigma} \tilde{u}_n v_n ds. \end{aligned} \tag{6.5}$$

Remind $w_n = u_n - v_n$ and consider

$$\begin{aligned} J_n := &\int_{\Omega\setminus\overline{D}} (\nabla w_n \nabla \tilde{w}_n - k^2 w_n \tilde{w}_n) dx - \int_{\partial D_3} i\sigma w_n \tilde{w}_n ds + \int_{\partial D_1} \tilde{w}_n \frac{\partial w_n}{\partial \nu} ds \\ = &- \int_{\Omega\setminus\overline{D}} (\nabla v_n \nabla \tilde{w}_n - k^2 v_n \tilde{w}_n) dx + \int_{\partial D_3} i\sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds. \end{aligned}$$

Then, we get

$$\int_{\Omega\setminus\overline{D}} (\nabla v_n \nabla \tilde{w}_n - k^2 v_n \tilde{w}_n) dx = -J_n + \int_{\partial D_3} i\sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds. \tag{6.6}$$

Inserting this expression into (6.5), and reminding (6.2), (6.3) leads to

$$
\begin{aligned}
&\langle \overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n}^{\sim}, f_n\rangle \\
&= -J_n + \int_{\partial D_3} i\sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx + \\
&\int_{\partial D_1} \frac{\partial \tilde{u}_n}{\partial \nu} v_n ds - \int_{\partial D_3} \widetilde{i\sigma} \tilde{u}_n v_n ds \\
&= -\int_{\Omega\backslash\overline{D}} (\nabla w_n \nabla \tilde{w}_n - k^2 w_n \tilde{w}_n) dx + \int_{\partial D_3} i\sigma w_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial w_n}{\partial \nu} ds + \\
&\int_{\partial D_3} i\sigma v_n \tilde{w}_n ds - \int_{\partial D_1} \tilde{w}_n \frac{\partial v_n}{\partial \nu} ds - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx + \\
&\int_{\partial D_1} \frac{\partial (\tilde{v}_n + \tilde{w}_n)}{\partial \nu} v_n ds - \int_{\partial D_3} \widetilde{i\sigma} (\tilde{v}_n + \tilde{w}_n) v_n ds \\
&= -\int_{\Omega\backslash\overline{D}} (|\nabla w_n|^2 - k^2 |w_n|^2) dx - \int_D (|\nabla v_n|^2 - k^2 |v_n|^2) dx + \\
&\int_{\partial D_3} \left(2 i\sigma v_n \tilde{w}_n + i\sigma |v_n|^2 + i\sigma |w_n|^2\right) ds + \\
&\int_{\partial D_1} \left[\left(v_n \frac{\partial \tilde{w}_n}{\partial \nu} - \tilde{w}_n \frac{\partial v_n}{\partial \nu} \right) + \left(v_n \frac{\partial \tilde{v}_n}{\partial \nu} - \tilde{w}_n \frac{\partial w_n}{\partial \nu} \right) \right] ds. \qquad (6.7)
\end{aligned}
$$

This expression will be used to identify ∂D_1 from ∂D. Now we prove the other expression (3.7) for $\langle \overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n}^{\sim}, f_n\rangle$, which applies the value of v_n and w_n on ∂D_2 and ∂D_3. This expression will be applied to distinguish ∂D_2 and ∂D_3 from ∂D furthermore.

By a straightforward calculation, we get

$$
\begin{aligned}
\int_{\Omega\backslash\overline{D}} |\nabla (u_n - v_n)|^2 dx = &\int_{\partial\Omega} u_n \frac{\partial (\tilde{u}_n - \tilde{v}_n)}{\partial \nu} ds - \int_{\partial D} u_n \frac{\partial (\tilde{u}_n - \tilde{v}_n)}{\partial \nu} ds + \\
&\int_{\Omega\backslash\overline{D}} k^2 u_n (\tilde{u}_n - \tilde{v}_n) dx - \int_{\Omega\backslash\overline{D}} \nabla v_n . \nabla (\tilde{u}_n - \tilde{v}_n) dx.
\end{aligned}
$$

From the definition of the Dirichlet-to-Neumann map, this generates

$$
\begin{aligned}
&\int_{\Omega\backslash\overline{D}} |\nabla w_n|^2 dx = \langle \overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0})f_n}^{\sim}, f_n\rangle - \int_{\partial D} u_n \frac{\partial (\tilde{u}_n - \tilde{v}_n)}{\partial \nu} ds + \\
&\int_{\Omega\backslash\overline{D}} k^2 u_n (\tilde{u}_n - \tilde{v}_n) dx - \int_{\Omega\backslash\overline{D}} \nabla v_n . \nabla \tilde{u}_n dx + \int_{\Omega\backslash\overline{D}} |\nabla v_n|^2 dx. \qquad (6.8)
\end{aligned}
$$

Applying $\nabla v_n \cdot \nabla \tilde{u}_n = \nabla \cdot (\tilde{u}_n \nabla v_n) - \tilde{u}_n \Delta v_n$ in this expression, we get

$$\begin{aligned} &- \int_{\Omega\setminus\overline{D}} \nabla v_n . \nabla \tilde{u}_n dx + \int_{\Omega\setminus\overline{D}} |\nabla v_n|^2 dx = - \int_{\partial\Omega} \tilde{u}_n \frac{\partial v_n}{\partial \nu} ds + \int_{\partial D} \tilde{u}_n \frac{\partial v_n}{\partial \nu} ds \\ &- k^2 \int_{\Omega\setminus\overline{D}} \tilde{u}_n v_n dx + \int_{\Omega} |\nabla v_n|^2 dx - \int_{D} |\nabla v_n|^2 dx. \end{aligned} \tag{6.9}$$

Also by applying $|\nabla v_n|^2 = \nabla \cdot (\tilde{v}_n \nabla v_n) - \tilde{v}_n \Delta v_n$ and noticing $w_n = u_n - v_n = 0$ on $\partial\Omega$ in (6.9), we get

$$\begin{aligned} - \int_{\Omega\setminus\overline{D}} \nabla v_n \cdot \nabla \tilde{u}_n dx + \int_{\Omega\setminus\overline{D}} |\nabla v_n|^2 dx = \int_{\partial D} \tilde{u}_n \frac{\partial v_n}{\partial \nu} ds - k^2 \int_{\Omega\setminus\overline{D}} \tilde{u}_n v_n dx - \\ \int_{\Omega} \tilde{v}_n \Delta v_n dx - \int_{D} |\nabla v_n|^2 dx. \end{aligned} \tag{6.10}$$

Now inserting (6.10) into (6.8) says

$$\begin{aligned} &\int_{\Omega\setminus\overline{D}} |\nabla w_n|^2 dx \\ &= \langle (\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) \tilde{f}_n, f_n \rangle + \int_{\partial D} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} - u_n \frac{\partial (\tilde{u}_n - \tilde{v}_n)}{\partial \nu} \right) ds + k^2 \int_{D} |v_n|^2 dx + \\ &k^2 \int_{\Omega\setminus\overline{D}} [u_n(\tilde{u}_n - \tilde{v}_n) - \tilde{u}_n v_n + v_n \tilde{v}_n] dx - \int_{D} |\nabla v_n|^2 dx, \end{aligned} \tag{6.11}$$

that is,

$$\begin{aligned} &\langle \widetilde{\overbrace{(\Lambda_{\partial D,\sigma} - \Lambda_{0,0}) f_n}}, f_n \rangle \\ &= \int_{\Omega\setminus\overline{D}} \left(|\nabla w_n|^2 - k^2 |w_n|^2 \right) dx + \int_{D} \left(|\nabla v_n|^2 - k^2 |v_n|^2 \right) dx - \\ &\quad \int_{\partial D} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} - u_n \frac{\partial (\tilde{u}_n - \tilde{v}_n)}{\partial \nu} \right) ds \\ &= \int_{\Omega\setminus\overline{D}} \left(|\nabla w_n|^2 - k^2 |w_n|^2 \right) dx + \int_{D} \left(|\nabla v_n|^2 - k^2 |v_n|^2 \right) dx - \\ &\quad \int_{\partial D_2} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} \right) ds - \int_{\partial D_3} \left(\tilde{u}_n \frac{\partial v_n}{\partial \nu} + u_n \frac{\partial \tilde{v}_n}{\partial \nu} - i\sigma |u_n|^2 \right) ds. \end{aligned} \tag{6.12}$$

6.2 Estimate for w

Here we prove the fourth estimate in Theorem 5.2.

For given needle $c \in \Omega \setminus \overline{D}$, put $x_0 = c(t) \in \Omega \setminus \overline{D}$ and let $a \in \partial D$ be the point at which the needle c first hits ∂D. Suppose x_0 is very near to a. Consider two families of functions $\{w(\cdot,x_0)\}$, $\{z(\cdot,x_0)\}$ depending on x_0 in some function space X. We denote by $w(\cdot,x_0)\sim z(\cdot,x_0)$ in X if $\{w(\cdot,x_0)-z(\cdot,x_0)\}$ is a bounded set in X for x_0 very near to a.

Let $G_0(x-x_0)=\frac{1}{4\pi|x-x_0|}$. Then it is easy to see that

$$(\partial_\nu+i\sigma)G(x-x_0)\sim(\partial_\nu+i\sigma)G_0(x-x_0)$$

in $L^2(\partial D)$, hence

$$w(\cdot,x_0)\sim w_0(\cdot,x_0)\qquad \text{in } H^1(\Omega\setminus\overline{D}), \tag{6.13}$$

where $w=w(\cdot,x_0)\in H^1(\Omega\setminus\overline{D})$ is the solution to (3.5) and $w_0=w_0(\cdot,x_0)\in H^1(\Omega\setminus\overline{D})$ is the solution to

$$\begin{cases} \Delta w_0+k^2w_0=0, & \text{in } \Omega\setminus\overline{D}\\ \left(\frac{\partial}{\partial\nu}+i\sigma\right)w_0=-\left(\frac{\partial}{\partial\nu}+i\sigma\right)G_0(\cdot-x_0), & \text{on } \partial D\\ w_0=0, & \text{on } \partial\Omega. \end{cases} \tag{6.14}$$

By the Sobolev embedding $H^{1/2}(\partial D)\hookrightarrow L^r(\partial D)$ with $2\le r\le 4$ and the Hölder inequality, for any $q(\frac{4}{3}\le q\le 2)$, there exists a constant $C>0$ such that

$$\begin{aligned} \left|\int_{\partial D} ib\sigma(x)G_0(x-x_0)\phi ds\right| &\le \|i\sigma G_0(\cdot-x_0)\|_{L^q(\partial D)}\|\phi\|_{L^r(\partial D)}\\ &\le C\|i\sigma G_0(\cdot-x_0)\|_{L^q(\partial D)}\|\phi\|_{H^{1/2}(\partial D)} \end{aligned} \tag{6.15}$$

for $\phi\in H^{1/2}(\partial D)$, where $\frac{1}{r}=1-\frac{1}{q}$ with $\frac{4}{3}\le q<2$.

Hence $i\sigma G_0(\cdot-x_0)\sim 0$ in $H^{-1/2}(\partial D)$, and by the well-posedness of our boundary value problem, this implies

$$w_0(\cdot,x_0)\sim w_1(\cdot,x_0)\qquad \text{in } H^1(\Omega\setminus\overline{D}), \tag{6.16}$$

where $w_1=w_1(\cdot,x_0)\in H^1(\Omega\setminus\overline{D})$ is the solution to

$$\begin{cases} \Delta w_1+k^2w_1=0, & \text{in } \Omega\setminus\overline{D}\\ \left(\frac{\partial}{\partial\nu}+i\sigma(x)\right)w_1=-\frac{\partial}{\partial\nu}G_0(\cdot-x_0), & \text{on } \partial D\\ w_1=0, & \text{on } \partial\Omega. \end{cases} \tag{6.17}$$

Now consider the solution $w_2=w_2(\cdot,x_0)\in H^1(\Omega\setminus\overline{D})$ to

$$\begin{cases} \Delta w_2=0, & \text{in } \Omega\setminus\overline{D}\\ \left(\frac{\partial}{\partial\nu}+i\sigma(x)\right)w_2=-\frac{\partial}{\partial\nu}G_0(\cdot-x_0), & \text{on } \partial D\\ w_2=0, & \text{on } \partial\Omega. \end{cases} \tag{6.18}$$

For this problem, we have

Claim 1: $i\sigma(x)w_2(\cdot,x_0)\sim 0$ in $H^{-1/2}(\partial D)$, $\quad w_2(x,x_0)\sim 0$ in $H^1(\Omega\setminus\overline{D})$.

Proof The proof given here also gives a more precise estimate for w_2, which will be used in the sequel.

Let $y = (y_1, y_2, y_3) = (y_1(x, x_0), y_2(x, x_0), y_3(x, x_0))$ be a boundary normal coordinates near point a such that

$$y(a) = 0, \qquad J(x) := \frac{\partial(y(x, x_0))}{\partial x} = I(\text{identity matrix})$$

at $x = x_0$ and $D_0 = \{y_1 < 0\}$ locally near point a. Also, let

$$A(x) := |J(x)|^{-1} J(x)(J(x))^T, \qquad x(y(x, x_0); x_0) = x,$$

$$\underline{A}(y) := A(x(y; x_0)), \qquad \underline{u}(y) := u(x(y; x_0)).$$

Then it is easy to see

(1) $\underline{A}(y) \in C^1$ near $y = 0$;

(2) $\Delta u = 0$ near point $a \iff \nabla.\underline{A}\nabla.\underline{u} = 0$ near 0;

(3) $\delta(x(y; x_0) - x_0) = \delta(y - y_0)$;

(4) $\partial_\nu = \partial_{y_1}$.

In order to simplify the description of our argument, from now on we extend $x(y; x_0)$ and $\underline{A}(y)$ to an open ball $V \subset R^3$ centered at $y = 0$ without destroying their regularities and positivity of $\underline{A}(y)$. By a direct estimate, we can easily see

$$\underline{G}_0(y; y_0) \sim G_0(y - y_0) \qquad \text{in } H^1(V),$$

where we have adopted the convention $y_0 = y(x_0; x_0)$.

Now consider the solution $\underline{w}_2^0 \in H^1(R^3_+)$ to

$$\begin{cases} \Delta \underline{w}_2^0 = 0, & \text{in } y_1 > 0 \\ \partial_{y_1} \underline{w}_2^0 = -\partial_{y_1} G_0(y - y_0) & \text{on } y_1 = 0 \end{cases} \tag{6.19}$$

and put $\underline{w}_2(y) := w_2(x(y, x_0))$. If we can prove

Claim 2: $\nabla \cdot (\underline{A}(y) - \underline{A}(y_0))\nabla \underline{w}_2^0 \sim 0$ in $(H_0^1(V \cap R^3_+))^*$.

Then we have

$$\underline{w}_2 \sim \underline{w}_2^0 \qquad \text{in } H^1(V \cap R^3_+) \tag{6.20}$$

by observing

$$\begin{cases} \nabla \cdot \nabla \underline{A} \nabla(\underline{w}_2 - \underline{w}_2^0) = -\nabla \cdot \nabla(\underline{A}(y) - \underline{A}(y_0))\nabla \underline{w}_2^0, & \text{in } V \cap R^3_+ \\ \partial_{y_1}(\underline{w}_2 - \underline{w}_2^0) = -\partial_{y_1} \underline{G}_0(y, y_0) + \partial_{y_1} G_0(y - y_0), & \text{on } y_1 = 0. \end{cases} \tag{6.21}$$

Proof for Claim 2 will be given in Section 6.3. Therein we also give a precise expression for $\widetilde{w}_2^0(y)$. Then, this expression and (6.20)imply Claim 1. The proof is complete.

Now we can see that

$$w_1(\cdot, x_0) \sim w_2(\cdot, x_0) \qquad \text{in } H^1(\Omega \setminus \overline{D}) \tag{6.22}$$

from Claim 1 and the well-posedness of our boundary value problem.

Now summing up (6.13), (6.16), (6.20) and (6.22), as well as the expression of $\widetilde{w}_2^0(y)$ given in the next subsection, we have

$$\int_{\partial D} |w(x,x_0)|^2 ds \le C\left(\int_{\partial D} |G(x-c(t))|^2 ds + 1\right),$$

which completes the proof of the fourth estimate in Theorem 5.2.

6.3 Proof for Claim 2

Proof: Let $y_0 = (y_{01}, y_{02}, y_{03}) = (y_{01}, y_0')$. Then it is well known that $H(y) = H(y; y_0) = G_0(y - y_0)$ can be given by

$$H(y) = \begin{cases} H_+(y) = H_+(y; y_0), & \text{in } y_1 > y_{01} \\ H_-(y) = H_-(y; y_0), & \text{in } y_1 < y_{01} \end{cases} \tag{6.23}$$

with the solution $H_\pm(y)$ to

$$\begin{cases} \Delta H_\pm(y) = 0, & \text{in } \pm(y_1 - y_{01}) > 0 \\ H_+(y)|_{y_1 = y_{01}+0} = H_-(y)|_{y_1 = y_{01}-0} \\ \partial_{y_1} H_+(y)|_{y_1=y_{01}+0} - \partial_{y_1} H_-(y)|_{y_1 = y_{01}-0} = -\delta(y' - y_0'). \end{cases} \tag{6.24}$$

Denote by $\Gamma_\pm(y_1, \eta')$ and $w(y_1, \eta')$ the Fourier transforms of $\hat{H}_\pm(y)$ and $\widetilde{w}_2^0(y)$ with respect to y', respectively. Then, $\Gamma_\pm' := e^{iy_0'\cdot\eta'}\Gamma_\pm$ and $w' := e^{iy_0'\cdot\eta'} w$ satisfy

$$\begin{cases} \left(\partial_{y_1}^2 - |\eta'|^2\right)\Gamma_\pm' = 0, & \text{in } \pm(y_1 - y_{01}) > 0 \\ \Gamma_+'|_{y_1 = y_{01}+0} = \Gamma_-'|_{y_1 = y_{01}-0} \\ \partial_{y_1}\Gamma_+'|_{y_1 = y_{01}+0} - \partial_{y_1}\Gamma_-'|_{y_1=y_{01}-0} = -1 \end{cases} \tag{6.25}$$

and

$$\begin{cases} \left(\partial_{y_1}^2 - |\eta'|^2\right) w' = 0, & \text{in } y_1 > 0 \\ \partial_{y_1} w' = -\partial_{y_1}\Gamma_-', & \text{on } y_1 = 0 \end{cases} \tag{6.26}$$

respectively. $\Gamma_\pm' = \Gamma_\pm'(y_1)$ is given by

$$\Gamma_\pm'(y_1) = 2^{-1}|\eta'|^{-1} e^{\mp(y_1 - y_{01})|\eta'|}.$$

Hence $w' = w'(y_1) = 2^{-1}|\eta'|^{-1}e^{-(y_1+y_{01})|\eta'|}$. Comparing these two formula, we have

$$\widetilde{w}_2^0(y) = H_+(y_1, y'; -y_{01}, y_0') = \frac{1}{4\pi\sqrt{(y_1+y_{01})^2 + |y'-y_0'|^2}}.$$

This completes the proof of Claim 2. □

Acknowledgments: The authors think the referees for their kind comments and suggestions, which led to an improved version of this paper. The first author is supported by NSF of China(No.19971016). The second author is partly supported by the Science Foundation of Southeast University(No.9207011148, No.3007011043). The third author is partly supported by Grand-in-Aid for scientific research, Ministry of Education, Science and Culture, Japan(No.12640153).

References

[1] J. Cheng, J.J. Liu, G .Nakamura, Reconstruction and uniqueness of an inverse scattering problem with impedance boundary, Ser.A, Vol.45, No.11, 1408-1419, 2002.

[2] J. Cheng, J.J. Liu, G. Nakamura, Recovery of the shape of an obstacle and the boundary impedance from the far-field pattern, to appear in J. Math. Kyto University.

[3] D.L. Colton, A. Kirsch, A simple method for solving inverse scattering problems in the resonance region, Inverse Problems 12 (1996), 383–393.

[4] D.L. Colton, A. Kirsch, The determination of the surface impedance of an obstacle from measurements of the far field pattern, SIAM J. Appl. Math. 41 (1981), 8–15.

[5] D.L. Colton, R. Kress, Integral Equation Methods in Scattering Theory, John Wiley & Sons, Inc., New York (1983).

[6] D.L.Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Springer-Verlag, Berlin (1998).

[7] M.A. Epton, B. Dembart, Multipole translation theory for the 3-D Laplace and Helmholtz equation, SIAM, J. Sci. Comput. 16 (1995), 865–897.

[8] M. Ikehata, Reconstruction of an obstacle from the scattering amplitude at a fixed frequency, Inverse Problems 14 (1998), 949–954.

[9] M. Ikehata, Reconstruction of the shape of the inclusion by boundary measurements, Commun. in PDE 23 (1998), 1459–1474.

[10] M. Ikehata, Reconstruction of obstacles from buondary measurements, Wave Motion 30 (1999), 949–954.

[11] M. Ikehata, G. Nakamura, Slicing of a 3-D object from boundary measurements, Inverse Problems 15 (1999), 1243–1253.

[12] M. Ikehata, G. Nakamura, K. Tanuma, Identification of the shape of the inclusion in anisotropic elastic body, Appl. Anal., 72 (1999), 17–26.

[13] A.W. Kedzierawski, The determination of the surface impedance of an obstacle, Proc. Edinburgh Math. Soc. 36 (1993), 1–15.

[14] R. Kress, A. Zinn, On the numerical solution of the three-dimensional inverse obstacle scattering problem, J. Comput. Appl. Math. 42 (1992), 49–61.

[15] R. Kress, The numerical solution of a hypersingular integral equation in scatterin theory, J. Comput. Appl. Math., Vol.61, No.3, 345-360, 1995.

[16] J.J. Liu, Inverse scattering for the problem with impedance-type boundary, J. Partial Differential Equations 13 (2000), 279–288.

[17] J.J. Liu, Recovery of boundary impedance coefficient in 2-D media, Chinese Journal of Numerical Mathematics and Applications 23 (2001), 99–112.

[18] L. Monch, A Newton method for solving the inverse scattering problem for a sound-hard obstacle, Inverse Problems 12 (1996), 309–323.

[19] R. Potthast, Stability estimates and reconstructions in inverse acoustic scattering scattering using singular sources, J. Comput. Appl. Maths.114 (2000), 247–274.

[20] R.T. Smith, An inverse acoustic scattering problem for an obstacle with an impedance boundary condition. J. Math. Anal. Appl. 105 (1985), 333–356.